4. ASYMPTOTICS FOR EUCLIDEAN FUNCTIONALS: THE UNIFORM CASE

The goal of this chapter is to provide a simple and natural approach to finding the asymptotics of the lengths of graphs associated with problems in geometric probability. This includes the classic problems described in the previous chapters as well as others which are less widely known. The structural properties developed in Chapter 3 facilitate our approach.

Much of this chapter and indeed the entire monograph is inspired by the pioneering paper of Beardwood, Halton, and Hammersley (1959) who showed that the TSP functional T satisfies

$$\lim_{n \to \infty} T(U_1, ..., U_n)/n^{(d-1)/d} = \alpha(T, d) \quad a.s. \tag{4.1}$$

Here and throughout $U_1, ..., U_n$ are i.i.d. with the uniform distribution on $[0,1]^d$ and $\alpha(T, d)$ is a positive constant depending only on the dimension d. Beardwood, Halton, and Hammersley actually proved much more and considered the behavior of the TSP functional on sequences of random variables more general than the uniform random variables; this generalization, while important, does not concern us here and we will return to it in Chapter 7.

The limit (4.1) is easily explained on intuitive grounds. Indeed, one expects that the average length of a typical edge in the tour on $U_1, ..., U_n$ would be of the order $n^{-1/d}$, the distance between a typical point and its nearest neighbor. Since there are n edges, one anticipates an average growth rate of $n^{(d-1)/d}$. The limit (4.1) makes this precise. In this chapter we show that (4.1) holds for general subadditive Euclidean functionals.

Borovkov (1962) was apparently the first to use (4.1) in the study of algorithms for the Euclidean TSP, an NP-complete problem (cf. Papadimitriou, 1978b). He used (4.1) to show the existence of a feasible tour whose length is a.s. within a small multiple of the minimal tour length.

Karp (1976, 1977) put (4.1) to striking use in the analysis of partitioning algorithms approximating the shortest tour. As mentioned earlier, Karp showed that in the stochastic setting we may use (4.1) to exhibit a polynomial time algorithm such that the algorithm a.s. provides a solution that is within a factor of $1 + \epsilon$ of the length of the minimal tour. Karp's algorithm involves a partitioning heuristic which consists of tying together $O(n/\log n)$ optimal tours on $O(n/\log n)$ subsquares to obtain a grand tour which suitably approximates the optimal tour. Rephrasing Karp's theorem, one often says that the partitioning heuristic is with probability one ϵ-optimal. Karp's results did much to stimulate interest in stochastic versions of combinatorial optimization problems. We will return to Karp's partitioning heuristic in Chapter 5.

The limit (4.1) and its generalizations also play an important role in the probabilistic evaluation of the performance of heuristic algorithms for vehicle routing problems. Haimovich and Rinnooy Kan (1985) use (4.1) to analyze heuristics for the capacitated vehicle routing problem. For a survey of these algorithms we refer to Karp and Steele (1985) and Haimovich et al. (1988).

In the sequel we will need to prove limit results in the sense of *complete convergence.* We recall that a sequence of random variables X_n, $n \geq 1$, converges completely (c.c.) to a constant C if and only if for all $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} P\{|X_n - C| > \epsilon\} < \infty.$$

The main attraction of complete convergence is not that it strengthens a.s. convergence, but that it provides convergence results for the two distinctly different ways to interpret the dependence of the functionals $L^p(X_1, ..., X_n)$ and $L^p(X_1, ..., X_n, X_{n+1})$. The dependence of these functionals on one another has not been made explicit. Given the functional $L^p(X_1, ..., X_n)$, one can increment the number of existing sample points by one to get the new functional $L^p(X_1, ..., X_n, X_{n+1})$; this is the so-called *incrementing model of problem generation.* However, one can also consider the functional which is based on a completely new sample of points $\{X'_1, ..., X'_n, X'_{n+1}\}$ to get the new functional $L^p(X'_1, ..., X'_n, X'_{n+1})$. This second method is the *independent model of problem generation.* The difference between the limit theory for the two models is analogous to the difference between the limit theory of sequences and triangular arrays of random variables. A.s. limit results for the independent model imply a.s. limits for the incrementing model, but without extra assumptions on L^p, the converse is false in general.

To prove a.s. limits for both models of problem generation we will show the complete convergence of $L^p(X_1, ..., X_n)/n^{(d-p)/d}$. Notice that the "hard" half of the Borel-Cantelli lemma shows that c.c. results are necessary if one is to obtain a.s. asymptotics in the context of the independent model. Weide (1978) was the first to recognize the need for complete convergence results in the probabilistic analysis of algorithms.

Thus, given an optimization problem in the form of a smooth subadditive Euclidean functional L^p, we henceforth will not specify the underlying model of problem generation since complete convergence handles both models alike.

4.1. Limit Theorems for Euclidean Functionals L^p, $1 \le p < d$

When $p = 1$ the limiting a.s. behavior of the TSP functional T and its cousins M and S is relatively well understood. This is due largely to the seminal work of Steele (1981a, 1988, 1990b). Steele (1981a) showed that the asymptotics (4.1) for the TSP could be generalized to a wide class of optimization functionals. He showed that if a subadditive Euclidean functional L is *monotone* in the sense that $L(F) \le L(F \cup x)$ for sets $F \subset \mathbb{R}^d$ and singletons $x \in \mathbb{R}^d$ then L enjoys the asymptotics

$$\lim_{n\to\infty} L(U_1, ..., U_n)/n^{(d-1)/d} = \alpha(L, d) \quad a.s., \tag{4.2}$$

where $\alpha(L, d)$ is a positive constant depending only on the functional L and the dimension d. Since the TSP is monotone, (4.2) obviously gives asymptotics for the TSP functional as a special case.

However, many Euclidean functionals, including the minimal spanning tree and minimal matching functionals, are not monotone. Inserting points may reduce the total edge length in the minimal spanning tree and minimal matching graphs. This source of bedevilment has been an obstacle to the development of asymptotics.

Rhee (1993b) overcame this obstacle. She recognized that (4.2) continues to hold whenever the Euclidean functional L is *smooth* in the sense of (3.8), namely whenever for all F, $G \subset [0,1]^d$ we have

$$|L^p(F \cup G) - L^p(F)| \le C_3(\mathrm{card} G)^{(d-p)/d}.$$

We recall from Lemma 3.5 that the TSP, MST, and minimal matching functionals are smooth and in the sequel we will see that many other Euclidean functionals are smooth. By using smoothness as a substitute for monotonicity, Rhee captures the a.s. asymptotics for the TSP, MST, and minimal matching functionals.

In the spirit of Rhee (1993b) we now formulate a general asymptotic result lying at the heart of our subject.

Theorem 4.1. *(basic limit theorem for Euclidean functionals L^p, $1 \le p < d$) If L_B^p is a smooth superadditive Euclidean functional of order p on $\mathbb{R}^d$, $1 \le p < d$, then*

$$\lim_{n\to\infty} L_B^p(U_1, ..., U_n)/n^{(d-p)/d} = \alpha(L_B^p, d) \quad c.c., \tag{4.3}$$

where $\alpha(L_B^p, d)$ is a positive constant. If L^p is a Euclidean functional of order p on $\mathbb{R}^d$, $1 \le p < d$, which is pointwise close to L_B^p, then

$$\lim_{n\to\infty} L^p(U_1, ..., U_n)/n^{(d-p)/d} = \alpha(L_B^p, d) \quad c.c. \tag{4.4}$$

Remark. Since $L^p(U_1, ..., U_n) \le C_3 n^{(d-p)/d}$ always holds for smooth L, it follows by the bounded convergence theorem that (4.3) and (4.4) also hold for the means of L^p_B and L^p.

Since the optimization functionals T^p, M^p, and S^p are smooth subadditive Euclidean functionals (Lemma 3.5) which are pointwise close to their respective boundary versions (Lemma 3.7), we immediately deduce our first asymptotic result for optimization functionals on random samples:

Corollary 4.2. *Let* $1 \le p < d$. *If* L^p *denotes either the TSP, MST, or minimal matching functional of order p, then*

$$\lim_{n\to\infty} L^p(U_1, ..., U_n)/n^{(d-p)/d} = \alpha(L^p_B, d) \quad c.c.$$

Later, we will see that the basic limit Theorem 4.1 admits an extension which allows us to replace the uniform random variables by arbitrary i.i.d. random variables X_i, $i \ge 1$. The only additional assumption required for this extension is closeness in mean (3.15) of L^p and L^p_B. See Theorems 7.1 and 7.5.

Proof of Theorem 4.1. (Sketch) We only prove a mean version of (4.3), namely

$$\lim_{n\to\infty} EL^p_B(U_1, ..., U_n)/n^{(d-p)/d} = \alpha(L^p_B, d). \tag{4.5}$$

Later, isoperimetric methods will show that the mean version is equivalent to the c.c. version (see Corollary 6.4 of Chapter 6).

To prove (4.5), fix $1 \le p < d$ and set $\phi(n) := EL^p_B(U_1, ..., U_n)$. Observe that the number of points from the sample $(U_1, ..., U_n)$ which fall in a given subcube of $[0,1]^d$ of volume m^{-d} is a binomial random variable $B(n, m^{-d})$ with parameters n and m^{-d}. It follows from the superadditivity of L^p_B, homogeneity (3.2), smoothness (3.8), and Jensen's inequality that

$$\begin{aligned}
\phi(n) &\ge m^{-p} \sum_{i \le m^d} \phi(B(n, m^{-d})) \\
&\ge m^{-p} \sum_{i \le m^d} \left(\phi(nm^{-d}) - C_3 E(|B(n, m^{-d}) - nm^{-d}|^{(d-p)/d})\right) \\
&\ge m^{-p} \sum_{i \le m^d} \left(\phi(nm^{-d}) - C_3 (nm^{-d})^{(d-p)/2d}\right).
\end{aligned}$$

Simplifying, we get

$$\phi(n) \ge m^{d-p}\phi(nm^{-d}) - C_3 m^{(d-p)/2} n^{(d-p)/2d}.$$

Dividing by $n^{(d-p)/d}$ and replacing n by nm^d yields the homogenized relation

$$\frac{\phi(nm^d)}{(nm^d)^{(d-p)/d}} \geq \frac{\phi(n)}{n^{(d-p)/d}} - \frac{C_3}{n^{(d-p)/2d}}. \tag{4.6}$$

Set $\alpha := \alpha(L^p_B, d) := \limsup_{n\to\infty} \phi(n)/n^{(d-p)/d}$ and note that $\alpha \leq C_3$ by the assumed smoothness. For all $\epsilon > 0$, choose n_o such that for all $n \geq n_o$ we have $C_3/n^{(d-p)/2d} \leq \epsilon$ and $\phi(n_o)/n_o^{(d-p)/d} \geq \alpha - \epsilon$. Thus, for all $m = 1, 2, ...$ it follows that

$$\phi(n_o m^d)/(n_o m^d)^{(d-p)/d} \geq \alpha - 2\epsilon.$$

To now obtain (4.5) we use the smoothness of L and a simple interpolation argument. For an arbitrary integer $k \geq 1$ find the unique integer m such that

$$n_o m^d < k \leq n_o(m+1)^d.$$

Then $|n_o m^d - k| \leq C n_o m^{d-1}$ and by smoothness (3.8) we therefore obtain

$$\begin{aligned}\frac{\phi(k)}{k^{(d-p)/d}} &\geq \frac{\phi(n_o m^d)}{(n_o(m+1)^d)^{(d-p)/d}} - \frac{(Cn_o m^{d-1})^{(d-p)/d}}{(m+1)^{d-p}\; n_o^{(d-p)/d}} \\ &\geq (\alpha - 2\epsilon)(\frac{m}{m+1})^{d-p} - \frac{(Cn_o m^{d-1})^{(d-p)/d}}{(m+1)^{d-p}\; n_o^{(d-p)/d}}.\end{aligned}$$

Since the last term in the above goes to zero as m goes to infinity, it follows that

$$\liminf_{k\to\infty} \phi(k)/k^{(d-p)/d} \geq \alpha - 2\epsilon. \tag{4.7}$$

Now let ϵ tend to zero to see that the liminf and the limsup of the sequence $\phi(k)/k^{(d-p)/d}$, $k \geq 1$, coincide, that is

$$\lim_{k\to\infty} \phi(k)/k^{(d-p)/d} = \alpha.$$

We have thus shown

$$\lim_{n\to\infty} EL^p_B(U_1, ..., U_n)/n^{(d-p)/d} = \alpha$$

as desired. We will see in Section 4.4 that α is positive. This completes the proof of (4.5).

The limit (4.4) is automatic and the proof of Theorem 4.1 is complete. □

The above proof takes advantage of the self-similarity properties of the Euclidean functional L^p. The proof is pleasantly simple and self-contained. Notice that the proof breaks down when $p \geq d$; for these values of p we will need an approach which is more delicate and which is discussed in the next section.

If $L^p,\ 1 \le p < d$, is a smooth Euclidean functional which satisfies the subadditivity condition

$$(4.8) \qquad L^p(F, [0,1]^d) \le \sum_{i=1}^{m^d} L^p(F \cap Q_i, Q_i) + C_1 m^{d-p},$$

where $Q_i,\ 1 \le i \le m^d$, denotes a partition of $[0,1]^d$ into m^d subcubes of edge length m^{-1}, then (4.4) follows. To see this, follow the proof of (4.5) verbatim and notice that (4.6) becomes

$$(4.9) \qquad \frac{\phi(nm^d)}{(nm^d)^{(d-p)/d}} \le \frac{\phi(n)}{n^{(d-p)/d}} + \frac{C_3}{n^{(d-p)/d}} + \frac{C_1 d^{1/2}}{n^{(d-p)/d}},$$

where $\phi(n) := EL(U_1, ..., U_n)$. The proof of (4.4) now follows the proof of (4.5).

We will henceforth not need the subadditivity condition (4.8) and instead we will rely upon the simpler subadditivity (3.4), which, as noted previously, implies (4.8) when m is a power of 2^d.

4.2. Limit Theorems for Euclidean Functionals $L^p,\ p \ge d$

The previous section developed a limit theorem for Euclidean functionals L^p when $1 \le p < d$. This section discusses the more delicate case $p \ge d$. For many Euclidean functionals L^p the dth power of the length of a typical edge in the graph associated with $L^p(U_1, ..., U_n)$ is of the order n^{-1}. When there are $O(n)$ edges in the graph we would expect that $L^d(U_1, ..., U_n)$ would behave like a constant for large n. The main point of this section is to show that this is indeed the case. This is less straightforward than might first appear.

For $p \ge d$ the methods of Section 4.1 break down since they introduce non-negligible constant terms in both the superadditive and subadditive relations (4.6) and (4.9) respectively. For these critical values of p, a more delicate approach is needed. By considering the probability theory of infinite trees, Aldous and Steele (1992) obtain L^2 asymptotics for the MST functional $M^d(U_1, ..., U_n)$. The Aldous and Steele approach spawned a number of interesting conjectures on infinite trees and their methods may possibly be useful in the context of the TSP and minimal matching functionals. We refer to Steele (1997) for a complete treatment.

In what follows we will continue to view optimization problems L as superadditive Euclidean functionals on the product space $\mathcal{F} \times \mathcal{R}(d)$. This approach delivers asymptotics for L^p in the critical case $p \ge d$ as well as in the case $1 \le p < d$.

The following limit theorem provides asymptotics for the mean of Euclidean functionals $L^p,\ p \ge d$, over a Poisson number N of points. Here $N := N(n)$ is an independent Poisson random variable with parameter n. More about the constants $\alpha(L_B^p, d)$ will follow in section 4.4.

Theorem 4.3. *(basic limit theorem for Euclidean functionals L^p, $p \geq d$) Let L_B^p be a superadditive Euclidean functional of order p on $\mathbb{R}^d$, where $p \geq d \geq 2$. If $EL_B^p(U_1, ..., U_n) \leq Cn^{(d-p)/d}$, then*

$$\lim_{n\to\infty} n^{(p-d)/d} EL_B^p(U_1, ..., U_N) = \alpha(L_B^p, d) \tag{4.10}$$

where $\alpha(L_B^p, d)$ is a positive constant. If L^p is close in mean to L_B^p in the sense that

$$E|L^p(U_1, ..., U_N) - L_B^p(U_1, ..., U_N)| = o(n^{(d-p)/d}), \tag{4.11}$$

then

$$\lim_{n\to\infty} n^{(p-d)/d} EL^p(U_1, ..., U_N) = \alpha(L_B^p, d). \tag{4.12}$$

To obtain asymptotics for the de-Poissonized versions of (4.10) and (4.12) we would normally appeal to smoothness (3.8). However, this smoothness is not sufficiently strong and we will instead consider a modified smoothness condition.

Definition 4.4. *(smooth in mean) For all $p > 0$ and $d \geq 2$ a Euclidean functional L^p is smooth in mean if there exists a constant $\gamma < 1/2$ such that for all $n \geq 1$ and $0 \leq k \leq n/2$ we have*

$$E|L^p(U_1, ..., U_n) - L^p(U_1, ..., U_{n\pm k})| \leq Ckn^{-p/d+\gamma}. \tag{4.13}$$

The next result is a de-Poissonized version of Theorem 4.3.

Theorem 4.5. *Let L_B^p be a superadditive Euclidean functional of order p on $\mathbb{R}^d$, where $p \geq d \geq 2$. Assume that $EL_B^p(U_1, ..., U_n) \leq Cn^{(d-p)/d}$. If L_B^p is close in mean (4.11) to L^p and smooth in mean (4.13) then*

$$\lim_{n\to\infty} n^{(p-d)/d} EL_B^p(U_1, ..., U_n) = \alpha(L_B^p, d) \tag{4.14}$$

and

$$\lim_{n\to\infty} n^{(p-d)/d} EL^p(U_1, ..., U_n) = \alpha(L_B^p, d). \tag{4.15}$$

In the remainder of this section we consider the proofs of Theorems 4.3 and 4.5.

Proof of Theorem 4.3. Let $N(\lambda)$ be a Poisson random variable with parameter $\lambda > 0$ and which is independent of the sequence U_i, $i \geq 1$. Set

$$\phi(\lambda) := EL_B^p(U_1, ..., U_{N(\lambda)}).$$

The proof of (4.10) centers around two functional inequalities for a scaled version of ϕ defined by

$$h(\lambda) := \phi(\lambda)/\lambda^{(d-p)/d}.$$

Notice first that $\sup_\lambda h(\lambda) \leq C$. To see this, note that by the assumed boundedness of $EL_B^p(U_1, ..., U_n)$, the assumed independence of N and $U_i,\ i \geq 1$, and Fubini's theorem we have

$$EL_B^p(U_1, ..., U_{N(\lambda)}) \leq CE\left((N(\lambda))^{(d-p)/d} \cdot 1_{\{N(\lambda)>0\}}\right).$$

The right side of the above is bounded by

$$CE\left((N(\lambda))^{(d-p)/d} \cdot 1_{\{0<N(\lambda)<\lambda/2\}}\right) + CE\left((N(\lambda))^{(d-p)/d} \cdot 1_{\{\lambda/2\leq N(\lambda)<\infty\}}\right)$$
$$\leq C\lambda^{(d-p)/d}$$

using exponential bounds for $P\{0 < N(\lambda) < \lambda/2\}$.

To derive our first inequality, observe that the superadditivity of L_B^p implies for all $0 < \delta < 1$

$$L_B^p(U_1, ..., U_{N(\lambda)}) \geq L_B^p\left(\{U_1, ..., U_{N(\lambda)}\} \cap [0, 1-\delta]^d, [0, 1-\delta]^d\right).$$

Taking expectations and scaling gives

$$\phi(\lambda) \geq (1-\delta)^p \phi(\lambda(1-\delta)^d).$$

Dividing by $\lambda^{(d-p)/d}$ we obtain

$$h(\lambda) \geq (1-\delta)^d h(\lambda(1-\delta)^d),$$

where $\lambda > 0$ and $0 < \delta < 1$. Using $\sup_\lambda h(\lambda) \leq C$, we easily obtain our first inequality for h:

$$h(\lambda) \geq h(\lambda(1-\delta)^d) - \delta C. \tag{4.16}$$

To derive a second functional relationship for h, partition $[0,1]^d$ into m^d disjoint subcubes $Q_1, ..., Q_{m^d}$ of edge length m^{-1}. Superadditivity of L_B^p gives

$$L_B^p(U_1, ..., U_{N(\lambda)}) \geq \sum_{i=1}^{m^d} L_B^p\left(\{U_1, ..., U_{N(\lambda)}\} \cap Q_i, Q_i\right).$$

Taking expectations and scaling yields

$$\phi(\lambda) \geq m^{d-p} \phi(\lambda m^{-d}).$$

Thus for all $\lambda > 0$ and all positive integers m we get

$$\phi(m^d\lambda) \geq m^{d-p}\phi(\lambda).$$

Dividing by $\lambda^{(d-p)/d}m^{d-p}$ we obtain our second inequality valid for all $m \in \mathbb{N}$ and all $\lambda > 0$:

$$h(m^d\lambda) \geq h(\lambda). \tag{4.17}$$

We now combine the functional relations (4.16) and (4.17) to deduce (4.10). Let $\limsup_{\lambda\to\infty} h(\lambda) := \beta$ and note that β is finite. For all $\epsilon > 0$ we may find $\lambda_o := \lambda_o(\epsilon)$ such that $h(\lambda_o) \geq \beta - \epsilon$. By (4.17) we thus have for all positive integers m

$$h(m^d\lambda_o) \geq \beta - \epsilon.$$

It remains to show that $\liminf_{\lambda\to\infty} h(\lambda) = \beta$. To do this, we examine how h fluctuates in the interval $[m^d\lambda_o, (m+1)^d\lambda_o)$. With ϵ fixed as above and $m \geq \epsilon^{-1}$, consider $t \in \mathbb{R}^+$ such that

$$m^d\lambda_o < t \leq (m+1)^d\lambda_o.$$

The relation (4.16) gives

$$h(t) \geq h(t(1-\delta)^d) - \delta C$$

for all $0 < \delta < 1$. Set $\delta := 1 - (\frac{m^d\lambda_o}{t})^{1/d}$. Note that since $t \leq m^d\lambda_o + Cm^{d-1}\lambda_o$ we obtain the estimate

$$\delta \leq 1 - (\frac{m}{m+C})^{1/d} \leq 1 - \frac{m}{m+C} \leq \frac{C}{m}.$$

We thus obtain

$$h(t) \geq h(m^d\lambda_o) - \delta C \geq h(m^d\lambda_o) - \frac{C}{m}.$$

Thus for all t between $m^d\lambda_o$ and $(m+1)^d\lambda_o$ we deduce from (4.17)

$$h(t) \geq h(m^d\lambda_o) - C\epsilon \geq \beta - \epsilon - C\epsilon.$$

Thus, $\liminf_{t\to\infty} h(t) \geq \beta - \epsilon - C\epsilon$. Let ϵ tend to zero and set $\beta := \alpha(L_B^p, d)$ to deduce (4.10). We will see in Section 4.4 that $\alpha(L_B^p, d)$ is positive. The limit (4.12) is immediate and the proof of Theorem 4.3 is complete. □

We now consider the proof of Theorem 4.5.

Proof of Theorem 4.5. We need to show that (4.10) implies the de-Poissonized limit (4.14). Let A denote the event $\{|N-n| \leq C(n\log n)^{1/2}\}$ and for all $k \in \mathbb{N}$ let $L(k) := L(U_1, ..., U_k)$. Write the decomposition

$$\begin{aligned} &n^{(p-d)/d}|EL_B^p(N) - EL_B^p(n)| \\ &\leq n^{(p-d)/d}E\left(|L_B^p(N) - L_B^p(n)| \cdot 1_{A^c}\right) + n^{(p-d)/d}E\left(|L_B^p(N) - L_B^p(n)| \cdot 1_A\right). \end{aligned}$$

For C large, the first term converges to zero as n tends to infinity by the assumed boundedness of $EL_B^p(n)$ as well as by the exponential tails for $N-n$. The second term is handled by a conditioning argument and by independence equals

$$
\begin{aligned}
&= n^{(p-d)/d} \sum_{|k|\le C(n\log n)^{1/2}} E(|L_B^p(n+k) - L_B^p(n)|)\, P\{N=n+k\} \\
&\le Cn^{(p-d)/d} \sum_{|k|\le C(n\log n)^{1/2}} kn^{-p/d+\gamma} P\{N=n+k\} \\
&\le Cn^{-1+\gamma} \sum_{k=-n}^{\infty} kP\{N-n=k\} \\
&\le Cn^{-1+\gamma}\, E|N-n| \\
&= o(1).
\end{aligned}
$$

Thus the second term goes to zero as well, showing that (4.10) implies (4.14). The proof that (4.12) implies (4.15) is similar. □

4.3. Applications to Problems in Combinatorial Optimization

Since the edges in the graphs of the minimal spanning tree and the shortest tour on n uniform random variables have an average length of the order $n^{-1/d}$, we would expect that the sum of the dth powers of the lengths of these edges would behave like a constant. The main point of this section is to justify these heuristics with the aid of the limit Theorems 4.3 and 4.5. This section is not essential to the sequel and may be skipped without loss of continuity.

Throughout, we adhere to the convention that U_i, $i \ge 1$, denotes an i.i.d. sequence of uniform random variables on $[0,1]^d$ and $N := N(n)$ denotes an independent Poisson random variable with parameter n.

Theorem 4.6. *(asymptotics for the power-weighted MST) For all $p \ge d \ge 2$ we have*

$$\lim_{n\to\infty} n^{(p-d)/d} EM^p(U_1,...,U_n) = \alpha(M_B^p, d). \tag{4.18}$$

Theorem 4.7. *(asymptotics for the power-weighted TSP) For all $d \ge 2$*

$$\lim_{n\to\infty} ET^d(U_1,...,U_n) = \alpha(T_B^d, d). \tag{4.19}$$

The basic limit Theorems 4.3 and 4.5 could also be used to capture the asymptotics for the minimal matching functional and related optimization problems. However we will not pursue this and instead we sketch the proof of Theorem 4.6; the somewhat involved proof of Theorem 4.7 appears in Yukich (1995a).

The interest of Theorems 4.6 and 4.7 derives from the fact that they hold for the critical case $p = d$, where the usual geometric subadditivity methods break down. As early as 1968 Gilbert and Pollak (1968) proved that the MST functional $M^2(F)$ is uniformly bounded over $F \subset [0,1]^2$, but they didn't address the issue of convergence. When $p = d$, Bland and Steele conjectured that (4.18) and (4.19) hold respectively.

When $p = d$, Aldous and Steele (1992) considered the probabilistic theory of infinite trees in $\mathbb{R}^d$ to obtain an L^2 version of (4.18), thus settling Bland's conjecture. Aldous and Steele use the "objective method" (cf. Steele, 1997) to prove an L^2 version of (4.18). They were motivated to use this approach since at the time it appeared that (4.18) was not within reach of the usual subadditivity methods. It is not clear whether the objective method can be used for the minimal matching and TSP functionals. Later, Yukich (1995a) used boundary functionals to establish (4.18) and (4.19), thus settling Steele's conjecture.

In order to prove Theorem 4.6 we will call upon the following lemma which shows that the MST functional satisfies the smooth in mean condition (4.13). We continue to write $M(n)$ for $M(U_1, ..., U_n)$.

Lemma 4.8. *(smoothness for the MST functional). For all $p > 0$ and $d \geq 2$ there is a $C := C(d,p)$ such that for all $n \geq 1$ and $0 \leq k \leq n/2$ we have*

$$|M_B^p(n) - M_B^p(n+k)| \leq Ck(\log n/n)^{p/d} \tag{4.20}$$

with high probability. Moreover, for all $0 \leq k \leq n/2$

$$|M_B^p(n) - M_B^p(n-k)| \leq Ck(\log n/n)^{p/d} \tag{4.21}$$

with high probability.

The precise meaning of the "high probability" statement (4.20) is as follows. For any prescribed $\beta > 0$ we can find $C := C(\beta)$ such that for any $0 \leq k \leq n/2$

$$P\{|M_B^p(n) - M_B^p(n+k)| > Ck(\log n/n)^{p/d}\} = O(n^{-\beta}).$$

A similar meaning is attached to the statement (4.21) as well as related high probability statements in this monograph.

Proof of Lemma 4.8. Notice that given the graph of the minimal spanning tree on $(U_1, ..., U_n)$, we can construct a feasible spanning tree on $(U_1, ..., U_n, U_{n+1})$ by inserting the edge of minimal length between U_{n+1} and the sample $(U_1, ..., U_n)$. Thus

$$M_B^p(n+1) \leq M_B^p(n) + d^p(U_{n+1},\ (U_i)_{i=1}^n),$$

where $d(x, F)$ denotes the Euclidean distance between the point x and the set F. In general, for $k \geq 1$ we have

$$M_B^p(n+k) \leq M_B^p(n) + \sum_{j=n+1}^{n+k} d^p(U_j,\ (U_i)_{i=1}^n).$$

For all $j \geq n+1$ we have by standard arguments that $d(U_j, (U_i)_{i=1}^n) \leq C(\log n/n)^{1/d}$ with high probability. We obtain for all $k \geq 0$ the high probability estimate

$$M_B^p(n+k) \leq M_B^p(n) + Ck(\log n/n)^{p/d}.$$

To complete the proof of (4.20) we need to show the reverse high probability inequality

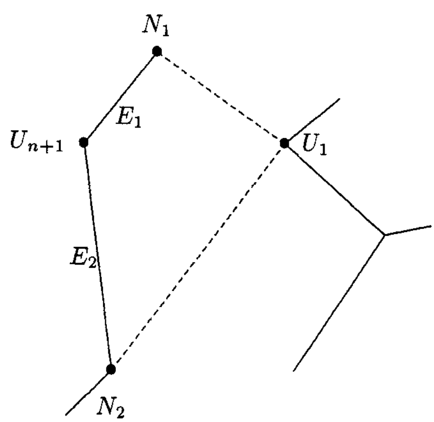
$$M_B^p(n) \leq M_B^p(n+k) + Ck(\log n/n)^{p/d}.$$

We will first show the high probability inequality

$$M_B^p(n) \leq M_B^p(n+1) + C(\log n/n)^{p/d} \tag{4.22}$$

and then iterate. Let $(N_j)_{j=1}^{M(d)} \subset \{U_1, ..., U_n\}$ denote the vertices which are linked to U_{n+1} by the minimal spanning tree on $U_1, ..., U_n, U_{n+1}$. $M(d)$ is finite since vertices in minimal spanning trees have bounded degree (see e.g. Melzak (1973)). With high probability there is a sample point, which without loss of generality we label U_1, such that U_{n+1} is within $C(\log n/n)^{1/d}$ of U_1. Replace all $M(d)$ edges E_i, $1 \leq i \leq M(d)$, having U_{n+1} as a vertex with edges E_i' leading to U_1 instead. See Figure 4.1.

Figure 4.1. Constructing a feasible spanning tree: replace edges E_1 and E_2 with the dashed edges



For each $1 \le i \le M(d)$, this may be achieved at a cost of at most

$$(|E_i| + C(\log n/n)^{1/d})^p \le C(\log n/n)^{p/d}$$

since with high probability $|E_i| \le C(\log n/n)^{1/d}$, a fact which we leave as an exercise. The resulting graph gives a feasible spanning tree on the pruned set $U_1, ..., U_n$ and shows the high probability bound

$$M_B^p(n) \le M_B^p(n+1) + C(\log n/n)^{p/d},$$

which is precisely (4.22). Iterating gives (4.20). The proof of (4.21) is similar. This completes the proof of Lemma 4.8. □

We are now ready for the

Proof of Theorem 4.6.

It is straightforward to show that the MST functional M_B^p is superadditive and satisfies $EM_B^p(U_1, ..., U_n) \le Cn^{(d-p)/d}$. By Theorem 4.3 we have

$$\lim_{n \to \infty} n^{(p-d)/d} EM_B^p(U_1, ..., U_N) = \alpha(L_B^p, d).$$

To show (4.18) we will show that M_B^p satisfies the conditions of Theorem 4.5. By Lemma 4.8 we already know that smoothness in mean (4.13) is satisfied. It remains to show closeness in mean, i.e., show for all $p \ge d \ge 2$ that

$$E|M^p(n) - M_B^p(n)| = o(n^{(d-p)/d}). \tag{4.23}$$

We rely upon a construction which consists of adding extra edges to the components in the graph $G_B := G_B^p(n)$ which realizes $M_B^p(n)$.

Enumerate the components of G_B by $T_1, ..., T_N$, where N is random and where each T_i represents a tree which is rooted to the boundary of the unit cube. Let the tree T_i meet the boundary of $[0,1]^d$ at the point B_i and let M_i denote the unique sample point which is rooted to B_i, $1 \le i \le N$. We now want to show that the sum of the pth powers of the lengths of the edges connected to the boundary is $o(n^{(d-p)/d})$. We will in fact establish the following high probability bound

$$\sum_{i=1}^{N} |M_i - B_i|^p \le Cn^{(d-p-1)/d}(\log n)^p. \tag{4.24}$$

To prove (4.24), we will use another partition of $[0,1]^d$. We begin by considering the subcube S of edge length $1 - 2n^{-1/d}$ and centered within $[0,1]^d$. Notice that with high probability there are at most $Cn^{(d-1)/d}$ points in the moat $[0,1]^d - S$ and these points contribute at most $Cn^{(d-p-1)/d}$ to (4.24).

To complete the proof of (4.24) we have to show

$$\sum_{M_i \in S} |M_i - B_i|^p \leq C \cdot n^{(d-p-1)/d} \cdot (\log n)^p.$$

Let F denote a face of $[0,1]^d$. Observe that S is the disjoint union of $Cn^{(d-1)/d}$ rectangular solids which are perpendicular to F, have height $1 - 2n^{-1/d}$, and have a base with diameter $n^{-1/d}$. See Figure 4.2. Geometric considerations show that every such solid contains at most one edge of the graph G_B which is rooted to F. Were there two or more such edges this would contradict optimality, as it would be more efficient to join the points rooted to F with a single edge. Thus the number of points in S which are joined to the boundary of $[0,1]^d$ is at most $Cn^{(d-1)/d}$.

Moreover, the point M_i must be the sample point in the solid which is closest to the boundary. By considering rectangular solids which are perpendicular to the remaining faces of $[0,1]^d$ we may easily conclude that given $M_i \in S$, there is a rectangular solid R such that among all sample points in R, M_i is the one closest to the boundary. Thus for all $M_i \in S$ we have the high probability estimate

$$|M_i - B_i|^p \leq C(\log n)^p n^{-p/d}.$$

Since there are as many points in S as there are solids, (4.24) follows.

We now add three types of edges to the trees $T_1, ..., T_N$ which comprise the graph G_B. For all $1 \leq i \leq N$ insert the edge F_i joining M_i to the nearest point in the grid $G := (G_i)_{i=1}^{n^{(d-1)/d}}$ of regularly spaced points on $\partial[0,1]^d$. See Figure 4.2.

Since each B_i is within $n^{-1/d}$ of a point in G, (4.24) and the triangle inequality imply the high probability estimate

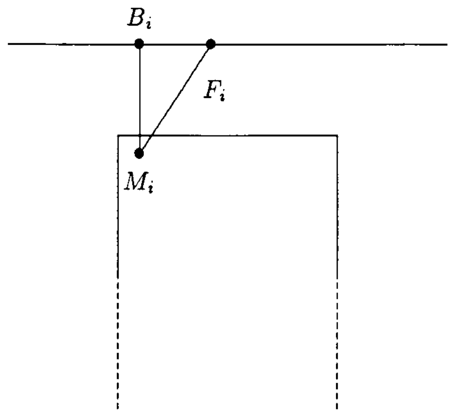
$$S_F^p := \sum_{i=1}^{N} |F_i|^p \leq Cn^{(d-p-1)/d}(\log n)^p. \tag{4.25}$$

Next, for all $1 \leq i \leq n^{(d-1)/d}$, consider the edge E_i joining G_i to the nearest sample point, say $U_{\sigma(i)}$, where σ is some function with domain $1, 2, ..., n^{(d-1)/d}$ and range contained in $1, 2, ..., n$. There are $n^{(d-1)/d}$ such edges and since each edge length satisfies the high probability bound $|E_i| \leq C(\log n/n)^{1/d}$, it follows that

$$S_E^p := \sum_{i=1}^{N} |E_i|^p \leq Cn^{(d-p-1)/d}(\log n)^p \tag{4.26}$$

with high probability.

Figure 4.2. Estimating the lengths of edges joined to the boundary



By inserting the two types of edges F_i, $1 \le i \le N$, and E_i, $1 \le i \le N$, we generate a boundary rooted tree on the the union $G \cup \{U_1, ..., U_n\}$; this tree has disjoint components, say $T_1, ..., T_L$, where $L \le N$. Given the grid G note that each grid point in G is centered in a $d-1$ dimensional cube on $\partial[0,1]^d$. Say that two components are *neighbors* if they contain grid points which are centered in adjacent grid cubes.

The triangle inequality implies that we may tie together any two neighboring components with a third type of edge H, with a length which may be bounded in terms of lengths of edges of the first two types. Let H_i, $i \ge 1$, denote an enumeration of all such edges. Then the H_i, $i \ge 1$, together with the edges in the graph G_B, form a global tree T' through the union $G \cup (U_1, ..., U_n)$. The triangle inequality, (4.25), and (4.26) imply that

$$S_H^p := \sum_{i=1}^{L} |H_i|^p \le C(S_E^p + S_F^p + n^{(d-p-1)/d}) \le Cn^{(d-p-1)/d}(\log n)^p, \tag{4.27}$$

where the last inequality holds with high probability. Moreover, by deleting all edges in T' which involve grid points, we form a feasible tree T through the pruned set $(U_1, ..., U_n)$, which shows

$$M^p(n) \le \sum_{e \in T} |e|^p \le M_B^p(n) + S_H^p.$$

Now (4.23) follows from the high probability bound (4.27). This completes the proof of Theorem 4.6. $\square$

4.4. Ergodic Theorems, Superadditive Functionals

Euclidean functionals by definition are invariant under translations and it is thus natural to look for an intrinsic ergodic structure yielding laws of large numbers. We pursue this line of inquiry in this section. We will see that a scaled a.s. version of (4.3) is essentially a consequence of a multiparameter subadditive ergodic theorem, one which generalizes Kingman's subadditive ergodic theorem (Kingman (1968,1973)). Such theorems hold for real-valued multiparameter functionals $L := (L(R),\ R \in \mathcal{R}(d))$ defined on a probability space $(\Omega, \mathcal{A}, P)$. We will use these theorems to help identify the limiting constants of Theorems 4.1 and 4.3. Before stating our results, we need a few definitions.

Say that the functional L is *stationary* if for all $m \geq 1$, $R_1, ..., R_m \in \mathcal{R}(d)$ and $u \in (\mathbb{R}^+)^d$, the joint distribution of $L(R_1), ..., L(R_m)$ and $L(R_1 + u), ..., L(R_m + u)$ are the same. Say that L is *bounded* if

$$\sup_n EL([0,n]^d)/n^d < \infty.$$

Say that L is *strongly superadditive* if

$$L(R) \geq \sum_{i=1}^{m} L(R_i),$$

where the rectangles R_i, $1 \leq i \leq m$, form a partition of the rectangle $R \in \mathcal{R}(d)$. Notice that strong superadditivity is stronger than the usual superadditivity (3.3). Strong superadditivity is critical in our search for ergodicity. All superadditive functionals encountered in this monograph are strongly superadditive.

Observe that L is not required to satisfy homogeneity (3.2). There are many examples of stationary strongly superadditive functionals beyond those defined by combinatorial optimization problems. One such example involves the number of clusters in the percolation model. More precisely, if $L(R)$, $R \in \mathcal{R}(d)$, denotes the number of clusters in the percolation model in the rectangle R, then the cluster functional $-L(R)$, $R \in \mathcal{R}(d)$, is strongly superadditive, as observed by Grimmett (1976). Note that $L(R)$, $R \in \mathcal{R}(d)$, is not homogeneous.

Hammersley (1974) provided another example of a strongly superadditive functional. His example, motivated by the statistical theory of liquid-vapor equilibrium, is as follows. Distribute points in $\mathbb{R}^d$ according to a stationary Poisson point process. Fix $r > 0$ and consider a sphere of radius r around each point. For $R \in \mathcal{R}(d)$ let $L(R)$ denote the volume of the spheres whose interiors are wholly in R (we count each element of volume once only, regardless of the number of spheres covering it). Then it is easily checked that $L(R)$ is a bounded, stationary strongly superadditive functional. It is not homogeneous and thus not a Euclidean functional.

A third example of a strongly superadditive functional consists of a modification of an example of Hammersley and Welsh (1965), which we learned from Smythe (1976). Let straight lines be distributed on the plane uniformly and independently at random (in other words their directions are uniformly and independently distributed between 0 and 2π, and their perpendicular distances from the origin are the points of a Poisson process on the positive reals). Let $L(R)$, $R \in \mathcal{R}(2)$, be the number of polygons of some given class (acute triangles, for example) which intersect R. Then $-L(R)$, $R \in \mathcal{R}(2)$, is strongly superadditive but not homogeneous.

To understand the behavior of stationary, strongly superadditive functionals on large cubes, we will appeal to the following theorem, stated without proof. This theorem is due to Akcoglu and Krengel (1981) although the L^1 part follows as in Smythe (1976, Theorem 1.1). This result, which is essentially a strong law of large numbers, generalizes Kingman's (1968) deep subadditive ergodic theorem. This generalized ergodic theorem has been used in statistical physics to analyze the behavior of long range spin systems; see e.g. Van Enter and Van Hemmen (1983). In this section $\mathcal{R}(d)$ denotes rectangles in $\mathbb{N}^d$.

Theorem 4.9. *(Akcoglu and Krengel) Let* $L := (L(R) : R \in \mathcal{R}(d))$ *be a stationary, bounded, strongly superadditive functional defined on* $(\Omega, \mathcal{A}, P)$. *Then*

$$\lim_{n\to\infty} L([0,n]^d)/n^d = f(L,d)$$

a.s. and in L^1, *where* $f(L,d) \in L^1(\Omega, \mathcal{A}, P)$. *Moreover,*

$$Ef(L,d) = \alpha(L,d) = \sup_R \frac{EL(R)}{volumeR}.$$

Clearly $\alpha(L,d)$ is positive whenever L is not identically zero. By the assumed boundedness of L, $\alpha(L,d)$ is finite.

We are now positioned to give a second proof of (4.5) for functionals L which are strongly superadditive. We will use Theorem 4.9 heavily.

Proof of (4.5). Let $\Pi := \Pi(1)$ denote a Poisson point process on $(\mathbb{R}^+)^d$ with intensity 1 and put

$$L_B^p(R) := L_B^p(\Pi \cap R, R), \quad R \in \mathcal{R}(d).$$

We need to verify that the functional $L_B^p(R)$, $R \in \mathcal{R}(d)$, satisfies the conditions of Theorem 4.9. Stationarity follows from translation invariance (3.1) and strong superadditivity follows by hypothesis. Since $\Pi \cap [0,n]^d \stackrel{d}{=} n(U_1, ..., U_N)$, where N is an independent Poisson random variable with parameter n^d, we see that boundedness results from homogeneity (3.2) and Jensen's inequality:

$$\begin{aligned} EL_B^p([0,n]^d) &= EL_B^p(n(U_1,...,U_N),\ [0,n]^d) \\ &= n^p EL_B^p(U_1,...,U_N,\ [0,1]^d) \\ &= Cn^p EN^{(d-p)/d} \\ &\le Cn^d. \end{aligned}$$

Thus Theorem 4.9 gives the existence of a function $f(L_B^p, d) \in L^1(\Omega, \mathcal{A}, P)$ such that

$$\lim_{n \to \infty} L_B^p([0, n]^d)/n^d = f(L_B^p, d) \tag{4.28}$$

a.s. and in L^1.

Since

$$\lim_{n \to \infty} EL_B^p([0, n]^d)/n^d = Ef(L_B^p, d) = \alpha(L_B^p, d)$$

we obtain by homogeneity (3.2) a Poissonized version of (4.5):

$$\lim_{n \to \infty} EL_B^p(U_1, ..., U_N)/n^{d-p} = \alpha(L_B^p, d).$$

By smoothness (3.8) the convergence is unaffected if N is replaced by n^d. Easy interpolation arguments and smoothness for L_B^p show that the argument n^d may be replaced by n, which yields precisely (4.5). Moreover, when p lies in the range $1 \le p < d$, (4.5) is equivalent to the c.c. version but we have to wait until Chapter 6 (Corollary 6.4) to give the details.

We may use Theorem 4.9 to identify the constant $\alpha(L_B^p, d)$ in Theorems 4.1 and 4.3 and to show that it is necessarily positive. Let Π denote a Poisson point process on $\mathbb{R}^d$ with intensity 1 and put

$$L(R) := L_B^p(\Pi \cap R, R),$$

where L_B^p is a smooth superadditive Euclidean functional of order p on $\mathbb{R}^d$. Then Theorems 4.1, 4.3, and 4.9 give the existence of the limit

$$\begin{aligned}\lim_{n \to \infty} EL(n \cdot [0,1]^d)/n^d &= \lim_{n \to \infty} EL_B^p\left(\Pi \cap (n \cdot [0,1]^d), n \cdot [0,1]^d\right)/n^d \\ &= \lim_{n \to \infty} EL_B^p(U_1, ..., U_{N(n)}, [0,1]^d)/n^{(d-p)/d}\end{aligned}$$

and show that it equals the *spatial constant*

$$\alpha(L_B^p, d) = \sup_{R \in \mathcal{R}(d)} \frac{EL_B^p(\Pi \cap R, R)}{\text{volume} R}. \tag{4.29}$$

By assumption $L_B^p(F, R) > 0$ when $\text{card} F > 0$ and thus (4.29) shows that the spatial constant $\alpha(L_B^p, d)$ is positive. Spatial constants are the multiparameter analogs of the time constants appearing in one-dimensional subadditive theory.

Remarks On the Limiting Constants:

(i) In the statement of Theorem 4.9, the number N of sample points in $[0,n]^d$ and the volume V of $[0,n]^d$ tend to infinity but in such a way that the particle number density N/V essentially remains finite. Borrowing a term from statistical mechanics, we say that $f(L_B^p, d)$ is the infinite volume limit or thermodynamic limit for the functional L_B^p.

(ii) Since the constant $\alpha(L,d)$ given by Theorem 4.9 is called the spatial constant for the functional L we interpret the constant $\alpha(L_B^p, d)$ of Theorem 4.1 as the spatial constant for the Euclidean functional L_B^p. We now review some of the estimates for the constants $\alpha(S^p, d)$, $\alpha(M^p, d)$, and $\alpha(T^p, d)$. Little is known concerning the exact values of these mysterious constants but perhaps changing the Euclidean norm to another norm may offer a path towards progress.

$\alpha(S^1, 2)$ is shown in Papadimitriou (1978a) to lie in the interval $[0.25, 0.40106]$ and based on Monte Carlo experiments it is conjectured that $\alpha(S^1, 2) = 0.35$. Bertsimas and van Ryzin (1990) use Crofton's method to show that

$$\begin{aligned}\frac{1}{2\pi^{1/2}}\Gamma(1/d+1)\Gamma(d/2+1)^{1/d} &\le \alpha(S^1, d)\\ &\le \frac{d}{(2d-1)\pi^{1/2}}\Gamma(1/d+1)\Gamma(d/2+1)^{1/d}2^{1/d}\end{aligned}$$

from which it follows that as $d \to \infty$

$$\alpha(S^1, d) \sim \frac{1}{2}(d/2\pi e)^{1/2}.$$

Concerning the MST, Gilbert (1965) shows that $\alpha(M^1, 2)$ is bounded by $2^{-1/2} \approx 0.707$ and he obtains $\alpha(M^1, 2) \approx 0.68$ based on experimental evidence. Bertsimas and van Ryzin (1990) show that

$$\frac{1}{\pi^{1/2}}\Gamma(1/d+1)\Gamma(d/2+1)^{1/d} \le \alpha(M^1, d) \le \frac{1}{\pi^{1/2}}\Gamma(1/d+1)\Gamma(d/2+1)^{1/d}2^{1/d}$$

from which it follows that as $d \to \infty$

$$\alpha(M^1, d) \sim (d/2\pi e)^{1/2}.$$

More generally they show for all $p < d$ that

$$\alpha(M^p, d) \sim (d/2\pi e)^{p/2}.$$

The constant $\alpha(M^p, d)$ is also treated by Avram and Bertsimas (1992), who identify $\alpha(M^p, d)$ with an infinite series. Using their infinite series identification, they show that $\alpha(M^1, 2)$ is bounded below by 0.600822 which improves upon the lower bound of 0.5 given by Bertsimas and van Ryzin (1990).

$\alpha(T^1, 2)$ is estimated by Marks (1948) and is known to be bounded by $0.62 < \alpha(T^1, 2) < 0.93$ (Beardwood, Halton, Hammersley (1959, p. 302, and Lemma 3.10)); it is found numerically as $\alpha(T^1, 2) = 0.749$ (Bonomi and Lutton (1984)). Rhee (1992) applies an estimate of Talagrand (1992) to show that

$$|\alpha(T^1, d) - (d/2\pi e)^{1/2}| \leq K(\log d/d)$$

which shows that as $d \to \infty$ we have $\alpha(T^1, d) \sim \alpha(M^1, d)$.

(iii) The identification $Ef(L, d) = \alpha(L, d)$ may be seen from Smythe's (1976) mean ergodic theorem, which proves a mean version of (4.28) using a slightly different form of superadditivity.

4.5. Concluding Remarks

This chapter describes a general framework for proving limit theorems for the classic problems of combinatorial optimization. We have focussed on the prototypical problems involving the TSP, MST, and minimal matching functionals. Our approach is useful for other problems, including the Steiner MST problem, the semi-matching problem (Chapter 8), the minimal triangulation problem (Chapter 9), and the k-median problem of Steinhaus (Chapter 10). The approach proves useful for finding asymptotics of functionals L^p with pth power-weighted edges, where p ranges over both the critical range $p \geq d$ as well as the usual range $1 \leq p < d$. The method described here consists of two main steps:

(i) use the superadditive structure of the canonical boundary functional L_B^p to deduce the asymptotics of L_B^p and

(ii) show that the boundary functional L_B^p is close to L^p to deduce identical asymptotics for L^p.

Notes and References

1. The difference between the incrementing model and independent model of problem generation was not explicitly recognized until Weide's thesis (1978). The classic a.s. limit results of Beardwood, Halton, and Hammersley (1959) and Steele (1981a) hold for the incrementing model. Steele (1981b) obtained the first limit result for the independent TSP model.

2. This chapter considered limit theorems for Euclidean functionals $L^p(X_1, ..., X_n)$ where X_i, $i \geq 1$, are i.i.d. with the uniform distribution on the unit cube $[0,1]^d$. In the sequel we will consider limit theorems for $L^p(X_1, ..., X_n)$, where the X_i, $i \geq 1$, have a general distribution. When the X_i, $i \geq 1$, have a "self similar" distribution, Lalley (1990) showed that the asymptotics of L^p may involve growth rates which are different from the usual rate of $n^{(d-p)/d}$ and which may involve periodic functions.

3. Returning to the context of Theorem 4.6, recall that $M^p(n) := M^p(U_1, ..., U_n)$. Kesten and Lee (1996) showed an asymptotic result which is a little more precise than (4.14). They showed

$$\lim_{n \to \infty} \left((n+1)^{p/d} EM^p(n+1) - n^{p/d} EM^p(n) \right) = C(d,p),$$

where $C(d,p)$ is a constant depending only on p and d. They also showed that $(M^p(n) - EM^p(n)) / n^{(d-2p)/2d}$ has a limiting normal distribution. See Lee (1997a,b) for more general results, especially central limit theorems over non-uniform samples. A deep and challenging open problem involves showing analogous limiting behavior for the TSP and minimal matching functionals.

4. The Akcoglu and Krengel Theorem 4.9 strengthens previous multiparameter superadditive ergodic theorems given earlier by Smythe (1976) and Nguyen (1979).

5. An interesting open problem involves using the Aldous and Steele objective method to prove Theorem 4.7 and thereby establish a theoretical value of the limiting constant $\alpha(T_B^d, d)$. McElroy (1997) uses the method of boundary functionals to prove the analog of Theorems 4.6 and 4.7 for the minimal matching and semi-matching functionals.

6. Krengel and Pyke (1987) proved a uniform version of Theorem 4.9 over general averaging sets. Yukich (1996b) used this to deduce uniform a.s. limit results for the TSP functional.

7. Classical multiparameter ergodic theorems (see e.g. Dunford (1951) and Zygmund (1951)) prove laws of large numbers for arrays of random variables $\{X_j\}_{j \in \mathbb{N}^d}$. Instead of assuming boundedness of L, these theorems require a moment condition of the form $E\left(|X_1|(\log|X_1|)^{d-1}\right) < \infty$. Proofs of these theorems (see e.g. Krengel (1985)) may be deduced by simple inductive arguments from the 1-parameter case. The first multiparameter pointwise ergodic theorems go back to Wiener (1939).