7. UMBRELLA THEOREMS FOR EUCLIDEAN FUNCTIONALS

7.1. The Basic Umbrella Theorem

In 1959 Beardwood, Halton, and Hammersley proved their celebrated result describing the asymptotic length of the shortest tour on a random sample. They showed that the shortest tour $T(X_1, ..., X_n)$ through i.i.d. random variables X_i, $i \geq 1$, with values in $[0,1]^d$ satisfies

$$(7.1) \qquad \lim_{n \to \infty} T(X_1, ..., X_n)/n^{(d-1)/d} = \alpha(d) \int_{[0,1]^d} f(x)^{(d-1)/d} dx \quad a.s.,$$

where f is the density of the absolutely continuous part of the law of X_1 and $\alpha(d)$ is a positive constant. This seminal limit law extends the limit law encountered in Chapter 4 to the non-uniform case.

Chapter 4 established (7.1) for the special case of uniformly distributed X_i, $i \geq 1$. One expects that the general case follows by suitably approximating general random variables by linear combinations of uniform random variables and then taking limits. Despite the lack of a suitable convergence theorem, this "standard approach" can be made to work, but only by calling upon the tools and methods developed in previous chapters. This is where isoperimetry and the complementary notions of subadditive and superadditive functionals play an indispensable role.

The asymptotics (7.1) for the TSP provide enticing evidence that the basic limit Theorems 4.1, 4.3, and 4.5 may be extended to the non-uniform setting. Under what conditions can the uniform random variables $U_1, ..., U_n$ of Theorems 4.1 and 4.3 be replaced by non-uniform random variables $X_1, ..., X_n$? Following Steele (1981a, 1988), and Rhee (1993b) we reformulate this question and ask: can the asymptotics (7.1) be usefully generalized in terms of an umbrella result which includes the solutions to a broad range of problems in Euclidean optimization? Such an umbrella result would ideally cover those Euclidean functionals lacking monotonicity as well as those having power-weighted edges.

The following general result, together with the basic limit Theorem 4.1, lie at the heart of this monograph. The constant $\alpha(L_B^p, d)$ is of course the same constant (4.5) appearing earlier.

Theorem 7.1. *(umbrella theorem for Euclidean functionals on compact sets) Let L^p and L_B^p be smooth subadditive and superadditive Euclidean functionals of order p, respectively. Assume that L^p and L_B^p are close in mean (3.15). Let $(X_i)_{i \geq 1}$ be i.i.d. random variables with values in $[0,1]^d$, $d \geq 2$. If $1 \leq p < d$ then*

$$(7.2) \qquad \lim_{n \to \infty} L^p(X_1, ..., X_n)/n^{(d-p)/d} = \alpha(L_B^p, d) \int_{[0,1]^d} f(x)^{(d-p)/d} dx \quad c.c.,$$

where f is the density of the absolutely continuous part of the law of X_1.

We will shortly see that Theorem 7.1 is truly an umbrella result in the sense that it captures c.c. asymptotics for a wide range of problems in geometric probability, including those in combinatorial optimization, operations research, and computational geometry. Notice that Theorem 7.1 provides the asymptotics for the archetypical problems (TSP, MST, and minimal matching) considered up to now. Indeed, recalling Lemma 3.5 we know that the TSP, MST, and minimal matching functionals T^p, M^p, and S^p, respectively, are smooth Euclidean functionals of order p. By Lemma 3.10, these functionals are close in mean to their respective boundary functionals T^p_B, M^p_B, and S^p_B. Therefore we have shown:

Corollary 7.2. *The Euclidean functionals T^p, M^p, and S^p, $1 \le p < d$, and their corresponding boundary functionals all exhibit the asymptotic behavior (7.2).*

Remarks.

(i) Corollary 7.2 is not exhaustive. We will show that other problems in geometric probability satisfy the hypotheses of Theorem 7.1. This will include the semi-matching problem and the k nearest neighbors problem (Chapter 8), the minimal triangulation problem (Chapter 9), and geometric location problems (Chapter 10).

(ii) Returning to (7.2), we clearly have $L^p(X_1, ..., X_n) = o(n^{(d-p)/d})$ a.s. when X_1 has a singular distribution. On the other hand, Jensen's inequality shows that the right side of (7.2) is largest when the density $f(x)$ equals $1_{[0,1]^d}(x)$, that is when the X_i, $i \ge 1$, have the uniform distribution on the unit cube. Thus Euclidean functionals L^p assume their largest value (in the asymptotic sense) when the underlying sample is uniformly distributed.

(iii) The TSP, MST, and minimal matching functionals may be defined on the torus T, defined as the unit cube equipped with the flat metric or Euclidean d-torus metric. The weight of an edge (x_i, x_j) is now $\|(x_i - x_j)(\text{mod}1)^d\|$. It is easily checked that the resulting functionals, which we call T^p_T, M^p_T, and S^p_T, respectively, are sandwiched between the boundary functional and the standard functional. In other words, $T^p_B \le T^p_T \le T^p$ and similarly for M_T and S_T. By Corollary 7.2 these functionals all exhibit the asymptotic behavior (7.2). Jaillet (1993b), using different methods, was the first to observe that the torus versions of our archetypical functionals satisfy the limit law (7.2).

(iv) For Euclidean functionals L^p of the generality described in Theorem 7.1, the use of boundary functionals L^p_B is critical in order to obtain the asymptotics (7.2). In the lucky event that L^p satisfies the extra side condition of monotonicity (that is $L^p(F) \le L^p(F \cup \{x\})$ for all sets F and singletons x) then it is possible to deduce (7.2) in a relatively straightforward way. Indeed, we first prove (7.2) when the law

μ of X_1 is absolutely continuous with a step function density, then when μ is the sum of a step function density and a singular part, and finally when μ is a mixture of absolutely continuous and singular laws (see Steele (1997)). For the last step, coupling methods are useful. However, without monotonicity of L^p, considerable effort is required to handle distributions having a singular part. In particular, it is especially difficult to obtain the lower bound implicit in (7.2). To see that there are real difficulties in proving the lower bound, the reader may try to prove (7.2) for the MST functional. Steele (1988) addresses and overcomes these difficulties.

However, with the use of boundary functionals the proof of (7.2) becomes much easier. Indeed, once we use the subadditivity of L^p to prove the upper bound in (7.2) then exactly *the same methods* may be used to show that the superadditive functional L_B^p satisfies the lower bound in (7.2). Since L^p and L_B^p are close in mean, (7.2) follows. This is a brief outline of the main idea of the proof of (7.2). The details and the complete proof of (7.2) are in the next section.

7.2. Proof of the Basic Umbrella Theorem

The proof of (7.2) depends upon two observations which greatly simplify the analysis. The first is that by Corollary 6.4, it is enough to show that (7.2) holds in expectation, namely it suffices to show

$$\lim_{n\to\infty} EL^p(X_1,...,X_n)/n^{(d-p)/d} = \alpha(L_B^p,d)\int_{[0,1]^d} f(x)^{(d-p)/d}dx. \tag{7.3}$$

The limit (7.3) is a statement about a sequence of scalars and is thus easier to prove than the limit (7.2).

The second observation is that in the presence of the assumed smoothness of L^p, it is enough to establish (7.3) for a special class of distributions which we call *blocked distributions*. These are distributions μ on $[0,1]^d$ with the form $\phi(x)dx+\mu_s$, where $\phi(x)$ is a simple non-negative function of the form $\sum_{i=1}^{m^d}\alpha_i 1_{Q_i}$, the measure μ_s is purely singular, m is a power of 2, and Q_i, $i\geq 1$, are the usual subcubes. More precisely, we have the following lemma which is due to Steele (1988).

Lemma 7.3. *(reduction to blocked distributions) Let L^p be a smooth subadditive Euclidean functional and suppose that for every sequence of i.i.d. random variables $(X_i)_{i\geq 1}$ distributed with a blocked distribution $\mu := \phi(x)dx + \mu_s$, we have that*

$$\lim_{n\to\infty} EL^p(X_1,...,X_n)/n^{(d-p)/d} = \alpha(L_B^p,d)\int_{[0,1]^d} \phi(x)^{(d-p)/d}dx. \tag{7.4}$$

We then have that

$$\lim_{n\to\infty} EL^p(Y_1,...,Y_n)/n^{(d-p)/d} = \alpha(L_B^p,d)\int_{[0,1]^d} f(x)^{(d-p)/d}dx \tag{7.5}$$

whenever $(Y_i)_{i\geq 1}$ are independent and identically distributed with respect to any probability measure on $[0,1]^d$ with an absolutely continuous part given by $f(x)dx$.

Proof. The proof evolves from a coupling argument which is discussed and proved by Steele (1988, Theorem 3). Assume that the distribution of Y has the form $\mu_Y := f(x)dx + \mu_s$, where μ_s is singular. For all $\epsilon > 0$ we may find a blocked approximation to μ_Y of the form $\mu_X := \phi(x)dx + \mu_s$ where $\phi := \phi_\epsilon$ approximates f in the L^1 sense:

$$\int_{[0,1]^d} |\phi(x) - f(x)|dx < \epsilon. \tag{7.6}$$

By standard coupling arguments there is a joint distribution for the pair of random variables (X, Y) such that $P\{X \neq Y\} \leq 2\epsilon$. Thus it follows that

$$\begin{aligned} |EL^p(X_1, ..., X_n) - EL^p(Y_1, ..., Y_n)| &\leq CE(\text{card}\{i \leq n : X_i \neq Y_i\}^{(d-p)/d}) \\ &\leq C(\epsilon n)^{(d-p)/d}. \end{aligned} \tag{7.7}$$

Thus by (7.4) we obtain

$$\lim_{n\to\infty} \left| \frac{EL^p(Y_1, ..., Y_n)}{n^{(d-p)/d}} - \alpha(L^p_B, d) \int_{[0,1]^d} \phi(x)^{(d-p)/d} dx \right| \leq C\epsilon^{(d-p)/d}. \tag{7.8}$$

For all $a,\ b \geq 0$ we have

$$|a^{(d-p)/d} - b^{(d-p)/d}| \leq |a - b|^{(d-p)/d}$$

and therefore by (7.6)

$$\begin{aligned} \left| \int f(x)^{(d-p)/d} dx - \int \phi(x)^{(d-p)/d} dx \right| &\leq \int |f(x) - \phi(x)|^{(d-p)/d} dx \\ &< \epsilon^{(d-p)/d}. \end{aligned} \tag{7.9}$$

Combining (7.8) and (7.9) and letting ϵ tend to zero gives the result (7.5) as desired. □

We now prove (7.3) for the blocked distributions

$$\mu(x) := \sum_{i=1}^{m^d} \alpha_i 1_{Q_i}(x)dx + \mu_s$$

and we set $\phi(x) := \sum_{i=1}^{m^d} \alpha_i 1_{Q_i}(x)$. We will follow an approach similar to that of Steele (1981a,1988) and Redmond and Yukich (1994). Fix $\epsilon > 0$ and assume without loss of generality that $m^{-1} < \epsilon$. We will assume that m is a power of 2 so that we can apply geometric subadditivity (3.5). Let E denote the singular support of μ and let λ denote Lebesgue measure on the cube.

We may assume that m is chosen so that:

(1) $E \subset A \cup B$, where A and B are disjoint, $\lambda(A) = 0$ and $\mu(A) \leq \epsilon$, and

(2) $B := \bigcup_{i \in J} Q_i$ for some $J \subset I := \{1, ..., m^d\}$ and moreover $\lambda(B) \leq \epsilon$.

By smoothness (3.8), property (1), Jensen's inequality, and geometric subadditivity (3.5) we have

$$\begin{aligned} EL^p(X_1, ..., X_n) &\leq EL^p(\{X_1, ..., X_n\} - A) + C_3(\epsilon n)^{(d-p)/d} \\ &\leq \sum_{i \in I - J} EL^p(\{X_1, ..., X_n\} - A \cap Q_i, Q_i) + \\ &\quad + \sum_{i \in J} EL^p(\{X_1, ..., X_n\} - A \cap Q_i, Q_i) + \\ &\quad + C_1 m^{d-p} + C_3(\epsilon n)^{(d-p)/d}. \end{aligned} \tag{7.10}$$

Letting $(U_k)_{k \geq 1}$ be i.i.d. with the uniform distribution on $[0,1]^d$ it follows by smoothness and homogeneity that the first sum in (7.10) is bounded by

$$m^{-p} \sum_{i \in I - J} \left(EL^p((U_k)_{k=1}^{[\alpha_i m^{-d} n]}) + C_3 \left(E|B(n, \alpha_i m^{-d}) - [n\alpha_i m^{-d}]|\right)^{(d-p)/d} \right)$$

since for $i \in I - J$ the number of points not in A and in the subcube Q_i is a binomial random variable $B(n, \alpha_i m^{-d})$ with parameters n and $\alpha_i m^{-d}$. By Jensen's inequality, the above is clearly bounded by

$$m^{-p} \sum_{i \in I - J} \left(EL^p((U_k)_{k=1}^{[\alpha_i m^{-d} n]}) + C(m) n^{(d-p)/2d} \right),$$

where $C(m)$ is a constant depending only on d, m, and p. We now consider the second sum in (7.10). The expected number of points in $Q_i - A$ is at most $n\mu(Q_i)$. By Jensen's inequality and the growth bounds of Lemma 3.3 the second sum in (7.10) is bounded by

$$\begin{aligned} C_2 \sum_{i \in J} m^{-p} (n\mu(Q_i))^{(d-p)/d} &= C_2 n^{(d-p)/d} \sum_{i \in J} (m^{-d})^{p/d} \mu(Q_i)^{(d-p)/d} \\ &\leq C_2 n^{(d-p)/d} \left(\sum_{i \in J} m^{-d}\right)^{p/d} \\ &= C_2 n^{(d-p)/d} (\lambda(B))^{p/d} \\ &\leq C_2 \, \epsilon^{p/d} n^{(d-p)/d} \end{aligned}$$

by Hölder's inequality and the estimate $\lambda(B) \leq \epsilon$.

Combining the above estimates and dividing (7.10) by $n^{(d-p)/d}$ we get

$$\begin{aligned}
&EL^p(X_1,...,X_n)/n^{(d-p)/d} \\
\le &\sum_{i\in I-J} m^{-p}([\alpha_i m^{-d}n]/n)^{(d-p)/d}\cdot EL^p\left((U_k)_{k=1}^{[\alpha_i m^{-d}n]}\right)/[\alpha_i m^{-d}n]^{(d-p)/d}+ \\
&+C(m)n^{(p-d)/2d}+C_2\epsilon^{p/d}+C_1 m^{(d-p)}/n^{(d-p)/d}+C_3\epsilon^{(d-p)/d}.
\end{aligned}$$

Since the right side involves the L^p functional over a sequence of uniform random variables, we may evaluate this by applying the basic limit theorems of Chapter 4. Therefore, letting n tend to infinity, applying Theorem 4.1, and recalling that $1 \le p < d$, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} EL^p(X_1,...,X_n)/n^{(d-p)/d} \\
\le &\sum_{i\in I-J}\alpha_i^{(d-p)/d}m^{-d}\alpha(L_B^p,d)+C_2\epsilon^{p/d}+C_3\epsilon^{(d-p)/d} \\
= &\ \alpha(L_B^p,d)\int_{\cup_{i\in I-J}Q_i}\phi(x)^{(d-p)/d}dx \ + \ C_2\epsilon^{p/d} \ + \ C_3\epsilon^{(d-p)/d}.
\end{aligned}$$

As ϵ tends to zero, we see that m tends to infinity and $\bigcup_{i\in I-J}Q_i \uparrow [0,1]^d$. We apply the monotone convergence theorem to conclude that

$$\limsup_{n\to\infty} EL^p(X_1,...,X_n)/n^{(d-p)/d} \le \alpha(L_B^p,d)\int \phi(x)^{(d-p)/d}dx. \tag{7.11}$$

We have now established the upper bound implicit in (7.3). Establishing the lower bound implicit in (7.3) does not involve any new ideas. We merely use the superadditivity of L_B in place of the subadditivity of L . This convenience is not accidental and in fact is one of the motivating reasons for considering boundary functionals in the first place.

Therefore, by the smoothness and the superadditivity of the boundary functional L_B^p, we obtain the lower estimate

$$\begin{aligned}
EL_B^p(X_1,...,X_n) \ge &\sum_{i\in I-J} EL_B^p(\{X_1,...,X_n\}-A\cap Q_i,Q_i) - \\
&- C_3(\epsilon n)^{(d-p)/d}.
\end{aligned}$$

Using this bound, *following the analysis of (7.10) through (7.11) verbatim*, and using Theorem 4.1 once more, we deduce the analogous lower bound

$$\liminf_{n\to\infty} EL_B^p(X_1,...,X_n)/n^{(d-p)/d} \ge \alpha(L_B^p,d)\int_{[0,1]^d}\phi(x)^{(d-p)/d}dx. \tag{7.12}$$

Combining the complementary estimates (7.11) and (7.12) and using the closeness in mean (3.15) we see that the limsup in (7.11) and the liminf in (7.12) coincide.

Therefore, the asymptotics (7.3) hold for blocked distributions, as desired. By the reduction Lemma 7.3, this concludes the proof of Theorem 7.1. □

7.3. Extensions of the Umbrella Theorem

The previous section proved a generalization of the remarkable Beardwood, Halton, and Hammersley (1959) theorem describing the shortest tour on random points. This general umbrella result covers a lot of ground. As we will soon see, it satisfactorily describes the asymptotic behavior of the lengths of graphs given by a wide range of problems in geometric probability. The only shortcoming is that the theorem is limited by the assumption that the underlying point sets have compact support.

In this section we remove this shortcoming and consider graphs on unbounded random point sets in $\mathbb{R}^d$, $d \geq 2$. Our main result, Theorem 7.6, is straightforward and easy to state, but the proof involves a rather lengthy and difficult computation. The reader may skip the proof without loss of continuity.

We draw our inspiration from Rhee (1993a), who determined the asymptotics for the TSP functional over unbounded domains. In the process she disproved a conjecture of Beardwood, Halton, and Hammersley (1959). Rhee's approach depends upon a definition.

Definition 7.4. Let A_o denote the ball in $\mathbb{R}^d$ centered at the origin and with radius 2. For all $k \geq 1$, let A_k denote the annular shell centered around A_o with inner radius 2^k and outer radius 2^{k+1}. Given $f \in L^1(\mathbb{R}^d)$ set

$$a_k(f) := 2^{dk/(d-1)} \int_{A_k} f(x)dx.$$

Rhee's (1993a) main theorem is as follows.

Theorem 7.5. *(asymptotics for the TSP on unbounded domains) Let $(X_i)_{i\geq 1}$ be i.i.d. random variables with an absolutely continuous distribution on $\mathbb{R}^d$ having a density $f(x)$. If $\int_{\mathbb{R}^d} f(x)^{(d-1)/d}dx < \infty$ and*

$$\sum_{k=1}^{\infty}(a_k(f))^{(d-1)/d} < \infty \tag{7.13}$$

then the TSP functional T satisfies

$$\lim_{n\to\infty} T(X_1, ..., X_n)/n^{(d-1)/d} = \alpha(T_B, d) \int_{\mathbb{R}^d} f(x)^{(d-1)/d}dx \quad a.s. \tag{7.14}$$

Remarks.

(i) Condition (7.13) is satisfied whenever the density f satisfies the moment condition $\int_{\mathbb{R}^d} |x|^r f(x)dx < \infty$ for some $r > d/(d-1)$. To see this, let $\epsilon > 0$ be fixed and consider the representation

$$\sum_{k=1}^{\infty}(a_k(f))^{(d-1)/d} = \sum_{k=1}^{\infty} 2^{-\epsilon k} \left(\int_{A_k} f(x)dx\right)^{(d-1)/d} 2^{(1+\epsilon)k}.$$

Applying Hölder's inequality yields the upper bound

$$\leq C \left(\sum_{k=1}^{\infty}(\int_{A_k} f(x)dx) \cdot 2^{\frac{(1+\epsilon)kd}{d-1}}\right)^{(d-1)/d}$$

$$\leq C \left(\sum_{k=1}^{\infty} \int_{A_k} |x|^{\frac{(1+\epsilon)d}{d-1}} f(x)dx\right)^{(d-1)/d}$$

$$= C \left(\int |x|^{\frac{(1+\epsilon)d}{d-1}} f(x)dx\right)^{(d-1)/d}.$$

Thus (7.13) holds whenever $\int |x|^r f(x)dx < \infty$ for $r > d/(d-1)$.

(ii) The best condition on the sequence $a_k(f)$ that will insure (7.14) is the condition (7.13). In other words, consider a sequence $a_k > 0$ such that $\sum_{k=1}^{\infty} a_k^{(d-1)/d} = \infty$. Then, as shown by Rhee (1993a), there is an f such that $\int_{\mathbb{R}^d} f(x)^{(d-1)/d}dx < \infty$ and $a_k(f) \leq a_k$ for which

$$\lim_{n\to\infty} T(X_1, ..., X_n)/n^{(d-1)/d} = \infty.$$

The condition (7.13) is however not necessary and it is unknown whether there is a meaningful necessary and sufficient condition for (7.14).

By following Rhee's approach in the context of sub and superadditive Euclidean functionals, we will prove a version of Theorem 7.1 which holds for random variables with unbounded support. Since we will work with the general case $1 \leq p < d$ this suggests redefining $a_k(f)$ by

$$a_k(f) := a_{k,p}(f) := 2^{dkp/d-p} \int_{A_k} f(x)dx.$$

Our generalized umbrella theorem takes the following form. By following the ideas of Remark (i) it is easy to check that the theorem applies to i.i.d. random variables having a density f satisfying the integrability condition $\int |x|^r f(x)dx < \infty$ for some $r > d/d-p$.

Notice that if L^p is a Euclidean functional and L^p_B is the canonical boundary Euclidean functional, then $L^p_B(F, \mathbb{R}^d)$ and $L^p(F, \mathbb{R}^d)$ coincide. We will assume that $L^p(F, \mathbb{R}^d) = L^p(F, R)$ whenever $F \cap R = F$. In the remainder of the chapter we write $L^p(F)$ and $L^p_B(F)$ for $L^p(F, \mathbb{R}^d)$ and $L^p_B(F, \mathbb{R}^d)$, respectively.

Theorem 7.6. *(umbrella theorem for Euclidean functionals on $\mathbb{R}^d$, $d \geq 2$) Let L^p and L^p_B be smooth subadditive and superadditive Euclidean functionals of order p, respectively. Assume that L^p satisfies simple subadditivity (2.2) and that L^p and L^p_B are close in mean (3.15). Let $(X_i)_{i\geq 1}$ be i.i.d. random variables with an absolutely continuous distribution on $\mathbb{R}^d$ having a density $f(x)$. If $1 \leq p < d$, $\int_{\mathbb{R}^d} f(x)^{(d-p)/d}dx < \infty$, and*

$$\sum_{k=1}^{\infty}(a_k(f))^{(d-p)/d} < \infty, \tag{7.15}$$

then the Euclidean functional L^p satisfies

$$\lim_{n\to\infty} L^p(X_1,...,X_n)/n^{(d-p)/d} = \alpha(L^p_B, d)\int_{\mathbb{R}^d} f(x)^{(d-p)/d}dx \quad a.s. \tag{7.16}$$

It is easy to see that the TSP functional T satisfies the conditions of Theorem 7.6 and thus Theorem 7.6 generalizes Theorem 7.5. Moreover, it is also easy to check that both the minimal spanning tree functional M and the minimal matching functional S satisfy the hypotheses of Theorem 7.6 and therefore (7.16) provides the asymptotics for the solutions to the archetypical problems of combinatorial optimization over samples of unbounded support. Later we will see that other problems in geometric probability satisfy the hypotheses of Theorem 7.6. This includes the k-median problem and the semi-matching problem among others.

Let us now turn to the proof of the umbrella Theorem 7.6. Throughout we will closely follow the proof of Rhee's Theorem 7.5. There are few new ideas and the proof is unfortunately rather long and technical. We have included it for completeness.

Lower bounds for Euclidean functionals are often the most difficult, but now to prove the lower bound implicit in (7.16), namely the bound

$$\liminf_{n\to\infty} L^p_B(X_1,...,X_n)/n^{(d-p)/d} \geq \alpha(L^p_B, d)\int_{\mathbb{R}^d} f(x)^{(d-p)/d}dx \quad a.s. \tag{7.17}$$

we have only to make a few elementary observations. Given $\epsilon > 0$, let $Q := Q(\epsilon)$ be a cube centered at the origin such that $\rho := P(Q) \geq 1-\epsilon$, where P is the law of X. Using superadditivity as a surrogate for monotonicity we have

$$\liminf_{n\to\infty} L^p_B(X_1,...,X_n)/n^{(d-p)/d} \geq \liminf_{n\to\infty} L^p_B(\{X_1,...,X_n\}\cap Q, Q)/n^{(d-p)/d}.$$

Now let Z denote the restriction of X to the cube Q. Then by the smoothness of L^p_B, the above is a.s. bounded below by

$$\liminf_{n\to\infty} L^p_B(Z_1,...,Z_{n\rho}, Q)/n^{(d-p)/d}. \tag{7.18}$$

Since the random variables Z have compact support, Theorem 7.1 implies that (7.18) is in turn bounded below by

$$\rho^{(d-p)/d}\alpha(L^p_B, d)\int_{\mathbb{R}^d} f_Z(x)^{(d-p)/d}dx \quad a.s., \tag{7.19}$$

where f_Z denotes the density for the random variable Z. Let ϵ go to zero in (7.19) and apply Fatou's lemma to deduce the lower bound (7.17).

The proof of the lower bound (7.17) is rather straightforward. The proof of (7.16) thus depends on the proof of the upper bound implicit in (7.16), i.e., the proof of

$$\limsup_{n\to\infty} L^p(X_1,...,X_n)/n^{(d-p)/d} \le \alpha(L_B^p,d)\int_{\mathbb{R}^d} f(x)^{(d-p)/d}dx \quad a.s. \tag{7.20}$$

The proof of (7.20) is broken into two steps and we will faithfully follow Rhee (1993a).

We set the stage by letting $s(n)$ be the largest $k \in \mathbb{N}$ such that $A_k \cap \{X_1,...,X_n\}$ is not empty; for all $1 \le q \le s(n)$ we let B_q be the ball of radius 2^q. We now bound $L^p(X_1,...,X_n)$ by considering the simple subadditivity (2.2) of the Euclidean functional L^p over the union of $A_{s(n)}$ and $\{\bigcup_{k=q}^{s(n)-1} A_k \cup B_q\}$. Simple subadditivity gives for fixed q

$$\begin{aligned} L^p(X_1,...,X_n) &\le L^p\left(\{X_1,...,X_n\}\cap A_{s(n)}\right) + \\ &+ L^p\left(\{X_1,...,X_n\}\cap\{\cup_{k=q}^{s(n)-1}A_k\cup B_q\}\right) + C_1(2^{s(n)+1})^p. \end{aligned}$$

Applying simple subadditivity to the second term on the right side of the above gives

$$\begin{aligned} &L^p\left(\{X_1,...,X_n\}\cap\{\bigcup_{k=q}^{s(n)-1} A_k\cup B_q\}\right) \\ &\le L^p\left(\{X_1,...,X_n\}\cap A_{s(n)-1}\right) + \\ &\quad + L^p\left(\{X_1,...,X_n\}\cap\{\bigcup_{k=q}^{s(n)-2} A_k\cup B_q\}\right) + C_1(2^{s(n)})^p. \end{aligned}$$

Repeatedly applying simple subadditivity and bounding the geometric series $\sum_{k=q}^{s(n)} 2^{kp}$ by $C(p)2^{ps(n)}$ we come to the starting point of the proof of (7.20), namely we arrive at

$$\begin{aligned} L^p(X_1,...,X_n) &\le L^p(\{X_1,...,X_n\}\cap B_q) + \\ &+ \sum_{q\le k\le s(n)} L^p(\{X_1,...,X_n\}\cap A_k) + C(p)2^{ps(n)}. \end{aligned} \tag{7.21}$$

The proof of (7.20) now follows from the following two steps. Since $L^p(F,\mathbb{R}^d) = L^p(F,R)$ if $F\cap R = F$, growth bounds for Euclidean functionals imply that if

$$N_k(n) := \mathrm{card}(\{X_1,...,X_n\}\cap A_k),$$

then $L^p(\{X_1,...,X_n\}\cap A_k) \le C2^{kp}(N_k(n))^{(d-p)/d}$.

Step 1. Show that

$$\limsup_{n\to\infty} \frac{L^p(\{X_1,...,X_n\}\cap B_q)}{n^{(d-p)/d}} \le \alpha(L^p_B, d) \int_{B_q} f(x)^{(d-p)/d} dx \quad a.s. \tag{7.22}$$

Step 2. Show that

$$\limsup_{n\to\infty} \left\{ C(p)2^{p\cdot s(n)} + \sum_{k=q}^{s(n)} 2^{kp}(N_k(n))^{(d-p)/d} \right\} / n^{(d-p)/d} \le t(p,q) \quad a.s., \tag{7.23}$$

where $\lim_{q\to\infty} t(p,q) = 0$.

Indeed, combining (7.21)-(7.23) and letting q tend to infinity yields the desired upper bound (7.20).

Let us now prove (7.22). Fix q and set

$$\int_{B_q} f(x)dx = 1 - \epsilon$$

where ϵ is between 0 and 1. We let Z denote the restriction of the random variable X to the ball B_q, we let f_Z be its density, and we let $B(n, 1-\epsilon)$ denote a binomial random variable with parameters n and $1-\epsilon$. By the assumed smoothness of L^p we have with probability one the estimate

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{L^p(\{X_1,...,X_n\}\cap B_q)}{n^{(d-p)/d}} \\
&= \limsup_{n\to\infty} \frac{L^p(Z_1,...,Z_{B(n,1-\epsilon)})}{n^{(d-p)/d}} \\
&\le \limsup_{n\to\infty} \frac{L^p(Z_1,...,Z_{n(1-\epsilon)})}{n^{(d-p)/d}} + \\
&\quad + \; C_3 2^{qp} \limsup_{n\to\infty} \frac{|B(n,1-\epsilon)-n(1-\epsilon)|^{(d-p)/d}}{n^{(d-p)/d}} \\
&= (1-\epsilon)^{(d-p)/d} \alpha(L^p_B, d) \int_{B_q} f_Z(x)^{(d-p)/d} dx,
\end{aligned}$$

where the last equality follows from the strong law of large numbers. The proof of (7.22) and Step 1 is completed by noting that $f_Z(x) = f(x)/(1-\epsilon)$ for all $x \in B_q$.

To prove Step 2 we only need to bound the tail of binomial distributions. This involves some lengthy computations and we will follow the approach used by Rhee (1993a) for the TSP and generalized by McGivney (1997).

To prove (7.23) we note that since $N_k(n)$ and $s(n)$ increase in n, it suffices to show with probability one that

$$\limsup_{r\to\infty} T(r,p,q) \le C(p)\sum_{k\ge q} 2^{-p\cdot t(k)}, \tag{7.24}$$

where

$$T(r,p,q) := \frac{1}{2^{r(d-p)/d}}\left\{C(p)2^{p\cdot s(2^r)} + \sum_{k\ge q} 2^{kp}(N_k(2^r))^{(d-p)/d}\right\},$$

and where $\sum_{k\ge1} 2^{-p\cdot t(k)}$ is a convergent sum.

To prove (7.24) we fix p, $1\le p<d$, and for $r\in\mathbb{N}$ define events E_r in such a way that on the infinite intersection $\bigcap_{r\ge m} E_r$, m arbitrary, we have

$$\limsup_{r\to\infty} T(r,p,q) \le C\sum_{k\ge q} 2^{-p\cdot t(k)} \tag{7.25}$$

and moreover

$$\lim_{m\to\infty} P(\bigcap_{r\ge m} E_r) = 1. \tag{7.26}$$

We will choose $E_r := \bigcap_{i=1}^3 E_{i,r}$, where the events $E_{1,r}, E_{2,r}$, and $E_{3,r}$ are defined as follows and where the integers A, B, Q, and R are to be chosen later, all as a function of the parameter r:

$$E_{1,r} := \{\forall k > B, N_k(2^r) = 0\}$$

and

$$E_{2,r} := \{\sum_{A\le k\le B} N_k(2^r) \le Q\}$$

and

$$E_{3,r} := \{\forall k < A, N_k(2^r) \le R\}.$$

Let us first consider how to arrange for (7.26). Set $m_k := \int_{A_k} f(x)dx$ so that

$$a_k(f) := 2^{kdp/(d-p)}m_k.$$

Clearly, if for each $1\le i\le 3$ we have $\sum_{r=1}^\infty P\{E_{i,r}^c\} < \infty$ then the limit (7.26) follows. We will now show that the sums $\sum_{r=1}^\infty P\{E_{i,r}^c\}, 1\le i\le 3$, are each bounded by an expression defined in terms of the $m_k, k\ge 1$, which we will in turn bound by terms from the sum $\sum_{k=1}^\infty 2^{-t(k)}$. We have by independence

$$\begin{aligned} P\{E_{1,r}\} &= P\left\{\bigcap_{k>B}\{X_1\notin A_k,\dots,X_{2^r}\notin A_k\}\right\} \\ &= \left(P\{\bigcap_{k>B}(X_1\notin A_k)\}\right)^{2^r} \\ &= (1-\sum_{k>B} m_k)^{2^r}. \end{aligned} \tag{7.27}$$

Considering $P\{E_{2,r}^c\}$, we have

$$P\{E_{2,r}^c\} = P\{\sum_{A\le k\le B} N_k(2^r) > Q\} \le P\{B(2^r, \sum_{A\le k\le B} m_k) \ge Q\}.$$

By standard estimates for the tail of a binomial random variable (see e.g. Shorack and Wellner (1986,Chapter 11)), the right side of the above is bounded by $\exp(-\alpha Q)$ if $Q \ge 2^{r+1}\sum_{A\le k\le B} m_k$. Here α is a generic positive constant whose value is not important to us. Thus

$$P\{E_{2,r}^c\} \le \exp(-\alpha Q) \tag{7.28}$$

if Q is at least as large as $2^{r+1}\sum_{A\le k\le B} m_k$. Considering $P\{E_{3,r}^c\}$ we have

$$P\{E_{3,r}^c\} \le \sum_{k<A} P\{B(2^r, m_k) > R\} \le \sum_{k<A} \exp(-\alpha R) \tag{7.29}$$

if R is at least as large as $2^{r+1}m_k$. Thus the three probabilities $P\{E_{i,r}^c\}, 1 \le i \le 3$, are each bounded by terms involving m_k.

We are thus motivated to search for bounds for the terms m_k, $k \ge 1$. Since $m_k := a_k(f)2^{-kdp/(d-p)}$ and since the behavior of the terms $a_k(f)$ may be irregular we will regularize the sequence $a_k(f)$, $k \ge 1$, and find a sequence $t(k)$, $k \ge 1$, of positive numbers which satisfies

$$a_k(f)^{(d-p)/d} \le 2^{-p\cdot t(k)}, \tag{7.30}$$

$$\sum_{k=1}^{\infty} 2^{-p\cdot t(k)} < \infty, \tag{7.31}$$

and

$$\forall j,k \in \mathbb{N} \quad |t(j) - t(k)| \le \frac{p|j-k|}{2d}. \tag{7.32}$$

One choice of sequence $t(k)$, $k \ge 1$, is that defined by

$$2^{-p\cdot t(k)} := \sum_{l=1}^{\infty} (a_l(f))^{(d-p)/d} 2^{-p^2|l-k|/2d}.$$

It is straightforward to verify that such a sequence satisfies (7.30)-(7.32); the details are provided at the end of this chapter.

By (7.30), we have $m_k \le 2^{-(k+t(k))dp/(d-p)}$ which suggests the natural definition

$$m_k' := 2^{-(k+t(k))dp/(d-p)}. \tag{7.33}$$

This immediately yields

$$m_{k+1} \le m_{k+1}' \le 2^{-p} m_k', \tag{7.34}$$

where the second inequality uses (7.32). Moreover, we have

$$\sum_{l=k}^{\infty} m_l \leq \sum_{l=k}^{\infty} m_l' \leq m_k' \sum_{l\geq 1} 2^{-pl} \leq 2m_k'. \tag{7.35}$$

Since $m_k \leq m_k' := 2^{-(k+t(k))dp/d-p}$, our search for bounds for m_k brings us to look for bounds for $k+t(k)$. For each $r \in \mathbb{N}$, denote by $k(r)$ the largest k such that

$$k + t(k) \leq r\frac{d-p}{dp}. \tag{7.36}$$

By (7.32) and (7.36) we have the estimate

$$\begin{aligned} k(r) + t(k(r)) \leq r\frac{d-p}{dp} &\leq k(r) + 1 + t(k(r)+1) \\ &\leq k(r) + t(k(r)) + 2, \end{aligned} \tag{7.37}$$

which implies that if $k(r) = k(r')$ then $\frac{d-p}{dp}|r - r'| \leq 2$. Using this observation and (7.31) it is easy to see that we must have

$$\sum_{r=1}^{\infty} 2^{-p\cdot t(k(r))} < \infty. \tag{7.38}$$

This convergent sum will serve us well in our ongoing attempt to bound the three sums $\sum_{r\geq 1} P\{E_{i,r}^c\}$, $i = 1,2,3$. More precisely, we will show for each $i = 1,2,3$ that

$$P\{E_{i,r}^c\} \leq C2^{-p\cdot t(k(r))}.$$

Let's first consider the case $i = 1$. From (7.27) it is clearly desirable to obtain a bound of the form

$$\sum_{k>B} m_k \leq 2^{-p\cdot t(k(r)) - r + C}, \tag{7.39}$$

for some appropriate choice of C. Now (7.35) gives

$$\sum_{k>B} m_k \leq 2m_{B+1}' = 2\cdot 2^{-(B+1+t(B+1))\frac{dp}{d-p}}. \tag{7.40}$$

Notice that if B is defined by

$$B := [k(r) + (1-b)t(k(r))], \tag{7.41}$$

where $0 < b < 1$ is a constant to be chosen later, then we can achieve the necessary small upper bound (7.39). Noting that (7.32) yields

$$|t(B+1) - t(k(r)+1)| \leq \frac{p(B-k(r))}{2d} \leq \frac{p\cdot t(k(r))}{2d}$$

we indeed find by (7.40) and (7.41) that

$$\sum_{k>B} m_k \le 2 \cdot 2^{-\left(k(r)+(1-b)t(k(r))+t(k(r)+1)-\frac{p(B-k(r))}{2d}\right)\frac{dp}{d-p}}$$
$$\le 2 \cdot 2^{-\left(k(r)+(1-b)t(k(r))+t(k(r)+1)-\frac{p\cdot t(k(r))}{2d}\right)\frac{dp}{d-p}}$$
$$\le 2 \cdot 2^{-\left(r(\frac{d-p}{dp})-1+(1-b-\frac{p}{2d})t(k(r))\right)\frac{dp}{d-p}},$$

where the last inequality follows by (7.37). Many values of b will achieve the bound (7.39); for specificity we choose $b := p/2d$ and obtain the rough estimate

$$\sum_{k>B} m_k \le 2 \cdot 2^{-r+\frac{dp}{d-p}-p\cdot t(k(r))}.$$

Returning to (7.27) and using the estimate $e^{-x} \ge 1-x$ for x small and positive gives

$$P\{E_{1,r}\} \ge 1 - C \cdot 2^{\frac{dp}{d-p}-p\cdot t(k(r))}$$

and therefore

$$P\{E_{1,r}^c\} \le C \cdot 2^{-p\cdot t(k(r))},$$

as desired.

Now we turn to an estimate for $P\{E_{2,r}^c\}$. We recall by (7.28) that

$$P\{E_{2,r}^c\} \le \exp(-\alpha Q)$$

if $Q := Q(r) \ge 2^{r+1}\sum_{A\le k\le B} m_k$. Observe by (7.35) that

$$2^{r+1}\sum_{A\le k\le B} m_k \le 2^{r+2}2^{-(A+t(A))\frac{dp}{d-p}}. \tag{7.42}$$

We would like to select A in such a way that we can choose $Q(r) \approx 2^{Ct(k(r))}$ and thus obtain

$$P\{E_{2,r}^c\} \le \exp(-\alpha Q) \le C \cdot 2^{-p\cdot t(k(r))}. \tag{7.43}$$

As with the choice of B, we see that if A is of the form

$$A := [k(r) - at(k(r))], \tag{7.44}$$

where $0 < a < 1$ is a constant to be chosen, then we can apply (7.37) and achieve the bound (7.42). Noting that (7.32) yields

$$|t(A) - t(k(r))| \le \frac{p(k(r)-A)}{2d} \le \frac{p\cdot t(k(r))}{2d},$$

we indeed find by (7.42) and (7.44) that

$$2^{r+1}\sum_{A\le k\le B} m_k \le 2^{r+2}2^{-\left(k(r)-at(k(r))+t(k(r))-\frac{p(k(r)-A)}{2d}\right)\frac{dp}{d-p}}.$$

By (7.37) we obtain $|k(r) + t(k(r)) - r\frac{d-p}{dp}| \leq 2$ and thus

$$2^{r+1} \sum_{A \leq k \leq B} m_k \leq C2^r 2^{-(r\frac{d-p}{dp} - (a+\frac{ap}{2d})t(k(r)))\frac{dp}{d-p}}$$
$$\leq C2^{2at(k(r))\frac{dp}{d-p}},$$

since $a + \frac{ap}{2d} \leq 2a$. We now see that if we set

$$Q := C2^{2at(k(r))\frac{dp}{d-p}},$$

then (7.28) gives the desired estimate

$$\sum_r P\{E_{2,r}^c\} \leq \sum_r \exp(-\alpha Q) \leq C \sum_r 2^{-p \cdot t(k(r))} < \infty,$$

since $\exp(-\alpha 2^{-ct}) \leq 2^{-pt}$ for t large enough.

To conclude the proof of (7.26) it only remains to verify that $\sum_r P\{E_{3,r}^c\} < \infty$. We will follow the ideas above. We recall by (7.29) that

$$P\{E_{3,r}^c\} \leq \sum_{k<A} \exp(-\alpha R)$$

if $R := R(r) \geq 2^{r+1} m_k \geq 2^{r+1} m'_k$. We may thus choose $R := 2^{r+1} m'_k$ and we now seek lower bounds on R in order to bound $\sum_{k<A} \exp(-\alpha R)$. Observe that iteration of the estimate $m'_k \geq 2^p m_{k+1}$ a total of $A - k$ times gives

$$m'_k \geq 2^{p(A-k)} m'_A.$$

Thus by (7.33) and (7.44) we have for $k < A$

$$m'_k \geq 2^{p(A-k)} 2^{-(A+t(A))\frac{dp}{d-p}}$$
$$\geq 2^{p(A-k)} 2^{-(k(r) - at(k(r)) + 1 + t(k(r)) + \frac{p(A-k(r))}{2d})\frac{dp}{d-p}}.$$

Arguing as in previous cases, we obtain from (7.37)

$$m'_k \geq 2^{p(A-k)} 2^{-(k(r) - at(k(r)) + 1 + t(k(r)) + \frac{p(1-at(k(r)))}{2d})\frac{dp}{d-p}}$$
$$\geq 2^{p(A-k)} 2^{-(r\frac{d-p}{dp} + 2 - t(k(r))(a+\frac{ap}{2d}))\frac{dp}{d-p}}$$
$$\geq C2^{p(A-k)} 2^{-r} 2^{(a+\frac{ap}{2d})\frac{dp}{d-p}}$$
$$\geq C2^{p(A-k)} 2^{-r+ap \cdot t(k(r))},$$

where we use $-p/2d > -1, a + ap/2d > a$, and $\frac{dp}{d-p} \geq p$. Thus

$$R \geq C2^{p(A-k)+ap \cdot t(k(r))}.$$

It follows that

$$\begin{aligned}P\{E_{3,r}^c\} &\le \sum_{k<A} \exp(-\alpha R)\\ &\le \sum_{k<A} \exp(-\alpha C 2^{p(A-k)} 2^{ap\cdot t(k(r))})\\ &\le C\exp(-\alpha C 2^{ap\cdot t(k(r))}).\end{aligned}$$

Now since $\sum_r 2^{-p\cdot t(k(r))} < \infty$ we have $\lim_{r\to\infty} t(k(r)) = \infty$ and thus if r is large enough

$$\exp(-\alpha C 2^{ap\cdot t(k(r))}) \le 2^{-p\cdot t(k(r))}$$

which gives

$$\sum_r P\{E_{3,r}^c\} \le C \sum_r 2^{-p\cdot t(k(r))} < \infty.$$

We have thus shown the equality (7.26) for $A := A(r) := [k(r) - at(k(r))]$, $B := [k(r) + (1 - \frac{p}{2d})t(k(r))]$, $Q := C2^{2at(k(r))\frac{dp}{d-p}}$, and $R := 2^{r+1} m_k'$, where $a > 0$ is still to be defined. We now show the inequality (7.25).

To see that (7.25) holds, we will consider the behavior of $T(r,p,q)$ on the events $E_{i,r}$, $i = 1,2,3$. On $E_{3,r}$ we have by choice of R that

$$\begin{aligned}&2^{-r\frac{d-p}{d}} \sum_{q\le k<A} 2^{kp}(N_k(2^r))^{(d-p)/d}\\ &\le 2^{-r\frac{d-p}{d}} \sum_{q\le k<A} 2^{kp}(2^{r+1}m_k')^{(d-p)/d}\\ &= 2^{-r\frac{d-p}{d}} \sum_{q\le k<A} 2^{kp}\left(2^{r+1}2^{-(k+t(k))\frac{dp}{d-p}}\right)^{(d-p)/d}\\ &\le C \sum_{q\le k<A} 2^{-p\cdot t(k)}.\end{aligned} \tag{7.45}$$

On the event $E_{2,r}$ we have by Hölder's inequality that

$$\begin{aligned}&2^{-r\frac{d-p}{d}} \sum_{A\le k\le B} 2^{kp}(N_k(2^r))^{(d-p)/d}\\ &\le 2^{-r\frac{d-p}{d}} \left(\sum_{A\le k\le B} 2^{dk}\right)^{p/d} \left(\sum_{A\le k\le B} N_k(2^r)\right)^{(d-p)/d}\\ &\le C2^{-r\frac{d-p}{d}} 2^{Bp} 2^{2ap\cdot t(k(r))},\end{aligned}$$

by the definition of Q. Recalling the definition of B, the above is bounded by

$$\begin{aligned}&\le C2^{-r(\frac{d-p}{d})} 2^{p(k(r)+(1-\frac{p}{2d})t(k(r)))+2ap\cdot t(k(r))}\\ &\le C2^{-r(\frac{d-p}{d})} 2^{p(r\frac{d-p}{dp} - (\frac{p}{2d}-2a)t(k(r)))}\\ &= C2^{-p(\frac{p}{2d}-2a)t(k(r))}.\end{aligned}$$

Now we choose $a > 0$ such that $\frac{p}{2d} - 2a > 0$. There is more than one value for a which is suitable and for specificity we choose $a := \frac{p}{5d}$, giving $\frac{p}{2d} - 2a = \frac{p}{10d}$. On $E_{2,r}$ this yields the upper bound

$$(7.46) \qquad 2^{-r\frac{d-p}{d}} \sum_{A \le k \le B} 2^{kp} \left(N_k(2^r)\right)^{(d-p)/d} \le C 2^{-\frac{p^2}{10d} t(k(r))}.$$

We note for future reference that the right side of (7.46) approaches zero as r tends to infinity.

Finally, on the event $E_{1,r}$ we have

$$(7.47) \qquad 2^{-r\frac{d-p}{d}} \sum_{k > B} 2^{kp} (N_k(2^r))^{(d-p)/d} = 0.$$

To estimate the remaining term $2^{p \cdot s(2^r)} 2^{-r\frac{d-p}{d}}$ in $T(r, p, q)$ we note that on $E_{1,r}$ we have $N_k(2^r) = 0$ for all $k > B$, implying $s(2^r) \le B$. Thus by the definition of B we obtain the upper bound

$$\begin{aligned} 2^{p \cdot s(2^r) - r\frac{d-p}{d}} &\le 2^{pB - r\frac{d-p}{d}} \\ &\le 2^{p[k(r) + (1 - \frac{p}{2d}) t(k(r))] - r\frac{d-p}{d}} \\ &= 2^{-\frac{p^2 t(k(r))}{2d}}, \end{aligned}$$

from which it follows that

$$(7.48) \qquad \lim_{r \to \infty} 2^{p \cdot s(2^r) - \frac{r(d-p)}{d}} = 0.$$

Collecting the estimates (7.45) - (7.48) we obtain on the set $\bigcap_{r \ge m} E_r$, $m \in \mathbb{N}$ and arbitrary, the upper bound

$$\limsup_{r \to \infty} 2^{-\frac{r(d-p)}{d}} \{C 2^{p \cdot s(2^r)} + \sum_{k \ge q} 2^k (N_k(2^r))^{(d-p)/d}\} \le C \sum_{q \le k < A} 2^{-p \cdot t(k)}.$$

This is precisely (7.24), which was to be shown. This completes the proof of Theorem 7.6. □

Notes and References

1. The proof of the umbrella Theorem 7.1 follows Redmond and Yukich (1994) which treats the case $p = 1$ and Redmond and Yukich (1996), which treats the general case $1 \le p < d$. The proof of the umbrella Theorem 7.6 follows approximately as in McGivney (1997). It would be worthwhile to find an umbrella theorem for values of p in the range $0 < p < 1$ and $p \ge d$.

2. Smoothness of L is not necessary in order to insure (7.2). Indeed, the bi-partite matching functional is not smooth but nonetheless satisfies (7.2); see Dobrić and Yukich (1995). See Yukich (1992) for asymptotics for bi-partite matching on $\mathbb{R}^d$.

3. We verify that the sequence $t(k)$, $k \geq 1$, satisfies conditions (7.30)-(7.32). To see that condition (7.30) is satisfied note simply that

$$\begin{aligned} 2^{-t(k)} &= \sum_{l=1}^{\infty} a_l(f)^{(d-p)/dp}\, 2^{-p|l-k|/2d} \\ &\geq a_k(f)^{(d-p)/dp}. \end{aligned}$$

To verify (7.31) we observe that

$$\begin{aligned} \sum_{k=1}^{\infty} 2^{-p \cdot t(k)} &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (a_l(f))^{(d-p)/d}\, 2^{-p^2|l-k|/2d} \\ &< \sum_{k=-\infty}^{\infty} \sum_{l=1}^{\infty} (a_l(f))^{(d-p)/d}\, 2^{-p^2|l-k|/2d} \\ &= \sum_{l=1}^{\infty} (a_l(f))^{(d-p)/d} \sum_{k=-\infty}^{\infty} 2^{-p^2|l-k|/2d} \\ &< \infty. \end{aligned}$$

Finally, to show (7.32) it suffices to show both

$$t(k) - t(j) \leq \frac{p|j-k|}{2d} \quad \text{and} \quad t(j) - t(k) \leq \frac{p|j-k|}{2d}.$$

We will show the first inequality; the second holds in a similar way. To show the first inequality, it suffices to show

$$2^{-t(j) - \frac{p|j-k|}{2d}} \leq 2^{-t(k)}.$$

Observe that this follows from

$$\begin{aligned} 2^{-t(k)} &= \sum_{l=1}^{\infty} (a_l(f))^{(d-p)/dp}\, 2^{-\frac{p|l-k|}{2d}} \\ &\geq \sum_{l=1}^{\infty} (a_l(f))^{(d-p)/dp}\, 2^{-\frac{p}{2d}(|l-j|+|j-k|)} \\ &= 2^{-\frac{p}{2d}|j-k|} \sum_{l=1}^{\infty} (a_l(f))^{(d-p)/dp}\, 2^{-\frac{p}{2d}|l-j|} \\ &= 2^{-\frac{p}{2d}|j-k|} 2^{-t(j)}. \end{aligned}$$

The second inequality follows similarly.