5. RATES OF CONVERGENCE AND HEURISTICS

In this chapter we show that boundary functionals L_B provide the ideal tools for finding rates of convergence of the means $EL(U_1, ..., U_n)$ as well as for analyzing the performance of easily computed partitioning heuristics L_H which approximate L. In both instances, boundary functionals provide a simple approach to problems which have been traditionally attacked by less powerful and less natural bare-hands methods.

5.1. Rates of Convergence in The Basic Limit Theorem

Chapter 4 provided basic limit theorems for superadditive Euclidean functionals. This chapter examines the rates of convergence of the means of these functionals.

Subadditivity of a functional L is not enough to give rates of convergence. Subadditivity only yields one sided estimates whereas rate results require two sided estimates. However, if the functional L can be made superadditive by appropriately modifying it then we can usually extract rates of convergence. This idea is widely known and discussed in Hammersley (1974), for example. It should be no surprise that boundary functionals L_B provide exactly the right type of modification of L which we are looking for.

The next result shows that if subadditive Euclidean functionals L^p are close in mean (3.16) to the associated superadditive Euclidean functional L^p_B, namely if

$$|EL^p(U_1, ..., U_n) - EL^p_B(U_1, ..., U_n)| \le C(n^{(d-p-1)/d} \vee 1), \tag{5.1}$$

then we may find rates of convergence for $EL^p(U_1, ..., U_n)$. Since the TSP, MST, and minimal matching functionals satisfy closeness in mean ($p \neq d-1, d \geq 3$) the following theorem immediately provides rates of convergence for our three prototypical examples.

Theorem 5.1. *(rates of convergence of means). Suppose that L^p and L^p_B are subadditive and superadditive Euclidean functionals of order p, respectively, and that they satisfy the close in mean approximation (5.1). If N is an independent Poisson random variable with parameter n, then for all $d \geq 2$ and $1 \leq p < d$ we have*

$$|EL^p(U_1, ..., U_N) - \alpha(L^p_B, d)n^{(d-p)/d}| \le C\left(n^{(d-p-1)/d} \vee 1\right). \tag{5.2}$$

Moreover, if L^p is smooth (3.8), then for $d \geq 2$ and $1 \leq p < d$ we have

$$|EL^p(U_1, ..., U_n) - \alpha(L^p_B, d)n^{(d-p)/d}| \le C\left(n^{(d-p)/2d} \vee n^{(d-p-1)/d}\right). \tag{5.3}$$

Proof. For all $n \in \mathbb{N}$ set $\phi(n) := EL^p(U_1, ..., U_{N(n)})$, where $N(n)$ is a Poisson random variable with parameter n and which is independent of the U_i, $i \geq 1$. It follows from translation invariance (3.1), homogeneity (3.2), and subadditivity (3.5) that whenever m is a power of 2 we have

$$\begin{aligned}\phi(nm^d) &\leq m^{-p} \sum_{i=1}^{m^d} \phi(n) + Cm^{d-p} \\ &= m^{d-p}\phi(n) + Cm^{d-p}.\end{aligned}$$

Dividing by $(nm^d)^{(d-p)/d}$ yields the homogenized relation

$$\frac{\phi(nm^d)}{(nm^d)^{(d-p)/d}} \leq \frac{\phi(n)}{n^{(d-p)/d}} + \frac{C}{n^{(d-p)/d}}.$$

As in (4.6) it follows that the limit of the left side as m tends to infinity exists and equals $\alpha(L_B^p, d)$. Thus

$$\frac{\phi(n)}{n^{(d-p)/d}} - \alpha(L_B^p, d) \geq \frac{-C}{n^{(d-p)/d}}$$

or simply

$$\phi(n) - \alpha(L_B^p, d)n^{(d-p)/d} \geq -C. \tag{5.4}$$

Setting $\phi_B(n) := EL_B^p(U_1, ..., U_{N(n)})$ and exploiting the superadditivity of L_B in the same way that we exploited the subadditivity of L, we effortlessly obtain the companion estimate to (5.4) where we may now let $C = 0$:

$$\phi_B(n) - \alpha(L_B^p, d)n^{(d-p)/d} \leq 0. \tag{5.5}$$

By the assumed closeness in mean (5.1), we have by Fubini's theorem and independence

$$\begin{aligned}|\phi_B(n) - \phi(n)| &\leq E_N|E_U L_B^p(U_1, ..., U_N) - E_U L^p(U_1, ..., U_N)| \\ &\leq E_N(N^{(d-p-1)/d} \vee 1) \\ &\leq C(n^{(d-p-1)/d} \vee 1),\end{aligned}$$

where E_N and E_U denote the expectation with respect to the random variables N and U, respectively. Now (5.1) follows from (5.4) and (5.5). Finally, the de-Poissonized version (5.3) is a simple consequence of smoothness:

$$\begin{aligned}|EL^p(U_1, ..., U_N) - EL^p(U_1, ..., U_n)| &\leq CE(|N - n|^{(d-p)/d}) \\ &\leq Cn^{(d-p)/2d}.\end{aligned}$$

This completes the proof of Theorem 5.1. □

5.2. Sharper Rates of Convergence in the Basic Limit Theorem

The proof of Theorem 5.1 is remarkably simple. If it has a shortcoming, it is only that de-Poissonizing introduces an extra error $Cn^{(d-p)/2d}$. We may remove this error whenever L satisfies what Steele calls an "add-one bound" of the type

$$|EL^p(U_1, ..., U_{n+1}) - EL^p(U_1, ..., U_n)| \leq Cn^{-p/d}. \tag{5.6}$$

The improved version of Theorem 5.1 takes the following form:

Theorem 5.2. *(rates of convergence of means). Suppose that L^p and L^p_B are subadditive and superadditive Euclidean functionals of order p, respectively, and that they satisfy the close in mean approximation* (5.1) *and the "add-one bound" (5.6). Then for all $d \geq 2$ and $1 \leq p < d$ we have*

$$|EL^p(U_1, ..., U_n) - \alpha(L^p_B, d)n^{(d-p)/d}| \leq C\left(n^{(d-p-1)/d} \vee 1\right). \tag{5.7}$$

Proof. Let N denote an independent Poisson random variable with parameter n and follow the proof of Theorem 5.1 up to the Poissonized estimates (5.4) and (5.5). Conditioning on N we show that the add-one bound (5.6) leads to the de-Poissonized estimate

$$|EL^p(U_1, ..., U_n) - EL^p(U_1, ..., U_N)| \leq Cn^{1/2-p/d}, \tag{5.8}$$

which will be enough to complete the proof of Theorem 5.2 since $n^{1/2-p/d} \leq n^{(d-p-1)/d}$ holds for all $n \geq 1$, $d \geq 2$, and $1 \leq p < d$.

Now to prove (5.8), we use the decomposition

$$\begin{aligned}
&|EL^p(U_1, ..., U_n) - EL^p(U_1, ..., U_N)| \\
&\leq \left|E(L^p(U_1, ..., U_n) - L^p(U_1, ..., U_N)) \cdot 1_{\{0 \leq N < n/2,\ N > 3n/2\}}\right| + \\
&\quad + \left|E(L^p(U_1, ..., U_n) - L^p(U_1, ..., U_N)) \cdot 1_{\{n/2 \leq N \leq 3n/2\}}\right| \\
&:= I + II.
\end{aligned}$$

Using the growth bound $L^p(U_1, ..., U_n) \leq C_2 n^{(d-p)/d}$ and the fact that $N - n$ has exponential tails, we obtain the bound $I = O(n^{1/2-p/d})$. By the assumed add-one bound (5.6), term II is bounded by

$$\begin{aligned}
&\leq E_N\left|E_U(L^p(U_1, ..., U_n) - L^p(U_1, ..., U_N)) \cdot 1_{\{n/2 \leq N \leq 3n/2\}}\right| \\
&\leq C\sum_{k=n+1}^{3n/2}(k-n)n^{-p/d}P\{N = k\} + C\sum_{k=n/2}^{n}(n-k)n^{-p/d}P\{N = k\} \\
&\leq Cn^{-p/d}E|N - n| \\
&\leq Cn^{1/2-p/d}.
\end{aligned}$$

Thus (5.6) leads to (5.8) and the proof of Theorem 5.2 is complete. □

Rate results for Euclidean functionals L thus follow once the "add-one bound" (5.6) is satisfied. When L is the MST functional, then (5.6) is satisfied, as shown by Redmond and Yukich (1996). However, if L is the minimal matching or TSP functional, then it is unclear whether (5.6) is satisfied.

On the other hand, in the absence of (5.6), one can still obtain rates of convergence. When $p = 1$ and $d = 2$, for example, Alexander (1994) shows directly that the minimal matching functional satisfies the de-Poissonization estimate

$$|EL^p(U_1, ..., U_n) - EL^p(U_1, ..., U_N)| = O(1)$$

and in this way obtains the rate result (5.3).

5.3. Optimality of Rates

In general, the rate results (5.1) and (5.3) cannot be improved. This is illustrated in dimension 2 by the following theorems, proved by Jaillet (1993) and Rhee (1994a), respectively. Rhee's theorem settles a conjecture attributed to Karp (1977). Jaillet (1995) presents a fine overview of rate results and we refer to this paper for complete details. Throughout this section, $(U_i)_{i\geq 1}$ denotes an i.i.d. sequence of uniformly distributed random variables with values in the unit square and $N := N(n)$ denotes an independent Poisson random variable with parameter n. It is not clear whether the following inequalities hold when N is replaced by its parameter n.

Theorem 5.3. *(optimal rates of convergence for the MST on the square). There is a universal constant C and $n_o \in \mathbb{N}$ such that for all $n \geq n_o$*

$$EM(U_1, ..., U_N)/n^{1/2} \geq \alpha(M_B, 2) + Cn^{-1/2}. \tag{5.9}$$

Theorem 5.4. *(optimal rates of convergence for the TSP on the square). There is a universal constant C and $n_o \in \mathbb{N}$ such that for all $n \geq n_o$*

$$ET(U_1, ..., U_N)/n^{1/2} \geq \alpha(T_B, 2) + Cn^{-1/2}. \tag{5.10}$$

The idea behind the estimate (5.9) involves showing that

$$EM(U_1, ..., U_{N(4n)}) \leq 2EM(U_1, ..., U_{N(n)}) - C \tag{5.11}$$

for some constant $C > 0$. This bound is somewhat surprising since heuristically one might expect that scaling arguments would show the approximate equivalence $EM(U_1, ..., U_{N(4n)}) \sim 2EM(U_1, ..., U_{N(n)})$.

Iterating (5.11) leads to

$$EM(U_1, ..., U_{N(4^m n)}) \leq 2^m EM(U_1, ..., U_{N(n)}) - C(2^m - 1).$$

Dividing by $(4^m n)^{1/2}$ and letting m tend to infinity gives (5.9). A similar approach yields (5.10). The complete proof of (5.9) and (5.10) is too involved to present here and we refer to Jaillet (1993) and Rhee (1994a), respectively, for the details. Steele (1997, Chapter 3) presents a fine exposition of the inequality (5.11).

It is not difficult to prove the analogs of Theorems 5.3 and 5.4 for the canonical boundary functionals M_B and T_B, respectively. In fact, by establishing the lower bound

$$EM_B(U_1, ..., U_{N(4n)}) \geq 2EM_B(U_1, ..., U_{N(n)}) + C \tag{5.12}$$

we may prove a companion result to Theorem 5.3:

Theorem 5.5. *(optimal rates of convergence for the boundary MST functional on the square). There is a universal constant C and $n_o \in \mathbb{N}$ such that for all $n \geq n_o$*

$$EM_B(U_1, ..., U_N)/n^{1/2} \leq \alpha(M_B, 2) - Cn^{-1/2}. \tag{5.13}$$

An interesting open question concerns rates of convergence for the MST functional on the torus, namely the unit square equipped with the flat metric. The MST functional M_T on the torus T is defined as follows: for $F \subset T$, $M_T(F)$ denotes the length of the minimal spanning tree through F, where distances are measured in terms of the flat metric. Since M_T is related to the boundary MST M_B and the standard MST M by $M_B \leq M_T \leq M$, M_T clearly admits the rate of convergence

$$|EM_T(U_1, ..., U_n) - \alpha(M_B, 2)n^{1/2}| \leq C. \tag{5.14}$$

However, it is not clear that this is the exact rate of convergence. In other words, are the rates of convergence expressed by (5.14) optimal?

5.4. Analysis of Partitioning Heuristics

As we saw in Chapter 4, the asymptotics for the TSP

$$\lim_{n\to\infty} T(U_1, ..., U_n)/n^{(d-1)/d} = \alpha(T, d) \quad a.s.$$

lead Karp (1976, 1977) to find efficient methods for approximating the length $T(U_1, ..., U_n)$ of the shortest path through i.i.d. uniformly distributed random variables $U_1, ..., U_n$ on the unit square. Karp developed the "fixed dissection algorithm" which provides a simple heuristic $T_H(U_1, ..., U_n)$ having the property that $T_H(U_1, ..., U_n)/T(U_1, ..., U_n)$ converges completely to 1 and which moreover has polynomial mean execution time.

The fixed dissection algorithm consists of dividing the unit cube $[0, 1]^d$ into m^d congruent subcubes $Q_1, ..., Q_{m^d}$, finding the shortest tour T_i of length $T(\{U_1, ..., U_n\} \cap Q_i)$ on each of the subcubes, constructing a tour T which links representatives from each T_i, and then deleting excess edges to generate a grand (heuristic) tour through $U_1, ..., U_n$ having length $T_H(U_1, ..., U_n)$. See Figure 5.1.

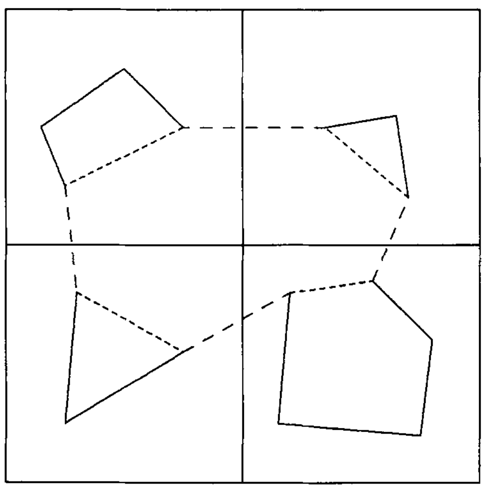
Using Karp's seminal work as a guide, Karp and Steele (1985) show via elementary methods that the partitioning heuristic T_H is ϵ-optimal with probability one:

Theorem 5.6. *(Karp and Steele, 1985) If $m^d := n/\sigma$, where σ is an unbounded increasing function of n, then for all $\epsilon > 0$*

$$\sum_{n=1}^{\infty} P\left\{\frac{T_H(U_1,...,U_n)}{T(U_1,...,U_n)} \geq 1+\epsilon\right\} < \infty.$$

Theorem 5.6 thus shows that the ratio of the lengths of the heuristic tour and the optimal tour converges completely to 1. Given the computational complexity of the TSP, it is remarkable that the optimal tour length is so well approximated by a sum of individual tour lengths, where the sum has polynomial mean execution time.

Figure 5.1. Karp's TSP heuristic: the grand tour is constructed from the local tours by inserting the dashed edges and deleting the dotted edges



The Karp and Steele result is an example of a general phenomenon: given a Euclidean functional L, there is a heuristic L_H which closely approximates it in the sense of Theorem 5.6. We show that the general framework of subadditive and superadditive Euclidean functionals L and L_B, respectively, permits a strikingly easy analysis of a partitioning heuristic L_H which shares many of the same pleasant features as Karp's partitioning heuristic for the TSP. The use of boundary functionals, especially their closeness property to the standard functional, provides a simplifying tool.

The results of this section essentially show that Euclidean functionals L^p can be approximately represented as a sum of i.i.d. local functionals $L^p(F \cap Q_i, Q_i)$, $1 \leq i \leq m^d$. The upcoming Lemma 5.8 shows that $L^p(F)$ may be approximated by the sum of the independent local functionals $L^p(F \cap Q_i, Q_i)$ on the subcubes Q_i, $1 \leq i \leq m^d$, plus a correction term which is deterministically small compared to $n^{(d-p)/d}$.

The correction term may be regarded as a way of measuring the interactions between the local functionals on the m^d subcubes, a point of view which is valuable in statistical mechanics. The correction term is small enough to yield a strong law of large numbers, but not quite small enough to yield a central limit theorem.

Let us now fix our ideas. Let L^p be a subadditive Euclidean functional associated with the solution to an optimization problem. We assume $L^p \geq L^p_B$. Given $F \subset [0,1]^d$ and $L^p(F)$, consider the feasible solution obtained by solving the optimization problem on the subcubes Q_i, $1 \leq i \leq m^d$, and then adding and deleting edges in the resulting graph to obtain a global solution on the set F. This feasible solution, which we call the canonical heuristic, has a length denoted by $L^p_H(F, m^d)$. $L^p_H(F, m^d)$ is the sum of the local functionals $L^p(F \cap Q_i, Q_i)$, $1 \leq i \leq m^d$, plus a correction term which we implicitly assume is bounded by $C_1 m^{d-p}$ whenever m is a power of 2. Thus we assume that L^p_H satisfies for all $F \subset [0,1]^d$

$$L^p(F) \leq L^p_H(F, m^d) \leq \sum_{i=1}^{m^d} L^p(F \cap Q_i, Q_i) + C_1 m^{d-p}. \tag{5.15}$$

When L is the TSP functional, for example, then L_H is Karp's (1976,1977) heuristic.

It will be convenient to let the number m^d of subcubes depend on the cardinality of F, which for brevity we denote by $|F|$. This approach is similar to that of Karp (1976,1977). To make this precise, let $\sigma := \sigma(n)$ denote a function of n such that $\sigma(n)$ increases up to infinity and $1 < \frac{n}{\sigma(n)} = 2^{d \cdot j(n)}$ for some non-decreasing sequence of integers $j(n)$, $n \geq 1$. For such functions σ we will consider heuristics $L^p_\sigma(F)$ of the form

$$L^p_\sigma(F) := L^p_H\left(F, \frac{|F|}{\sigma(|F|)}\right).$$

Thus the heuristic $L^p_\sigma(F)$ subdivides the unit cube into $\frac{|F|}{\sigma(|F|)}$ subcubes. If we set $m^d := \frac{|F|}{\sigma(|F|)}$ we obtain from (5.15):

$$L^p_\sigma(F) \leq \sum_{i=1}^{|F|/\sigma(|F|)} L^p(F \cap Q_i, Q_i) + C_1 \left(\frac{|F|}{\sigma(|F|)}\right)^{(d-p)/d}. \tag{5.16}$$

In order to proceed we will need to make a closeness assumption on the Euclidean functional L. Suppose that L^p is a Euclidean functional which is pointwise close to its boundary functional L^p_B in the sense that for all $F \subset [0,1]^d$ we have

$$|L^p(F) - L^p_B(F)| \leq C\left((\mathrm{card} F)^{(d-p-1)/(d-1)} \vee \log(\mathrm{card} F)\right). \tag{5.17}$$

By Lemma 3.7 we know that the TSP, MST, and minimal matching functionals satisfy (5.17).

We now state a deterministic result which shows that the heuristic solutions $L^p_\sigma(F)$ have a length which is close to that of the optimal solution.

Lemma 5.7. *Assume that L^p and L_B^p are subadditive and superadditive Euclidean functionals of order p, respectively, and that they satisfy pointwise closeness (5.17). Then the heuristic L_σ^p satisfies*

$$\sup_{|F|=n} |L^p(F) - L_\sigma^p(F)| = o(n^{(d-p)/d}). \tag{5.18}$$

Proof. Consider $F \subset [0,1]^d$, where $\mathrm{card} F = n$. For ease of notation set $m = (n/\sigma(n))^{1/d}$ and note that m is a power of 2 by hypothesis, i.e., $m = 2^{j(n)}$. For ease of presentation we take $p < d-1$; therefore the log term in (5.17) may be ignored and we obtain

$$|L^p(F,[0,1]^d) - L_B^p(F,[0,1]^d)| \le Cn^{(d-p-1)/(d-1)}. \tag{5.19}$$

Since m is a power of 2 we have by (5.16) and (5.19)

$$\begin{aligned} L^p(F) &\le L_\sigma^p(F) \\ &\le \sum_{i=1}^{m^d} L^p(F\cap Q_i, Q_i) + C_1 m^{d-p} \\ &\le \sum_{i=1}^{m^d} \left(L_B^p(F\cap Q_i, Q_i) + m^{-p}C(\mathrm{card}(F\cap Q_i))^{(d-p-1)/(d-1)}\right) + C_1 m^{d-p} \\ &\le L^p(F) + m^{-p}\sum_{i=1}^{m^d} C\left(\mathrm{card}(F\cap Q_i))^{(d-p-1)/(d-1)}\right) + C_1 m^{d-p}. \end{aligned}$$

Thus we have

$$|L^p(F) - L_\sigma^p(F)| \le m^{-p}\sum_{i=1}^{m^d} C\left(\mathrm{card}(F\cap Q_i))^{(d-p-1)/(d-1)}\right) + C_1 m^{d-p}. \tag{5.20}$$

The above sum is largest when $\mathrm{card}(F\cap Q_i) = n/m^d$, $1 \le i \le m^d$. It is thus bounded by $Cm^{p/(d-1)}n^{(d-p-1)/(d-1)}$. Using the definition of m we arrive at the estimate

$$|L^p(F) - L_\sigma^p(F)| \le C\sigma^{-p/d(d-1)}n^{(d-p)/d} + C_1\sigma^{(p-d)/d}n^{(d-p)/d}. \tag{5.21}$$

Thus (5.21) tells us that the heuristic $L_\sigma^p(F)$ is larger than $L^p(F)$ by a quantity which is deterministically small compared to $n^{(d-p)/d}$. This completes the proof of Lemma 5.7. □

By Lemma 5.7 the asymptotic behavior of the scaled heuristic

$$\frac{L_\sigma^p(X_1,...,X_n)}{n^{(d-p)/d}} \tag{5.22}$$

coincides with the asymptotic behavior of the scaled Euclidean functional

$$\frac{L^p(X_1, ..., X_n)}{n^{(d-p)/d}}, \tag{5.23}$$

where X_i, $i \geq 1$, are i.i.d. random variables with values in $[0, 1]^d$. While we have not treated the asymptotics of Euclidean functionals over non-uniform samples we will show in Chapters 6 and 7 that the functional (5.23) converges completely to a positive constant whenever the law of X_1 has a continuous part. It will thus follow that (5.22) also converges completely to a constant. Admitting the complete convergence of (5.22) we can now extend Karp and Steele's result to general Euclidean functionals over general sequences of random variables:

Theorem 5.8. *(the heuristic L^p_σ is ϵ-optimal) Let L^p and L^p_B be subadditive and superadditive Euclidean functionals of order p, respectively. Assume that they are pointwise close (5.17). Then for all $\epsilon > 0$ and all i.i.d. sequences X_i, $i \geq 1$, of random variables with a continuous part, the heuristic L^p_σ is ϵ-optimal:*

$$\sum_{n=1}^{\infty} P\left\{\frac{L^p_\sigma(X_1, ..., X_n)}{L^p(X_1, ..., X_n)} \geq 1 + \epsilon\right\} < \infty. \tag{5.24}$$

Proof. We will assume that (5.23) converges completely to a positive constant C, a fact which will be proved in Chapters 6 and 7. By Lemma 5.7 it follows that the scaled heuristic (5.22) converges completely to C as well. It therefore follows by standard arguments that the ratio of (5.22) to (5.23) converges completely to 1. This is precisely (5.24). □

We now show that the expected execution time for the partitioning heuristic L^p_σ is polynomially bounded under some weak assumptions. This feature, together with (5.19), show that L^p_σ has all the properties of Karp's heuristic.

The time required to compute the heuristic $L^p_\sigma(U_1, ..., U_n)$ is bounded by

$$T_n := \sum_{i=1}^{\frac{n}{\sigma(n)}} f(N_i),$$

where $N_i := \text{card}\{Q_i \cap \{U_1, ..., U_n\}\}$, $1 \leq i \leq n/\sigma(n)$, and where $f(N)$ denotes a bound on the time needed to compute $L^p(F)$, $\text{card}F = N$. If we assume that f exhibits polynomial growth, say $f(x) = Ax^B 2^x$ for some constants A and B, as would be the case for the TSP, then since the N_i, $1 \leq i \leq n/\sigma(n)$, are binomial random variables, straightforward calculations show that

$$ET_n \leq 4An(\sigma(n))^{B-1}\exp(\sigma(n))$$

see e.g. Karp and Steele (1985).

We now want to choose $\sigma(n)$ in such a way that we minimize the expected computation time T_n. We notice that there is a non-decreasing sequence $j = j(n),\ n \geq 1$, such that

$$C \log n \leq \frac{n}{2^{dj}} \leq \log n$$

for some constant $C < 1$. Here and henceforth we set $\sigma(n) = \frac{n}{2^{dj}}$ so that $\sigma(n) \leq \log n$. This value of $\sigma(n)$ implies that the expected execution time for the heuristic L^p_σ is $O(n^2 \log^{B-1} n)$.

Using this value of $\sigma(n)$ in (5.21), the above discussion may be summarized by the following general result, which applies to the TSP and other optimization functionals. As before we write $|F|$ to denote card$F$.

Theorem 5.9. *With $\sigma := \sigma(n)$ as above, let L^p_σ denote the heuristic associated with the subadditive Euclidean functional L^p. If L^p is pointwise close (5.17) to the boundary functional L^p_B, then the heuristic L^p_σ closely approximates L^p*

$$|L^p(F) - L^p_\sigma(F)| \leq C \left((\log |F|)^{-p/d(d-1)} + (\log |F|)^{(p-d)/d} \right) |F|^{(d-p)/d}$$

and is ϵ-optimal (5.24) with probability one. Moreover, if $L^p(F)$ may be computed in time bounded by $A(|F|)^B \cdot 2^{|F|}$, then the expected execution time for $L^p_\sigma(U_1, ..., U_n)$ is $O\left(n^2 (\log n)^{B-1}\right)$.

Clearly, by making a different choice of σ, e.g. $\sigma = \sigma(n) = \log \log n$, we can reduce the expected execution time at the expense of increasing the bound for $|L^p(F) - L^p_\sigma(F)|$.

5.5. Concluding Remarks, Open Questions

In this chapter we have already indicated a few open problems. We close with a list of several more problems and remarks.

1. When $p = 1$ and L is the TSP functional, the rate (5.3) was conjectured by Beardwood, Halton, and Hammersley (1959). Alexander (1994) obtained a result similar to (5.3) without the simplifying use of boundary functionals. He instead requires that L satisfy several hypotheses beyond the minimal ones (3.1)-(3.3). These hypotheses are shown to give a form of superadditivity similar to (3.3), but with a non-negligible correction term. Alexander applies his results to the TSP functional and in this way was the first to settle the rate conjecture of Beardwood, Halton, and Hammersley.

2. We have already seen from (5.11) that the estimate

$$|EM(U_1, ..., U_{N(4n)}) - 2EM(U_1, ..., U_{N(n)})| = O(1)$$

cannot be improved. Thus one wouldn't expect that the estimate

$$|M(U_1, ..., U_{N(4n)}) - \sum_{i=1}^{4} M(U_1, ..., U_{N(4n)} \cap Q_i)| = O_P(1)$$

could be improved, where Q_1, Q_2, Q_3, Q_4 represents the usual partition of $[0,1]^2$ into four congruent subsquares. Yet it is conceivable that there is a constant ψ such that

$$|M(U_1, ..., U_{N(4n)}) - \sum_{i=1}^{4} M(U_1, ..., U_{N(4n)} \cap Q_i) - \psi| = o_P(1).$$

Clearly a similar estimate might hold for the TSP functional.

The following intriguing conjecture, due to Rhee (1994a), formalizes these remarks. Her conjecture is based on an analysis of the behavior of the TSP functional near the boundary of the square. Let Π be a Poisson point process of constant intensity λ on the unit square.

Conjecture 5.10. *(Rhee, 1994a) There is a constant ψ such that given $\epsilon > 0$ for large enough λ we have*

$$P\{|T(\Pi) - \sum_{i=1}^{4} T(\Pi \cap Q_i, Q_i) - \psi| \geq \epsilon\} \leq \epsilon. \tag{5.25}$$

Were it true, this conjecture would show that the centered and Poissonized TSP functional $T(\Pi) - ET(\Pi)$ is asymptotically normal, i.e., satisfies a central limit theorem. To see this, one only needs to iterate (5.25) k times to approximately represent $T(\Pi) - ET(\Pi)$ as a sum of 2^k i.i.d. random variables.

3. Karp (1977) anticipates the general rate results of this chapter and observes that in dimension 2 the TSP functional satisfies

$$ET(U_1, ..., U_n) - \alpha(T_B, 2)n^{1/2} \leq C.$$

This, combined with the simple estimate (5.4), leads to the correct convergence rate for the TSP in dimension 2.

4. Section 5.4 extends and strengthens the results in Yukich (1995b, section 4), which is motivated in part by discussions with D. Bertsimas.

5. It should be possible to find rates of convergence for the power weighted functional L^p, when $p = d$. This would involve using the methods of Chapter 4 and improved subadditive bounds for L^p.

6. Is it possible to refine the rate results (5.2) and (5.3) by finding a function of n, say $\gamma(n)$, such that

$$|EL(U_1, ..., U_N) - \alpha(L_B^1, d)n^{(d-1)/d} - \gamma(n)| = o(1)?$$