Module 02 – Extra Class

Basic Probability

Nguyen Quoc Thai
MSc in Computer Science

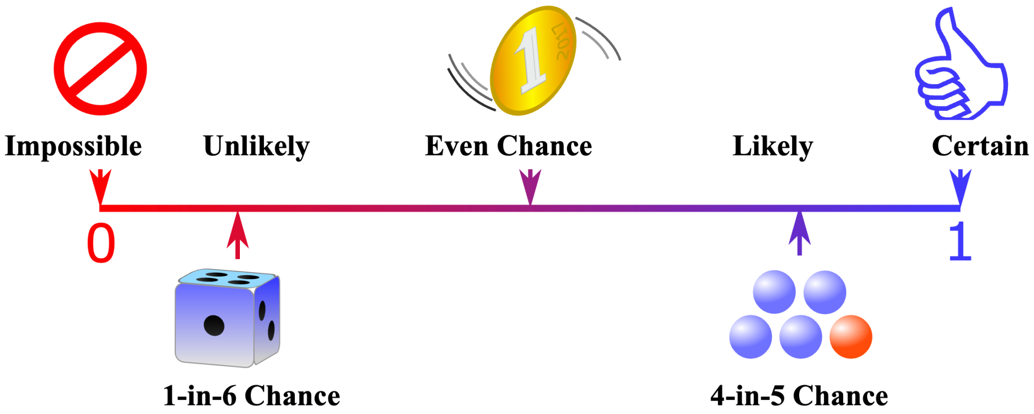
Objectives

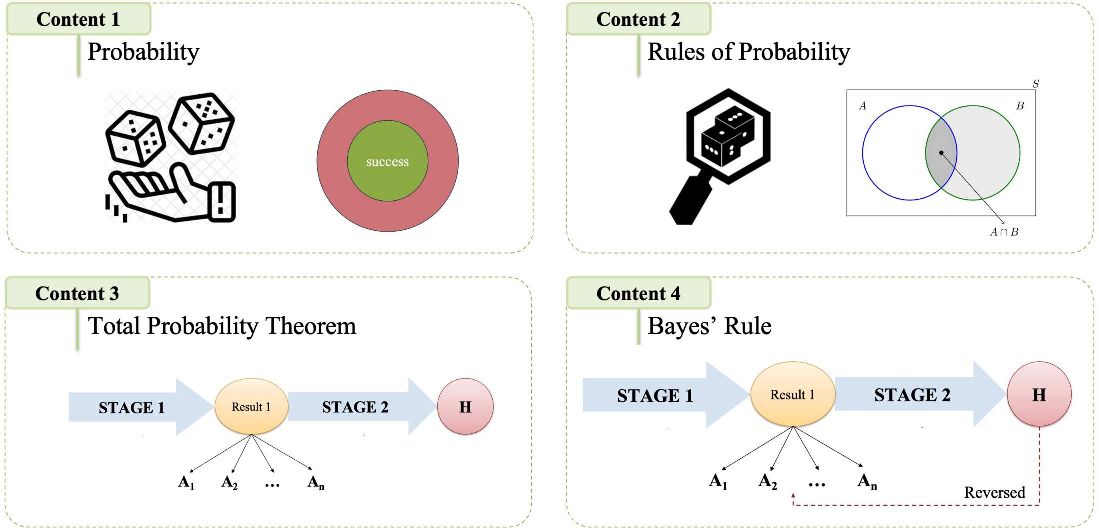
Introduction

- Experiment
- Event
- Operations on Events
- Relations of Events

Probability

- Definition
- Rule of Probability
- Total Probability
- Bayes’ Rule

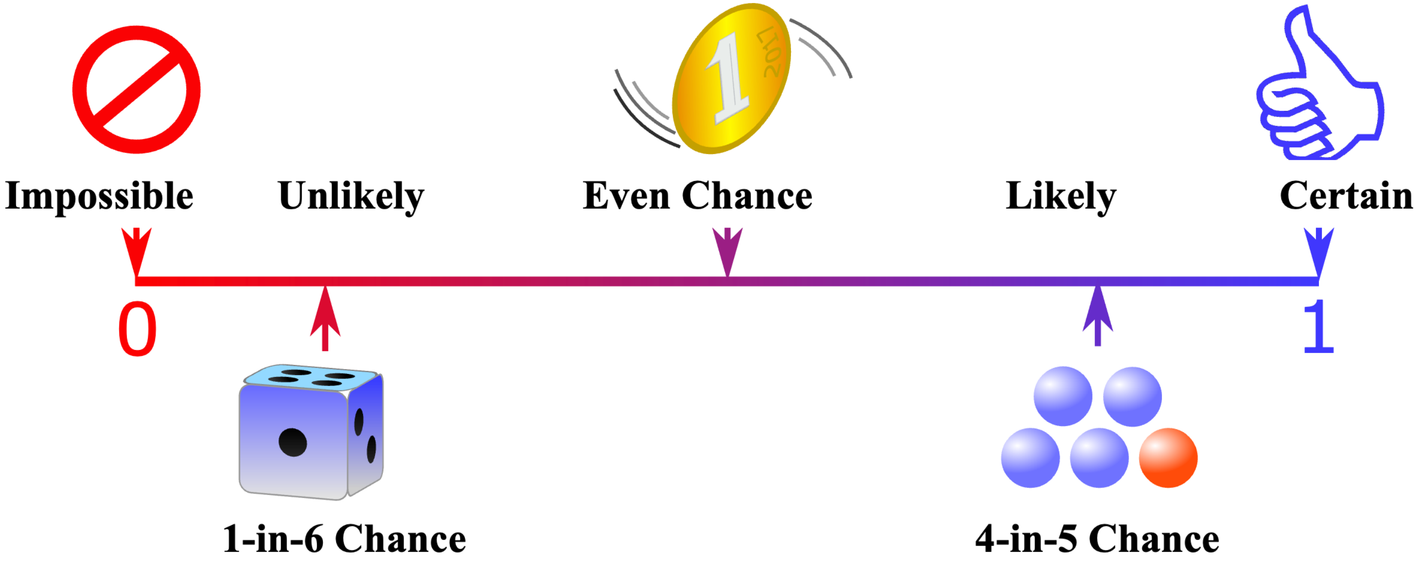




Outline

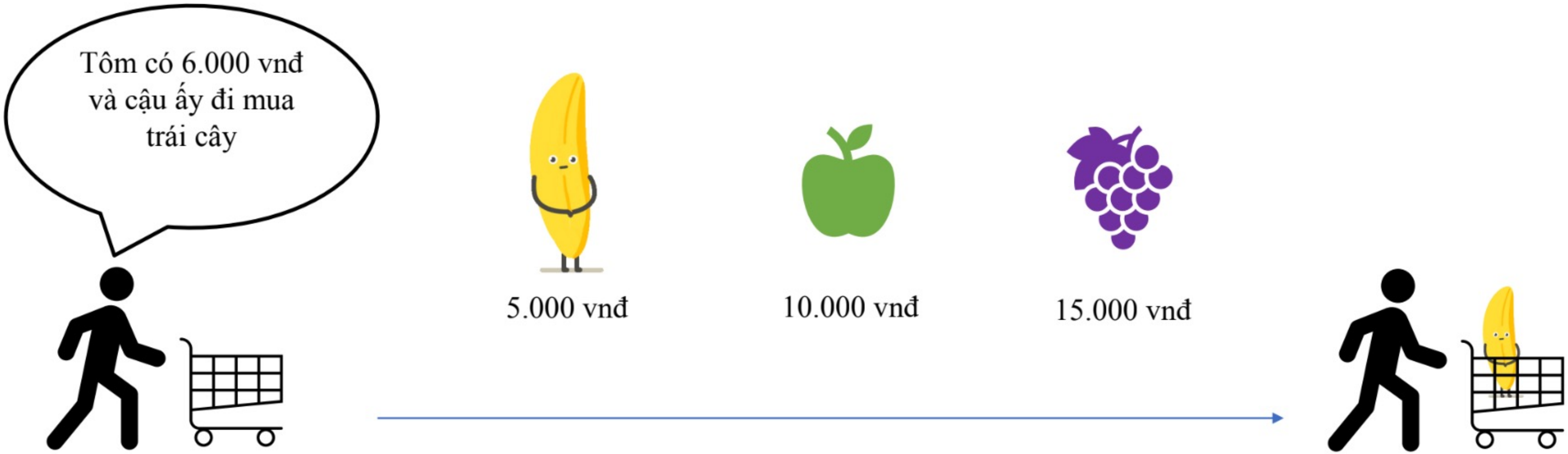
SECTION 1
Introduction

SECTION 2
Probability



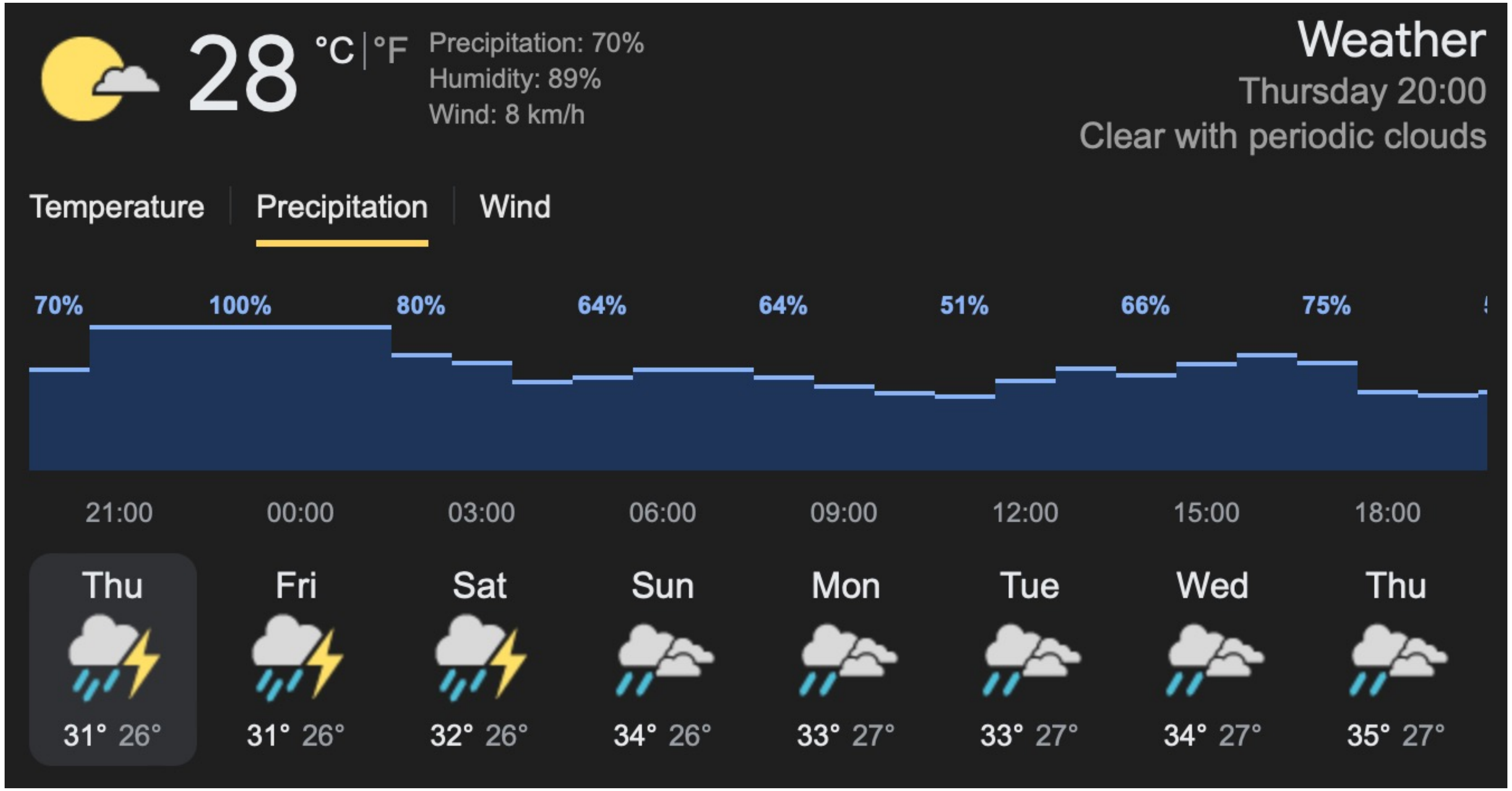
Introduction

Probability



Introduction

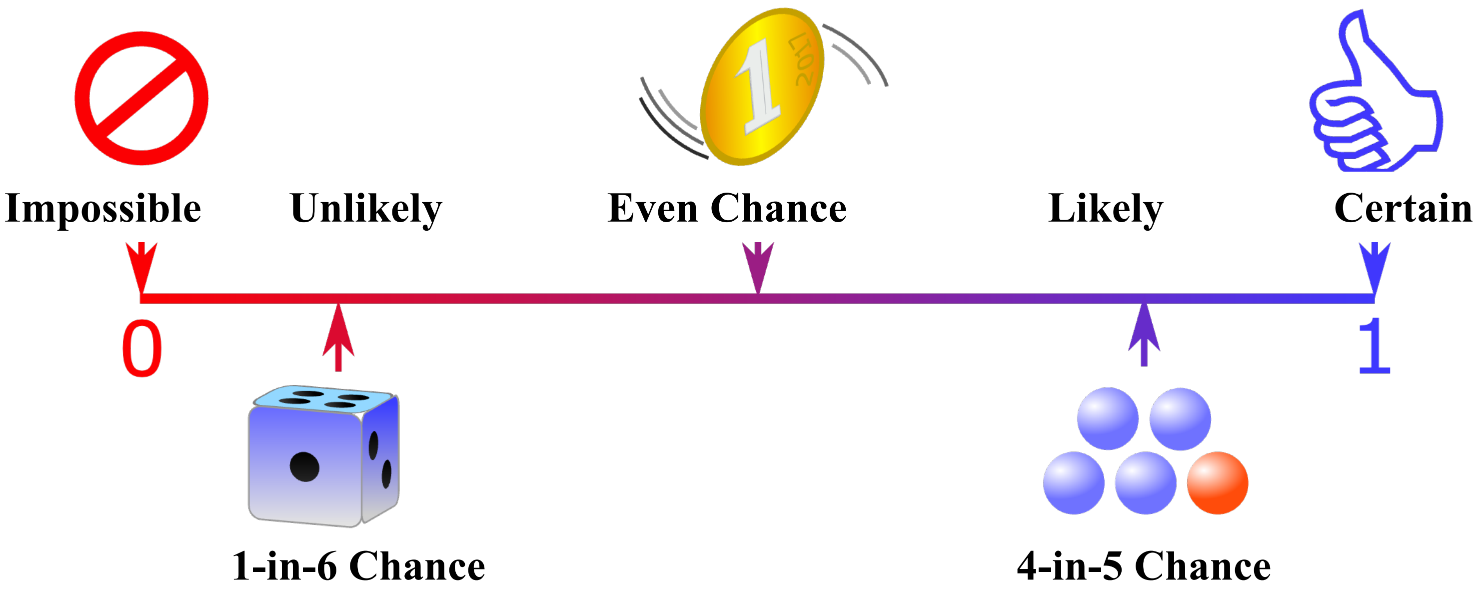
Probability



Introduction

Introduction

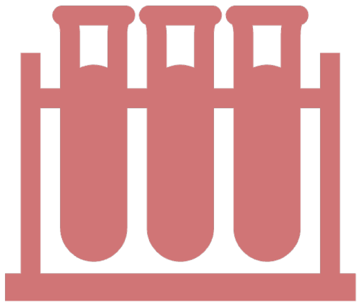
- Measure to the likelihood of an event occurring

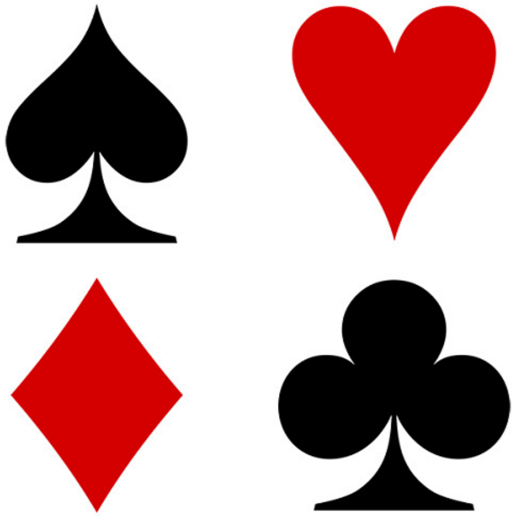


Introduction

Experiment & Event

- Experiment: implementation of set of basic conditions for observing a certain phenomenon
- An outcome: a result of an experiment
- A sample space: the set of all possible outcomes
- An event: a subset of the sample space





Introduction

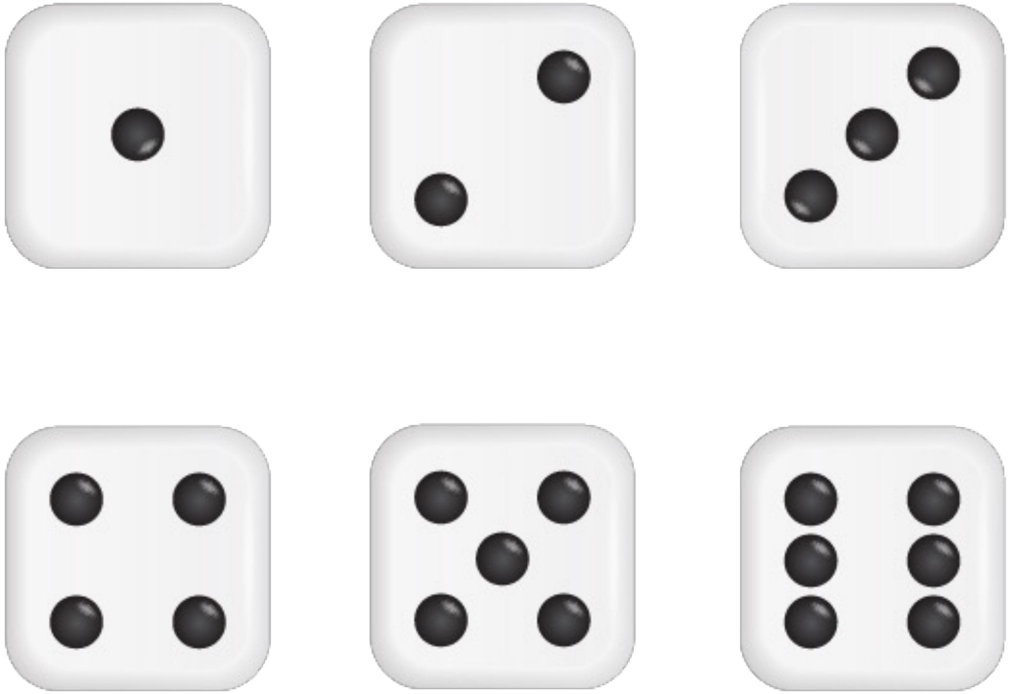
Experiment & Event

Toss a coin

Sample space: S = {*heads, tails*}

Roll a dice

Sample space: S = {1, 2, 3, 4, 5, 6}

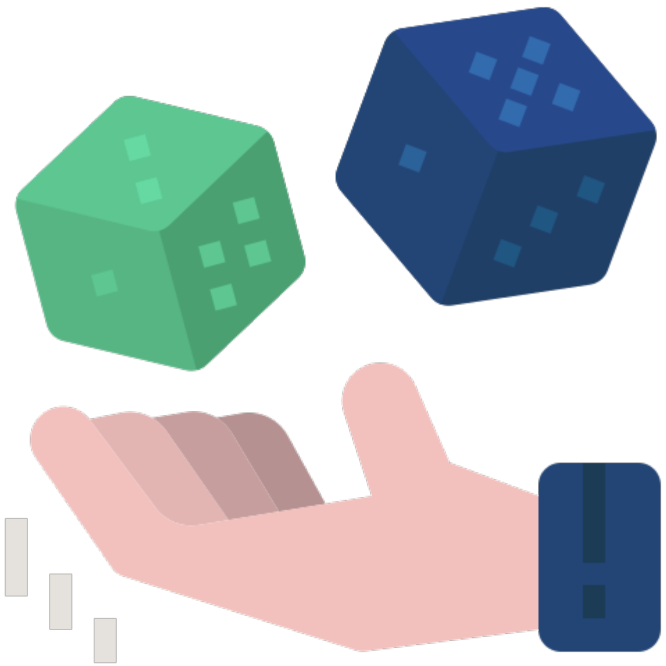


Introduction

Relation of Events

Assume A and B: two events in the same experiment

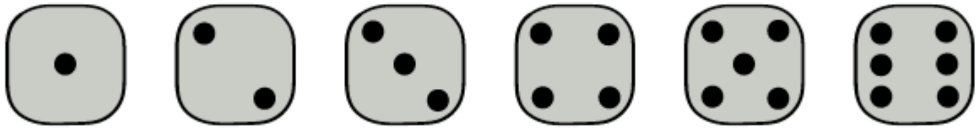
- Implication
  - "event A implies event B": if event A occurs, then event B occurs
  - $A \Rightarrow B$ means that $A \subseteq B$
- Equivalent
  - "event A equal event B": if $A \Rightarrow B$ and $B \Rightarrow A$
  - $A \Leftrightarrow B$



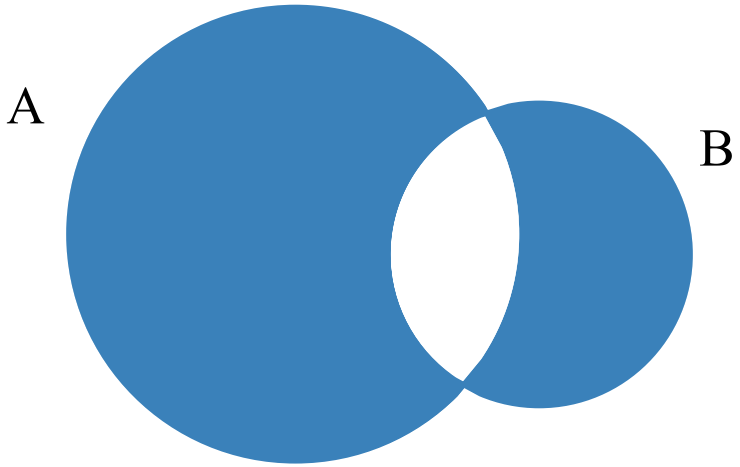
Introduction

Operations on Events

- **Intersection of events (A ∩ B)**
  - In the experiment of rolling a single die



  - Event A: “the number rolled is even”
  - Event B: “the number rolled is divisible by 3”
  - Find the intersection of A and B?



Introduction

Operations on Events

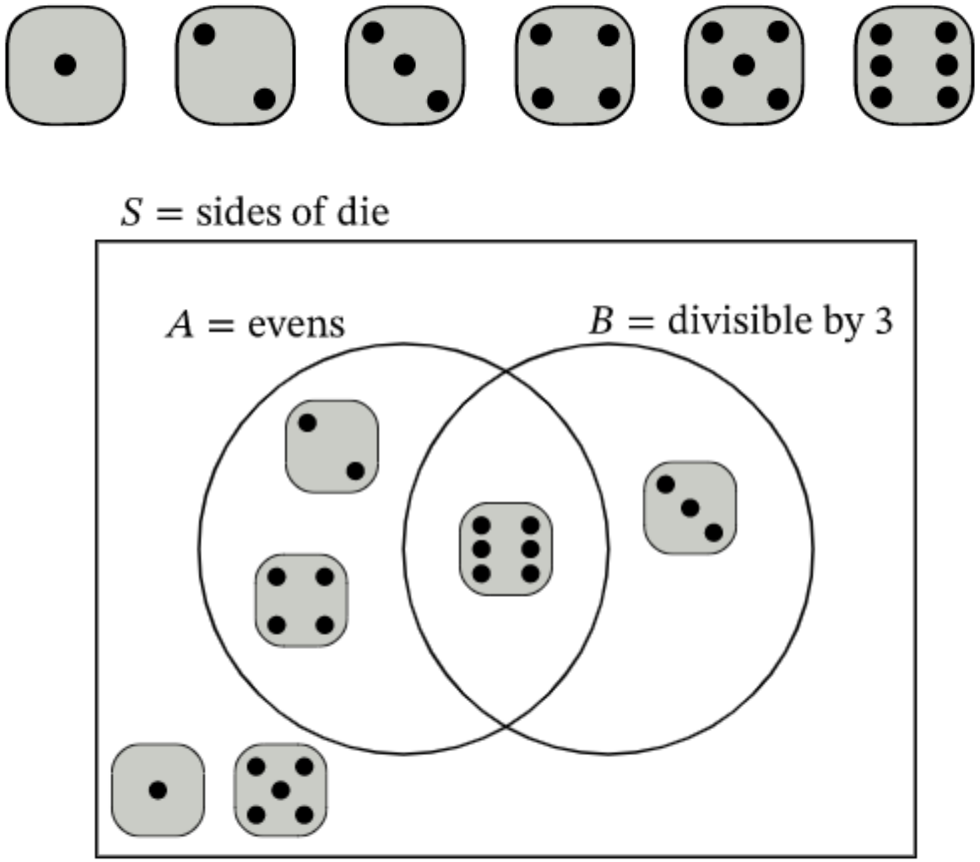
- **Intersection of events (A ∩ B)**
  - In the experiment of rolling a single die
  - Event A: "the number rolled is even"

=> A = {2, 4, 6}

  - Event B: "the number rolled is divisible by 3"

=> B = {3, 6 }

  - Find the intersection of A and B?



Introduction

Operations on Events

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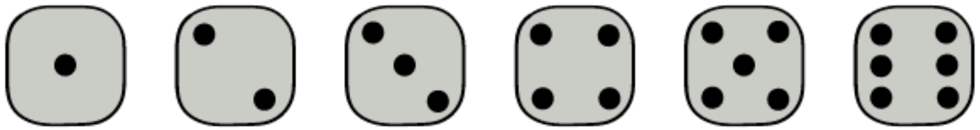
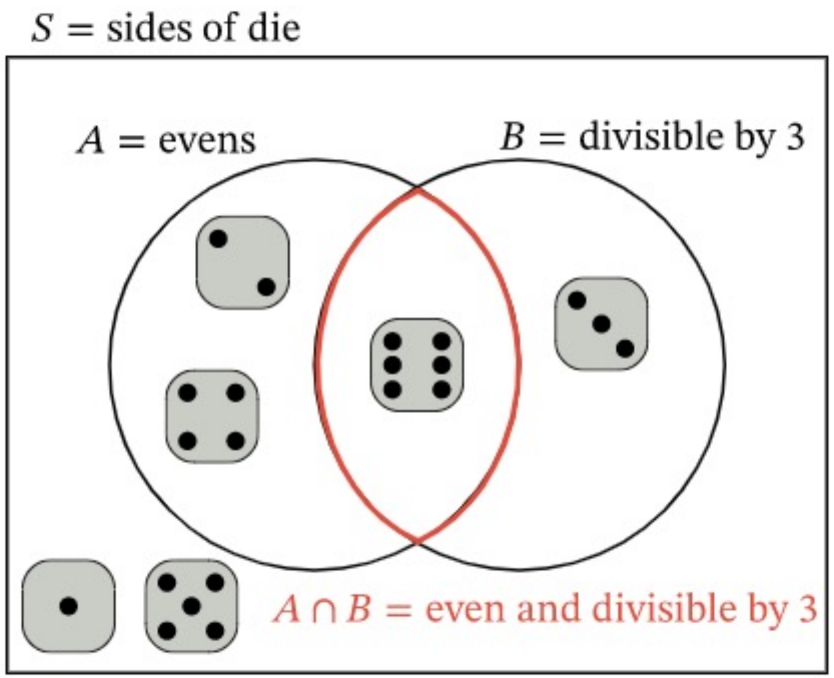
=> B = {3, 6 }

  - A ∩ B = {6}

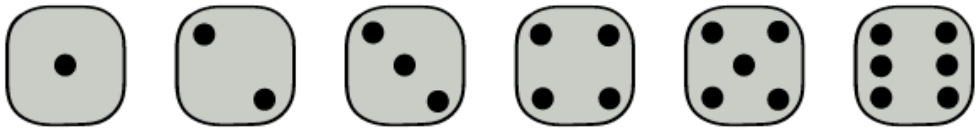


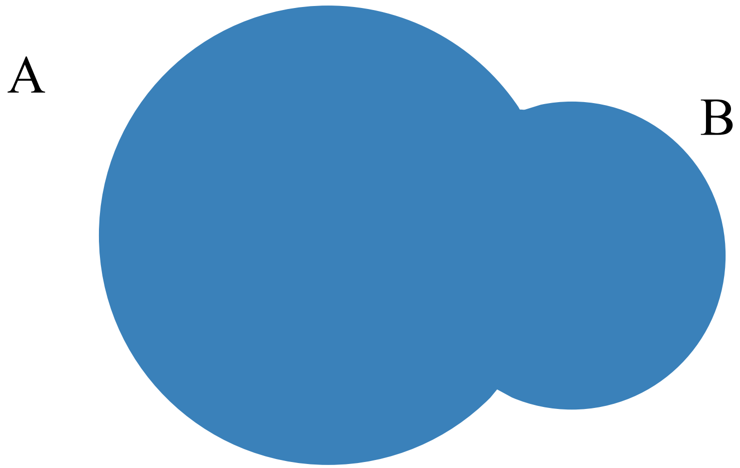


Introduction

Operations on Events

- **Union of events (A ∪ B)**
  - In the experiment of rolling a single die
  - Event A: “the number rolled is even”
  - Event B: “the number rolled is divisible by 3”
  - Find the union of A and B?





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Operations on Events

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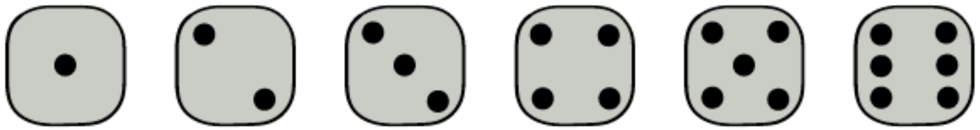
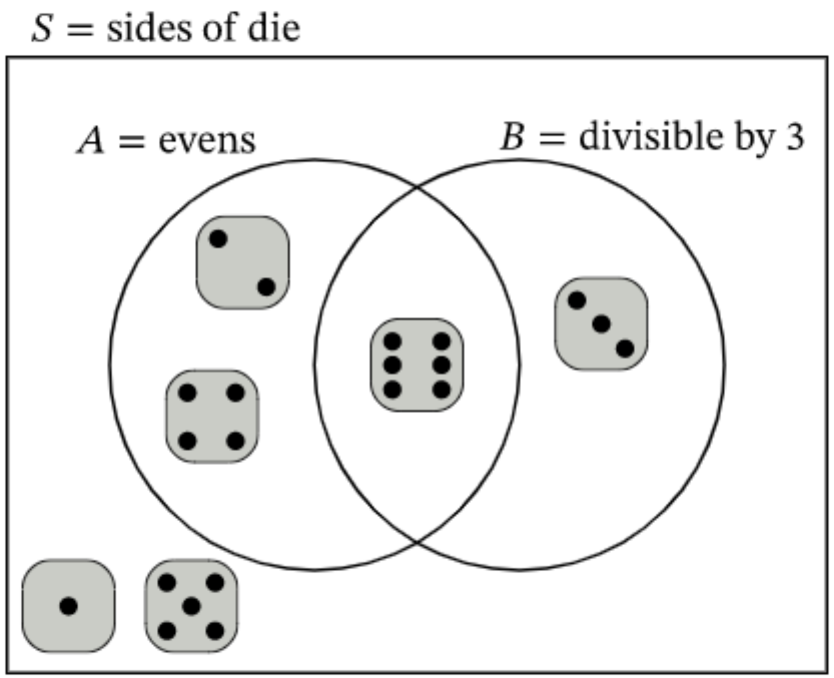
  - Find the union of A and B?





Introduction

Operations on Events

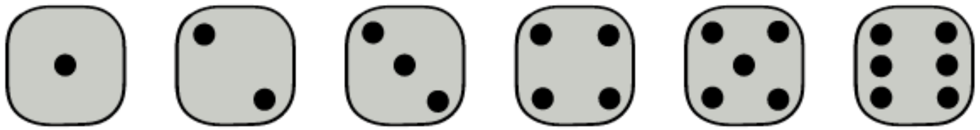
- **Union of events (A ∪ B)**
  - In the experiment of rolling a single die
  - Event A: “the number rolled is even”

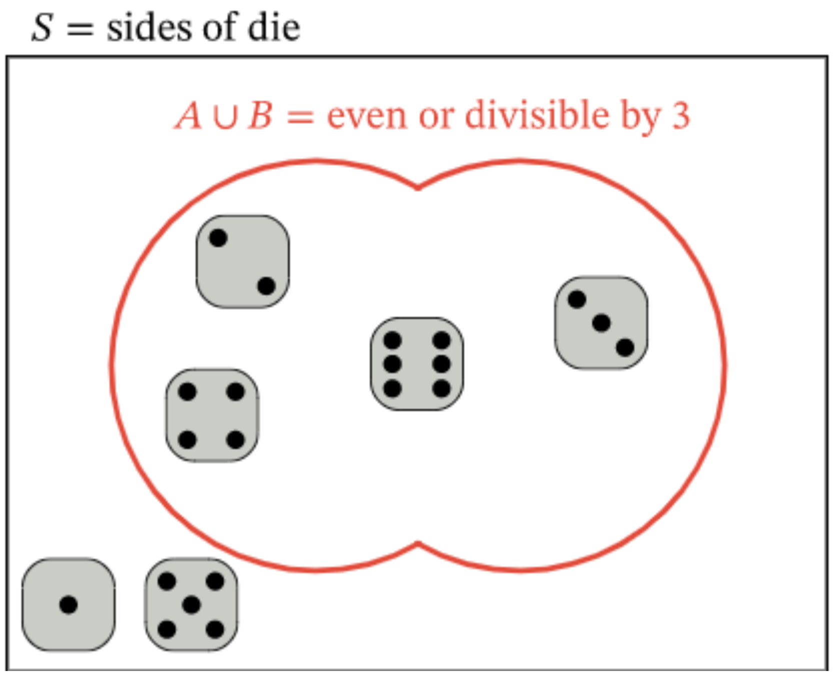
=> A = {2, 4, 6}

  - Event B: “the number rolled is divisible by 3”

=> B = {3, 6 }

  - A ∪ B = {2, 3, 4, 6}



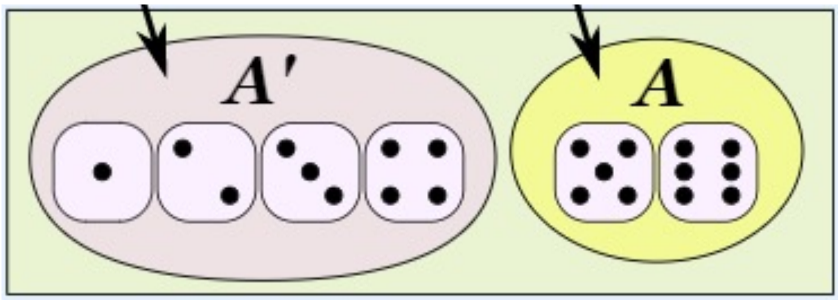


Introduction

Operations on Events

- **Complements**
  - The complement of an event A in a sample space S, denoted A’ (A^c)
  - The collection of all outcomes in S that are not elements of the set A
  - Corresponds to negating any description in words of the event A.
  - $A' \cup A = \Omega$

Complement of an event A An event A



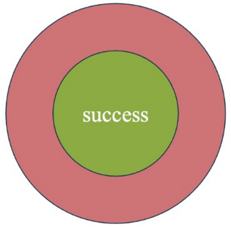
Outline

SECTION 1

Introduction

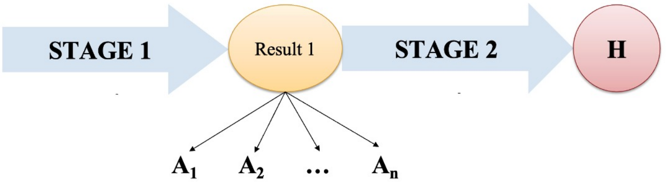
Content 1

Probability



Content 3

Total Probability Theorem

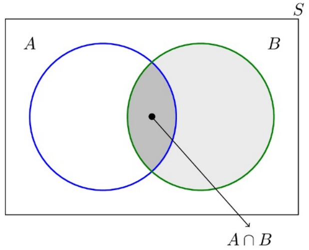


SECTION 2

Probability

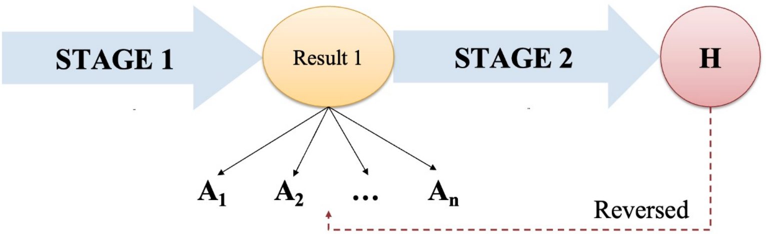
Content 2

Rules of Probability



Content 4

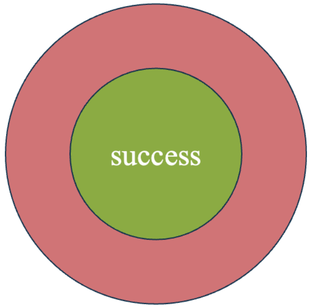
Bayes' Rule



Probability

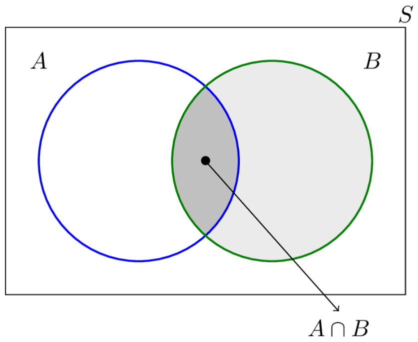
Content 1

Probability



Content 2

Rules of Probability



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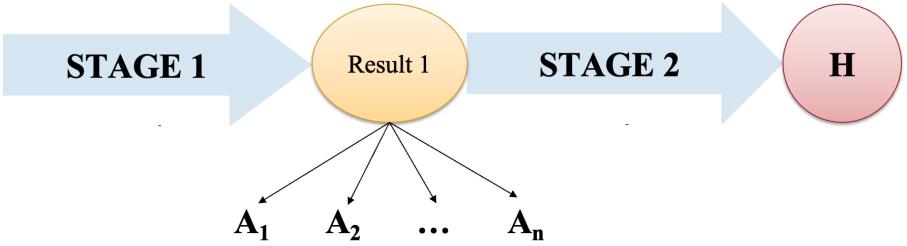
Total Probability Theorem



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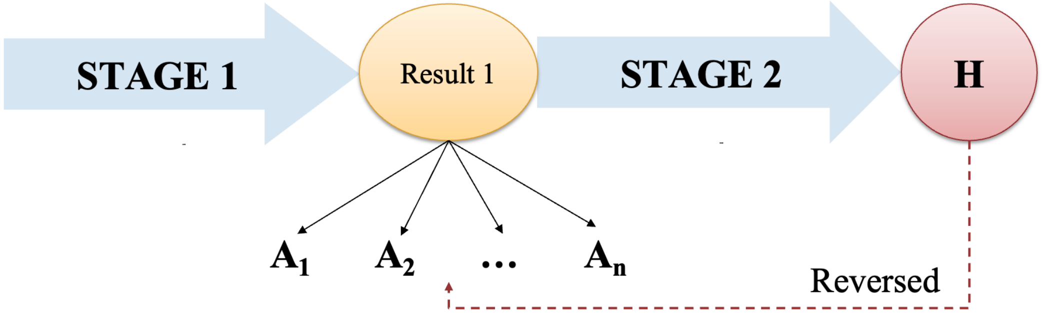
Bayes’ Rule



Probability

Classical Probability

- Measure to the likelihood of an event occurring

$$P(A) = \frac{number\ of\ favorable\ outcomes}{total\ number\ of\ possible\ outcomes} = \frac{n_A}{n_\Omega}$$

Example

What is the probability of rolling a number is even on a regular dice?

- There are 6 faces on a fair die, numbered 1 to 6 => $n(\Omega) = 6$
- A : "even number" => A = {2, 4, 6} => n(A) = 3

=> P(A) = 3/6 = 0.5

Probability

Geometric Probability

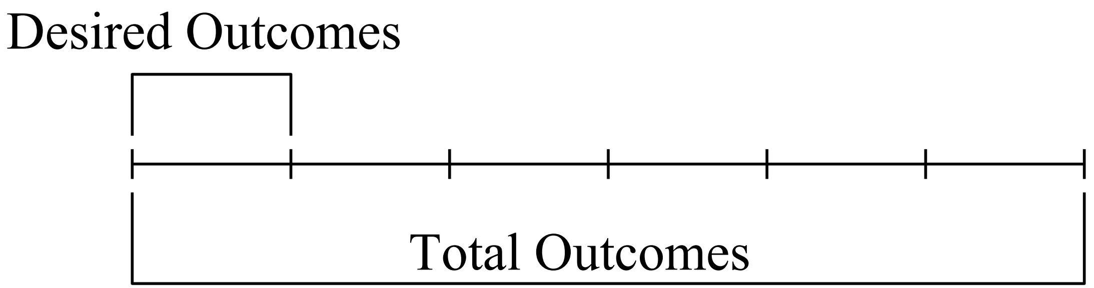
$$P(A) = \frac{measure\ of\ domain\ A}{measure\ of\ domain\ \Omega}$$

1-D Geometric probability

X is a random real number between 0 and 3. What is the probability X is closer to 0 than it is to 1?

=> A: "X is closer to 0 than to 1"

=> Measure: length in this 1D case: $P(A) = \frac{length\ of\ segment\ where\ 0<X<0.5}{length\ of\ segment\ where\ 0<X<3} = \frac{0.5}{3} = \frac{1}{6}$



Probability

Geometric Probability

$$P(A) = \frac{measure\ of\ domain\ A}{measure\ of\ domain\ \Omega}$$

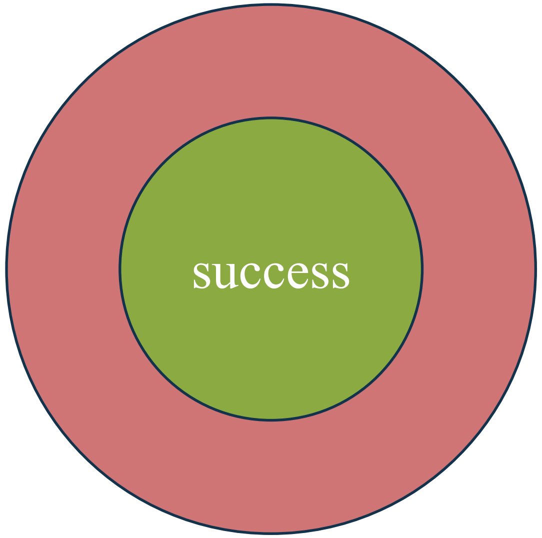
2-D Geometric probability

A dart is thrown at a circular dartboard such that it will land randomly over the area of the dartboard. What is the probability that it lands closer to the center "success" than to the edge?

=> A: "closer to center than edge"

=> Measure: area in this 2D case:

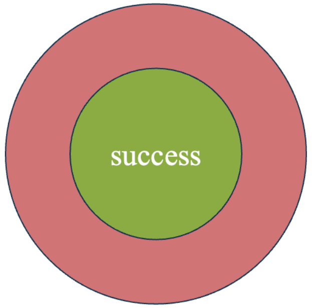
$$P(A) = \frac{area\ of\ desired\ outcomes}{area\ of\ total\ outcomes} = \frac{\frac{\pi r^2}{4}}{\pi r^2} = \frac{1}{4}$$



Probability

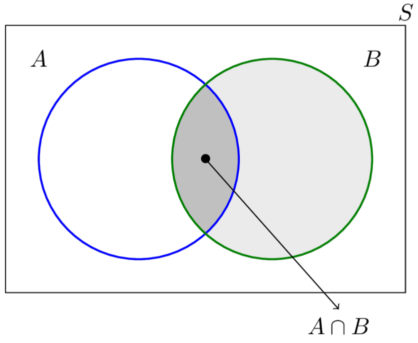
Content 1

Probability



Content 2

Rules of Probability



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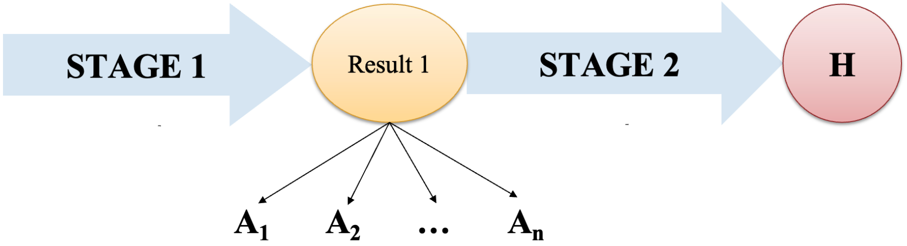

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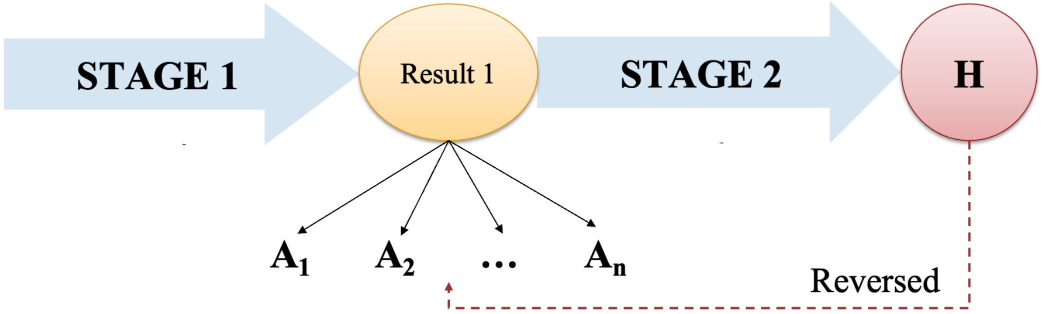


Probability

Rules of Probability

Addition

- Mutually exclusive events

$$P(A+B) = P(A) + P(B), \text{ where A and B mutually exclusive}$$

$$P(A \text{ or } B) = P(A) + P(B)$$

- In general

$$P(A+B) = P(A) + P(B) - P(AB)$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Probability

Rules of Probability

Addition

- Rolling a fair die. What is the probability of $A = \{1, 5\}$?
  - The problem states that the die is fair => all six possible outcomes are equally likely:

    $P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\})$
  - The events $\{1\},..\{6\}$ are disjoint:

    $1 = P(S) = P(\{1\}) + P(\{2\}) + \ldots + P(\{6\}) = 6P(\{1\})$

    $\Rightarrow P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = 1/6$
  - Since $\{1\}$ and $\{5\}$ are disjoint:

    $P(A) = P(\{1, 5\}) = P(\{1\}) + P(\{5\}) = 2/6 = 1/3$

Probability

Rules of Probability

Addition

- For any event A

$$P(A^c) = 1 - P(A)$$

$$P(A) = 1 - P(A^c)$$

Find the probability that when we roll a dice we get a number different than 1 and 6?

- Let's A: "Getting the number 1 and 6" => $A = \{1, 6\}$

  "Getting a number different than 1 and 6" = A^c

  Since, $P(A) = P(1) + P(6) = 1/6 + 1/6 = 2/6 = 1/3$

  P("Getting a number different than 1 and 6") = $1 - P(A) = 1 - 1/3 = 2/3$

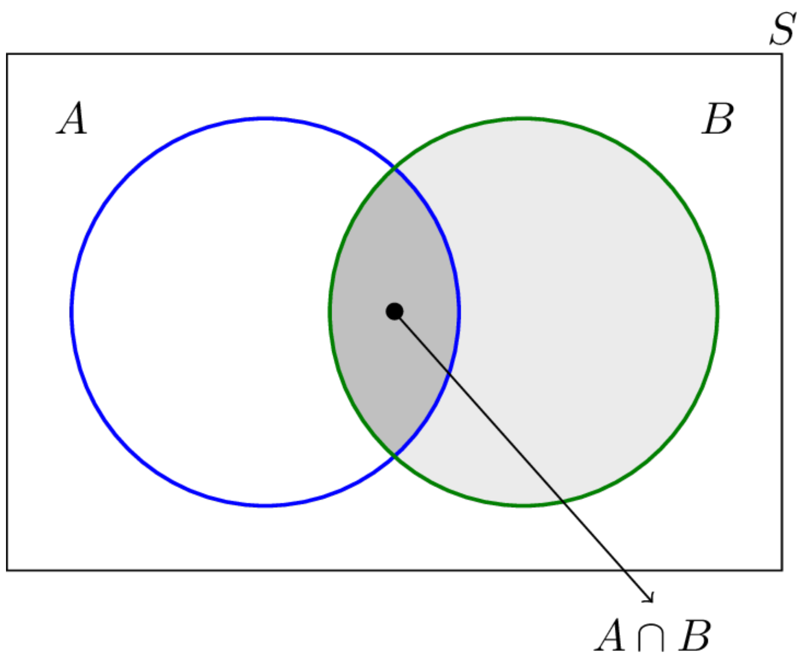
Probability

Rules of Probability

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Probability that A occurs given that B has already occurred



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Probability

Rules of Probability

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

A fair die is rolled

a) Find the probability that the number rolled is a five, given that it is odd.

b) Find the probability that the number rolled is odd, given that it is a five.

Probability

Rules of Probability

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

A fair die is rolled

- Sample space S = {1, 2, 3, 4, 5, 6}, consisting of 6 equally likely outcomes
- A: "a five is rolled" => A = {5} => P(A) = 1/6
- B: "an odd number is rolled" => B = {1, 3, 5} => P(B) = 3/6 = ½

=> A and B = {5} => P(A and B) = 1/6

a) Find the probability that the number rolled is a five, given that it is odd.

P(A|B) = P(A and B)/P(B) = (1/6)/(1/2) = 1/3

Probability

Rules of Probability

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

A fair die is rolled

- Sample space S = {1, 2, 3, 4, 5, 6}, consisting of 6 equally likely outcomes
- A: “a five is rolled” => A = {5} => P(A) = 1/6
- B: “an odd number is rolled” => B = {1, 3, 5} => P(B) = 3/6 = ½

=> A and B = {5} => P(A and B) = 1/6

b) Find the probability that the number rolled is odd, given that it is a five.

P(B|A) = P(B and A)/P(A) = P(A and B)/P(A) = (1/6)/(1/6) = 1

Probability

Rules of Probability

Multiplication

$$P(AB) = P(A).P(B|A) = P(B).P(A|B)$$
$$P(A_1A_2\ldots A_n) = P(A_1).P(A_2|A_1).P(A_3|A_1A_2)\ldots P(A_n|A_1A_2\ldots A_{n-1})$$

- In a factory there are 100 units of a certain product, 5 of which are defective. We pick three units from the 100 units at random. What is the probability that none of them are defective?

Probability

Rules of Probability

Multiplication

- Let's A_i as the event i^{th} chosen unit is not defective, for i =1, 2, 3

  => Compute $P(A_1A_2A_3)$

  - $P(A_1) = 95/100$
  - Given that the first chosen item was good, the second item will be chosen from 94 good units and 5 defective units, thus: $P(A_2|A_1) = 94/99$
  - Given that the first and second chosen items were okay, the third item will be chosen from 93 good units and 5 defective units, thus: $P(A_3|A_2A_1) = 93/98$

  => $P(A_1A_2A_3) = 95/100.94/99.93/98 = 0.8560$

Probability

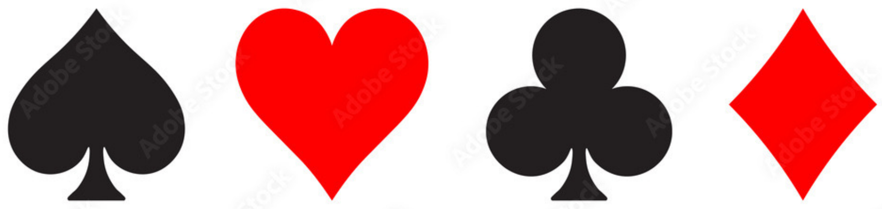
Rules of Probability – Practice

Multiplication

$$P(AB) = P(A).P(B|A) = P(B).P(A|B)$$

$$P(A_1A_2\ldots A_n) = P(A_1).P(A_2|A_1).P(A_3|A_1A_2)\ldots P(A_n|A_1A_2\ldots A_{n-1})$$

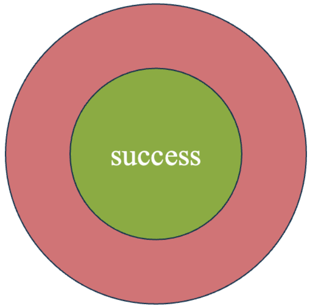
- Suppose you take out two cards from a standard pack of cards one after another, without replacing the first card. What is probability that the first card is the ace of spades, and the second card is a heart?



Probability

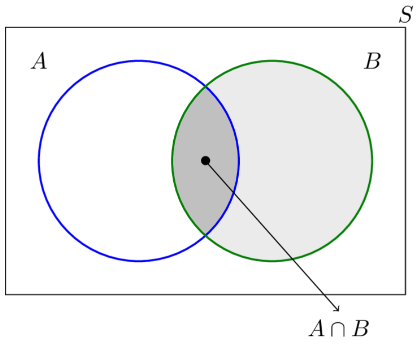
Content 1

Probability



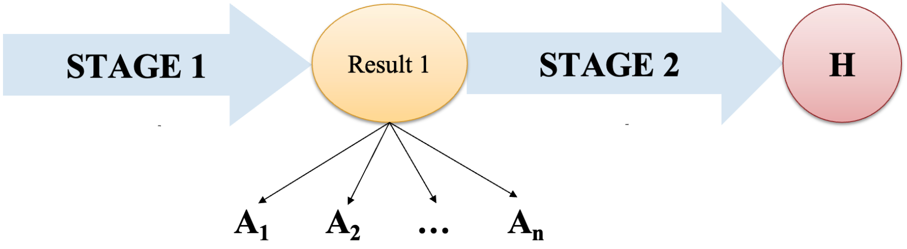
Content 2

Rules of Probability



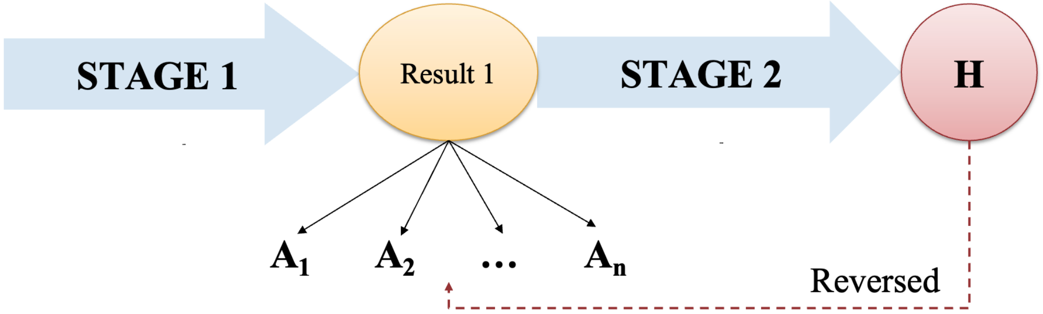
Content 3

Total Probability Theorem



Content 4

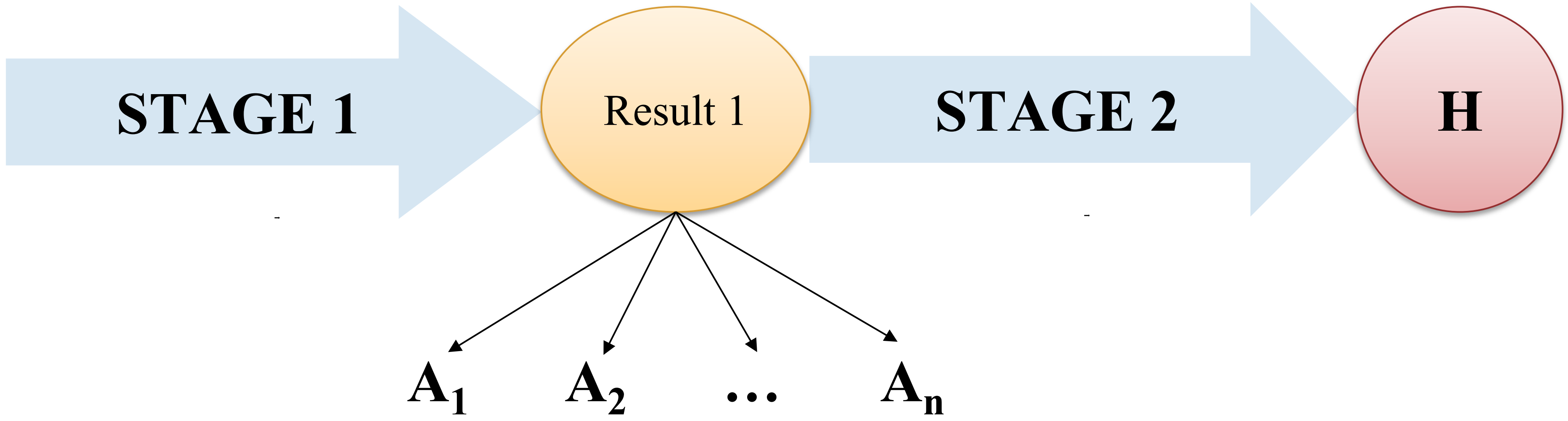
Bayes' Rule



Total Probability

Total Probability Theorem

- The result of stage 2 depends on the result of stage 1. The results of step 1 are divided into n sets A_i, each of which will contain a number of outcomes that have the same effect on the probability of H occurring.

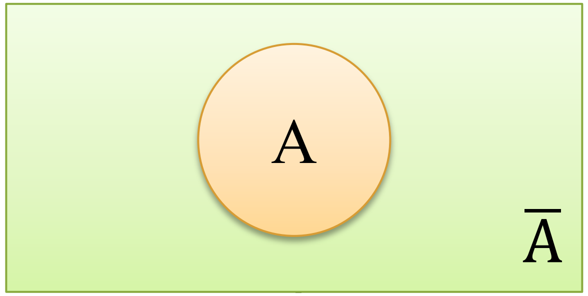


Total Probability

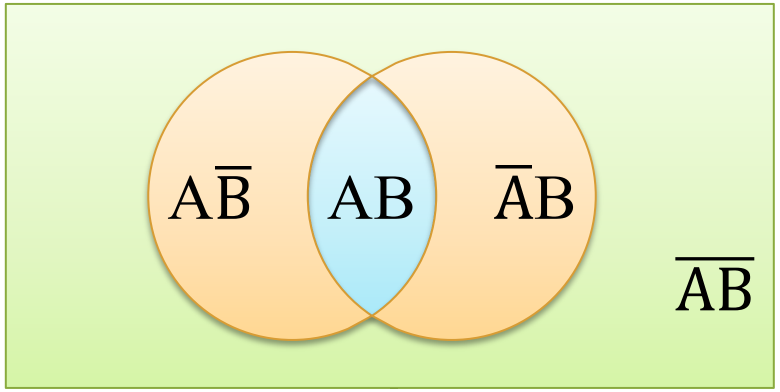
Total Probability Theorem

- Complete system of events
- Events A1, A2,…, An (n 2) of a trial is called complete system if:
  - $A_i \,.\, A_j = \emptyset, \forall\, i \neq j$
  - $\sum_{i=1}^{n} A_i = A_1 + A_2 + A_3 + \cdots + A_n = \Omega$
  - $P(A_1) + P(A_2) + \ldots + P(A_n) = 1$

$\{A, \bar{A}\}$



$\{AB, \bar{A}B, A\bar{B}, \overline{AB}\}$

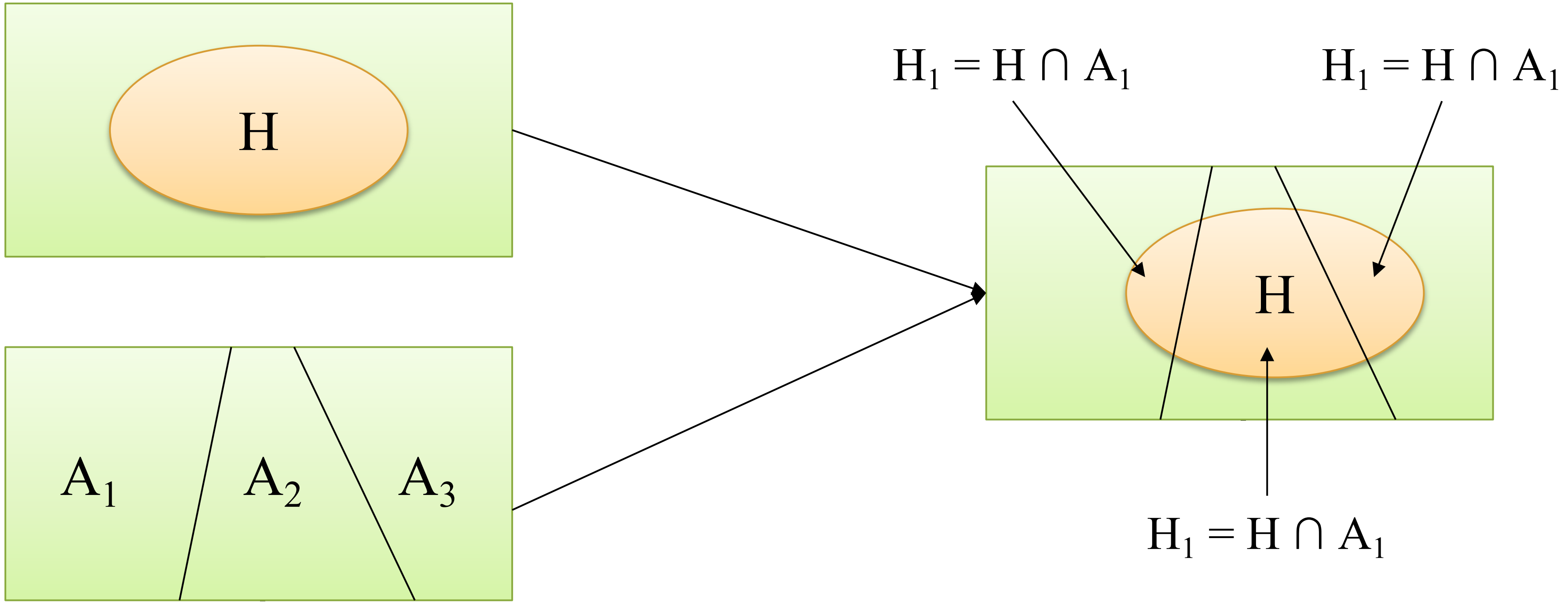


Total Probability

Total Probability

- Let A_1, A_2, …, An – complete system of events and assume. Consider any event H such that H occurs only when one of the events $A_1, A_2, \ldots, A_n$ occurred:

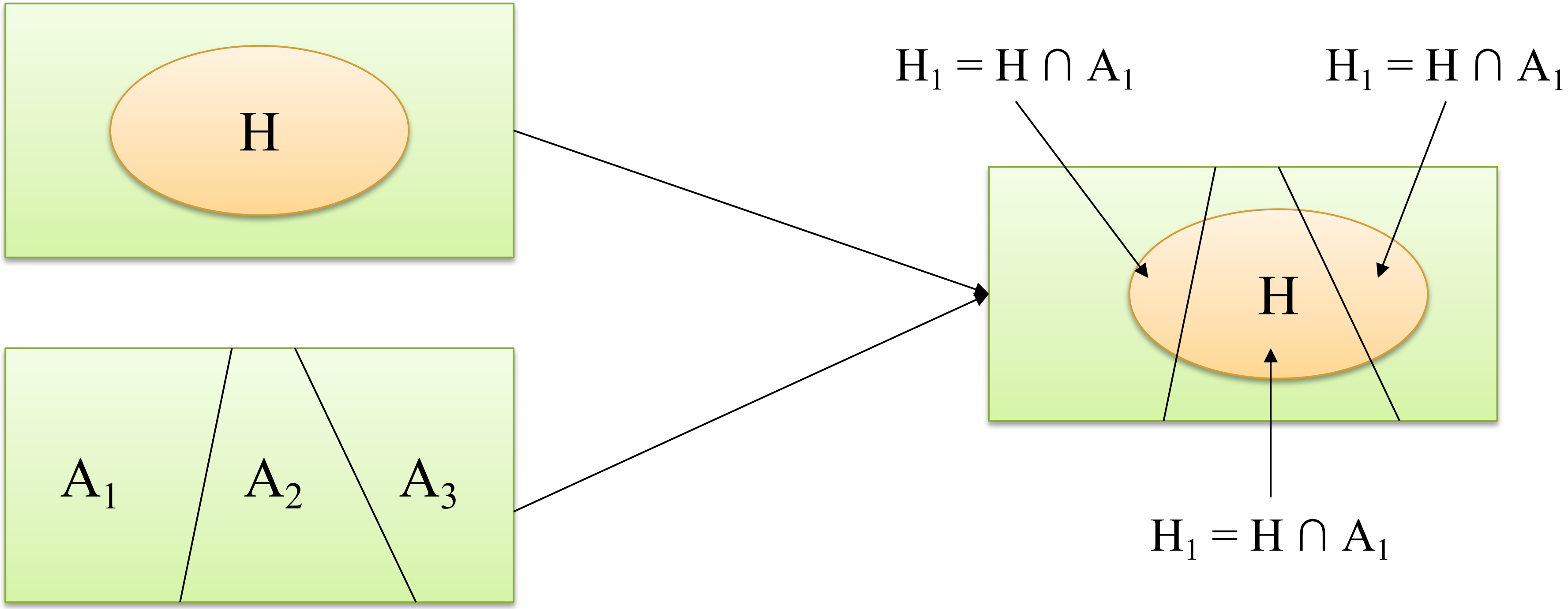
$$P(H) = P(H_1) + P(H_2) + P(H_3)$$
$$= P(A_1).P(H|A_1) + P(A_2).P(H|A_2) + P(A_3).P(H|A_3)$$



Total Probability

Total Probability

$$P(H) = \sum_{i=1}^{n} P(A_i).P(H|A_i)$$



Total Probability

Example: Widget Detection

- Company M supplies 80% of widgets for a car shop and only 1% of their widgets turn out to be defective. Company N supplies the remaining 20% of widgets for the car shop and 3% of their widgets turn out to be defective. If a customer randomly purchases a widget from the car shop, what is the probability that it will be defective?

Total Probability

Example: Widget Detection

$$P(H) = \sum_{i=1}^{n} P(A_i).P(H|A_i)$$

H: “Widget being defective”

A_M: “Widget came from company M”, A_N: “Widget came from company N”

Events A_M and A_N : complete system of events

=> $P(A_M) = 0.8$; $P(A_N) = 0.2$; $P(H|A_M) = 0.01$; $P(H|A_N) = 0.03$

The probability that it will be defective:

$P(H) = P(H|A_M).P(A_M) + P(H|A_N).P(A_N)$

$= 0.01*0.8 + 0.03*0.2$

$= 0.014$

Company M	Supplies 80% of widgets 1% are defective
Company N	Supplies 20% of widgets 3% are defective

Total Probability

Example: Pick Marble

- I have three bags that each contain 100 marbles:
  - Bag 1 has 75 red and 25 blue marbles
  - Bag 2 has 60 red and 40 blue marbles
  - Bag 3 has 45 red and 55 blue marbles.

I choose one of the bags at random and then pick a marble from the chosen bag, also at random. What is the probability that the chosen marble is red?

Total Probability

Example: Pick Marble

- I have three bags that each contain 100 marbles:
  - Bag 1 has 75 red and 25 blue marbles
  - Bag 2 has 60 red and 40 blue marbles
  - Bag 3 has 45 red and 55 blue marbles.

Total Probability

Example: Pick Marble

$$P(H) = \sum_{i=1}^{n} P(A_i).P(H|A_i)$$

R: "the event that the chosen marble is red"

B_i: "the event that I choose Bag *i*"

Events B_1 , B_2 and B_3 : complete system of events

=> $P(B_1) = 1/3$; $P(B_2) = 1/3$, $P(B_3) = 1/3$; $P(R|B_1) = 0.75$; $P(R|B_2) = 0.60$; $P(R|B_3) = 0.45$

The probability that the chosen marble is red:

$P(R) = P(R|B_1).P(B_1) + P(R|B_2).P(B_2) + P(R|B_3).P(B_3)$

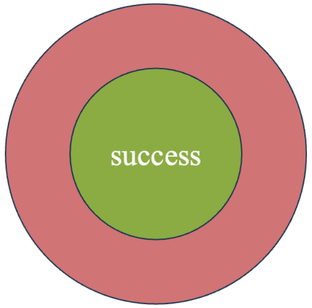
$= 0.75 * 1/3 + 0.60 * 1/3 + 0.45 * 1/3$

$= 0.60$

Probability

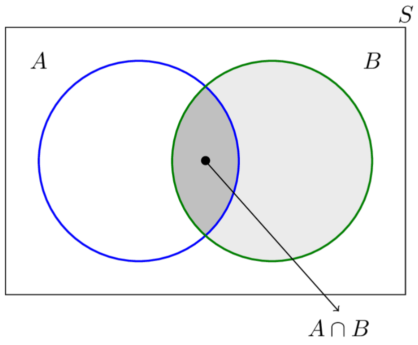
Content 1

Probability



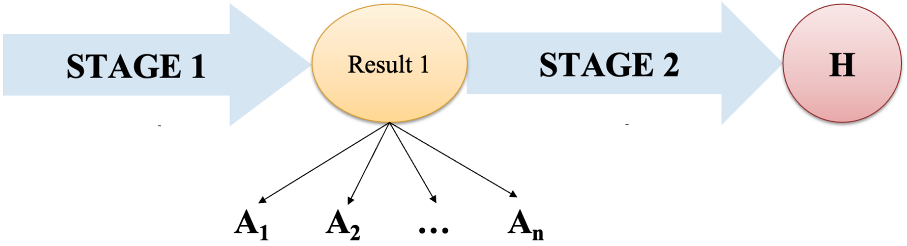
Content 2

Rules of Probability



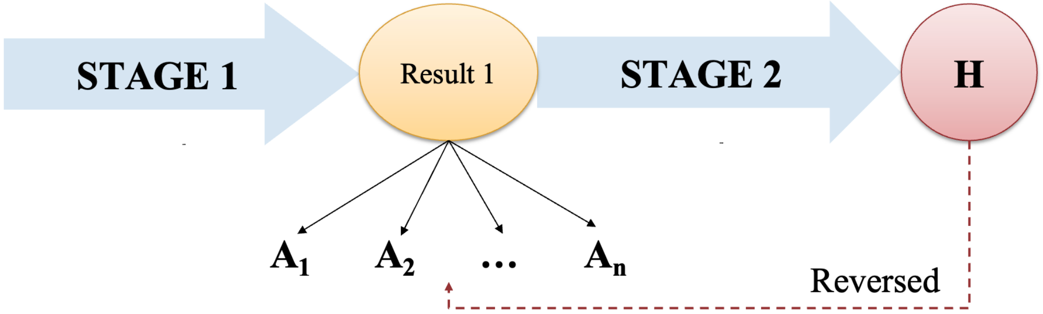
Content 3

Total Probability Theorem



Content 4

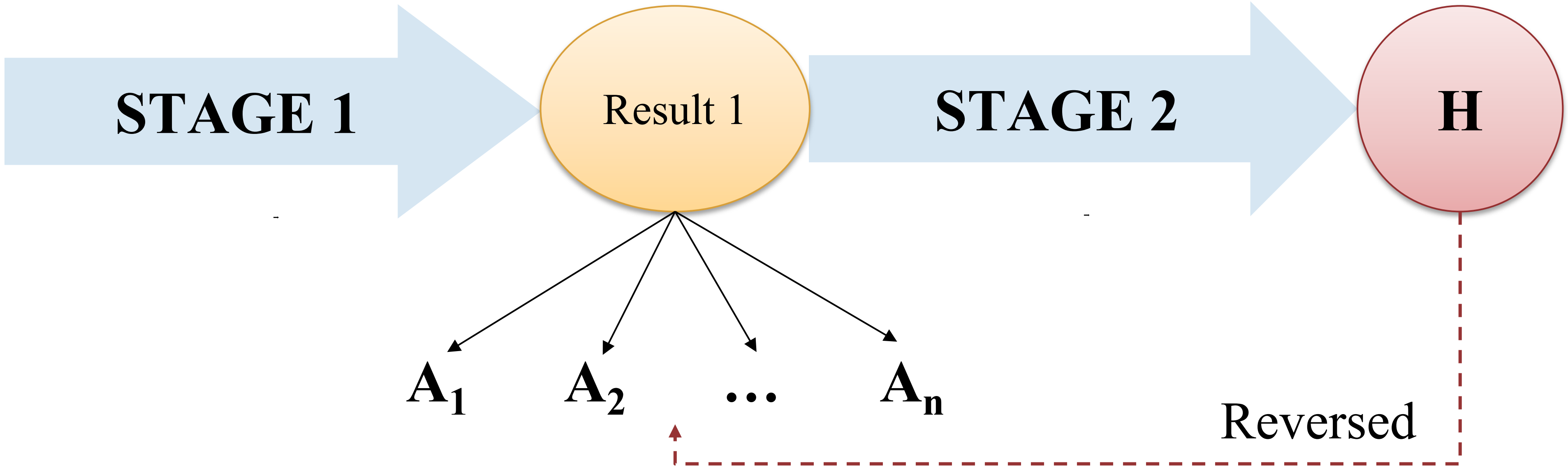
Bayes' Rule



Bayes' Rule

Bayes' Rule

- Based on the outcome H, calculate the probability that the i^th cause occur: $P(A_i|H)$



Bayes' Rule

Bayes' Rule

LIKELIHOOD
The probability of "X" being True. Given "c" True

PRIOR
The probability of "c" being True. This is the knowledge

$$\mathbf{P(c|X)} = \frac{\mathbf{P(X|c).P(c)}}{\mathbf{P(X)}}$$

POSTERIOR
The probability of "c" being True. Given "X" True

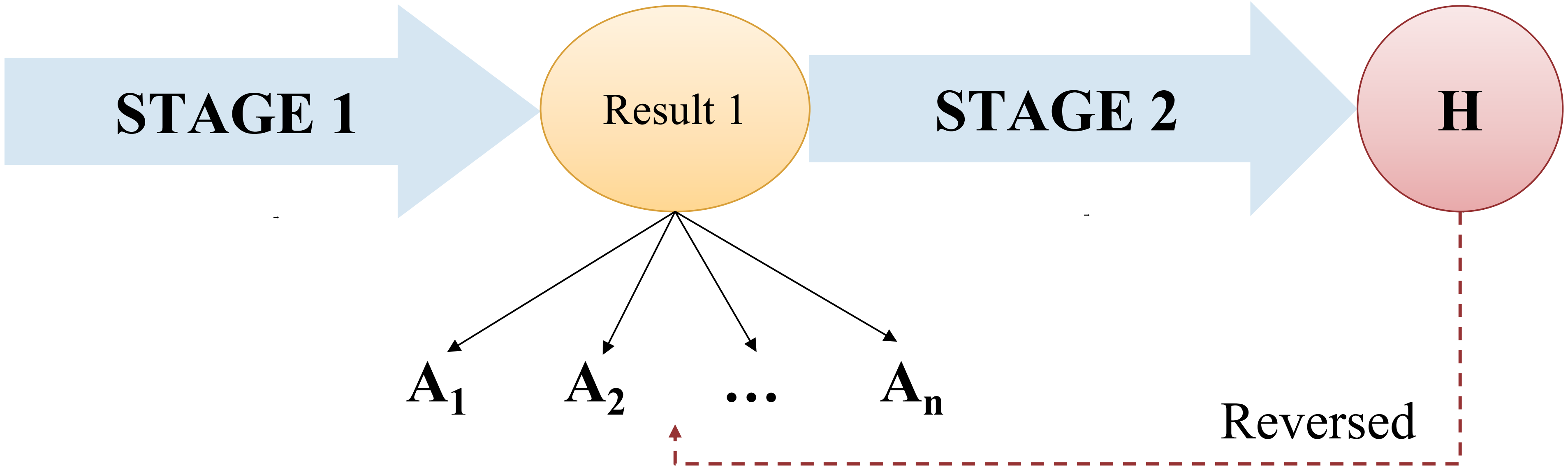
MARGINALIZATION
The probability of "X" being True.

Bayes' Rule

Bayes' Rule

- If $A_1, A_2, .. A_n$: complete system of events and H is any event with $P(A) \neq 0$:

$$P(A_i|H) = \frac{P(A_i)P(H|A_i)}{P(H)} = \frac{P(A_i)P(H|A_i)}{\sum_{j=1}^{n} P(A_j)P(H|A_j)}, i = 1, 2, \dots, n$$



Bayes' Rule

Example: Pick Marble

- I have three bags that each contain 100 marbles:
  - Bag 1 has 75 red and 25 blue marbles
  - Bag 2 has 60 red and 40 blue marbles
  - Bag 3 has 45 red and 55 blue marbles.

Suppose we observe that the chosen marble is red. What is the probability that Bag 1 was chosen?

Bayes' Rule

Example: Pick Marble

$$P(H) = \sum_{i=1}^{n} P(A_i).P(H|A_i)$$

R: "the event that the chosen marble is red"

B_i: "the event that I choose Bag *i*"

Events B_1 , B_2 and B_3 : complete system of events

=> $P(B_1) = 1/3$; $P(B_2) = 1/3$, $P(B_3) = 1/3$; $P(R| B_1) = 0.75$; $P(R| B_2) = 0.60$; $P(R| B_3) = 0.45$

The probability that the chosen marble is red:

$P(R) = P(R|B_1).\ P(B_1) + P(R|B_2).\ P(B_2) + P(R|B_3).\ P(B_3) = 0.60$

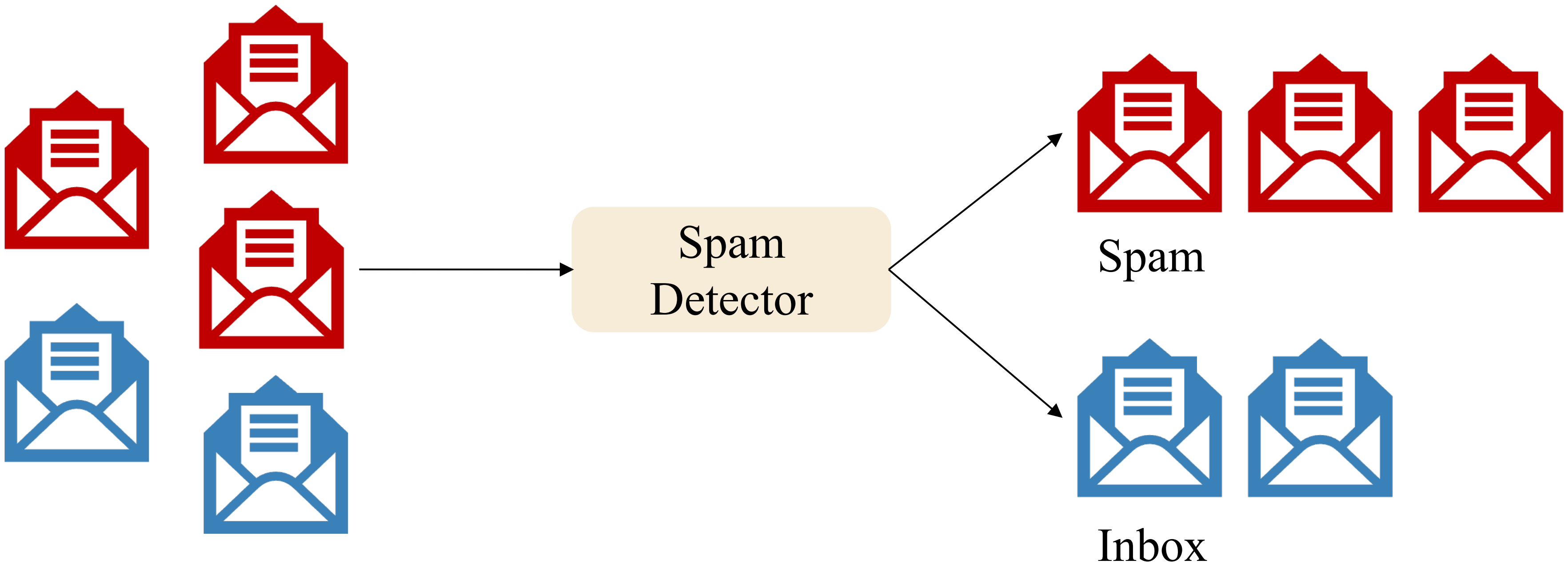
The probability that the chosen marble is red, Bag 1:

$$P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R)} = \frac{0.75 * 1/3}{0.6} = \frac{5}{12}$$

Bayes' Rule

Detect Spam E-Mail

- Assume that the word 'offer' occurs in 80% of the spam messages in my account. Also, let's assume 'offer' occurs in 10% of my desired e-mails. If 30% of the received e-mails are considered as a scam, and I will receive a new message which contains 'offer', what is the probability that it is spam?



Bayes' Rule

Detect Spam E-Mail

- Assume that the word 'offer' occurs in 80% of the spam messages in my account. Also, let's assume 'offer' occurs in 10% of my desired e-mails. If 30% of the received e-mails are considered as a scam, and I will receive a new message which contains 'offer', what is the probability that it is spam?

100 e-mails

100*0.3=30 → Spam

100*0.7=70 → Desired

Spam → Contains 'offer' 30*0.8=24

Spam → NOT contains 'offer' 30*0.2=6

Desired → Contains 'offer' 70*0.1=7

Desired → NOT contains 'offer' 70*0.9=63

Bayes' Rule

Detect Spam E-Mail

- **Detect Spam E-Mail (Simple NLP problem)**

  Let A_1: "Spam", A_2: "Not spam" => A_1, A_2 : complete system of events

  H: "contains the word 'offer'"

  If a new message which contains 'offer', the probability that it is spam is:

$$P(A_1|H) = \frac{P(A_1)P(H|A_1)}{P(H)}$$

  $P(H|A_1) = 0.8$; $P(A_1) = 0.3$, $P(A_2) = 1 - P(A_1) = 0.7$, $P(H|A_2) = 0.1$

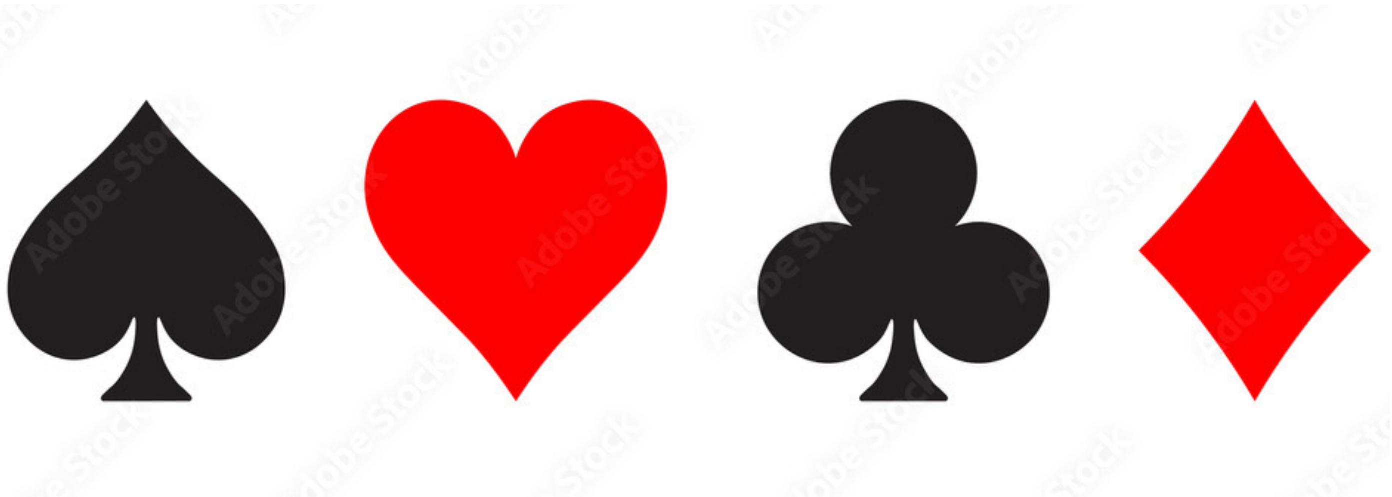
  $P(H) = P(A_1).P(H|A_1) + P(A_2).P(H|A_2) = 0.3*0.8 + 0.7*0.1 = 0.31$

  => $P(A_1|H) = (0.8*0.3)/(0.31) = 0.774$

Bayes’ Rule

Practice:

- A card is lost from a pack of 52 cards. From the remaining cards two are drawn randomly and found to be both clubs. Find the probability that the lost card is also a clubs.



QUIZ TIME

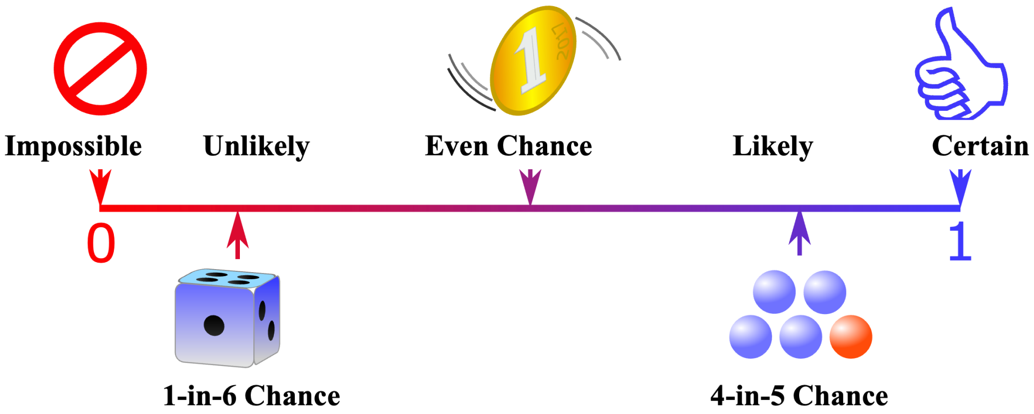
Objectives

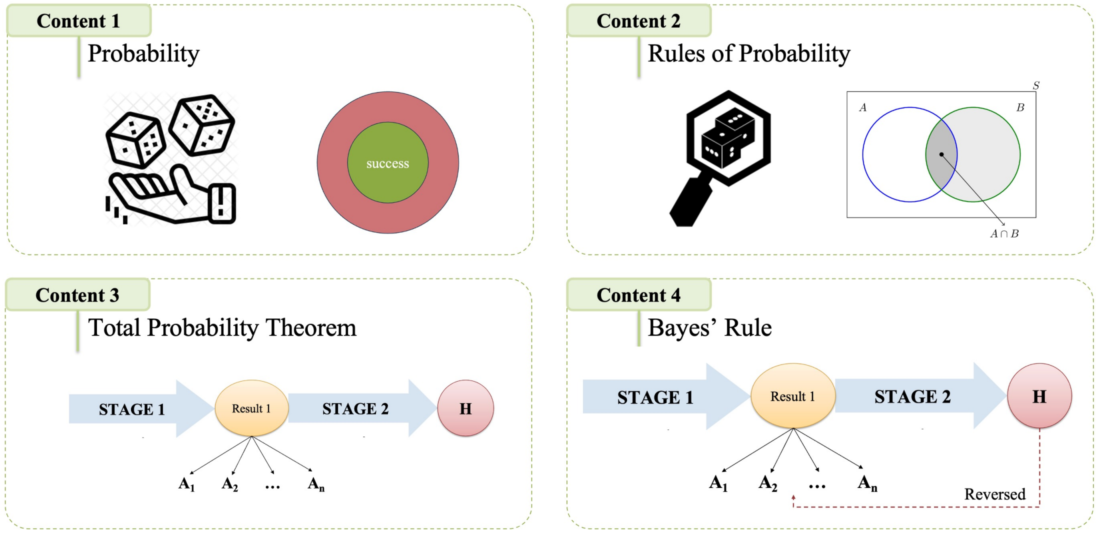
Introduction

- Experiment
- Event
- Operations on Events
- Relations of Events

Probability

- Definition
- Rule of Probability
- Total Probability
- Bayes' Rule





Thanks!

Any questions?