3. SUBADDITIVE AND SUPERADDITIVE EUCLIDEAN FUNCTIONALS

3.1. Definitions

In this chapter we lay the foundation for some of the basic limit results in geometric probability. As is often the case in mathematics, we search for a structure which is general enough to encompass a wide variety of problems, yet strong enough to yield results of interest. With this in mind, we formulate the idea of subadditive and superadditive Euclidean functionals and explore their crucial properties. The definitions and properties set forth in this chapter will be used over and over in the sequel.

We have previously seen that subadditive (respectively, superadditive) functionals $L^p(\cdot, R)$, $R \in \mathcal{R}$, enjoy a subadditive (respectively, superadditive) structure over the parameter set of d-dimensional rectangles. The following conditions endow the functional $L^p(F, R)$, $(F, R) \in \mathcal{F} \times \mathcal{R}$, with a *Euclidean structure* as well:

$$\forall\, y \in \mathbb{R}^d, \quad R \in \mathcal{R}, \quad F \subset R \qquad L^p(F, R) = L^p(F + y, R + y) \tag{3.1}$$

and

$$\forall\, \alpha > 0, \quad R \in \mathcal{R}, \quad F \subset R \qquad L^p(\alpha F, \alpha R) = \alpha^p L^p(F, R). \tag{3.2}$$

Conditions (3.1) and (3.2) express the *translation invariance* and *homogeneity of order p* of L^p, respectively.

It is easy to check that many functionals defined on point sets in Euclidean space satisfy translation invariance and homogeneity. This is especially true of functionals describing the length of a given graph. The TSP, MST, and minimal matching functionals are but a few examples. Functionals describing the length of a graph with pth power weighted edges are in general homogeneous of order p. These are not the only functionals which are homogeneous of order p. The probabilistic Plateau functional, essentially the minimal triangulation problem in three dimensions, does not describe edge lengths but instead describes surface area and in this way is homogeneous of degree 2. This is considered in Chapter 9.

If a functional $L^p(F, R)$, $(F, R) \in \mathcal{F} \times \mathcal{R}$, is superadditive over rectangles and has a Euclidean structure over $\mathcal{F} \times \mathcal{R}$, then we will say that L^p is a *superadditive Euclidean functional.* Formally we have the following definition, which is central to all that follows:

Definition 3.1. Let $L^p(\emptyset, R) = 0$ for all $R \in \mathcal{R}$ and suppose L^p satisfies (3.1) and (3.2). If L^p satisfies

$$L^p(F, R) \geq L^p(F \cap R_1, R_1) + L^p(F \cap R_2, R_2), \tag{3.3}$$

whenever $R \in \mathcal{R}$ is partitioned into rectangles R_1 and R_2 then L^p *is a superadditive Euclidean functional of order p. Subadditive Euclidean functionals of order p* satisfy (3.1), (3.2), and geometric subadditivity

$$L^p(F, R) \leq L^p(F \cap R_1, R_1) + L^p(F \cap R_2, R_2) + C_1(\mathrm{diam} R)^p. \tag{3.4}$$

Remarks.

(i) As noted in Chapter 2, if $\{Q_i\}_{i=1}^{2^{dj}}$ is a partition of $[0,1]^d$ into 2^{dj} subcubes of edge length 2^{-j}, then repeated applications of (3.4) yield for $0 < p < d$

$$L^p(F, [0,1]^d) \leq \sum_{i=1}^{2^{dj}} L^p(F \cap Q_i, Q_i) + C_1 2^{(d-p)j}, \tag{3.5}$$

for a new value of the constant C_1. We will make frequent use of this estimate.

(ii) We tacitly assume that $L^p(F, R)$ takes values in $(0, \infty)$ if $\mathrm{card} F > 1$. We will also assume that $L^p(F, R)$ is a measurable function from $(\mathbb{R}^d)^n$ to $[0, \infty)$, where $F \subset \mathbb{R}^d$ and $\mathrm{card} F = n$. This assumption is rather benign, since essentially all Euclidean functionals are continuous functions of the input F in the usual sense: small changes in F with respect to the Euclidean distance produce small changes in $L^p(F, R)$.

We will also tacitly assume the finiteness condition $\sup_{y \in [0,1]^d} L^p(\{y\}, [0,1]^d) < \infty$. This assumption is highly non-restrictive and is satisfied by essentially all Euclidean functionals.

(iii) The superadditive Euclidean functionals considered here are always the canonical boundary functionals associated with some standard Euclidean functional L. We therefore will henceforth designate superadditive Euclidean functionals as L_B.

It is not clear from the definition that there are many subadditive and superadditive Euclidean functionals. The following lemma, an immediate consequence of Chapter 2, tells us that the classic Euclidean optimization problems yield subadditive and superadditive Euclidean functionals.

Lemma 3.2. *For all $p > 0$, the functionals T^p, M^p, and S^p are subadditive Euclidean functionals of order p. For all $p \geq 1$, their respective boundary versions T_B^p, M_B^p, and S_B^p are superadditive Euclidean functionals of order p.*

In the remainder of this chapter we describe the salient features of subadditive and superadditive Euclidean functionals. These properties are critical and will be used throughout the monograph.

3.2. Growth Bounds

Geometric subadditivity (3.4) leads to several non-trivial consequences for the functional L^p. It is rather surprising that subadditivity leads to growth bounds for L^p. This observation was first noticed by Rhee (1993b) and has a wide range of applications. By using (3.5) with $j = 1$ and induction arguments, we may show the following growth estimate for L^p, which was first proved by Rhee (1993b) for the case $p = 1$.

Lemma 3.3. *(growth bounds) Let L^p be a subadditive Euclidean functional of order p, $0 < p < d$. Then there exists a finite constant $C_2 := C_2(d,p)$ such that for all cubes R and all $F \subset R$ we have*

$$L^p(F,R) \le C_2(\mathrm{diam}R)^p(\mathrm{card}F)^{(d-p)/d}. \tag{3.6}$$

Proof. We follow Rhee (1993b). By homogeneity we may without loss of generality assume that R is the unit cube $[0,1]^d$. If $\{Q_i\}_{i=1}^{2^d}$ is a partition of $[0,1]^d$ into congruent subcubes of edge length $1/2$ then subadditivity (3.5) implies that

$$L^p(F,[0,1]^d) \le \sum_{i=1}^{2^d} L^p(F \cap Q_i, Q_i) + C_1',$$

where $C_1' := C_1 \cdot 2^{d-p}$. This simple subadditive estimate will be useful in the proof, which proceeds via induction on $\mathrm{card}F$.

To formulate the inductive proof, set $a := \sup_{y\in[0,1]^d} L^p(\{y\},[0,1]^d)$ and note that a is finite by assumption. Let $a_2 := \frac{C_1'}{2^{d-p}-1}$ and let $a_1 := a + \frac{d}{p}2^{d-p}a_2$. We wish to show for all $F \subset [0,1]^d$ that

$$L^p(F,[0,1]^d) \le a_1(\mathrm{card}F)^{(d-p)/d}.$$

As for the induction hypothesis itself, we will assume that the stronger bound

$$L^p(F,[0,1]^d) \le a_1(\mathrm{card}F)^{(d-p)/d} - a_2$$

holds whenever $\mathrm{card}F < n$. Note that $a \le a_1 - a_2$ and so the induction hypothesis holds when $\mathrm{card}F = 1$.

Consider the partition $\{Q_i\}_{i=1}^{2^d}$ of $[0,1]^d$. By assumption, F is not contained in any of the subcubes Q_i. Therefore for all $1 \le i \le 2^d$ we have $n_i := \mathrm{card}(F \cap Q_i) < n$ and thus by the induction hypothesis and homogeneity we have

$$L^p(F \cap Q_i, Q_i) \le 2^{-p}(a_1 n_i^{(d-p)/d} - a_2).$$

By geometric subadditivity and the assumption that $L^p(\emptyset, R) = 0$ for all rectangles R we obtain

$$L^p(F,[0,1]^d) \le 2^{-p} \sum_{i:n_i>0} (a_1 n_i^{(d-p)/d} - a_2) + C_1'.$$

Letting $m := \mathrm{card}\{i : n_i > 0\}$, Hölder's inequality implies

$$\begin{aligned} L^p(F,[0,1]^d) &\le 2^{-p} a_1 m^{p/d} n^{(d-p)/d} - 2^{-p} a_2 m + C_1' \\ &= 2^{-p} a_1 m^{p/d} n^{(d-p)/d} - 2^{-p} a_2 m + (2^{d-p} - 1) a_2, \end{aligned}$$

where the last equality uses the definition of a_2.

It suffices to show that the right side of the above is at most $a_1 n^{(d-p)/d} - a_2$, or, equivalently, that

$$a_2(2^{d-p} - 2^{-p} m) \le a_1 n^{(d-p)/d} \left(1 - 2^{-p} m^{p/d}\right).$$

Since $n \ge 1$ and $a_1 \ge \frac{d}{p} 2^{d-p} a_2$ it suffices to show

$$2^d - m \le \frac{d}{p} 2^d \left(1 - 2^{-p} m^{p/d}\right).$$

If we choose x so that $m = x2^d$ then this reduces to showing the inequality

$$1 - x \le \frac{d}{p}(1 - x^{p/d})$$

for $0 < x \le 1$ and $0 < p < d$. This last inequality follows from the observation that the graph of the function $\frac{d}{p}(1 - x^{p/d}) + x - 1$ has a minimum at $(1, 0)$. This verifies the induction step when $\mathrm{card} F = n$ and completes the proof of Lemma 3.3. $\square$

For some Euclidean functionals, there are other ways to obtain growth bounds without appealing to Lemma 3.3. For example when $p = 1$, easy arguments based on the pigeonhole principle give a direct proof of (3.6) for the standard TSP T. When L^p is the power-weighted TSP functional T^p, the bound (3.6) is also a consequence of the space filling curve heuristic, as shown by Steele (1990). In fact, using this heuristic we may easily obtain for all $p > 0$ and $d \ge 1$ the growth bound

$$L^p(F) \le C\left((\mathrm{card} F)^{(d-p)/d} \vee 1\right) \tag{3.7}$$

which is valid when L is either the TSP, MST, or minimal matching functional. (Here and elsewhere $x \vee y$ denotes the maximum of the real numbers x and y.) The interest of Rhee's bound (3.6) is that it holds for all subadditive Euclidean functionals of order p, $0 < p < d$.

The space filling curve heuristic is elegantly described by Steele (1997) and for the sake of completeness we briefly recall its applicability to the TSP. We follow the exposition of Steele (1997).

Let $\{x_1, ..., x_n\}$ be a point set in $[0,1]^d$, $d \geq 2$. To bound the length $T^p(x_1, ..., x_n)$ we consider a continuous function ϕ from $[0,1]$ *onto* $[0,1]^d$ that is Lipschitz of order $1/d$, that is for all $0 \leq s, t \leq 1$ we have

$$\|\phi(s) - \phi(t)\| \leq C|s-t|^{1/d}.$$

To find the length of a short feasible tour through $\{x_1, ..., x_n\}$ we use the space filling function ϕ and follow this recipe:

- compute a set of points $\{t_1, ..., t_n\} \subset [0,1]$ such that $\phi(t_i) = x_i$, $1 \leq i \leq n$,
- order the t_i so that $t_{(1)} \leq t_{(2)} \leq ... \leq t_{(n)}$, and
- define a permutation $\sigma : [1,n] \to [1,n]$ by requiring that $x_{\sigma(i)} = \phi(t_{(i)})$.

The feasible tour which visits $x_1, ..., x_n$ in the order of $x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}$ satisfies the length estimate

$$\begin{aligned}\sum_{i=1}^{n-1} |x_{\sigma(i)} - x_{\sigma(i+1)}|^p &= \sum_{i=1}^{n-1} |\sigma(t_{(i)}) - \sigma(t_{(i+1)})|^p \\ &\leq C^p \sum_{i=1}^{n-1} |t_{(i)} - t_{(i+1)}|^{p/d} \\ &\leq C^p(n^{1-p/d} \vee 1),\end{aligned}$$

where for $0 < p < d$ we use Hölder's inequality and for $p \geq d$ we use the estimate $\sum |t_{(i)} - t_{(i+1)}|^{p/d} \leq \sum |t_{(i)} - t_{(i+1)}| \leq 1$. Thus

$$T^p(x_1, ..., x_n) \leq C\left(n^{(d-p)/d} \vee 1\right)$$

as desired.

The space filling curve heuristic thus provides an easy way to obtain the growth bound (3.7) even for the delicate case $p \geq d$. Without the heuristic, the proof of the bound (3.7) for this last case is no easy task.

3.3. Smoothness

We will soon see that geometric superadditivity (3.3) and subadditivity (3.4) are powerful tools when used together. They gain further strength in the presence of smoothness conditions on the functional L^p, which describe the variation of L^p as points are added and deleted. Functionals which are monotone in the sense that $L(F) \leq L(F \cup x)$ for all point sets F and singletons x have variations which are easy to describe. This is the case with the TSP functional, for example. Most functionals are not so easy to describe, however, and this is where smoothness conditions become important. Smoothness, along with subadditivity and superadditivity, occupies a central role in framing the limit theorems in this monograph.

Many Euclidean functionals enjoy the following smoothness condition, forms of which were used by Steele (1988) and Avis, Davis, and Steele (1988) in the context of the MST and the greedy matching heuristic, respectively. Later, Rhee (1993b) investigated smoothness conditions for the minimal matching functional. She was the first to recognize that smoothness conditions yield isoperimetric inequalities which show that a Euclidean functional is close to its mean value. This insight is addressed more fully in Chapter 6.

Definition 3.4. (smoothness) A Euclidean functional L^p of order p is *smooth of order* p if there is a finite constant $C_3 := C_3(d,p)$ such that for all sets $F,\ G \subset [0,1]^d$ we have

$$|L^p(F \cup G) - L^p(F)| \le C_3(\mathrm{card} G)^{(d-p)/d}. \tag{3.8}$$

There are some simple consequences of smoothness (3.8), which we interpret as a Hölder continuity condition. First we notice that smoothness implies

$$|L^p(F) - L^p(G)| \le 2C_3(\mathrm{card}(F \mathbin{\Delta} G))^{(d-p)/d},$$

where $F \mathbin{\Delta} G$ denotes the symmetric difference of the sets F and G. For point sets F and G in a general rectangle R, homogeneity gives

$$|L^p(F \cup G, R) - L^p(F, R)| \le C_3(\mathrm{diam} R)^p(\mathrm{card} G)^{(d-p)/d}$$

for a possibly different choice of constant C_3. We will make frequent use of these smoothness inequalities.

The next result suggests that smoothness is rather ubiquitous. This result is not exhaustive and we will see that smoothness is a property common to many Euclidean functionals.

Lemma 3.5. *When $0 < p \le d$ the subadditive Euclidean functionals T^p, M^p, and S^p are smooth of order p. When $1 \le p \le d$ their superadditive canonical boundary versions T_B^p, M_B^p, and S_B^p are smooth of order p.*

Proof. There is no single approach which yields smoothness (3.8) simultaneously for all three functionals and we therefore prove this on a case by case basis. Still there are some common ideas. For instance, letting L denote either of the functionals of Lemma 3.5, simple subadditivity (2.2) and the growth bounds (3.7) imply that for all sets $F,\ G \subset [0,1]^d$ we have $L^p(F \cup G) \le L^p(F) + (C_1 + C_2 d^{p/2})(\mathrm{card} G)^{(d-p)/d} \le L^p(F) + C(\mathrm{card} G)^{(d-p)/d}$. Thus, (3.8) follows once we show the reverse inequality

$$L^p(F \cup G) \ge L^p(F) - C(\mathrm{card} G)^{(d-p)/d}. \tag{3.9}$$

To show (3.9) for the minimal spanning tree functional M^p, let T denote the graph of the minimal spanning tree on $F \cup G$. Remove the edges in T which contain a vertex in G. Since each vertex has bounded degree, say D, this generates a subgraph $T_1 \subset T$ which has at most $D \cdot \mathrm{card}G$ components. Choose one vertex from each component and form the minimal spanning tree T_2 on these vertices. Since the union of the trees T_1 and T_2 is a feasible spanning tree on F, it follows that

$$M^p(F) \le \sum_{e \in T_1 \cup T_2} |e|^p \le M^p(F \cup G) + C(D \cdot \mathrm{card}G)^{(d-p)/d}$$

by Lemma 3.3. Thus (3.8) holds for the MST functional M^p.

To see that (3.9) holds for the minimal matching functional S^p, we argue as follows. From the global matching on $F \cup G$, remove the edges which are incident to a vertex in G. This yields a collection $\mathcal{V}$ of unmatched vertices in F, where $\mathrm{card}\mathcal{V} \le \mathrm{card}G$. By simple subadditivity of S^p we have

$$S^p(F) \le S^p(F \cup G) + S^p(\mathcal{V}) \le S^p(F \cup G) + C(\mathrm{card}G)^{(d-p)/d}$$

and thus S^p satisfies (3.9) and is smooth.

Finally, we show that (3.9) holds for the TSP functional T^p. When $p = 1$, (3.9) follows at once from monotonicity, that is from the relation $T(F) \le T(F \cup \{x\})$ for all subsets $F \subset \mathbb{R}^d$ and singletons $x \in \mathbb{R}^d$. When $p > 1$ we need a more involved approach.

From the minimal tour on $F \cup G$, remove the edges which are incident to a vertex in G. This generates at most $\mathrm{card}G$ disconnected paths; letting $\mathcal{E}$ denote the collection of the endpoints of these paths we observe that $\mathrm{card}\mathcal{E} \le 2\mathrm{card}G$. Consider $S^p(\mathcal{E})$, the length of the minimal matching on $\mathcal{E}$. This matching, together with the disconnected paths, generates a collection of tours $\{G_i\}_{i=1}^N$ on F, where N is random. From each tour G_i, $1 \le i \le N$, select an edge E_i which is generated by the minimal matching. Let e_i denote one endpoint of E_i. Let $\mathcal{E}' := \{e_i\}_{i=1}^N$ and consider $T^p(\mathcal{E}')$, the length of the minimal tour through $\mathcal{E}'$. The minimal tour consists of oriented edges $\{H_i\}_{i=1}^N$.

We now construct a tour through F by replacing the edges $\{H_i\}_{i=1}^N$ and $\{E_i\}_{i=1}^N$ according to the following simple rule. Observing that each edge $H \in \{H_i\}_{i=1}^N$ leads to one endpoint of an edge $E := E(H) \in \{E_i\}_{i=1}^N$, we replace the pair of edges H and E by the single edge joining the tail of H to the other endpoint of E. We may perform this replacement operation for all edges $H \in \{H_i\}_{i=1}^N$ at an extra cost of at most

$$C(S^p(\mathcal{E}) + T^p(\mathcal{E}')).$$

Moreover, this construction generates a feasible tour through F. We have thus shown that

$$T^p(F) \le T^p(F \cup G) + C(S^p(\mathcal{E}) + T^p(\mathcal{E}')).$$

Since both $S^p(\mathcal{E})$ and $T^p(\mathcal{E}')$ are both bounded above by $C(\mathrm{card}\mathcal{E})^{(d-p)/d}$ and $\mathrm{card}\mathcal{E} \le 2\mathrm{card}G$, the estimate (3.9) follows as desired.

We have thus shown that the functionals T^p, M^p, and S^p are smooth of order p. The proof that the boundary functionals T_B^p, M_B^p, and S_B^p are smooth of order p follows verbatim. This completes the proof of Lemma 3.5. $\square$

3.4. Closeness of a Functional to Its Canonical Boundary Functional

Superadditivity (3.3) and subadditivity (3.4) become especially useful when the boundary functional L_B^p is close to the standard functional L^p. Closeness of the two functionals can be measured in a deterministic (sup norm) sense or in a probabilistic sense. The first way of measuring closeness, given by the following definition, is especially valuable in the analysis of partitioning heuristics and large deviations, discussed in Chapters 5 and 6, respectively.

Definition 3.6. Say that L^p and L_B^p are *pointwise close* if for all subsets $F \subset [0,1]^d$ we have

$$|L^p(F,[0,1]^d) - L_B^p(F,[0,1]^d)| = o\left((\mathrm{card}F)^{(d-p)/d}\right). \tag{3.10}$$

Many functionals are pointwise close to the canonical boundary functional. The following lemma, which is not exhaustive, illustrates this.

Lemma 3.7. *The TSP, MST, and minimal matching functionals are pointwise close to their respective boundary functionals for $1 \le p < d$ and in fact for all $F \subset [0,1]^d$ satisfy the estimate*

$$|L^p(F,[0,1]^d) - L_B^p(F,[0,1]^d)| \le \begin{cases} C(\mathrm{card}F)^{(d-p-1)/(d-1)}, & 1 \le p < d-1, \\ C\log(\mathrm{card}F), & p = d-1 \ne 1, \\ C, & d-1 < p < d, \\ & p = d-1 = 1. \end{cases}$$

Proof. We will sketch the proof for the minimal matching functional S^p. When $p = 1$ it clearly suffices to prove

$$S(F,[0,1]^d) \le S_B(F,[0,1]^d) + C(\mathrm{card}F)^{(d-2)/(d-1)}. \tag{3.11}$$

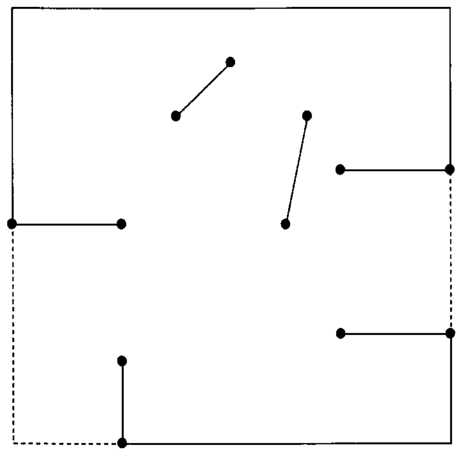
Given $S_B(F,[0,1]^d)$, let $\mathcal{B}$ be the set of points where the edges in the graph which realizes $S_B(F,[0,1]^d)$ meet the boundary of $[0,1]^d$. Let $S(\mathcal{B})$ denote the length of a minimal matching on $\mathcal{B}$ with edges lying on $\partial[0,1]^d$. By simple subadditivity and Lemma 3.3 we have

$$S(F,[0,1]^d) \le S_B(F,[0,1]^d) + S(\mathcal{B}) \le S_B(F,[0,1]^d) + C(\mathrm{card}\mathcal{B})^{(d-2)/(d-1)}$$

since $\mathcal{B}$ lies on the boundary, which has dimension $d-1$. See Figure 3.1. Since $\mathrm{card}\mathcal{B} \le \mathrm{card}F$, this proves (3.11).

Similar methods establish pointwise closeness for the MST and TSP functionals when $p = 1$. For more general p, $1 < p < d$, we require estimates for the sum of the lengths of the edges in $S_B^p(F)$ which meet the boundary. The following lemma is helpful.

Figure 3.1. Match the boundary points with the dotted edges to construct a feasible matching



Lemma 3.8. *Let $F \subset [0,1]^d$, $\mathrm{card} F = n$, and consider the graph realizing the boundary minimal matching functional $S_B^p(F)$. Let $1 \le p \le d-1$. The sum of the pth powers of the lengths of the edges connecting vertices in F with the boundary of $[0,1]^d$ is bounded by $C(n^{(d-p-1)/(d-1)} \vee \log n)$. For $d-1 < p < d$ the sum is bounded by a constant C. The same estimates hold for the boundary MST functional M_B^p and the boundary TSP functional T_B^p.*

Proof of Lemma 3.8. We first prove the lemma for the TSP functional T^p, which is the more difficult case. The proof depends upon a dyadic subdivision of $[0,1]^d$. Let Q_0 be the cube of edge length $1/3$ and centered within $[0,1]^d$. Let Q_1 be the cube of edge length $2/3$, also centered within $[0,1]^d$. Partition $Q_1 - Q_0$ into subcubes of edge length $1/6$; it is easy to verify that the number of such subcubes is bounded by $C6^{d-1}$.

Continue with the subdivision recursively, so that at the jth stage we define cube Q_j of edge length $1 - 2(3 \cdot 2^j)^{-1}$ and partition $Q_j - Q_{j-1}$ into subcubes of edge length $(3 \cdot 2^j)^{-1}$. The number of such subcubes is at most $C3^{d-1}(2^j)^{d-1}$. Carry out this recursion until the kth stage, where k is the unique integer chosen so that

$$2^{(k-1)(d-1)} \le n < 2^{k(d-1)}.$$

This procedure produces nested cubes $Q_1 \subset Q_2 \subset \dots \subset Q_k$. It produces a dyadic covering of the cube until the moat $[0,1]^d - Q_k$ has a width of the order $n^{-1/(d-1)}$. We use these properties to prove Lemma 3.8 as follows.

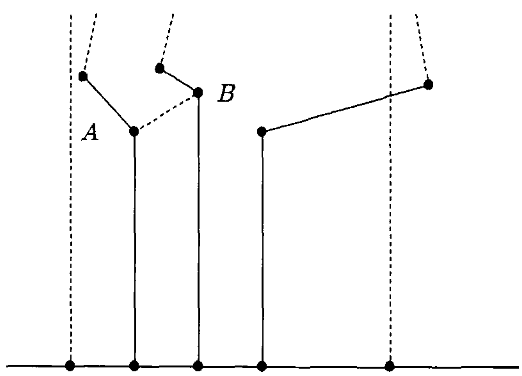
This dyadic subdivision partitions the largest cube Q_k into at most

$$\sum_{j=0}^{k} C3^{d-1}2^{j(d-1)} \le Cn$$

subcubes, each with an edge length equal to the distance between the subcube and the boundary of $[0,1]^d$. Furthermore, by partitioning each subcube of this partition into $2^{\ell d}$ congruent subcubes, where ℓ is the least integer satisfying $2^\ell \geq d^{1/2}$, we obtain a partition $\mathcal{P}$ of Q_k consisting of at most Cn subcubes with the property that the diameter of each subcube is less than the distance between it and the boundary.

Observe that in an optimal boundary tour on F each subcube Q in $\mathcal{P}$ contains at most two points in F which are rooted to the boundary. Indeed, were three or more points in $F \cap Q$ rooted to the boundary, then by minimality it would be more efficient to link two of these three points with an edge, since the diameter of the subcube is less than the distance to the boundary. See Figure 3.2.

Figure 3.2 Bounding the number of edges joined to the boundary: it is more efficient to insert edge AB



The sum of the pth powers of the lengths of the edges connecting vertices in $F \cap (Q_j - Q_{j-1})$ with the boundary is thus bounded by the product of the number of subcubes in $Q_j - Q_{j-1}$ and the pth power of the common diameter of the subcubes, namely

$$C3^{d-1}2^{j(d-1)} \cdot (3 \cdot 2^j)^{-p}.$$

Summing over all $1 \leq j \leq k$ gives a bound for the sum of the pth powers of the lengths of the edges connecting points in $F \cap Q_k$:

$$\sum_{j=1}^{k} C3^{d-1}2^{j(d-1)}(3 \cdot 2^j)^{-p} \leq \begin{cases} C(n^{(d-p-1)/(d-1)} \vee \log n), & 1 \leq p \leq d-1 \\ C, & d-1 < p < d. \end{cases} \tag{3.12}$$

The $\log n$ term is needed to cover the case $p = d-1$. The sum of the pth powers of the lengths of the edges connecting vertices in $F \cap ([0,1]^d - Q_k)$ with the boundary is at most the product of $n := \mathrm{card} F$ and the pth power of the width of the moat $[0,1]^d - Q_k$, i.e, at most

$$Cn \cdot n^{-p/(d-1)} = Cn^{(d-p-1)/(d-1)}. \tag{3.13}$$

Combining (3.12) and (3.13) establishes Lemma 3.8 for the TSP functional.

The proof for the analogous estimates involving the MST and minimal matching functionals is identical, save for the observation that there is *at most one vertex* in each subcube of $\mathcal{P}$ which is joined to the boundary. This concludes the proof of Lemma 3.8. □

We now conclude the proof of Lemma 3.7 for the minimal matching functional S^p, $1 < p < d$. Recalling that $n = \text{card}F$, we need to show

$$(3.14) \qquad S^p(F,[0,1]^d) \leq \begin{cases} S^p_B(F,[0,1]^d) + Cn^{(d-p-1)/(d-1)}, & 1 \leq p < d-1 \\ S^p_B(F,[0,1]^d) + C\log n, & p = d-1 \\ S^p_B(F,[0,1]^d) + C, & d-1 < p < d. \end{cases}$$

Consider the minimal boundary matching which realizes $S^p_B(F,[0,1]^d)$ and let $F' \subset F$ denote those vertices which are rooted to the boundary. Let $\mathcal{B} \subset \partial[0,1]^d$ denote the set of points where the rooted edges meet the boundary. Thus $\text{card}F' = \text{card}\mathcal{B}$. Our goal is to construct a feasible matching on F'.

Construct the graph G which realizes the optimal matching $S^p(\mathcal{B},[0,1]^d)$ and which has edges on the boundary of $[0,1]^d$. By (3.6) the edges in G have a total length of at most $C(n^{(d-p-1)/(d-1)} \vee 1)$. Using these edges we construct a natural pairing of points in F', which by the triangle inequality and Lemma 3.8, is achieved at a cost which is at most $C(n^{(d-p-1)/(d-1)} \vee \log n)$ for $1 < p \leq d-1$ and which is at most C for $d-1 < p < d$. This produces a feasible matching of F and therefore has a length which is clearly greater than $S^p(F)$. This shows the estimate (3.14) and concludes the proof of Lemma 3.7 for the minimal matching functional. The proof for the MST functional M^p is similar. To prove Lemma 3.7 for the TSP functional we may modify the proof of Lemma 3.10 below. This concludes the proof of Lemma 3.7. □

The pointwise closeness of functionals (3.10) expresses an estimate which is usually more than sufficient for most approximation purposes. There is a second useful way to measure closeness, one which will be sufficient for finding asymptotics and rates of convergence of means.

Definition 3.9. (close in mean) Let L^p be a Euclidean functional and L^p_B the boundary functional, $1 \leq p < d$. L^p and L^p_B are *close in mean* if

$$(3.15) \qquad |EL^p(U_1,...,U_n) - EL^p_B(U_1,...,U_n)| = o(n^{(d-p)/d}).$$

Pointwise closeness clearly implies closeness in mean. However, the approximation error associated with closeness in mean is usually smaller than the corresponding approximation error associated with pointwise closeness. The following lemma shows that this is the case for our prototypical Euclidean functionals; later we will see that Lemma 3.10 holds for many other Euclidean functionals as well.

Lemma 3.10. *Let $1 \le p < d$. The pth power weighted TSP, MST, and minimal matching functionals are close in mean to their respective boundary functionals and satisfy the approximation*

$$|EL^p(U_1, ..., U_n) - EL^p_B(U_1, ..., U_n)| \le M(d, p, n), \tag{3.16}$$

where $M(d,p,n) := Cn^{(d-p-1)/d}$ for $1 \le p < d-1$, $M(d,p,n) := C \log n$ for $p = d-1$, and $M(d,p,n) := C$ for $d-1 < p < d$.

Proof. We will prove Lemma 3.10 for the TSP functional; the proofs for the MST and minimal matching functionals are similar. Since $T^p_B \le T^p$, it suffices to show

$$ET^p(U_1, ..., U_n,) \le ET^p_B(U_1, ..., U_n) + M(d, p, n). \tag{3.17}$$

Let F denote one of the faces of $[0,1]^d$. Letting $\mathcal{U}_F \subset \{U_1, ..., U_n\}$ be the set of points that are rooted to F by T^p_B, we first show that $E\text{card}\, \mathcal{U}_F \le Cn^{(d-1)/d}$. For all $\epsilon > 0$ and $x \in F$, let $C(\epsilon, x)$ denote the cylinder in $[0,1]^d$ determined by the ϵ disk in F centered at x. We now make the crucial observation that in the part of $C(\epsilon, x)$ which is at a distance greater than ϵ from F, there are at most two sample points which are joined to F by T^p_B. Were there three or more points, then two of these points could be joined with an edge, which would result in a cost savings, contradicting optimality (recall Figure 3.2). Since F can be covered with $O(\epsilon^{-(d-1)})$ disks of radius ϵ, we have the bound

$$E\text{card}\, \mathcal{U}_F \le E\text{card}\{x \in (U_i)_{i \le n} : d(x, F) \le \epsilon\} + C\epsilon^{-(d-1)},$$

where $d(x, F)$ denotes the distance between the point x and the set F. The above is bounded by $n\epsilon + C\epsilon^{-(d-1)}$ and so putting $\epsilon = n^{-1/d}$ gives the desired estimate $E\text{card}\, \mathcal{U}_F \le Cn^{(d-1)/d}$. If $\mathcal{U} \subset \{U_1, ..., U_n\}$ denotes the set of sample points which are rooted to any face of the boundary, then

$$E\text{card}\, \mathcal{U} \le Cn^{(d-1)/d}. \tag{3.18}$$

To prove (3.17), consider the boundary tour T which achieves $T^p_B(U_1, ..., U_n, [0,1]^d)$. Let $\mathcal{B}$ denote the set of points where T meets the boundary; each point in $\mathcal{B}$ is thus the endpoint of some rooted path through some subset of $\{U_1, ..., U_n\}$. Let $S^p(\mathcal{B})$ denote the length of the minimal matching on $\mathcal{B}$ whose edges lie on $\partial[0,1]^d$. Since $\mathcal{B}$ lies in a subset of dimension $d-1$, it follows from the growth bounds (3.6) that $S^p(\mathcal{B}) \le C\left((\text{card}\mathcal{B})^{(d-p-1)/(d-1)} \vee 1\right)$. By the estimate (3.18) and Jensen's inequality it follows that

$$ES^p(\mathcal{B}) \le C(n^{(d-p-1)/d} \vee 1). \tag{3.19}$$

The matching G which realizes $S^p(\mathcal{B})$ takes paths rooted to the boundary and forms a collection of closed tours $(T_i)_{i=1}^N$ on the sample $\{U_1, ..., U_n\}$, where N is random. From each tour T_i, $1 \leq i \leq N$, select one edge E_i, $E_i \in G$. Let e_i denote one endpoint of E_i. Let $\mathcal{E} := \{e_i\}_{i=1}^N$ and let $T^p(\mathcal{E})$ be the length of the minimal tour through $\mathcal{E}$ with edges on $\partial[0,1]^d$. The minimal tour consists of oriented edges, say $\{H_i\}_{1\leq i\leq N}$. Note that as in (3.19)

$$ET^p(\mathcal{E}) \leq C(n^{(d-p-1)/d} \vee 1). \tag{3.20}$$

We now produce a tour through $\{U_1, ..., U_n\} \cup \mathcal{B}$ by replacing the edges $\{H_i\}_{i=1}^N$ and $\{E_i\}_{i=1}^N$ according to the following natural rule. Noting that each edge $H \in \{H_i\}_{1\leq i\leq N}$ leads to one endpoint of an edge $E := E(H) \in \{E_i\}_{i=1}^N$, replace H and E by the edge F joining the tail of H to the other endpoint of E. Thus F joins two points in $\mathcal{B}$. Performing this replacement operation for all edges H in $\{H_i\}_{1\leq i\leq N}$ incurs a total cost of at most $C(S^p(\mathcal{B}) + T^p(\mathcal{E}))$. Moreover, this construction generates a tour T' through $\{U_1, ..., U_n\} \cup \mathcal{B}$.

We may generate a tour through $\{U_1, ..., U_n\}$ from the tour T' by replacing each edge F according to the following natural scheme: if F has endpoints f_1 and f_2 then consider the three edges in T' which are joined to either f_1 or f_2. The union of these edges is a path of the form a, f_1, f_2, b and we may replace the three edges with the *single* edge ab at an additional cost of at most

$$C(\|f_1 - f_2\|^p + \|a - f_1\|^p + \|b - f_2\|^p).$$

We have thus shown

$$T^p(U_1, ..., U_n) \leq T_B^p(U_1, ..., U_n) + C(S^p(\mathcal{B}) + T^p(\mathcal{E}) + \Sigma^p(\mathcal{B})), \tag{3.21}$$

where $\Sigma^p(\mathcal{B})$ denotes the sum of the pth powers of the lengths of the edges in T which meet the boundary. A variation of Lemma 3.8 (choose k so that $2^{(k-1)d} \leq n < 2^{kd}$) shows that $E\Sigma^p(\mathcal{B}) \leq M(d,p,n)$. Taking expectations in (3.21) and applying (3.19)-(3.20) gives the estimate (3.17), as desired. This concludes the proof of Lemma 3.10. □

There is a second way to prove Lemma 3.10 which is similar to the proof of Lemma 3.8: we simply use a dyadic covering of the cube until the moat has width of the order $n^{-1/d}$. See Redmond and Yukich (1994) for details.

Notes and References

1. Condition (3.16) suggests that the functionals L^p and L^p_B are nearly additive, thus prompting Redmond and Yukich (1994) to use the appellation "quasi-additive".

2. The random version of the TSP received attention prior to the celebrated paper of Beardwood, Halton, and Hammersley (1959). We mention the work of Mahalanobis (1940), Jessen (1942), Marks (1948), and Few (1955).

3. We will not be concerned with the numerical value of the constant C_2 in (3.6). There has been nonetheless considerable work in this area. When L is the TSP functional, $p = 1$, and R is the unit square, for example, Few (1962) showed that C_2 can be taken to be $(4/3)^{(1/4)} + \epsilon$ for all sets F having cardinality larger than $N(\epsilon)$.

4. The space filling curve heuristic is described at length in Steele (1997). We have followed parts of his exposition very closely.

5. We expect that the $\log n$ term in the estimate (3.16) can be removed and that (3.16) can be improved to

$$|EL^p(U_1, ..., U_n) - EL^p_B(U_1, ..., U_n)| \leq C(n^{d-p-1)/d} \vee 1).$$

This is the case if L is the MST functional (Redmond and Yukich (1996)).