IITM-CS4410 : Topics in Algorithmic Combinatorics and Graph Theory Given on: Feb 18, 20:00

Problem Set #2 Due on : Mar 09 , 09:59 Arjun Bharat, CS17B006

- Turn in your solutions electronically at the institute moodle (courses.iitm.ac.in) page. Those who are not familiar with latex, may submit handwritten solutions to the instructor or TAs. Use LaTeX.
- You are expected to obtain the solutions independently.
- Any form of plagiarism will be reported to the Institute Disciplinary committee
- You are discouraged from using any source (internet, textbooks etc) for obtaining the solution. In case referred to, it should be indicated.
- Kindly use this latex file to type in your solutions.
- Late submission penalty: 10% of the earned points with will be deducted if the submission is made within the first two days of the deadline. 20% of the earned points with will be deducted if the submission is made after the first two days and within the first four days the deadline. No submissions will be accepted after four days.

1. (10 points) A planar graph G is called *outer planar* if and only if G has a planar embedding such that every vertex lies on the boundary of the outer face.
   (a) (4 points) We know that the number of edges in a planar graphs is bounded by $3n-6$. What if the graph is outer planar? Can you give a better bound on the number of edges? If not, obtain a sequence $G = (G_n)_{n\geq 0}$ of outer planar graphs such that G_n has n vertices and $3n-6$ edges. Prove your answers.
   (b) (3 points) Show that if G is outer planar, then every minor of G is outer planar.
   (c) (3 points) Using the above, show that an outer planar graph cannot have a K_4 as a minor.
   (d) (**Bonus question: 5 points**) Show that a graph is outer planar if and only if it does not have a K_4 or a $K_{2,3}$ as a minor.

Solution:

(a) For an outer planar graph,we observe that each face has at least 3 edges bounding it. However, since all vertices lie on the outer face, the outer face must be bounded by exactly n edges. Additionally, since each edge is common to exactly 2 faces of G, we have,

$3(f-1)+n \leq 2m$ (counting all faces of G, each edge is assessed twice.)

$\Rightarrow f \leq (2m - n + 3)/3$

Using Euler's identity we have,

$n - m + f = 2 \Rightarrow 2 + m - n \leq (2m - n)/3$

$\Rightarrow$

$$\boxed{m \leq 2n - 3}$$

We now have an improved bound for a graph to be outer planar.

(b) Assume G is an outer planar graph. Therefore, there exists a plane drawing of G where all vertices lie on the outer face. Consider this plane drawing of G.

If we contract any edge of G to generate a graph G', we observe that all vertices will continue to lie on the outer face of G' in this drawing.

If we contract an edge lying on the boundary of the outer face, it is trivial to observe that the drawing still remains planar, and hence G' is outer planar.

If we contract one chord of the outer face, we observe that the polygon P constituting the outer face boundary is divided into two polygons P_1 and P_2 which share one vertex. Because we began with an outer planar drawing, there must have been no edge whose arc began from a vertex in P_1 and ended in a vertex in P_2. This is argued as follows:

The arc could not have been entirely inside P as it would have intersected the contracted edge. It could not have lied entirely outside P either, because then the drawing would cease to be outer planar as some vertices are now hidden from the outer face. Naturally, the arc could not have left P_1 and re-entered P_2 as this would violate planarity of the drawing.

Thus,all edges of G are confined to strictly P_1 or P_2, and this is preserved in G'. Thus G' has a planar drawing where every vertex lies on the outer face, and is outer planar.

If we repeatedly contract the edges of G', we may use the same argument. Deletion of edges trivially preserves planarity.

Thus, it follows that any minor of G is outer planar.

(c) We know that K_4 has $n = 4$ and $m = 6$. Clearly, the condition $m(= 6) \leq 2n - 3(= 5)$ fails and hence, K_4 is not outer planar. From part (b), we know that every minor of an outer planar graph must be outer planar. Thus, K_4 cannot be a minor of an outer planar graph.

(d) Forward direction:

Assume the graph G is outer planar. We can add an extra vertex in the outer face and add all edges to the vertices of G on the outer face boundary to give G' which is still planar.Since G' is planar, by Kuratowski's theorem, it contains neither $K_{3,3}$ or K_5 as a minor. If G had K_4 as a minor, we can build a K_5 minor in G' by using the new vertex. If G hasd $K_{2,3}$ as a minor, we would have $K_{3,3}$ as a minor of G' by deleting two edges from the new vertex to the $K_{2,3}$ minor. Both cases lead to a contradiction.

Backward Direction:

Assume G has no K_4 or $K_{2,3}$ minor. Adding a new vertex to G and connecting it to all other vertices, we have a graph which does not contain K_5 or $K_{3,3}$ minors and is planar by Kuratowski's theorem. From the theorem proved in the Mid Semester exam, G is planar $\Leftrightarrow$ $G\backslash\{v\}$ is outer planar for v having degree $n-1$.

G must be outer planar.

2. (5 points) Let G_1 and G_2 be two graphs such that $|V(G_1) \cap V(G_2)| \leq 1$.
Let $G = G_1 \cup G_2$, with $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$.
Show that $\mathcal{C}(G)$ has a sparse cycle basis if and only if both $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ have one.

Solution: We use the following theorem proved in class:

Theorem 2: G has a sparse cycle basis $\Leftrightarrow$ G is planar.

Lemma 2.1: $C(G)$ has a sparse cycle basis $\Rightarrow$ $C(G_1)$ and $C(G_2)$ have a sparse cycle basis.

Proof:

Assume $C(G)$ has a sparse cycle basis. By Theorem 2, G is planar. If we take 2 subgraphs G_1 and G_2, consider 2 cases:

Case 1: $V(G_1)$ and $V(G_2)$ are disjoint

Clearly,since G has a planar drawing, we can simply partition the drawing into two planar sub drawings which correspond to G_1 and G_2. Since G_1 and G_2 are planar, their cycle spaces have a sparse basis.

Case 2: $V(G_1)$ and $V(G_2)$ have one common vertex v

The aforesaid arguments holds again since G_1 and G_2 are after all subgraphs of G.

Lemma 2.2 $C(G_1)$ and $C(G_2)$ have a sparse cycle basis $\Rightarrow$ $C(G)$ has a sparse cycle basis

Proof:

Assume the left hand side.From Theorem 2, G_1 and G_2 are both planar.

From the properties G_1 and G_2 satisfy, it is clear that v is a cut vertex of the graph. One can always consider drawings of G_1 and G_2 where the vertex lies on the outer face boundary. These two drawings can be superimposed to give a planar drawing for G, which hence has a sparse cycle basis.

From the two lemmas, we have our result.

3. (5 points) Show that $K_{1,3}$ is extremal without a path of length three.

Solution: We show that $K_{1,3}$ is maximal and maximum with respect to absence of a path of length 3 for all graphs on 4 vertices.

Clearly, the maximum path length should be 2.

Claim: If a graph has a path length of at most 2, each component must be a star.

Proof: Clearly the graph contains no cycles, because otherwise one can always find a path of length 3 (either a triangle or a path belonging to a larger cycle) which is a contradiction. Thus the graph is a forest. Now each component of the forest must have a diameter of at most 2, which clearly implies that a star graph is the only possibility.

$K_{1,3}$ is the largest star on 4 vertices, which makes it maximal. From the claim, it follows that a complete star is the maximum graph not having a path of length 3, and hence $K_{1,3}$ is also maximum.

4. (5 points) Show that $t_{r-1}(n) \leq \frac{1}{2}n^2\frac{r-2}{r-1}$.

Solution: We prove the theorem using induction on the value of n for a fixed r.

Since the value of r is generic, r may be replaced by $r+1$ for convenience.

To prove:

$$t_r(n) \leq \frac{1}{2}n^2\frac{r-1}{r}$$

Base Case: When n = r+1, we may have a complete graph on $r+1$ vertices with an edge removed,containing $\binom{r+1}{2}$ - 1 edges. We observe that:

$$r \geq 1 \Rightarrow -2r \leq -r-1 \Rightarrow r^3+r^2-2r \leq r^3+r^2-r-1$$

$$\Rightarrow \frac{r^2+r-2}{2} \leq \frac{r^3+r^2-r-1}{2r} \Rightarrow \binom{r+1}{2} - 1 \leq \frac{(r^2-1)(r+1)}{2r}$$

$$\Rightarrow E \leq \frac{1}{2}n^2\frac{r-1}{r}$$

Hence the hypothesis holds.

Inductive step:

Consider a maximal graph G on n vertices which does not have K_{r+1} as a subgraph. By definition, this should have $t_r(n)$ edges.

By the maximality of the graph, we assert that it contains K_r as a subgraph. We partition the vertex sets into two portions, A (which contains only the vertices of K_r) and B(which has the rest).

Any vertex in B cannot have more than $r-1$ neighbours in A, otherwise G would contain K_{r+1} as a subgraph(by taking K_r along with this vertex), contradicting the property of G.

The total number of edges in $G = E(A) + E(B) + E(A, B)$, where

$E(A)$ are the edges induced by A,$E(B)$ are the edges induced by B, and $E(A, B)$ are the edges of the induced bipartition.

Now,

$$E(A, B) \leq (n-r)(r-1)$$

$$\Rightarrow E \leq \binom{r}{2} + E(B) + (n-r)(r-1)$$

By the inductive hypothesis, we assume that the hypothesis holds for the graph B on $n-r$ vertices.

$$\Rightarrow E \leq \frac{r(r-1)}{2} + \frac{1}{2}(n-r)^2\frac{r-1}{r} + (r-1)(n-r)$$

Simplifying,we obtain

$$E \leq \frac{1}{2}\frac{r-1}{r}(r^2 + n^2 + r^2 - 2nr + 2nr - 2r^2)$$

$$\boxed{E \leq \frac{1}{2}n^2\frac{r-1}{r}}$$

Hence, by the principle of strong induction, the statement holds for all n.

5. (5 points) Without using Turán's theorem, show that the maximum number of edges in a triangle free graph is at most $n^2/4$.

Solution: Consider any edge (v_1, v_2) of a graph G that is triangle free.

v_1 and v_2 cannot have any common neighbours, because this would result in a triangle which is a contradiction. Thus, v_1 and v_2 both have a total of at most $n-2$ neighbours in G, excluding themselves.

Hence, we have, $d_{v_1} + d_{v_2} \leq n$, where d_{v_i} denotes the degree of a vertex v_i.

(d_{v_1} accounts for all neighbours of v_1 along with v_2, similar for d_{v_2})

Summating this over all edges in G, we have:

$$\sum d_{v_i} + d_{v_j} \leq nE, \forall i < j$$

Now on the left hand side, we observe that since each edge is counted once, d_{v_i} appears d_{v_i} times. This gives us,

$$\boxed{\sum_{i=1}^{n} d_{v_i}^2 \leq nE, \forall 1 \leq i \leq}$$

We know that

$$\sum_{i=1}^{n} d_{v_i} = 2E$$

Applying the Cauchy-Schwarz inequality to two n dimensional vectors,

$$A = \langle d_{v_1}, d_{v_2}, ..d_{v_n} \rangle$$

$$B = \langle 1, 1, ...1 \rangle$$

$$(\vec{A}.\vec{B})^2 \leq |\vec{A}|^2 |\vec{B}|^2$$

$$\Rightarrow (\sum_{i=1}^{n} d_{v_i})^2 \leq n \sum_{i=1}^{n} d_{v_i}^2$$

$$\Rightarrow \boxed{\frac{4E^2}{n} \leq \sum_{i=1}^{n} d_{v_i}^2}$$

Using our boxed results we have,

$$\frac{4E^2}{n} \leq nE$$

$$\Rightarrow \boxed{E \leq \frac{n^2}{4}}$$