IITM-CS4410 : Topics in Algoritmic Combinatorics and Graph Theory Given on: Jan 29, 17:00

Problem Set #1 Arjun Bharat, CS17B006 Evaluation due: Feb 25, 17:00

- Turn in your solutions electronically at the institute moodle (courses.iitm.ac.in) page. Those who are not familiar with latex, may submit handwritten solutions to the instructor or TAs. Use LaTeX.
- You are expected to obtain the solutions independently.
- Any form of plagiarism will be reported to the Institute Disciplinary committee
- You are discouraged from using any source (internet, textbooks etc) for obtaining the solution. In case referred to, it should be indicated.
- Kindly use this latex file to type in your solutions. begin(solution) end(solution).

1. (6 points) Recall the definition of a cut and a bond in a graph. Show that a cut in a connected graph G is a bond if and only if the corresponding bipartition of $V(G)$ are connected in G.

Solution: We attempt to prove both directions one by one.

Lemma 1.1: Given a bond G, the corresponding bipartition V(G) is connected.

Consider an edge cut E, and let the induced bipartition be (A, B). Assume that the bipartition (A, B) is not connected in the graph G. Let the components of the bipartition be C_1, C_2,...C_n, for $n \geq 2$.

If the aforesaid was possible, then one could simply remove only the edges in G[C_i] to disconnect the graph, violating the **minimality** of the bond. Thus, we have a contradiction, and must have $n = 1$. This implies that the bipartition is connected, and hence, G is connected.

Lemma 1.2: Given that the bipartition formed by an edge cut E is connected in G, E is a bond.

Let the bipartition produced by the edge cut E be (A, B). Consider C_1,C_2,...C_n as the n components in the bipartition.

Observation: $\forall\ 1 \leq i < j \leq n$, $\forall\ u \in A \cap C_i$, $v \in A \cap C_j$, $w \in B \cap C_i$,$x \in B \cap C_j$, $\exists\ u - v$ and $w - x$ paths in $G \backslash E$. This is necessary for the connectivity of G.

Claim: No proper subset of E can disconnect G, and E disconnects G.

This holds due to the observation made above. If even one edge (u, v) of E described by C_i remains, then there exists a path $w - -u - v - -x\ \forall\ 1 \leq j \leq k \leq n$, $w \in A \cap C_i$,$x \in B \cap C_k$, which ensures connectivity of $G \backslash E'$, $\forall\ E' \subset E$. Clearly, the removal of E produces two disconnected components A, B in G. Hence, E is a bond of G. From Lemmas 1.1 and 1.2, we have proved the required condition.

2. (6 points) Show that the cycle space of a graph is spanned by its induced cycles.

Solution: We prove this theorem by induction on the number of bounded faces f of G.

Basis Step: When $f = 0$, the graph is a tree which has no induced cycles, and the property is trivially true.

Inductive Step: Assume that the hypothesis holds for all graphs having at most f bounded faces. We consider a graph G having $f + 1$ bounded faces. We observe that each bounded face of the graph describes an induced cycle by means of its boundary.

To facilitate our inductive argument,we begin by reducing the number of faces in G by deleting appropriate edges. Two cases are to be considered here:

Case 1: Removing a face of G by deleting edges that are shared with another face

This reduces the number of induced cycles by 1 as well. By the inductive hypothesis, the resulting graph G' has induced cycles spanning its cycle space.

Let us say that the induced cycles C_1 and C_2 of G result in an induced cycle C in G',upon removal of edges. We observe that,

$\mathrm{C}(G) = \mathrm{C}(G') \cup \{C_1, C_2\}$

C_1 and C_2 span themselves in G.

Also, $C = C_1 \oplus C_2$ (this is trivial) may be substituted into the spanning formulas for all common cycles of G and G',which proves the result.

Case 2: Removing one face of G by erasing edges that are unique to itself

This results in the loss of one induced cycle in the new graph G'. By the hypothesis, all induced cycles of G span its cycle space. We note that

$\mathrm{C}(G) = \mathrm{C}(G') \cup \{C_1\}$, where C_1 is the induced cycle of G that was lost.

Since C_1 spans itself and can be omitted in all other spanning formulas for other cycles of G, we have our required result.

By the principle of Mathematical Induction, the theorem holds for all values of $f \geq 0$.

From the above analysis, we have proved that the set of induced cycles of G spans its cycle space.

3. (8 points) Let G be a connected graph and T be a spanning tree of G. For an edge $e \in G \backslash T$, let C_e be the fundamental cycle of e with respect to T and for an edge $f \in T$ let D_f denote the fundamental cut of f with respect to T. Show that $e \in D_f \iff f \in C_e$

Solution: We prove the theorem one direction at a time.

Tree property: Any two vertices in a tree have a unique path connecting them. Removing any edge of a tree results in two components being created.

Lemma 3.1: $e \in D_f \Rightarrow f \in C_e$

Consider a spanning tree T of graph G, edge $f \in T$,edge $e \in G \backslash T$. Let A, B be the two components in $T \backslash \{f\}$. Let us say $e = (u, v)$, and $f = (w, x)$, $u, w \in A$, $v, x \in B$. Because $e \in D_f$,$e \neq f$.

We notice that the graph $(T \backslash \{f\}) \cup \{e\}$ is a tree, having just one trivial $u - v$ path,which is the edge (u, v) Therefore the graph $T \cup \{e\} = ((T \backslash \{f\}) \cup \{e\}) \cup \{f\}$ has a unique cycle which consists of u, v and w, x as neighbours. This is also the fundamental cycle C_e, and hence $f = (w, x) \in C_e$.

Lemma 3.2: $f \in C_e \Rightarrow e \in D_f$

Consider a spanning tree T of G,$e = (u, v) \in G \backslash T$.

$T \cup \{e\}$ has a fundamental cycle C_e,with vertex sequence $uv_1v_2v_3..v_nvu$. Consider $f \in C_e \backslash \{e\}$. Clearly $f \in E(T)$.

The graph $T \backslash \{f\}$ will be disconnected, and u, v will be in two separate components, since the only $u{-}v$ path in T does not exist in $T \backslash \{f\}$. Thus, we have $e = (u, v) \in D_f$, by definition of D_f.

From Lemmas 3.1 and 3.2, $e \in D_f \iff f \in C_e$.

4. (10 points) Suppose $k \geq 2$ and G be a k-connected graph. Show that for any $S \subseteq V$ with $|S| = k$, G has a cycle containing all of the vertices in S.

Solution: We make use of Menger's theorem for this problem.

Menger's theorem: A graph G is $k - connected$ iff for $A, B \subset V(G)$,$A \cap B = \Phi$ there exist k vertex disjoint paths between A and B.

We prove the theorem by induction on the connectivity of G.

Basis Step: For a 2-connected graph G,consider any pair of vertices $u, v \in G$. By Menger's theorem applied to $A = \{u\}, B = \{v\}$, there exist 2 vertex disjoint paths, say P_1 and P_2 between u and v. We have a cycle $C = E(P_1) \cup E(P_2)$,which contains both u and v. Hence, the theorem holds good.

Inductive Step: Assume that the inductive hypothesis holds good for all k-connected graphs. We consider a $(k + 1)$-connected graph G and try to prove the validity by an inductive argument.

Trivially, a $(k + 1)$-connected graph is also a k-connected graph. Hence, by the inductive hypothesis, G has a cycle containing k vertices. Call this cycle C.

Let the chosen set of vertices be $v_1, v_2, ...v_{k+1}$. Without loss of generality, let us label the vertices of C as $v_1, v_2, ..v_k$ in that order. Our goal is to extend the cycle C to accommodate the vertex v_{k+1} as well. We define two sets A, B as follows:

$A = \{v_1, v_2, ...v_k\}, B = \{v_{k+1}\}$

The cycle C has at least k vertices. Thus,two cases are of interest:

Case 1: C has exactly k vertices

In this case, we apply Menger's theorem to G on A and B,taking the graph to be k-connected. This tells us that there are k vertex disjoint paths between A and B. Since the paths are vertex disjoint, each path should originate in v_{k+1} and end in a different vertex of the k vertex cycle. Let P_i = Path from v_{k+1} to v_i, $1 \leq i \leq k$. One can choose any pair of paths P_i, P_{i+1}, $1 \leq i \leq k-1$, to create a cycle C' having a cyclic path $v_{k+1} - -P_i - -v_i - v_{i-1}..v_1 - v_2... - v_{i+1} - -P_{i+1} - -v_{k+1}$.

This cycle contains v_i, $\forall 1 \leq i \leq k+1$, and hence, proves the theorem.

Case 2: C has more than k vertices

In this case, we apply Menger's theorem on A and B taking G to be $(k+1)$-connected. Let us partition the cycle C into disjoint regions as follows,

R_i = Set of vertices between v_i and v_{i+1} including only v_{i+1}, $1 \leq i \leq k-1$. and additionally, R_k = Set of vertices between v_k and v_1 including only v_1.

There are $(k+1)$ vertex disjoint paths from B to A. Now, each path must have its end vertex in some region R_i, $1 \leq i \leq k$. There are k such regions, and $k+1$ vertex disjoint paths. Thus, by the pigeonhole principle, treating the paths as pigeons and the regions as the pigeonholes, there must be at least 2 disjoint paths P_u and P_v, ending at some vertices $u, v \in R_i$, $1 \leq i \leq k$. Let the longer $u - v$ path lying on C be called P_w.

We observe that there is a cycle C' having a cyclic path

given by $v_{k+1} - -P_u - -u - -P_w - -v - -P_v - -v_{k+1}$.

Clearly, cycle C' contains v_i $\forall$ $1 \leq i \leq k+1$,proving the theorem.

Hence, by the principle of Mathematical induction, any set $S \subseteq V(G)$ of a k-connected graph G with $|S| = k$ is contained in a cycle of G.