9. MINIMAL TRIANGULATIONS

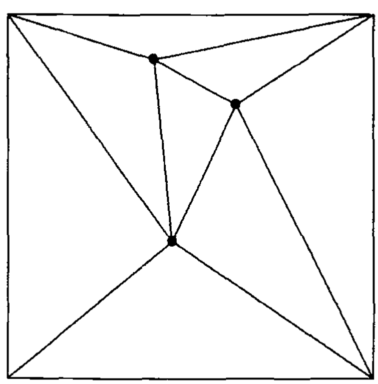
9.1. Introduction

This chapter examines the asymptotic behavior of the total edge length of the minimal triangulation of n points which are independently and identically distributed on the unit square.

Minimal triangulations arise naturally in many areas of mathematics and have particular importance in computational geometry. They have applications to surface interpolation, geometric searching techniques, and the finite-element method. See Preparata and Shamos (1985) and Bern and Eppstein (1992) for thorough treatments. Steele (1982) took the first important steps in the study of the triangulation functional on random points in the unit square.

We now formulate the minimal triangulation problem precisely. Given a finite set F in $[0,1]^2$, a triangulation of F is a decomposition of $[0,1]^2$ into triangles whose vertices coincide with F and the four corners of $[0,1]^2$. In general, a set F admits more than one triangulation, which need not be a simplex. The *total edge length* of a triangulation is the sum of the lengths of the edges in the triangulation. See Figure 9.1.

Figure 9.1. A triangulation of a three point set



Given $1 < \delta < \infty$, a "δ-triangulation" of F is a triangulation in which all triangles have aspect ratios which are less than δ, that is for all triangles the ratio of the radii of the circumscribed ball to the inscribed ball is less than δ. Such triangulations have a number of applications and motivations, including some in computational learning theory (Salzberg et al., 1991). Let S_δ denote a function which assigns to each set $F \subset [0,1]^2$ a δ-triangulation of F which has the least total edge length. Let $S_\delta(F)$ denote the graph of this triangulation (possibly empty depending on the choice of F and δ) and let $|S_\delta(F)|$ denote the total edge length of $S_\delta(F)$. We will occasionally write $S(F)$ for $S_\delta(F)$.

Figure 9.2. A triangulation of collinear points (no Steiner points allowed)

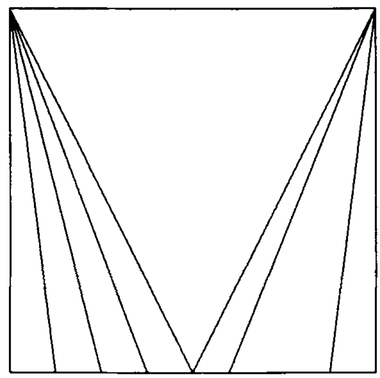
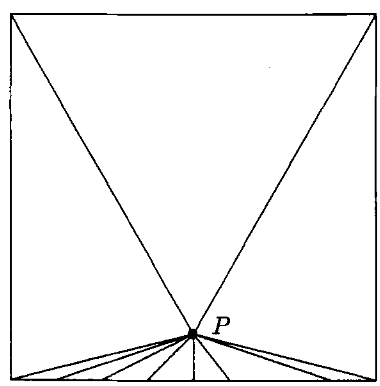


Figure 9.3. Adding a Steiner point P decreases total edge length



If $G \subset [0,1]^2$ is an additional set of points distinct from F (i.e., a Steiner set), then $S_\delta(F \cup G)$ denotes the graph of a *Steiner* δ-triangulation of F. We tacitly assume throughout and without further mention that $\delta > 1$ is chosen large enough such that for every set F there is a Steiner set G for which $S_\delta(F \cup G)$ exists. We define the *length of the minimal Steiner δ-triangulation* of F by

$$\sigma_\delta(F) := \inf_G |S_\delta(F \cup G)|,$$

where G ranges over all finite sets of Steiner points including the empty set. It is conceivable that adding more and more Steiner points may decrease the total length of the triangulation (see Bern and Eppstein (1992)). Thus is it unclear whether the infimum is realized by a set G and it is thus an open problem whether a "minimal δ-triangulation" actually exists. See Figures 9.2 and 9.3.

Without loss of generality we assume that G ranges over points in $[0,1]^2$ with rational coordinates. The set of admissible Steiner points of a set F thus has the cardinality of the countable set $\bigcup_{n=1}^{\infty}(\mathbb{Q}\times\mathbb{Q})^n$. This technical remark ensures the measurability of $\sigma_\delta(X_1,...,X_n)$, where $X_i,\ i\geq 1,$ are random variables.

The main goals of this chapter are to develop some basic deterministic properties of triangulations and to use these to determine the asymptotic behavior of $\sigma_\delta(X_1,...,X_n)$, where $X_i,\ i\geq 1,$ are i.i.d. random variables.

Based on our experience with optimization problems, we would expect that the minimal triangulation length σ_δ conforms to the asymptotics of Theorem 7.1. The proof is a bit more challenging since σ_δ is apparently not "simply subadditive" (2.2) and since not much is known about σ_δ. The main goal of this chapter is develop some basic properties of stochastic triangulations and in this way show that σ_δ fits naturally within the framework of Theorem 7.1. Showing that triangulations conform to the conditions of Theorem 7.1 involves methods combining probability and geometry. The arguments depend upon a judicious definition of a superadditive "boundary triangulation functional". By verifying that subadditive triangulations and superadditive boundary triangulations satisfy the conditions of Theorem 7.1 when $p=1$ and $d=2$, we prove:

Theorem 9.1. *(asymptotics for minimal triangulations) Let $X_i,\ i\geq 1,$ be i.i.d. random variables with values in $[0,1]^2$. Fix $1<\delta<\infty$ and consider the minimal triangulation length σ_δ. Then*

$$\lim_{n\to\infty}\sigma_\delta(X_1,...,X_n)/n^{1/2}=\alpha(\sigma_\delta)\int_{[0,1]^2}f(x)^{1/2}dx \quad c.c., \tag{9.1}$$

where $\alpha(\sigma_\delta)$ is a positive constant and f denotes the density of the absolutely continuous part of the law of X_1.

By placing the triangulation functional in the context of Euclidean functionals, we may moreover derive rates of convergence in a natural way. It is not clear how much improvement can be made in these rates. Let $U_1,U_2,...$ be i.i.d. with the uniform distribution on $[0,1]^2$.

Theorem 9.2. *Fix $1<\delta<\infty$. The mean of σ_δ satisfies*

$$|E\sigma_\delta(U_1,...,U_n)-\alpha(\sigma_\delta)n^{1/2}|\leq C(n\log n)^{1/4}, \tag{9.2}$$

where $C:=C(\delta)$ is a constant depending only on δ.

The above two-dimensional results have a natural three-dimensional analog. Given a finite set F in $[0,1]^3$, a tetrahedralization of F is a decomposition of $[0,1]^3$ into tetrahedra whose vertices coincide with the points in F and the corners of the cube $[0,1]^3$. In general, F admits more than one tetrahedralization. The *total surface area* of a tetrahedralization is the sum of the areas of the triangular faces.

Given $1 < D < \infty$, a D-tetrahedralization is one in which the tetrahedra have aspect ratios less than D, that is the ratio of the radii of the circumscribed sphere to the inscribed sphere is less than D for all tetrahedra. This regularity condition insures that the cube of the length of a tetrahedral edge is bounded by a constant multiple of the volume of the tetrahedron, a fact which will be useful in the sequel.

Given $1 < D < \infty$, let T_D denote a function which assigns to each set $F \subset [0,1]^3$ a D-tetrahedralization of F having the least total surface area. Let $T_D(F)$ denote the graph of the tetrahedralization and let its total surface area be denoted by $|T_D(F)|$. If $G \subset [0,1]^3$ is a Steiner set, then we let $T_D(F \cup G)$ denote the graph of the corresponding Steiner D-tetrahedralization of F. We tacitly assume that $D > 1$ is chosen large enough so that for every set $F \subset [0,1]^3$, there is a Steiner set G for which $T_D(F \cup G)$ exists. Analogously to $\sigma_\delta(F)$, define the area $\tau_D(F)$ of the *minimal* Steiner D-tetrahedralization of F by

$$\tau_D(F) := \inf_G |T_D(F \cup G)|,$$

where G ranges over all Steiner sets. $\tau_D(F)$ may be thought of as the *discrete Plateau functional* for the point set F.

As in the definition of σ_δ, we may without loss of generality restrict attention to Steiner points with rational coordinates. This ensures the measurability of $\tau_D(X_1, ..., X_n)$, where X_i, $i \geq 1$, are random variables.

In section 9.6 we will show that the Steiner tetrahedralization functional satisfies the conditions of Theorem 7.1 with $p = 2$ and $d = 3$. Notice that the order is 2 since τ_D involves sums of surface areas. Steiner tetrahedralizations furnish a natural example of a functional in geometric probability which has an order larger than 1. By applying Theorem 7.1 we will prove the following analog of Theorem 9.1. This makes progress on a question raised by Beardwood, Halton, and Hammersley (1959).

Theorem 9.3. *(asymptotics for minimal tetrahedralizations) Let X_i, $i \geq 1$, be i.i.d. random variables with values in $[0,1]^3$. Then for each fixed $1 < D < \infty$ we have*

$$\lim_{n\to\infty} \tau_D(X_1, ..., X_n)/n^{1/3} = \alpha(\tau_D) \int_{[0,1]^3} f(x)^{1/3}\, dx \quad c.c., \tag{9.3}$$

where $\alpha(\tau_D)$ is a positive constant and where f denotes the density of the absolutely continuous part of the law of X_1.

We anticipate that Theorems 9.1 and 9.3 admit extensions to higher dimensions. In this way we could perhaps find asymptotics for the randomized version of the problem of Douglas (1939) which considers minimal surfaces in higher dimensions. Beardwood, Halton, and Hammersley (1959) were apparently the first to consider such a problem. In their somewhat cryptic remarks they recognize the potential applicability of subadditivity methods but do not develop the necessary mathematics. For a full treatment of the problem of Douglas we refer to Courant and Schiffer (1950, Chapter 4) and Douglas (1939). We will not consider generalizations to higher dimensions in this monograph.

To facilitate the exposition and lighten the notation, we will henceforth omit mention of δ when referring to δ-triangulations. Moreover, when it is clear from the context, we will often simply write σ for σ_δ and τ for τ_D.

9.2. The Boundary Triangulation Functional

In the previous chapters "boundary functionals" played an important role in establishing the intrinsic superadditivity of problems in Euclidean optimization. The purpose of this section is to appropriately define the "boundary δ-triangulation functional" and to use it to prove Theorems 9.1 and 9.2.

Given any subset $F \subset \mathbb{R}^2$ whose convex hull $\mathrm{co}(F)$ contains $[0,1]^2$, consider a δ-triangulation of $\mathrm{co}(F)$. Such a δ-triangulation partitions $\mathrm{co}(F)$ into triangles whose vertices coincide with F and whose aspect ratios are bounded by δ. A "boundary δ-triangulation" of $[0,1]^2$ with respect to F, denoted here by $S_{B,\delta}(F,[0,1]^2)$, is obtained by considering the intersection of $[0,1]^2$ and the graph of a δ-triangulation of $\mathrm{co}(F)$. A boundary δ-triangulation of $[0,1]^2$ thus partitions $[0,1]^2$ into the usual triangles as well as perhaps quadrilaterals, pentagons, and even hexagons. When the context is clear, we will omit mention of δ and refer to boundary δ-triangulations as simply boundary triangulations.

By the "total edge length" $|S_{B,\delta}(F,[0,1]^2)|$ of the boundary δ-triangulation $S_{B,\delta}(F,[0,1]^2)$ we mean the sum of the lengths of the edges in the triangulation which lie in the interior of $[0,1]^2$. Analogously to $\sigma(F) := \sigma_\delta(F)$, define for all $F \subset [0,1]^2$ the length of the "minimal boundary δ-triangulation" of F with respect to $[0,1]^2$ by

$$\sigma_{B,\delta}(F,[0,1]^2) := \inf_G |S_{B,\delta}(F \cup G,[0,1]^2)|,$$

where G ranges over finite sets of Steiner points with the property that the convex hull of $F \cup G$ contains $[0,1]^2$. Without loss of generality we will assume that the points in G have rational coordinates. Minimal boundary triangulations may fail to exist for the same reasons that the standard minimal triangulation may not exist. We will occasionally condense notation and write σ_B for $\sigma_{B,\delta}$ and S_B for $S_{B,\delta}$. It is clear from the definitions that $\sigma_B(F,[0,1]^2) \le \sigma(F)$ for all $F \subset [0,1]^2$.

Given any convex polygon Δ and a point set F such that $\mathrm{co}(F) \supset \Delta$, we extend the above definitions and define a boundary δ-triangulation of Δ with respect to F as the intersection of Δ and the graph of a δ-triangulation of $\mathrm{co}(F)$. When $F \subset [0,1]^2$, $\sigma_B(F,\Delta) := \sigma_{B,\delta}(F,\Delta)$ denotes the length of the minimal boundary δ-triangulation of F with respect to Δ and is defined analogously to $\sigma_B(F,[0,1]^2)$; notice that $\sigma_B(F,\Delta)$ does not count the lengths of edges lying on the boundary of Δ.

The minimal boundary triangulation length σ_B enjoys geometric superadditivity (3.3): for all $F \subset [0,1]^2$ we have

$$\sigma_B(F,[0,1]^2) \geq \sum_{i=1}^{m^2} \sigma_B(F \cap Q_i, Q_i), \tag{9.4}$$

where Q_i, $1 \leq i \leq m^2$, denotes the usual partition of $[0,1]^2$ into subsquares of edge length m^{-1}.

To see this, find a sequence G_n of Steiner sets such that the boundary triangulations $S_B(F \cup G_n, [0,1]^2)$, $n \geq 1$, have lengths $|S_B(F \cup G_n, [0,1]^2)|$ decreasing down to $\sigma_B(F,[0,1]^2)$ as n goes to infinity. For each $1 \leq i \leq m^2$, let $S_B^i(F \cup G_n, [0,1]^2)$ denote the boundary triangulation of Q_i generated by the intersection of $S_B(F \cup G_n, [0,1]^2)$ with subsquare Q_i. Then for each $n \geq 1$ we have

$$|S_B(F \cup G_n, [0,1]^2)| \geq \sum_{i=1}^{m^2} |S_B^i(F \cup G_n, [0,1]^2)| \geq \sum_{i=1}^{m^2} \sigma_B(F \cap Q_i, Q_i),$$

where the last inequality follows by minimality of σ_B. Let n tend to infinity to deduce superadditivity (9.4). Summarizing, we have shown:

Lemma 9.4. *σ_B is superadditive.*

Having defined the minimal triangulation lengths σ and σ_B we are positioned to prove Theorem 9.1. We must verify that σ and σ_B are smooth subadditive and superadditive Euclidean functionals of order 1, respectively, and that they satisfy the closeness condition (3.15) with $p = 1$ and $d = 2$. This is shown in the remainder of the chapter.

9.3. Minimal Triangulations are Subadditive and Smooth

We will verify that the length $\sigma := \sigma_\delta$ of the minimal triangulation is a subadditive Euclidean functional of order 1. Throughout, let $1 < \delta < \infty$ be arbitrary but fixed and write σ for σ_δ. In the sequel we show that the boundary triangulation length $\sigma_B := \sigma_{B,\delta}$ is a superadditive Euclidean functional of order 1.

It will be helpful to consider the triangulation of a set in a region other than the unit square. We thus enlarge the definition of triangulations in the following way.

Definition 9.5. *Let $F \subset \Delta$ be a finite set, where $\Delta \subset [0,1]^2$ is a convex polygon. Consider all Steiner δ-triangulations $S_\delta(F \cup G, \Delta)$, $G \subset \Delta$, of Δ, i.e. all decompositions of Δ into triangles whose vertices coincide with $F \cup G$ and the corners of Δ and whose aspect ratios are bounded by δ. Let $S_\delta(F \cup G, \Delta)$ have total edge length $|S_\delta(F \cup G, \Delta)|$. Define*

$$\sigma_\delta(F,\Delta) := \inf_G |S_\delta(F \cup G, \Delta)|,$$

where G ranges over all finite sets in Δ. We call $\sigma(F,\Delta) := \sigma_\delta(F,\Delta)$ the length of the minimal δ-triangulation of F with respect Δ. When $\Delta = [0,1]^2$, we will simply write $\sigma(F)$ for $\sigma(F,[0,1]^2)$.

Notice that σ, considered as a function on pairs (F, R) is Euclidean. σ also satisfies geometric subadditivity (3.5) with no error term:

Lemma 9.6. *(subadditivity of minimal triangulations) For every $F \subset [0,1]^2$ we have*

$$\sigma(F) \leq \sum_{i=1}^{m^2} \sigma(F \cap Q_i, Q_i). \tag{9.5}$$

Proof. For each $1 \leq i \leq m^2$, find a sequence of Steiner sets $G_n := G_{n,i} \subset Q_i$, $n \geq 1$, such that the triangulations $S_\delta\left((F \cup G_n) \cap Q_i, Q_i\right)$, $n \geq 1$, have lengths $|S_\delta((F \cup G_n) \cap Q_i, Q_i)|$ which decrease down to the length of the minimal triangulation $\sigma(F \cap Q_i, Q_i)$ as n tends to infinity. For each $n \geq 1$, the union of the local triangulations $S_\delta((F \cup G_n) \cap Q_i, Q_i)$, $1 \leq i \leq m^2$, is a feasible Steiner triangulation of $[0,1]^2$. Minimality implies that for all $n \geq 1$

$$\sigma(F) \leq \sum_{i=1}^{m^2} |S_\delta((F \cup G_n) \cap Q_i, Q_i)|.$$

Now let n tend to infinity to deduce (9.5). □

When Q_i and Q_j are adjacent subsquares the Steiner points on the boundary of Q_i need not coincide with the Steiner points on the boundary of Q_j. We notice therefore that (9.5) would fail if we restricted attention to triangulations which were simplicial complexes.

Geometric subadditivity implies that σ satisfies the growth bounds of Lemma 3.3. Moreover, we may obtain growth bounds for $\sigma_\delta(F, \Delta)$, $\Delta \subset [0,1]^2$ a convex polygon, by approximating Δ by the union of inscribed subsquares and applying growth bounds on the individual subsquares. This argument, whose details are left to the reader, shows:

Lemma 9.7. *There is a finite constant C such that for all convex polygons $\Delta \subset [0,1]^2$ and all non-empty sets $F \subset \Delta$*

$$\sigma_\delta(F, \Delta) \leq C(\mathrm{card}F)^{1/2}. \tag{9.6}$$

Finally, we may verify that σ is smooth of order 1, namely

$$|\sigma(F \cup G) - \sigma(F)| \leq C(\mathrm{card}G)^{1/2}. \tag{9.7}$$

For all $\varepsilon > 0$ and all $F \subset [0,1]^2$ let $G_\varepsilon := G_\varepsilon(F) \subset [0,1]^2$ be (uniquely defined) Steiner sets such that

$$|S_\delta(F \cup G_\varepsilon)| \leq \sigma_\delta(F) + \varepsilon. \tag{9.8}$$

Here and henceforth, let $\Delta^\varepsilon(F)$ denote the collection of triangles defined by the triangulation $S_\delta(F \cup G_\varepsilon)$ and let $\mathcal{E}^\varepsilon := \mathcal{E}^\varepsilon(F)$ denote the collection of edges of these triangles. By assumption, the aspect ratios of the triangles in $\Delta^\varepsilon(F)$ are uniformly bounded by δ. It follows that the square of the length of an edge of a triangle is bounded by a constant multiple of the triangular area. Since the sum of the areas of the triangles in $\Delta^\varepsilon(F)$ is just the area of the unit square, it follows that

$$\sum_{E \in \mathcal{E}^\varepsilon} |E|^2 \le C \tag{9.9}$$

for some universal constant $C := C(\delta)$ which doesn't depend on ε. Here and elsewhere, $|E|$ denotes the Euclidean length of the edge E.

To show smoothness, it suffices by (9.9) to show

$$\sigma(F) \le \sigma(F \cup G) \le \sigma(F) + C \left(\sum_{E \in \mathcal{E}^\varepsilon} |E|^2 \right)^{1/2} (\mathrm{card} G)^{1/2}. \tag{9.10}$$

Notice that the first inequality in (9.10) is a consequence of the intrinsic monotonicity of σ. To show (9.7), it thus suffices to prove the second inequality in (9.10).

Given G, we may assume that $G \cap F = \emptyset$. The points in G are located in triangles $\Delta_1, ..., \Delta_J$, $J := J(G) < \infty$, belonging to $\Delta^\varepsilon(F)$ (if a point in G lies on an edge in $\mathcal{E}^\varepsilon(F)$, then it belongs to *two* triangles in $\Delta^\varepsilon(F)$). Let E_i, $1 \le i \le J$, be the longest edge of triangle Δ_i, $1 \le i \le J$.

Observe that $\sigma(F \cup G)$ is bounded by $|S_\delta(F \cup G_\varepsilon)|$ and the sum of the lengths of the minimal triangulations of $G \cap \Delta_i$ with respect to Δ_i, $1 \le i \le J$. In other words

$$\begin{aligned} \sigma(F \cup G) &\le |S_\delta(F \cup G_\varepsilon)| + \sum_{i=1}^{J} \sigma(G \cap \Delta_i, \Delta_i) \\ &\le |S_\delta(F \cup G_\varepsilon)| + C \sum_{i=1}^{J} |E_i| \ (\mathrm{card}(G \cap \Delta_i))^{1/2} \end{aligned} \tag{9.11}$$

by scaling and Lemma 9.7. Hölder's inequality implies

$$\sigma(F \cup G) \le |S_\delta(F \cup G_\varepsilon)| + C \left(\sum_{E \in \mathcal{E}^\varepsilon} |E|^2 \right)^{1/2} (\mathrm{card} G)^{1/2},$$

which together with (9.8)–(9.9) gives smoothness (9.7) as desired.

9.4. Boundary Minimal Triangulations

By Lemma 9.4 we know that the boundary triangulation functional $\sigma_B := \sigma_{B,\delta}$ is superadditive where $1 < \delta < \infty$ is arbitrary but fixed. It is also clear that σ_B is Euclidean. It remains to verify smoothness

$$|\sigma_B(F \cup G, [0,1]^2) - \sigma_B(F, [0,1]^2)| \le C (\mathrm{card} G)^{1/2}. \tag{9.12}$$

This will follow from a slight modification of the proof of smoothness of σ.

We first clarify the terminology. Let $1 < \delta < \infty$ be arbitrary but fixed. As before, for all $\varepsilon > 0$ and all $F \subset [0,1]^2$, let $G_\varepsilon := G_\varepsilon(F) \subset \mathbb{R}^2$ be (uniquely defined) Steiner sets such that

$$|S_B(F \cup G_\varepsilon)| \leq \sigma_B(F) + \varepsilon.$$

Without loss of generality we may assume that $\mathrm{co}(F \cup G_\varepsilon)$ is contained in a large square $Q \supset [0,1]^2$ where the edge length of Q is at most $C := C(\delta)$.

Let $\Delta_B^\varepsilon(F)$ denote the collection of polygons formed by the boundary triangulation $S_B(F \cup G_\varepsilon)$. Let $\mathcal{E}_B^\varepsilon := \mathcal{E}_B^\varepsilon(F)$ denote the set of all edges of these polygons. Since these edges form a subset of the edges of triangles contained in the square Q, it follows as in (9.9) that the sum of the squares of their lengths is bounded by a constant multiple of the area of Q, that is

$$\sum_{E \in \mathcal{E}_B^\varepsilon} |E|^2 \leq C$$

where $C := C(\delta)$. If Δ is a polygon in $\Delta_B^\varepsilon(F)$ with diameter D and if G is a set of points in Δ then $\sigma(G, \Delta)$ is bounded by $C \cdot D \cdot (\mathrm{card} G)^{1/2}$. D is bounded by the sum of the lengths of the edges of Δ and D^2 is bounded by a constant multiple of the sum of the squares of the lengths of the edges of Δ. The proof of smoothness (9.12) follows exactly as in the proof of the smoothness (9.7) of σ.

9.5. Closeness in Mean

We have now verified that the minimal triangulation lengths σ and σ_B are smooth subadditive and superadditive Euclidean functionals of order 1, respectively. We conclude the proof of Theorem 9.1 by showing closeness in mean (3.15) of $\sigma := \sigma_\delta$ and $\sigma_B := \sigma_{B,\delta}$. Letting $U_1, ..., U_n$ denote i.i.d. random variables with the uniform distribution on $[0,1]^2$, we will actually establish the stronger bound

$$|E\sigma(U_1, ..., U_n) - E\sigma_B(U_1, ..., U_n)| \leq C(n \log n)^{1/4} \tag{9.13}$$

which will be useful in obtaining the rate (9.2).

Given $\varepsilon > 0$ and the random variables $U_1, ..., U_n$ we recall that $G_\varepsilon := G_\varepsilon(U_1, ..., U_n)$ are the uniquely defined Steiner sets with the property that

$$|S_B(\{U_1, ..., U_n\} \cup G_\varepsilon)| \leq \sigma_B(U_1, ..., U_n) + \varepsilon.$$

Let $\Delta_B^\varepsilon(U_1, ..., U_n)$ denote the collection of polygons generated by the boundary triangulations $S_B(\{U_1, ..., U_n\} \cup G_\varepsilon)$ and let $\mathcal{E}^\varepsilon(U_1, ..., U_n)$ be the collection of edges formed from the intersection of $\Delta_B^\varepsilon(U_1, ..., U_n)$ and the interior of $[0,1]^2$. To prove (9.13) we first bound the lengths of the edges in $\mathcal{E}^\varepsilon(U_1, ..., U_n)$. This edge length bound implies that with high probability the collection $\Delta_B^\varepsilon(U_1, ..., U_n)$ contains neither hexagons nor those pentagons with a side linking opposite sides of $[0,1]^2$. Thus with high probability there are at most four pentagons in $\Delta_B^\varepsilon(U_1, ..., U_n)$.

Lemma 9.8. *(edge length bounds) With high probability all edges* $E \in \mathcal{E}^\varepsilon(U_1,...,U_n)$ *have a length* $|E|$ *satisfying*

$$|E| \le C(\log n/n)^{1/2}. \tag{9.14}$$

The meaning of the high probability statement (9.14) is: for any prescribed $\alpha > 0$ we can find $C := C(\alpha) > 0$ and a set Ω_0 with $P\{\Omega_0^c\} = O(n^{-\alpha})$ such that on Ω_0 all edges $E \in \mathcal{E}^\varepsilon(U_1,...,U_n)$ satisfy the bound

$$|E| \le C(\log n/n)^{1/2}.$$

The following argument can be modified to show this precise statement.

Proof. (sketch) The proof is a simple consequence of the fact that the aspect ratios of the polygons in $\Delta_B^\varepsilon(U_1,...,U_n)$ are bounded and therefore if an edge E belongs to $\mathcal{E}^\varepsilon(U_1,...,U_n)$, then there is a ball of radius $C|E|$ which is contained in $[0,1]^2$ and which does not contain any sample points, where $C := C(\delta)$ is a constant depending only on δ.

Indeed, for all $x \in [0,1]^2$ and $r > 0$, let $B(x,r)$ designate the ball centered at x with radius r and let $E_n(r)$ denote the event that there is an edge $E \in \mathcal{E}^\varepsilon(U_1,...,U_n)$ whose length exceeds r. Given $E_n(r)$, the bounded aspect ratio assumption implies the existence of a ball of radius at least Cr which is contained entirely within a polygon and thus does not contain any sample points. This is clearly true for edges E which do not meet the boundary. For edges E meeting the boundary there are several cases which may be checked in a straightforward fashion. In any case, there is a $C < \infty$ such that

$$E_n(r) \subset \{\exists\ x \in [0,1]^2 :\ B(x,Cr) \subset [0,1]^2,\ B(x,Cr) \cap \{U_i\}_{i\le n} = \emptyset\}. \tag{9.15}$$

Thus, $P\{E_n(r)\}$ is bounded by the probability that there is "hole" of radius at least Cr in the sample $\{U_1,...,U_n\}$. It is well-known and easy to show that with high probability holes with radius larger than $C(\log n/n)^{1/2}$ do not exist. Thus, with high probability, edges in $\mathcal{E}^\varepsilon(U_1,...,U_n)$ have length less than $C(\log n/n)^{1/2}$. □

We require one more auxiliary result before proving (9.13). To simplify the notation, write $\sigma(n)$ for $\sigma(U_1,..,U_n)$ and likewise for $\sigma_B(n)$ and $\mathcal{E}^\varepsilon(n)$. Let $S_B^\varepsilon(n) := S_B(\{U_1,...,U_n\} \cup G_\varepsilon)$. Consider the edges in $\mathcal{E}^\varepsilon(n)$ which meet the boundary of $[0,1]^2$ and let $\Sigma^\varepsilon(n) := \Sigma^\varepsilon(U_1,...,U_n)$ denote the sum of the lengths of these edges. The following lemma gives a crude yet sufficient upper bound for $\Sigma^\varepsilon(n)$.

Lemma 9.9. *For all* $0 < \varepsilon < 1$, $E\Sigma^\varepsilon(n) \le C(n\log n)^{1/4}$.

Proof. Decompose $[0,1]^2$ into a subsquare R_1 and a moat $R_2 := [0,1]^2 - R_1$; choose R_1 so that it has side length $1 - C(\log n/n)^{1/2}$ and is centered within $[0,1]^2$. Let $|S_B^\varepsilon(n) \cap R_1|$ denote the sum of the lengths of the edges in $S_B^\varepsilon(n) \cap R_1$ and similarly for $|S_B^\varepsilon(n) \cap R_2|$. By Lemma 9.8 we have that $\Sigma^\varepsilon(n) \le |S_B^\varepsilon(n) \cap R_2|$ with high probability. It will thus be enough to show

$$E|S_B^\varepsilon(n) \cap R_2| \le C(n \log n)^{1/4}.$$

Since $S_B^\varepsilon(n) \cap R_1$ is a feasible boundary triangulation of $\{U_1, ..., U_n\} \cap R_1$ with respect to R_1 we have

$$\begin{aligned} |S_B^\varepsilon(n)| &= |S_B^\varepsilon(n) \cap R_1| + |S_B^\varepsilon(n) \cap R_2| \\ &\ge \sigma_B(\{U_1, ..., U_n\} \cap R_1, R_1) + |S_B^\varepsilon(n) \cap R_2| \end{aligned}$$

where the inequality follows by the minimality of σ_B.

The number of sample points in R_1 is a binomial random variable $B(n,p)$ with parameters n and p, $p :=$ area R_1. Taking expectations and scaling we get

$$E|S_B^\varepsilon(n)| \ge \left(1 - C(\log n/n)^{1/2}\right) E\sigma_B(\{U_1, ..., U_{B(n,p)}\}) + E|S_B^\varepsilon(n) \cap R_2|.$$

By definition we have $E|S_B^\varepsilon(n)| \le E\sigma_B(n) + \varepsilon$ and thus

$$\begin{aligned} E|S_B^\varepsilon(n) \cap R_2| &\le E\sigma_B(n) + \varepsilon - \left(1 - C(\log n/n)^{1/2}\right) E\sigma_B(B(n,p)) \\ &\le E\left(\sigma_B(n) - \sigma_B(B(n,p))\right) + \varepsilon + C(\log n/n)^{1/2} E(B(n,p)^{1/2}) \end{aligned}$$

by the growth bound (9.6). By the smoothness of σ_B, the above is bounded by

$$\begin{aligned} &\le CE(|n - B(n,p)|^{1/2}) + \varepsilon + C(\log n/n)^{1/2}(np)^{1/2} \\ &\le C\left(E|B(n, 1-p)|\right)^{1/2} + C(\log n)^{1/2} \\ &\le C\left(n(1-p)\right)^{1/2} + C(\log n)^{1/2}. \end{aligned}$$

Since $p := (1 - C(\log n/n)^{1/2})^2 \ge 1 - C(\log n/n)^{1/2}$, we see that $n(1-p) \le C(n \log n)^{1/2}$, completing the proof of Lemma 9.9. □

We are now positioned to establish the estimate (9.13) and thus conclude the proof of Theorem 9.1.

Lemma 9.10. *(closeness in mean) We have*

$$|E\sigma(U_1, ..., U_n) - E\sigma_B(U_1, ..., U_n)| \le C(n \log n)^{1/4}.$$

Proof. It suffices to show for all $0 < \varepsilon < 1$ that

$$\sigma_B(n) \le \sigma(n) \le \sigma_B(n) + \Sigma^\varepsilon(n) + C.$$

Lemma 9.10 follows since it is enough to take expectations and apply Lemma 9.9. Let $0 < \varepsilon < 1$ and $S_B^\varepsilon(n)$ be as above. We claim that

$$\sigma_B(n) \le \sigma(n) \le |S_B^\varepsilon(n)| + \Sigma^\varepsilon(n) + C.$$

The first inequality follows by the definition of σ_B. To prove the second, we need to show that there is a feasible triangulation of $\{U_1, ..., U_n\}$ whose total length is bounded by $|S_B^\varepsilon(n)| + \Sigma^\varepsilon(n) + C$. Such a triangulation is obtained by triangulating the quadrilaterals and pentagons in the graph described by $S_B^\varepsilon(n)$ (recall that with high probability the graph contains no hexagons and at most four pentagons). We triangulate the quadrilaterals by adding their diagonals, the sum of whose lengths is at most the sum of $\Sigma^\varepsilon(n)$ and the perimeter of the unit square. A triangulation of the pentagons may be achieved with a cost bounded by a constant since there are at most four pentagons. Thus we have shown the claim. Since $|S_B^\varepsilon(n)| \leq \sigma_B(n) + \varepsilon$, the result follows. □

We have now proved Theorem 9.1 and turn to the proof of Theorem 9.2. The proof depends on the following general rate result, which follows from a slight modification of the proof of Theorem 5.2.

Theorem 9.11. *(rates of convergence) Let L be a smooth subadditive Euclidean functional of order 1 on $[0,1]^2$ such that the following "add-one bound" is satisfied:*

$$|EL(U_1, ..., U_n) - EL(U_1, ..., U_{n+1})| \leq C(\log n/n)^{1/2}. \tag{9.16}$$

If $|EL(U_1, ..., U_n) - EL_B(U_1, ..., U_n)| \leq \beta(n)$ where $\beta(n)$ is a function of n, then

$$|EL(U_1, ..., U_n) - \alpha(L)n^{1/2}| \leq C\left(\beta(n) \vee (\log n)^{1/2}\right). \tag{9.17}$$

Proof of Theorem 9.2. To apply Theorem 9.11 to triangulations, we must show that σ satisfies the estimate (9.16). For all $\varepsilon > 0$ and $U_1, ..., U_n$ we recall that $G_\varepsilon := G_\varepsilon(U_1, ..., U_n)$ are (uniquely defined) Steiner sets with the property that

$$|S(\{U_1, ..., U_n\} \cup G_\varepsilon)| \leq \sigma(U_1, ..., U_n) + \varepsilon.$$

Recall that $\Delta^\varepsilon(U_1, ..., U_n)$ denotes the collection of triangles generated by $S(\{U_1, ..., U_n\} \cup G_\varepsilon)$. Notice that

$$\begin{aligned}\sigma(U_1, ..., U_n) &\leq \sigma(U_1, ..., U_{n+1}) \\ &\leq |S(\{U_1, ..., U_n\} \cup G_\varepsilon)| + 3D(\varepsilon, n) \\ &\leq \sigma(U_1, ..., U_n) + 3D(\varepsilon, n) + \varepsilon, \end{aligned} \tag{9.18}$$

where $D(\varepsilon, n)$ is the diameter of the *random* triangle in $\Delta^\varepsilon(U_1, ..., U_n)$ which contains the point U_{n+1}. By Lemma 9.8 we have $D(\varepsilon, n) \leq C(\log n/n)^{1/2}$ with high probability where C doesn't depend upon ε. Letting $\varepsilon = (\log n/n)^{1/2}$ gives the high probability bound

$$|\sigma(U_1, ..., U_n) - \sigma(U_1, ..., U_{n+1})| \leq C(\log n/n)^{1/2}$$

and therefore the add-one bound

$$|E\sigma(U_1, ..., U_n) - E\sigma(U_1, ..., U_{n+1})| \leq C(\log n/n)^{1/2}.$$

Thus, by (9.13) and (9.17) we obtain

$$|E\sigma(U_1, ..., U_n) - \alpha(\sigma)n^{1/2}| \leq C(n \log n)^{1/4}$$

which is the desired estimate (9.2). This concludes the proof of Theorem 9.2. □

9.6. The Probabilistic Plateau Functional

To establish Theorem 9.3 we may adapt the above approach to the three dimensional setting. We fix D once and for all, $1 < D < \infty$. It is first helpful to enlarge the definition of $\tau := \tau_D$ in the following way. Let $F \subset \Delta$ be a finite set, where $\Delta \subset [0,1]^3$ is a convex polyhedron. Given D, a tetrahedralization $T_D(F, \Delta)$ is a decomposition of Δ into tetrahedra with aspect ratios bounded by D such that the tetrahedral vertices coincide with F and the corners of Δ. Analogously to $\tau_D(F)$, we define

$$\tau_D(F, \Delta) := \inf_G |T_D(F \cup G, \Delta)|,$$

where $|T_D(F \cup G, \Delta)|$ denotes the total surface area and where G ranges over finite sets in Δ.

We now consider the properties of the Plateau functional $\tau := \tau_D$. Notice that $\tau(\alpha F) = \alpha^2 \tau(F)$ and thus τ is homogeneous of order 2. Modifications of the proof of Lemma 9.6 show that τ is subadditive on $\mathbb{R}^3$. Also, for any $F \subset [0,1]^3, F \neq \emptyset$, we have $\tau(F) \leq C(\text{card}F)^{1/3}$ by Lemma 3.3 (here and in all that follows, $C := C(D)$ denotes a constant depending only on D and whose value may vary from line to line).

To see that τ is a smooth subadditive Euclidean functional on $\mathbb{R}^3$ of order two, it remains only to verify smoothness

$$|\tau(F \cup G) - \tau(F)| \leq C(\text{card}G)^{1/3}.$$

We will closely follow the approach used to verify the smoothness of σ. For all $\varepsilon > 0$ and $F \subset [0,1]^3$ let $G_\varepsilon := G_\varepsilon(F)$ be Steiner sets such that

$$|T_D(F \cup G_\varepsilon)| \leq \tau_D(F) + \varepsilon.$$

Let $T^\varepsilon(F) := T_D(F \cup G_\varepsilon)$. Let $\Delta^\varepsilon(F)$ denote the collection of tetrahedra defined by $T^\varepsilon(F)$ and let $\mathcal{E}^\varepsilon := \mathcal{E}^\varepsilon(F)$ denote the collection of tetrahedral edges. Since the tetrahedra have uniformly bounded aspect ratios, the sum of the cubes of the edges in $\mathcal{E}^\varepsilon$ is bounded by a finite constant C, $C := C(D)$. Using the growth bound $\tau(F) \leq C(\text{card}F)^{1/3}$, we easily establish the analog of (9.11), namely

$$\tau(F \cup G) \leq |T^\varepsilon(F)| + C \sum_{i=1}^{J} |E_i|^2 (\text{card}(G \cap \Delta_i))^{1/3},$$

where E_i is the longest edge of tetrahedron Δ_i, $\Delta_i \in \Delta^\varepsilon(F)$. Hölder's inequality, together with the bound $\sum_{i=1}^{J} |E_i|^3 \leq C$, completes the proof of smoothness.

Given a convex polyhedron $\Delta \subseteq [0,1]^3$ and $F \subset \mathbb{R}^3$ such that $\mathrm{co}(F) \supset \Delta$, we next define a "boundary tetrahedralization" of Δ with respect to F in the same way that we defined boundary triangulations. Given $1 < D < \infty$ fixed, consider a D-tetrahedralization of $\mathrm{co}(F)$. Such a tetrahedralization partitions $\mathrm{co}(F)$ into tetrahedra with aspect ratios bounded by D and whose vertices coincide with F. The boundary tetrahedralization $T_{B,D}(F,\Delta)$ of Δ with respect to F is obtained by considering the intersection of Δ and the D-tetrahedralization of $\mathrm{co}(F)$. The boundary tetrahedralization of Δ thus generates the usual tetrahedra as well as polyhedra with faces contained in the boundary of Δ. We let $|T_{B,D}(F)|$ denote the total surface area of the faces of the polyhedra in the interior of Δ. We let

$$\tau_{B,D}(F,\Delta) := \inf_G |T_{B,D}(F \cup G, \Delta)|,$$

where G ranges over all finite sets in $\mathbb{R}^3$ with the property that $\mathrm{co}(F \cup G) \supset \Delta$. We will suppress mention of D and henceforth write $\tau_B(F)$ for $\tau_{B,D}(F,[0,1]^3)$ and $\tau_B(F,\Delta)$ instead of $\tau_{B,D}(F,\Delta)$.

Given this extended definition of τ_B, observe that τ_B is a smooth superadditive Euclidean functional on $\mathbb{R}^3$ of order two. To verify smoothness

$$|\tau_B(F \cup G,[0,1]^3) - \tau_B(F,[0,1]^3)| \le C(\mathrm{card}G)^{1/3}$$

we may follow the approach of section four.

We may also show closeness in mean (3.15) of the functionals τ and τ_B with $p = 2$ and $d = 3$ there. To do this, we will follow the approach of section five. For all $\varepsilon > 0$ and $U_1, ..., U_n$, we let $G_\varepsilon(U_1, ..., U_n)$ be uniquely defined Steiner sets such that

$$|T_B(\{U_1, ..., U_n\} \cup G_\varepsilon)| \le \tau_B(U_1, ..., U_n) + \varepsilon. \tag{9.19}$$

Let $T_B^\varepsilon(n) := T_B(\{U_1, ..., U_n\} \cup G_\varepsilon)$. Let Δ_B^ε denote the collection of polyhedra generated by the tetrahedralization $T_B^\varepsilon(n)$. Let $\mathcal{E}^\varepsilon := \mathcal{E}^\varepsilon(U_1, ..., U_n)$ be the collection of edges formed from the intersection of the edges in Δ_B^ε and the interior of $[0,1]^3$ and let $\mathcal{F}^\varepsilon := \mathcal{F}^\varepsilon(U_1, ..., U_n)$ be the collection of faces formed by intersecting the faces of Δ_B^ε and the interior of $[0,1]^3$. The following estimate is the analog of Lemma 9.8. Observe that the proof of (9.20) below follows the proof of (9.14) with small modifications. Note that (9.21) follows from (9.20) and the bounded aspect ratio property of the tetrahedra.

Lemma 9.12. *With high probability all edges $E \in \mathcal{E}^\varepsilon$ satisfy the length bound*

$$|E| \le C(\log n/n)^{1/3} \tag{9.20}$$

and all faces $F \in \mathcal{F}^\varepsilon$ satisfy the area bound

$$\mathrm{area}F \le C(\log n/n)^{2/3}. \tag{9.21}$$

Consider the faces in $\mathcal{F}^\varepsilon$ which meet the boundary of $[0,1]^3$ and let $\Sigma^\varepsilon(n) := \Sigma^\varepsilon(U_1, ..., U_n)$ denote the sum of their areas. Exactly as in Lemma 9.9, we may find a rough estimate for $\Sigma^\varepsilon(n)$.

Lemma 9.13. *For all* $0 < \varepsilon < 1$ *we have* $E\Sigma^{\varepsilon}(n) \le Cn^{2/9}(\log n)^{1/9}$.

Proof. We will follow the proof of Lemma 9.9. Decompose $[0,1]^3$ into a subcube Q_1 centered within $[0,1]^3$ and a moat $Q_2 := [0,1]^3 - Q_1$. Let the edge length of Q_1 be $1 - C(\log n/n)^{1/3}$. Let $|T_B^{\varepsilon}(n) \cap Q_1|$ denote the sum of the areas of the faces in $T_B^{\varepsilon}(n) \cap Q_1$ and similarly for $|T_B^{\varepsilon}(n) \cap Q_2|$. By Lemma 9.13 we have $\Sigma^{\varepsilon}(n) \le |T_B^{\varepsilon}(n) \cap Q_2|$ with high probability. It will be enough to show that

$$E|T_B^{\varepsilon}(n) \cap Q_2| \le Cn^{2/9}(\log n)^{1/9}.$$

Now as in the proof of Lemma 9.9 we have

$$|T_B^{\varepsilon}(n)| \ge \tau_B(\{U_1, ..., U_n\} \cap Q_1, Q_1) + |T_B^{\varepsilon}(n) \cap Q_2|$$

and taking expectations we get

$$E|T_B^{\varepsilon}(n)| \ge (1 - C(\log n/n)^{1/3})^2 E\tau_B(\{U_1, ..., U_{B(n,p)}\}) + E|T_B^{\varepsilon}(n) \cap Q_2|,$$

where $B(n,p)$ denotes a binomial random variable with parameters n and $p :=$ volume of $Q_1 := (1 - C(\log n/n)^{1/3})^3$. Thus by scaling and the growth bounds for τ_B we have

$$\begin{aligned}
E|T_B^{\varepsilon}(n) \cap Q_2| &\le E\tau_B(n) + \varepsilon - (1 - C(\log n/n)^{1/3})^2 E\tau_B(B(n,p)) \\
&\le C(E|n - B(n,p)|^{1/3}) + \varepsilon + C(\log n/n)^{1/3}(np)^{1/3} \\
&\le C(n(1-p))^{1/3} + C(\log n)^{1/3} \\
&\le (nC(\log n/n)^{1/3})^{1/3} + C(\log n)^{1/3} \\
&\le Cn^{2/9}(\log n)^{1/9}.
\end{aligned}$$

This completes the proof of Lemma 9.13 □

Let $\tau(n) := \tau(U_1, ..., U_n)$. The analog of Lemma 9.10 becomes

Lemma 9.14. *(closeness in mean) We have*

$$|E\tau(U_1, ..., U_n) - E\tau_B(U_1, ..., U_n)| \le Cn^{2/9}(\log n)^{1/9}.$$

Proof. It suffices to show for all $0 < \varepsilon < 1$ that

$$\tau_B(n) \le \tau(n) \le \tau_B(n) + C(\Sigma^{\varepsilon}(n) + 1).$$

Lemma 9.14 follows by taking expectations and applying Lemma 9.13.

Let $0 < \varepsilon < 1$ and $T_B^{\varepsilon}(n)$ be as in (9.19). We claim that

$$\tau_B(n) \le \tau(n) \le |T_B^{\varepsilon}(n)| + C(\Sigma^{\varepsilon}(n) + 1). \tag{9.22}$$

The first inequality follows by definition. To prove the second, we need to show that there is a feasible tetrahedralization of $\{U_1, ..., U_n\}$ whose total surface area is bounded by $|T_B^\epsilon(n)| + C(\Sigma^\epsilon(n) + 1)$. We observe that the polyhedra given by $T_B^\epsilon(n)$ which meet the boundary are convex and may be tetrahedralized at a cost bounded by a constant multiple of the sum of the area of their faces. The combined areas of their faces is the sum of $\Sigma^\epsilon(n)$ and the area of the boundary of $[0,1]^3$. This proves (9.22). Combining (9.22) and (9.19) we obtain Lemma 9.14. □

Lemma 9.14 establishes that the Plateau functional τ and its boundary version τ_B are close in mean (3.15) when $p = 2$ and $d = 3$. We have therefore shown that τ and τ_B satisfy all the conditions of Theorem 7.1 with $p = 2$ and $d = 3$. Theorem 9.3 follows.

It is a simple matter to find rates of convergence for the mean of τ. Since we are in dimension three we can avoid appealing to Theorem 9.11. We may use the subadditivity of τ, the superadditivity of τ_B, and Lemma 9.14 to obtain the rate estimate

$$|E\tau(U_1, ..., U_n) - \alpha(\tau)n^{1/3}| \leq Cn^{2/9}(\log n)^{1/9}.$$

It is not clear whether these rates are optimal.

Notes and References

1. This chapter is based heavily on Yukich (1997a). Theorem 9.1 adds to Steele (1982) who considers the case $\delta = \infty$, i.e. the case involving no restrictions on the aspect ratios of the triangles. Steele uses geometric subadditivity of triangulations to establish (9.1) for σ_∞ in the special case that $X_1, ..., X_n$ are uniformly distributed on $[0,1]^2$. According to Steele (1982), Theorem 9.1 addresses a problem of György Turán.

2. Using the theory of smooth subadditive and superadditive Euclidean functionals, we have provided the asymptotics for the Steiner triangulation functional as well as its three-dimensional counterpart, the discrete Plateau functional. With regard to the former, we have extended the work of Steele (1982) under the regularity condition that the triangles have bounded aspect ratios. Whether one can remove or relax this condition remains open. Additional questions which merit investigation include:

(i) is there a suitable analog of Theorems 9.1 and 9.3 for triangulation and tetrahedralization functionals which do not use Steiner points?

(ii) are there asymptotics for triangulation and tetrahedralization functionals which are defined in terms of simplicial complexes?