CS 6100

Lecture - 6 + 7

Basics of probability Theory

Ref :- Heiko Röglin's notes on Prob. anal. of algorithms. (available in the google shared folder).

Recall

Analysis of 2-opt with manhatten-metric (instead of Euclidean)

→ random points distributed on $[0,1]^2$, $[0,1]^d$ $d \geq 2$

$$\Delta(S) = |x_1 - x_2| + \ldots$$

$$\Pr[\Delta(S) \leq \Delta_{min}]$$

Probability Spaces

Discrete Probability Space (Ω, P).

- Ω : samplespace. finite or countable set.
  - set of all possible outcomes.

- $p : \Omega \to [0,1]$.

* $\sum_{x \in \Omega} p(x) = 1$

An event $S \subseteq \Omega$.

$P(S) = \sum_{x \in S} p(x)$.

die with six faces.

$\{1, \dots 6\}$

$\Omega = \{1, \dots 6\}$.

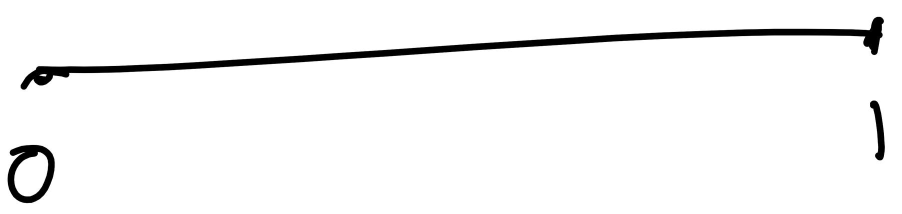
$p(1) = p(2) = \dots p(6) = \frac{1}{6}$

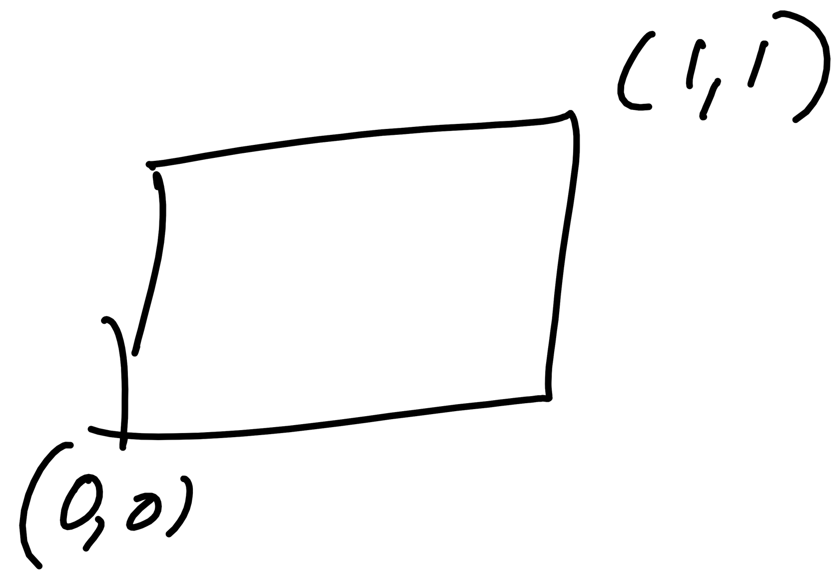
event - outcome is even no

$S = \{2, 4, 6\}$.

$p(S) = \frac{1}{2}$

$[0,1]$





$[-\epsilon, \epsilon]$.

Continuous case

A probability space is $(\Omega, \mathcal{F}, P)$ s.t

1. Ω : sample space
2. $\mathcal{F} \subseteq 2^{\Omega}$ with the following properties.
   (i) $\Omega \in \mathcal{F}$
   (ii) $\mathcal{F}$ should be closed under complementation
   ie $X \in \mathcal{F} \Rightarrow \Omega \setminus X \in \mathcal{F}$.

(iii) $\mathcal{F}$ is closed under countable unions

ie if $X_1, X_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_i X_i \in \mathcal{F}$.

{ie $\mathcal{F}$ should be a σ-algebra).

3. P is a probability measure

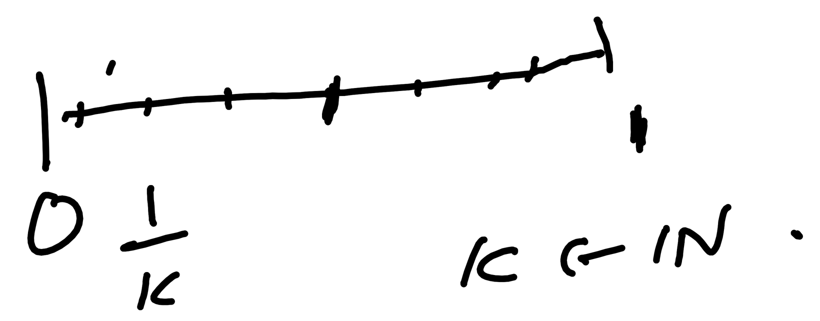
ie $P : \mathcal{F} \Rightarrow [0,1]$

(i) $P(\Omega) = 1$

(ii) if $X_1, X_2, \ldots \in \mathcal{F}$ s.t $X_i \cap X_j = \phi$ (disjoint)

then $P(\bigcup_i X_i) = \sum_i P(X_i)$

$[0,1]$



$\frac{1}{1}, \frac{1}{2}, \ldots,$

example of $\mathcal{F}$: Borel σ-algebra.

set of all intervals in $\mathbb{R}$.

& their unions.

Lemma: $X_1, X_2 \ldots X_n$ are events in $(\Omega, \mathcal{F}, P)$.

then $P[X_1 \cup X_2 \cup \cdots \cup X_n] \leq \sum_{i=1}^{n} P(X_i)$

$\rightarrow$ Union bound

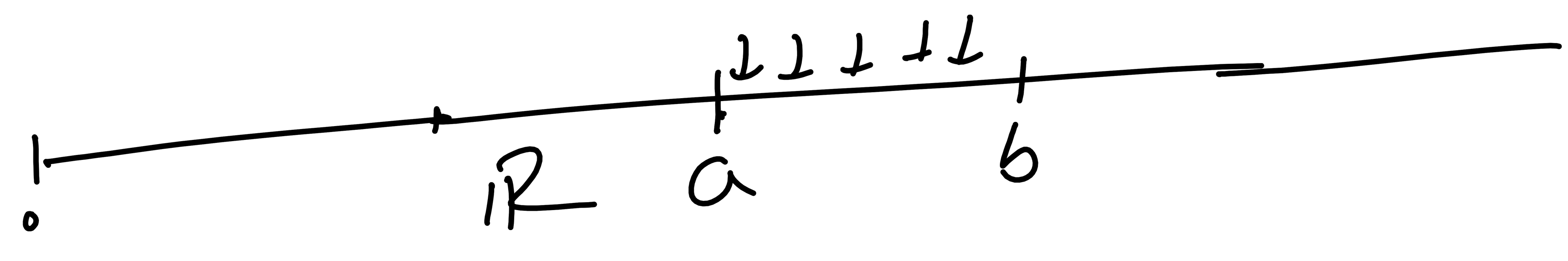
Random variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ - prob. space.

A random variable is a fn $X : \Omega \to \mathbb{R}$. X: continuos fn
(real valued).

S.t for an $a \leq b$.
$$X^{-1}([a,b]) = \{ z \in \Omega \mid X(z) \in [a,b] \} \in \mathcal{F}.$$

X



$\Omega = [0,1]$ $\qquad$ $X : \Omega \to \mathbb{R}$. $\qquad$ $P_1 \cdots P_n$
$\alpha \mapsto \alpha^2$

eg:- cost of a TSP tour
value of $\Delta(S)$
$\Delta(S) \geq \cdots$
$\Delta(S) = -$

X

$\Pr[X = x]$: probability that r.v X takes the value x.

$\Pr[X = x] \overset{\Delta}{=} P[X^{-1}(\{x\})]$ $\qquad$ $\Pr[X \leq t] \overset{\Delta}{=} P[X^{-1}([-\infty, t])]$

Cumulative Distribution Function (CDF) & Probability density function (PDF).

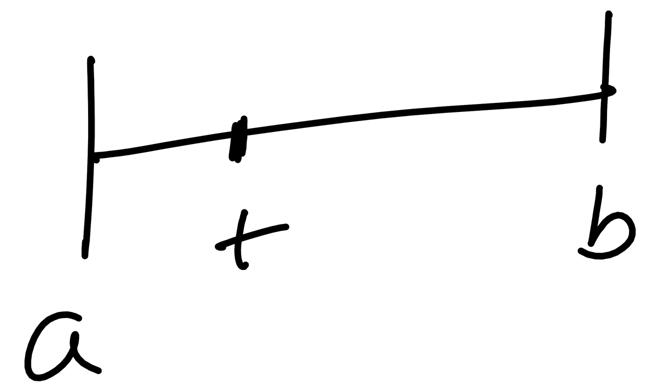
Let X be a continus random variable. The CDF of X is the fn $F_X : \mathbb{R} \to \mathbb{R}_{\geq 0}$ given by
$$\forall t \quad F_X(t) = \Pr[X \leq t]$$

A funtion $f_X : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is called PDF of X if.

$$\boxed{\Pr[X \in [a,b]] = \int_a^b f_X(t)\,dt}$$

Examples :

① X be a random variable that is uniformly distributed in the interval $[a,b]$



$$\Pr[X \le t] = \frac{t-a}{b-a}.$$

$$F_X(t) = \Pr[X \le t] = \begin{cases} 0 & t < a \\ \frac{t-a}{b-a}. & \\ 0 & t > b \end{cases}$$

$$f_X(t) = \begin{cases} 0 & t > b \text{ or } t < a \\ \frac{1}{b-a}. & \end{cases}$$

Gaussian / Normal distribution

Parameter : μ : mean.

σ : standard deviation.

The PDF of a normal distribution with mean μ & std. dev σ is given by

$$f_X(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{(t-\mu)^2}{2\sigma^2}\right)} \longrightarrow (*)$$

Lecture 7

Exercise.

→ More examples of random variables

→ Expectation and variance.

→ Conditional probability.

→ Interval lemmas

Obs:- $f_X(t) = \frac{d(F_X(t))}{dt}$

Expected value (mean)

Suppose X is a discrete random variable taking values from S.

Then the expectation of X

$$E[X] \triangleq \sum_{x \in S} x \cdot \Pr[X = x]$$

For X continuous

$$\boxed{E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx.}$$

Example:- X be uniformly distributed in $[0, 1]$

$$f_X(t) = 1$$

$$E[X] = \int_0^1 x f(x) dx = \int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

X uniform in $[a, b]$

$$E[X] = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

Ref: Sheldon M Ross.

② X be s.t the PDF is given by

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^1 x \cdot 2x\,dx = \left[\frac{2x^3}{3}\right]_0^1$$

$$= 2/3$$

③ Let X be given by

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

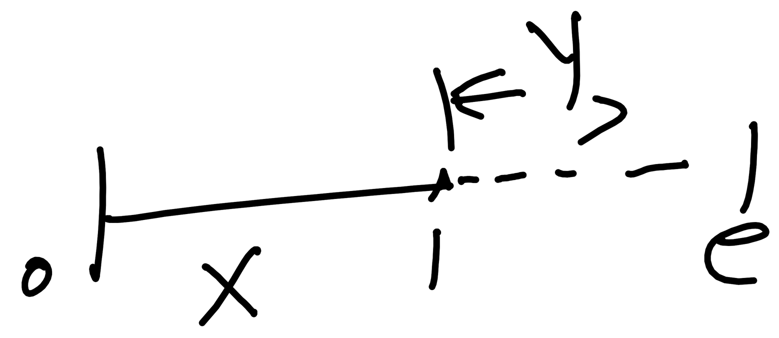
X

e^X, x^2,

$x^3 - x^2 + X$

To compute $E[e^X]$

$Y = e^X$. Y is a random variable.



$$F_Y(x) = \Pr[Y \le x] = \Pr[e^X \le x] = \Pr[X \le \log x]$$

$$= \int_0^{\log x} f_X(t)\,dt = \int_0^{\log x} 1\,dt$$

$$= \log x$$

~~Now~~ ie $F_Y(x) = \log x$

$$f_Y(x) = \frac{d F_Y(x)}{dx} = \frac{d \log x}{dx} = \frac{1}{x}.$$

for $\phi \le x \le e$

Then $\mathbb{E}[Y] = \int_1^e x \cdot f_Y(x)\, dx = \int_1^e x \cdot \frac{1}{x}\, dx$

$= \underline{e-1}$

Exercies : Compute $\mathbb{E}[X^3 - X^2]$

$\mathbb{E}[(X+1)^2]$

Lemma:- Let X be a non-negative random variable.

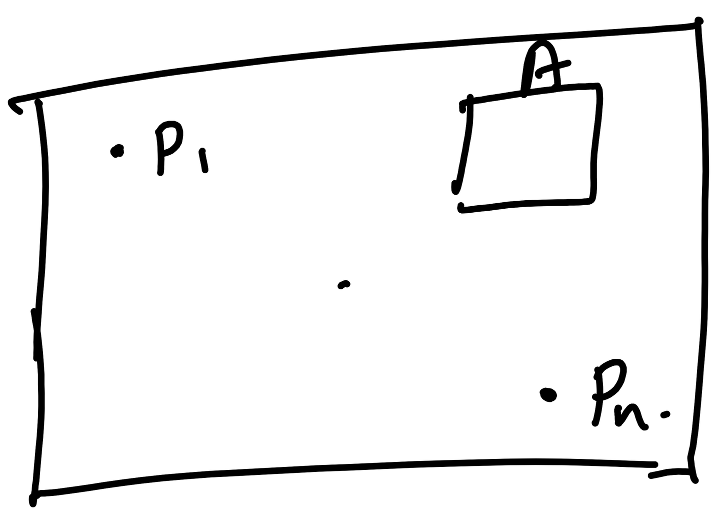
Then $\mathbb{E}[X] = \int_0^\infty \Pr[X \geq x]\, dx.$

Ex:- Prove this

Lemma: Let g be any function. Then

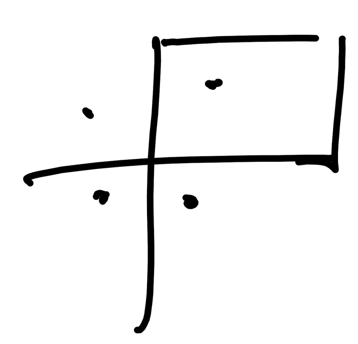
$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \cdot dx.$

Ex:- ① A be a square of area a in $[0,1]^2$



Suppose n points are sampled uniformly and independenty at random from $[0,1]^2$.

What is the expected number of points inside A ?



② Suppose $P_1, \cdots P_n$ are n points in $[0,1]^2$

$\xi_1, \cdots, \xi_n$ be n- points distributed independ. according to Gaussian with $\mu = 0$ & s.dev σ. Consider $P_1+\xi_1, P_2+\xi_2, \cdots, P_n+\xi_n$.

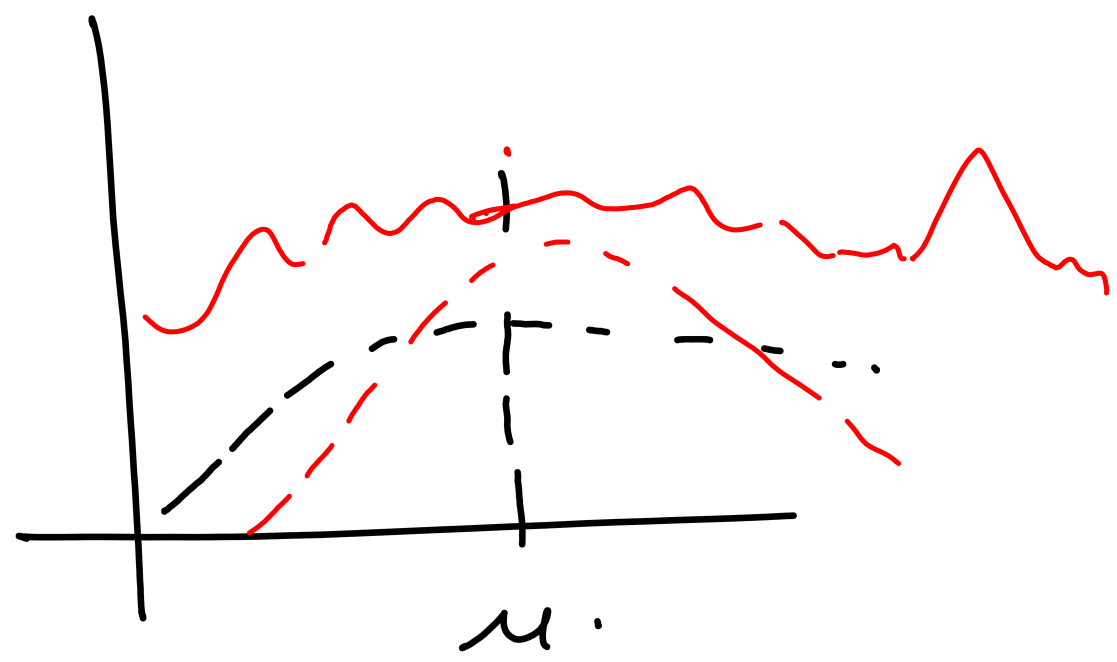
Compute: the expected # of points that lie outside $[0,1]^2$.

<u>Lemma</u> [Linearity of Expectation] Let X & Y be two random variables

then $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

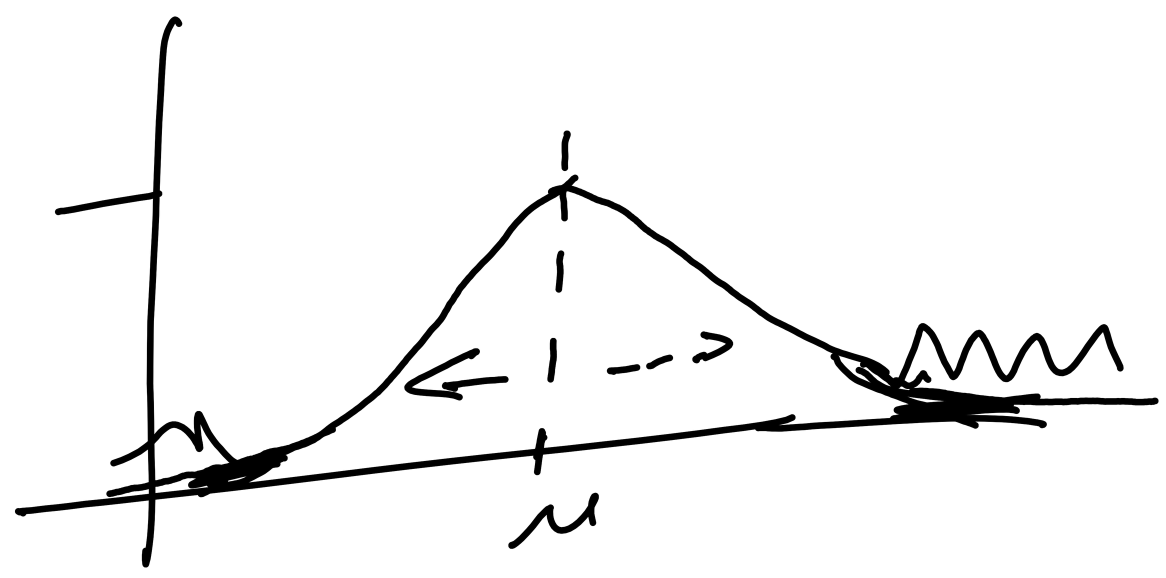
<u>Defn</u>:- [Variance].

$$\mathrm{var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$



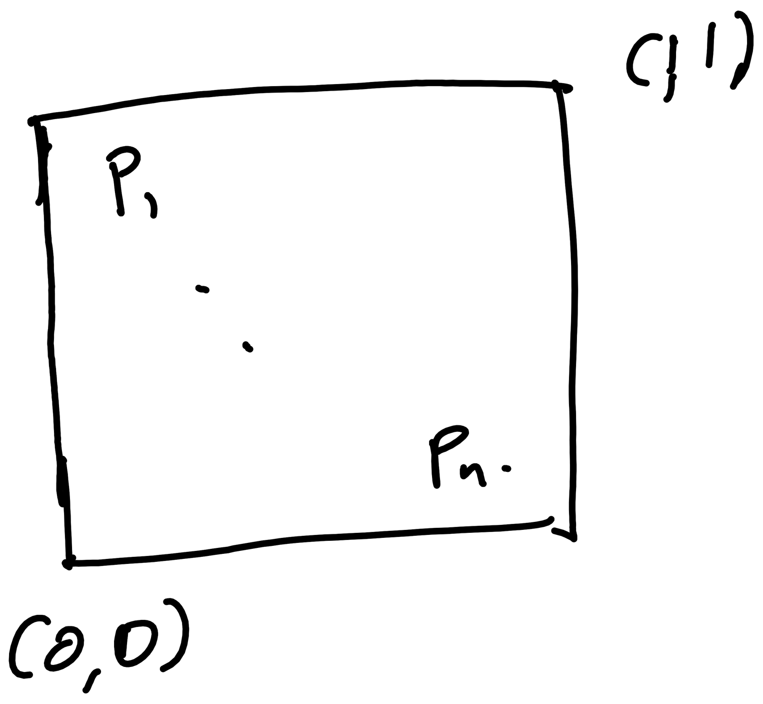
$Y = X - \mathbb{E}[X]$

<u>Obs</u>:- $\mathrm{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$



Ex:- ① Let $p_1, \dots p_n \in [0,1]^2$.

$T = \{p_1, \dots p_n\}$.



Show that

$$TSP(T) = O(\sqrt{n}).$$

② Show that the above bound is tight.