IITM-CS6100 : Topics in Design and analysis of Algorithms Given on: Mar 23

Problem Set #2 Arjun Bharat, CS17B006 Evaluation Due on : May 30

- Turn in your solutions electronically at the moodle page. The submission should be a pdf file typeset either using LaTeX or any other software that generates pdf. No handwritten solutions are accepted.
- Collaboration is encouraged, but all write-ups must be done individually and independently. For each question, you are required to mention the set of collaborators, if any.
- Submissions will be checked for **plagiarism**. Each case of plagiarism will be reported to the institute disciplinary committee (DISCO).

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1. (8 points) Let $P = \{p_1, \ldots, p_n\}$ be a set of n points from the unit square $[0,1]^2$. A triangulation τ of P is a maximal planar graph with P as the vertex set (i.e., the locations of points in P should be an embedding of the graph). A minimum weight triangulation (denoted by mwt) is a triangulation with minimum total edge weight. Let MWT denote the corresponding Euclidean functional. The convex hull (denoted by CH) of P is the smallest convex set (i.e. polygon) containing P. Let $|CH(P)|$ denote the number of vertices in the convex hull of P. Suppose Q_1, Q_2, Q_3 and Q_4 be a partition of $[0,1]^2$ of equal sized squares. Show that

$$MWT(P, [0,1]^2) \leq \sum_{i=1}^{4} MWT(P \cap Q_i, Q_i) + \sum_{i=1}^{4} O(|CH(P \cap Q_i)|).$$

Solution:

Collaborators: Shriram C(CS18B007). He gave me hints on how to start.
Claim: For a minimum weight triangulation of a set of points, the vertices on the boundary of the outer face of the maximal planar graph will constitute a minimum convex hull.
Proof: Consider the outer face of the minimum weight triangulation. It is a polygon enclosing all points, hence, it is a valid convex hull. If there was a smaller convex hull, optimal triangulation of the same could result in a smaller weighted triangulation which would contradict the assumption we made. Thus, the outer face must be the smallest convex hull possible.
Consider the minimum weight triangulations formed by $MWT(P \cap Q_i, Q_i)$. We consider the points on the outer faces of each region, and add edges only between them so as to maximally triangulate $[0,1]^2$.(the inner triangulations within each region would not be disturbed by this.) Each edge so added has a length of at most

$\sqrt{2}$(which is $O(1)$), and there are $\sum_{i=1}^{4} |CH(P \cap Q_i)|$ such vertices among which we try to construct a planar graph. Hence, we add at most $3 \cdot \sum_{i=1}^{4} |CH(P \cap Q_i)| - 6$ such edges (because the maximum number of edges in a planar graph on n vertices is $3n - 6$). Thus, the cost incurred to merge all sub-triangulations is $\sum_{i=1}^{4} O(|CH(P \cap Q_i)|)$, and the resulting triangulation is an upper estimate of $MWT(P, [0,1]^2)$. Hence, we have $MWT(P, [0,1]^2) \leq \sum_{i=1}^{4} MWT(P \cap Q_i, Q_i) + \sum_{i=1}^{4} O(|CH(P \cap Q_i)|)$

2. (16 points) In the vehicle routing problem, we are given a set of depots $D = \{d_1, \ldots, d_k\}$ from $[0,1]^d$ and a set of n points $P = \{p_1, \ldots, p_n\}$ (called customers) which are again points from $[0,1]^d$. The job is to compute minimum cost k vertex disjoint cycles such that every cycle has exactly one depot and every depot is in exactly one cycle. Let MDP denote the corresponding Euclidean functional.
   (a) (7 points) Formally define the functional MDP. Prove that MDP is subadditive but not superadditive.
   (b) (9 points) Define a canonical boundary functional MDP_B for MDP and show that it is superadditive. Show that the boundary functional is also a smooth Euclidean functional.

Solution: Collaborators:Naveen L S (CS18B031). He helped me with the second part by giving me an idea on how to define the functional.
(a) We assume that $n \geq 2k$, so that we get cycles having at least 3 vertices each. We define the functional as follows: $MDP(D, A) =$

$$\min_{\{i_1,i_2,\ldots i_n\} \in perm(\{1,2,\ldots n\}), l_0=0, l_j \geq 2, \sum_{j=1}^{k} l_j = n} \sum_{x=1}^{k} \left(dis(d_x, a_{i_{(l_x+v)}}) + dis(d_x, a_{i_{v+1}}) + \sum_{y=1}^{l_x-1} dis(a_{i_{(y+v)}}, a_{i_{(y+v+1)}}) \right) \tag{1}$$

where dis is the euclidean distance, $v = \sum_{m=0}^{x-1} l_m$ for each term of the summation $perm(\{1,2,\ldots n\})$ denotes all possible permutations of the set $\{1,2,\ldots n\}$, $D = \{d_1, d_2, \ldots d_k\}$ and $A = \{a_1, a_2, \ldots a_n\}$. We have created a cycle having depot d_i and any l_i customers. This ensures that all possible valid k disjoint cycles are considered.

Clearly, partitioning a region R into rectangular regions R_1, R_2 would result in a sub-solution for the individual regions, having x and y disjoint cycles respectively,where $x+y = k$. Retaining these solutions as such would be one estimate of the value of the least possible tour which is $MDP(D,A,R)$. Hence, $MDP(D,A,R) \leq MDP(D \cap R_1, A \cap R_1, R_1) + MDP(D \cap R_2, A \cap R_2, R_2)$. Since we are using euclidean distance, we can clearly see that $\forall y \in R^2, MDP(D+y, A+y) = MDP(D,A)$, and $\forall \alpha, MDP(\alpha D, \alpha A) = \alpha \cdot MDP(D,A)$ (euclidean distance is unaffected by a universal shift and is also linearly scaled), and hence this establishes subadditivity.

Consider $[0,1]^2$ where there are 2 depots and 4 customers. Divide the square into 4 equal squares. Place one depot each in the second and fourth quadrant, and 2 customers each in the first and third quadrant. The optimal MDP consists of two triangles formed on either side of the line $x = 0.5$. But the line $y = 0.5$ can result in longer sub-tours in each region that violate superadditivity.

(b) Assume that $n \geq 2k$. We notice a striking similarity to the TSP, where each of the k cycles must contain exactly one depot and every customer must be covered by exactly one cycle. Since it is easier to analyze TSP_B,we define the boundary functional as follows:

$$MDP_B(D,A) = \min(MDP(D,A), \min_{\bigcup_{i=1}^{k} A_i = A, |A_i| \geq 2, A_i \cap A_j = \phi} \sum_{i=1}^{k} TSP_B(A_i \cup \{d_i\}))$$

We have considered all possible partitions of customers among the k depots and use the boundary functional of the TSP to compute a minimal cost tour for each of the depots.

Since we have ensured that every cycle has at least 2 customers and one depot, the cycles are meaningful and of length at least 3. Therefore, by the superadditivity of TSP_B, we conclude that MDP_B is superadditive. This can be broken if we consider that cycles of length 2 are allowed: For example, $d_1 = (0.25, 0.55).d_2 = (0.75, 0.45), a_1 = (0.75, 0.55), a_2 = (0.25, 0.45)$, the regions R_1 and R_2 being the two regions formed by the line $y = 0.5$. This would violate superadditivity as $MDP_B(D,A) = 0.4$ whereas $MDP_B(D \cap R_1, A \cap R_1, R_1) = MDP(D \cap R_2, A \cap R_2, R_2) = 0.6$.

Let us fix the depots and consider an additional set of customers A'. Let the incremental partitions(that is, the vertices added to each of the existing optimum partitions to create the new optimum) created by A' be $A'_1, A'_2, \ldots A'_k$, and the original optimum partitions be $A_1, A_2, \ldots A_k$.Then,we clearly have

$|MDP_B(D, A \cup A') - MDP_B(D,A)| \leq \sum_{i=1}^{k} |TSP_B(A_i \cup A'_i) - TSP_B(A_i)| \leq c \cdot |A'|^{\frac{d-p}{d}}$ (by the smoothness of TSP_B, assuming a p order functional). This proves the smoothness of MDP_B.

3. (6 points) Refer to phase 3 of the algorithm for computing Hamiltonian cycle given in Page 6 of Frieze's survey. Show that for any tractable graph, the phase 3 of the algorithm always succeeds in computing a Hamiltonian cycle, if exists.

Solution: Assume that we can always apply either step 1 or step 2. Suppose G has a Hamilton cycle and neither step 1 not step 2 can be applied further, and we have $|V(C)| < n$. Then, this implies that no two neighbours on the cycle can share a common vertex outside the cycle, and no two neighbouring vertices can have one neighbour each outside the cycle that are connected to them.