

CS6100

Lecture - 6

Basics of probability Theory

Ref:- Heiko Röglin's
notes on Prob. anal.
& algorithms.
(available in the
google shared
folder).

Recall

Analysis of \mathcal{J} -opt with
manhattan-metric
(instead of Euclidean)

→ random points distributed on $[0,1]^2$, $[0,1]^d$ $d \geq 2$

$$\Delta(S) = |x_1 - x_2| + \dots$$

$$\Pr[\Delta(S) \leq \Delta_{\min}]$$

Probability Spaces

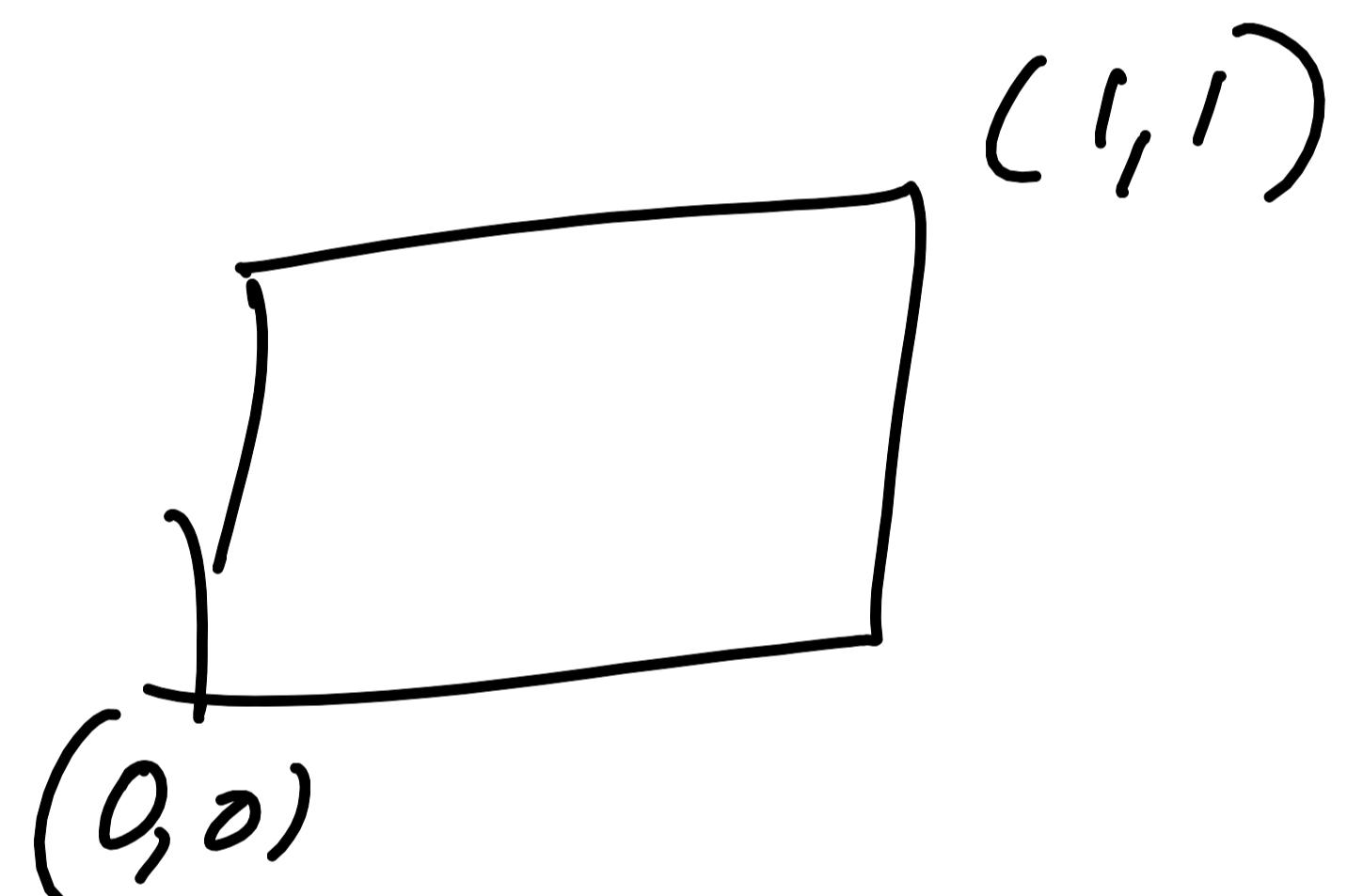
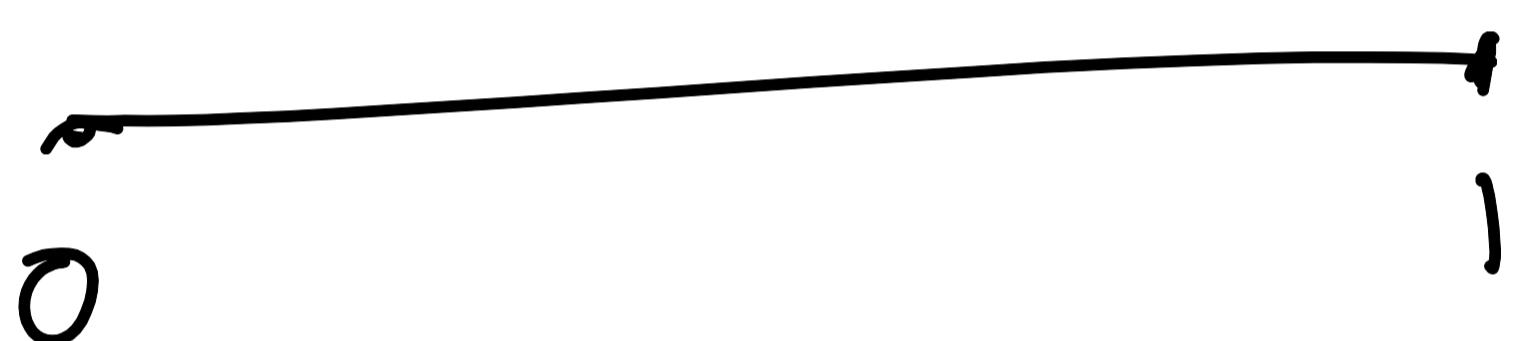
Discrete Probability Space

$$(\Omega, P)$$

- Ω : samplespace : finite or countable set.
 - set of all possible outcomes.

- $p: \Omega \rightarrow [0, 1]$.
- * $\sum_{x \in \Omega} p(x) = 1$
- An event $S \subseteq \Omega$.
- $P(S) = \sum_{x \in S} p(x)$.
- die with six faces
 $\{1, \dots, 6\}$
- $\Omega = \{1, \dots, 6\}$.
- $p(1) = p(2) = \dots = p(6) = \frac{1}{6}$
- event
outcome is even no
- $S = \{2, 4, 6\}$.
- $p(S) = \frac{1}{2}$
-

$[0, 1]$



$[-\epsilon, \epsilon]$.

Continuous case

A probability space is (Ω, \mathcal{F}, P) s.t

1 Ω : sample space

2 $\mathcal{F} \subseteq 2^\Omega$ with the following properties.

(i) $\Omega \in \mathcal{F}$

(ii) \mathcal{F} should be closed under complementation

i.e. $X \in \mathcal{F} \Rightarrow \Omega \setminus X \in \mathcal{F}$.

(iii) \mathcal{F} is closed under countable unions
 i.e if $X_1, X_2, \dots \in \mathcal{F} \Rightarrow \bigcup_i X_i \in \mathcal{F}$.
 (ie \mathcal{F} should be a σ -algebra).

3. P is a probability measure
 i.e $P: \mathcal{F} \rightarrow [0, 1]$

(i) $P(\Omega) = 1$

(ii) if $X_1, X_2, \dots \in \mathcal{F}$ s.t $X_i \cap X_j = \emptyset$
 (disjoint)

(iii) if $X_1, X_2, \dots \in \mathcal{F}$

$$\text{then } P\left(\bigcup_i X_i\right) = \sum_i P(X_i)$$



$$\frac{1}{1}, \frac{1}{2}, \dots$$

example of \mathcal{F} : Borel σ -algebra.
 set of all intervals in \mathbb{R} .
 & their unions.

Lemma: X_1, X_2, \dots, X_n are events in (Ω, \mathcal{F}, P) .

$$\text{then } P[X_1 \cup X_2 \cup \dots \cup X_n] \leq \sum_{i=1}^n P(X_i)$$

Union bound

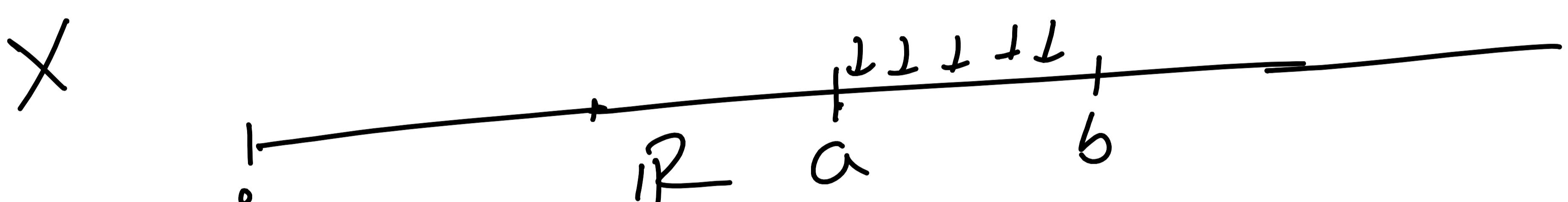
Random variables

Let (Ω, \mathcal{F}, P) -prob. space.

A random variable is a fn $X: \Omega \rightarrow \mathbb{R}$.
 $X: \underline{\text{continuous fn}}$
(real valued).

s.t for an $a \leq b$.

$$X^{-1}([a, b]) = \left\{ \omega \in \Omega \mid X(\omega) \in [a, b] \right\} \in \mathcal{F}.$$



$$\Omega = [0, 1]$$

$$X: \Omega \rightarrow \mathbb{R}$$

$$\omega \mapsto \omega^2$$

$$\underline{P_1}, \dots, \underline{P_n}$$

e.g:- Cost of a TSP tour

value of D(S)

$$X$$

$P_r[X=x]$: probability that
r.v X takes the value x .

$$\Delta(S) \geq \dots$$

$$\underline{\Delta(S)} = -$$

$$P_r[X=x] \stackrel{def}{=} P[X^{-1}(\{x\})]$$

$$P_r[X \leq t] \stackrel{def}{=} P[X^{-1}([- \infty, t])]$$

Cumulative Distribution Function (CDF) & Probability density function (PDF).

Let X be a continuous random variable. The CDF of X is the fn $F_X: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ given by

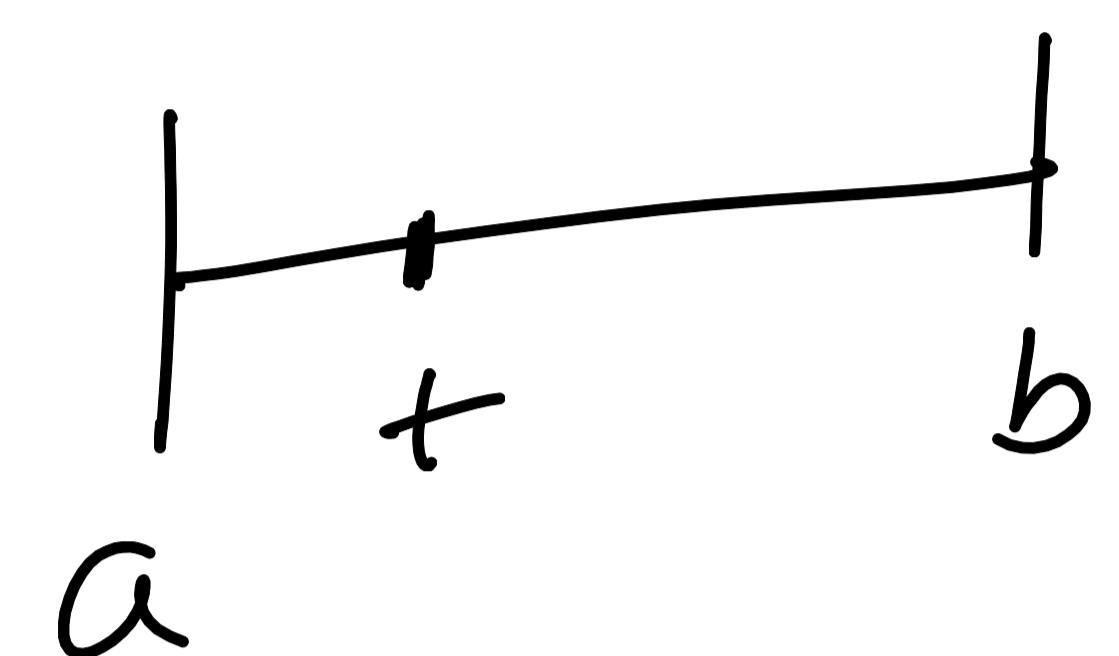
$$\forall t \quad F_X(t) = \underline{\Pr[X \leq t]}$$

A function $f_X: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ is called PDF of X if.

$$\Pr[X \in [a, b]] = \int_a^b f_X(t) dt$$

Example

① X be a random variable that is uniformly distributed in the interval $[a, b]$



$$\Pr[X \leq t] = \frac{t-a}{b-a}$$

$$F_X(t) = \Pr[X \leq t] = \begin{cases} 0 & t < a \\ \frac{t-a}{b-a} & a \leq t \leq b \\ 1 & t > b \end{cases}$$

$$f_X(t) = \begin{cases} 0 & t > b \text{ or } t < a \\ \frac{1}{b-a} & a \leq t \leq b \end{cases}$$

Gaussian / Normal distribution

Parameters: μ : mean
 σ : standard deviation.

The PDF of a normal distribution with mean μ & std. dev σ is given by

$$f_X(t) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

