CS6100

Lecture 9 + 10

Knapsack Problem.

n- items labelled $1, \dots, n$. $I = \{1, \dots n\}$.

$p_1, \dots p_n \in \mathbb{R}_{\geq 0}$ $\quad p_i$ - profit for ith item.

$w_1, \dots w_n \in \mathbb{R}_{\geq 0}$ $\quad w_i$ - weight of ~~the~~ ith item.

W - capacity.

Goal: obtain $x_1, \dots x_n \in \{0,1\}$ s.t

maximize $p_1 x_1 + \dots + p_n x_n$

subject to $w_1 x_1 + \dots + w_n x_n \leq W$.

$x_1, \dots x_n \in \{0,1\}$.

$x = (x_1, \dots x_n) \in \{0,1\}^n$.

$p = (p_1, \dots p_n) \in \mathbb{R}^n_{\geq 0}$

$w = (w_1, \dots w_n) \in \mathbb{R}^n_{\geq 0}$

max. $p \cdot x^T$ $\quad (p_1 \dots p_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

s.t $w x^T \leq W$.

& $x \in \{0,1\}^n$

Pareto optimality.

0 1 0 0 1

0 0 1 1 1

Let x & $y \in \{0,1\}^n$

We say that solution y dominates x if.

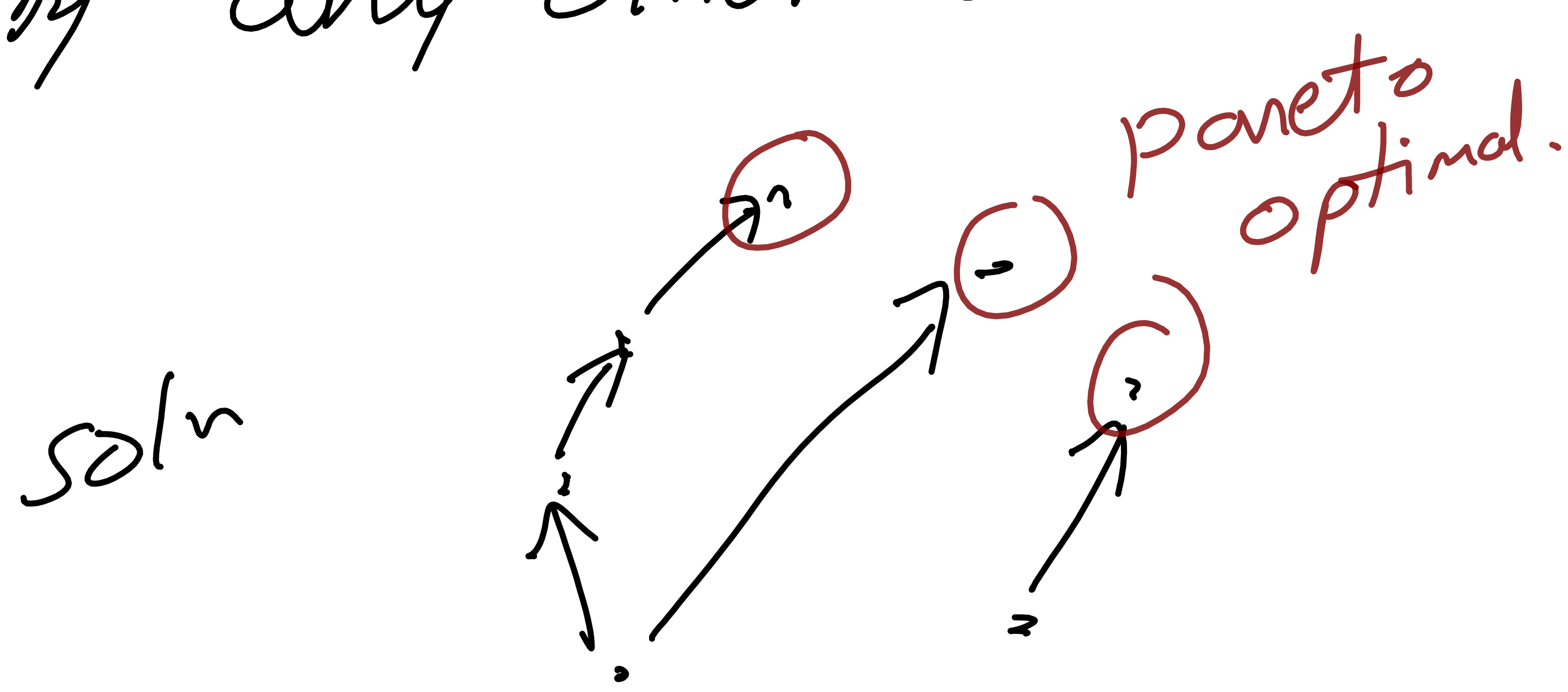
① $p y^T \geq p x^T$ (ie y has at least as much profit as x)

&

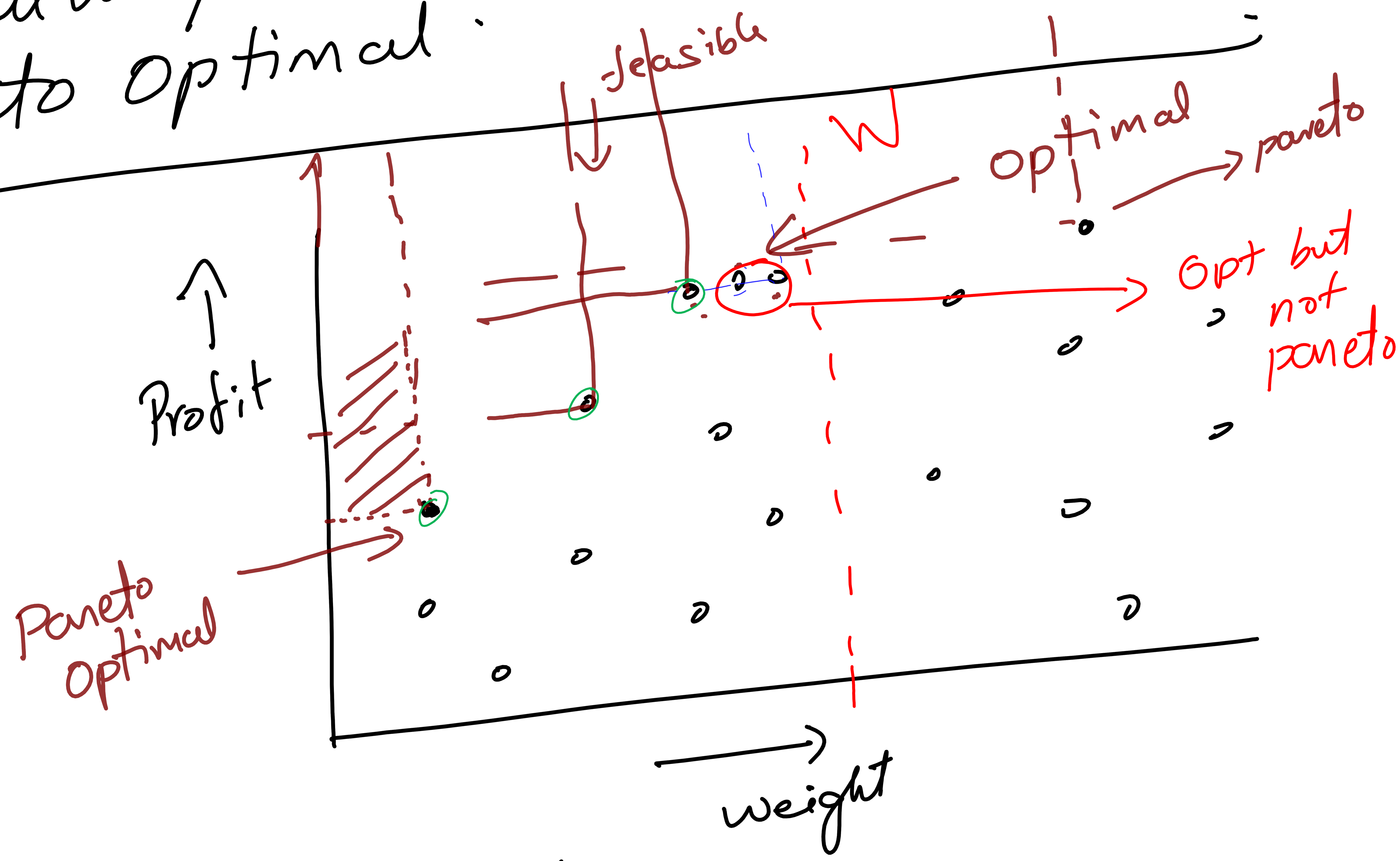
② $w y^T \leq w x^T$

and at least one of these is a strict inequality.

Defn: A solution y is said to be "pareto optimal" if it is not dominated by any other solution z.



Lemma :- There is always an optimal solution which is also pareto optimal.



Pf: Let x be an optimal solution.
Suppose x is not pareto, then there must be y that dominates x.
ie $py^T \geq px^T$ & $wy^T \leq wx^T$ one of the ineq strict.

Since x is opt, $py^T = px^T$

$\therefore \; wy^T < wx^T$

repeat the argument with y, if y is not pareto. Since there are only finitely many solution, repeating the argument gives an opt soln that is also pareto optimal.

for a given Knapsack instance (I, p, w, W)

Let P - denote the set of all pareto optimal solutions.

Obs:- If P is known, then an optimal solution can be computed in time $\underline{O(|P|)}$.

Then we immediately have an algo for knapsack with time $O(|P|)$ + time for computing P.

Question:-

① How ~~large~~ large can P be?

② Can P be computed efficiently? ie in time poly$(n, |P|)$ ← YES (Nemhauser & Ullmann)

Next:-

① Describe Nemhauser-Ullman.

② Show that $|P|$ is poly$(n, \cdot)$ when instances are perturbed

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Task:- To construct P.

$I = \{1 \dots n\}$

$I_i = \{1 \dots i\}$.

$S_i = \left\{ x \in \{0,1\}^n \;\middle|\; \begin{array}{l} x_j = 0 \\ \forall j > i \end{array} \right\}$

$P_i = \left\{ x \;\middle|\; \begin{array}{l} x \text{ is a} \\ \text{pareto optimal} \\ \text{soln for } I_i \end{array} \right\}$

$x \in S_{i-1}$.

x^{+i} is a solution obtained by adding i to x ie

$$x_j^{+i} = \begin{cases} x_j & j < i \\ 1 & j = i \\ 0 & \text{o.w.} \end{cases}$$

for any $T \subseteq \{0,1\}^n$ $\quad T^{+i} = \left\{ y \in \{0,1\}^n \;\middle|\; \begin{array}{l} \exists x \in T \\ y = x^{+i} \end{array} \right\}$.

Lemma: $\forall\, i \in \{1, \dots n\}$, $\quad P_i \subseteq P_{i-1} \cup P_{i-1}^{+i}$

pf:- Consider $x \in P_i$ ie x is a pareto optimal soln for $I_i := \{1 \dots i\}$.

$x = \underbrace{x_1 \dots x_i}\ \underset{0}{x_{i+1} \dots x_n}$

Case 1:- $x_i = 0$ $\quad x \in P_i$

then Claim: $x \in P_{i-1}$. Suppose not. $\quad | \ y \in P_{i-1} \subseteq S_{i-1} \subseteq S_i$

ie $\exists\, y \in P_{i-1}$ that dominates x.

ie y is a candidate soln to I_i

$\therefore$ by definitn $y \in S_i$

$\therefore$ x cannot be pareto optimal for I_i as y dominates it $\rightarrow$ contradiction to $x \in P_i$

$\therefore$ $x \in P_{i-1}$

Case 2:- $x_i = 1$. Claim: $x \in P_{i-1}^{+i}$ $\quad S_i = S_{i-1} \cup S_{i-1}^{+i}$

$x = \underbrace{x_1 \dots x_i}\ \overset{0}{x_{i+1}} \dots \overset{0}{x_n} \in S_i$

$\therefore$ $\exists\, y \in S_{i-1}$ s.t $x = y^{+i}$

Enough to show $y \in P_{i-1}$ ie y is pareto optimal for I_{i-1}.

Suppose not say there is a $z \in S_{i-1}$ s.t

z dominates y $\qquad pz^T \geq py^T$, $wz^T \leq wy^T$ one of them strict. $\rightarrow$ (*)

$x' = z^{+i}$

then $px'^T \geq px^T$ & $wx'^T \leq wx^T$ with one of them strict as $pz^T + p_i \geq py^T + p_i$ & $wz^T + w_i \leq wy^T + w_i$ from (*)

$\therefore$ $\exists\, x' \in S_i$ that dominates x

$\therefore \; x \notin P_i$ — contradiction

$\therefore \; y \in P_{i-1}$ where $x = y^{+i}$

$\therefore \; x \in P_{i-1} \cup P_{i-1}^{+i}$

Algo- $P_1 \quad P_2 \quad P_3 \quad \cdot \quad — \quad P_n = P$.