CS 6100

Lecture-6+7+8

Basics of probability Theory

Ref :- Heiko Röglin's notes on Prob. anal. of algorithms. (available in the google shared folder).

Recall

Analysis of 2-opt with manhatten-metric (instead of Euclidean)

→ random points distributed on $[0,1]^2$, $[0,1]^d$ $d \geq 2$

$\Delta(S) = |x_1 - x_2| + \ldots$

$Pr[\Delta(S) \leq \Delta_{min}]$

Probability Spaces

Discrete Probability Space (Ω, p).

- Ω : samplespace. finite or countable set.
  - set of all possible outcomes.

- $p : \Omega \to [0, 1]$.

* $\sum_{x \in \Omega} p(x) = 1$

An event $S \subseteq \Omega$.

$P(S) = \sum_{x \in S} p(x)$.

die with six faces.

$\{1, \dots 6\}$

$\Omega = \{1, \dots 6\}$.

$p(1) = p(2) = \dots p(6) = \frac{1}{6}$

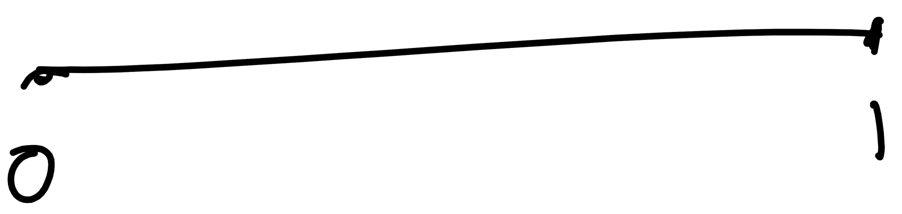
event -

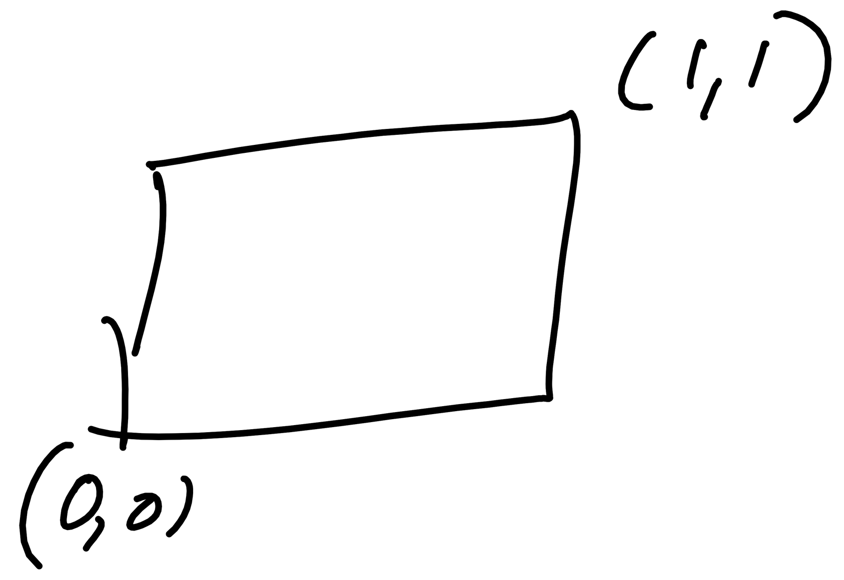
outcome is even no

$S = \{2, 4, 6\}$.

$p(S) = \frac{1}{2}$

$[0, 1]$





$[-\epsilon, \epsilon]$.

Continuous case

A probability space is $(\Omega, \mathcal{F}, P)$ s.t

1 Ω : sample space

2 $\mathcal{F} \subseteq 2^{\Omega}$ with the following properties.

(i) $\Omega \in \mathcal{F}$

(ii) $\mathcal{F}$ should be closed under complementation

ie $X \in \mathcal{F} \Rightarrow \Omega \setminus X \in \mathcal{F}$.

(iii) $\mathcal{F}$ is closed under countable unions

ie if $X_1, X_2, \dots \in \mathcal{F} \Rightarrow \bigcup_i X_i \in \mathcal{F}$.

(ie $\mathcal{F}$ should be a σ-algebra).

3. P is a probability measure

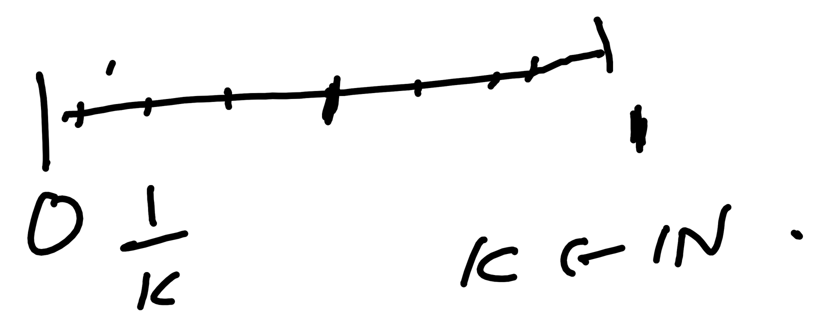
ie $P : \mathcal{F} \to [0,1]$

(i) $P(\Omega) = 1$

(ii) if $X_1, X_2, \dots \in \mathcal{F}$ s.t $X_i \cap X_j = \phi$ (disjoint)

then $P(\bigcup_i X_i) = \sum_i P(X_i)$

$[0,1]$



$\frac{1}{1}, \frac{1}{2}, \dots\dots,$

example of $\mathcal{F}$: Borel σ-algebra.

set of all intervals in $\mathbb{R}$.

& their unions.

Lemma : $X_1, X_2 \dots X_n$ are events in $(\Omega, \mathcal{F}, P)$.

then $P[X_1 \cup X_2 \cup \dots \cup X_n] \le \sum_{i=1}^{n} P(X_i)$

$\rightarrow$ Union bound

Random variables

Let $(\Omega, \mathcal{F}, P)$ - prob. space.

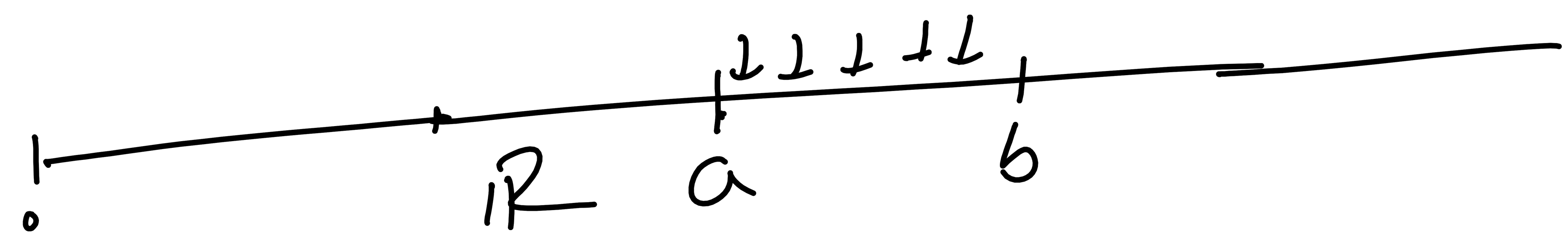
A random variable is a fn $X : \Omega \to \mathbb{R}$.
(real valued).

X: continuous fn

s.t for an $a \leq b$.

$$X^{-1}([a,b]) = \{ z \in \Omega \mid X(z) \in [a,b] \} \in \mathcal{F}.$$

X



$\Omega = [0,1]$

$X : \Omega \to \mathbb{R}$.
$\alpha \mapsto \alpha^2$

$P_1 \cdots P_n$

eg:- Cost of a TSP tour

value of $\Delta(S)$

$\Delta(S) \geq \cdots$

$\Delta(S) = -$

X

$\Pr[X = x]$: probability that r.v X takes the value x.

$\Pr[X = x] \overset{\Delta}{=} P[X^{-1}(\{x\})]$ $\quad \Pr[X \leq t] \overset{\Delta}{=} P[X^{-1}((-\infty, t])]$

Cumulative Distribution Function (CDF) & Probability density function (PDF).

Let X be a continuous random variable. The CDF of X is the fn $F_X : \mathbb{R} \to \mathbb{R}_{\geq 0}$ given by

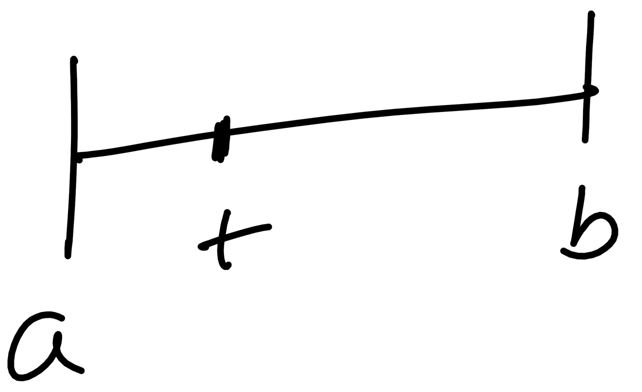
$$\forall t \quad F_X(t) = \Pr[X \leq t]$$

A funtion $f_X : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is called PDF of X if.

$$\boxed{\Pr[X \in [a,b]] = \int_a^b f_X(t)\,dt}$$

Examples :

① X be a random variable that is uniformly distributed in the interval $[a,b]$



$$\Pr[X \le t] = \frac{t-a}{b-a}.$$

$$F_X(t) = \Pr[X \le t] = \begin{cases} 0 & t < a \\ \frac{t-a}{b-a}. & \\ & t > b \\ 0 & \end{cases}$$

$$f_X(t) = \begin{cases} 0 & t > b \text{ or } t < a \\ \frac{1}{b-a}. & \end{cases}$$

Gaussian / Normal distribution

Parameters: μ : mean.

σ : standard deviation.

The PDF of a normal distribution with mean μ & std. dev σ is given by

$$f_X(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{(t-\mu)^2}{2\sigma^2}\right)} \longrightarrow (*)$$

Lecture 7

Exercise.

→ More examples of random variables

→ Expectation and variance.

→ Conditional probability.

→ Interval lemmas

Obs:- $f_X(t) = \frac{d(F_X(t))}{dt}$

Expected value (mean)

Suppose X is a discrete random variable taking values from S.

Then the expectation of X
$$E[X] \triangleq \sum_{x \in S} x \cdot \Pr[X = x]$$

For X continuous
$$\boxed{E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx.}$$

Example:- X be uniformly distributed in $[0,1]$

$f_X(t) = 1$

$E[X] = \int_0^1 x f(x) dx = \int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1$
$= \frac{1}{2} - 0$
$= \frac{1}{2}.$

X uniform in $[a,b]$

$E[X] = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$

Ref: Sheldon M Ross.

(2) X be s.t the PDF is given by

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_0^1 x \cdot 2x\, dx = \left[\frac{2x^3}{3}\right]_0^1$$

$$= 2/3$$

(3) Let X be given by

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

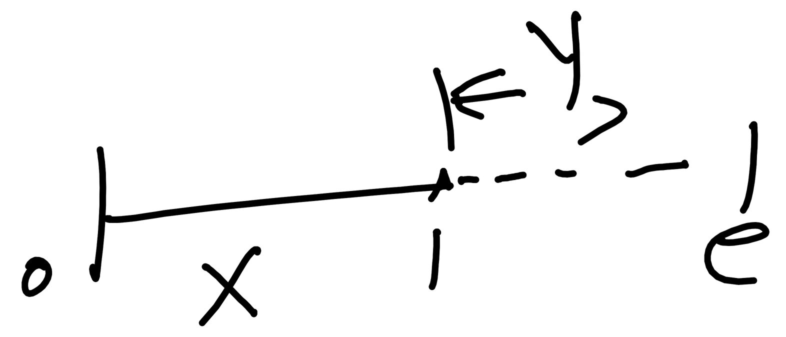
X

e^X, x^2,

$x^3 - x^2 + X$

To compute $E[e^X]$

$Y = e^X$. Y is a random variable.



$$F_Y(x) = \Pr[Y \le x] = \Pr[e^X \le x] = \Pr[X \le \log x]$$

$$= \int_0^{\log x} f_X(t)\, dt = \int_0^{\log x} 1\, dt$$

$$= \log x$$

~~Now~~ ie $F_Y(x) = \log x$

$$f_Y(x) = \frac{d\, F_Y(x)}{dx} = \frac{d \log x}{dx} = \frac{1}{x}.$$

for $\phi \le x \le e$

Then $E[Y] = \int_1^e x \cdot f_Y(x)\,dx = \int_1^e x \cdot \frac{1}{x}\,dx$

$= \underline{e-1}$

Exercies : Compute $E[X^3 - X^2]$

$E[(X+1)^2]$

Lemma:- Let X be a non-negative random variable.

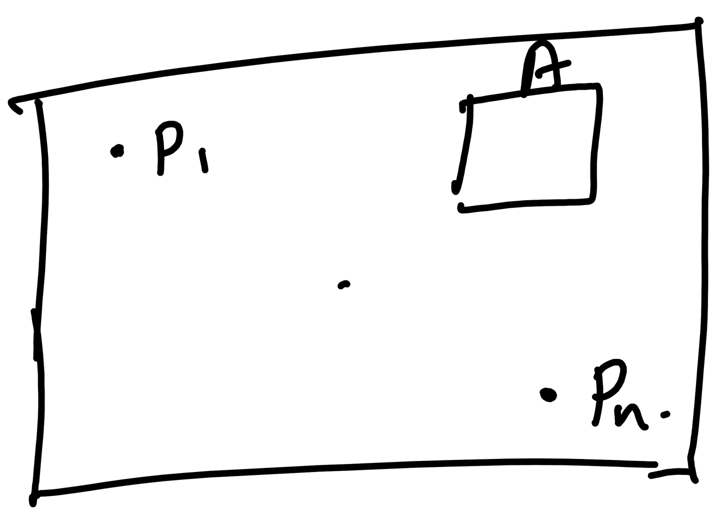
Then $E[X] = \int_0^\infty \Pr[X \geq x]\,dx.$

Ex:- Prove this

Lemma: Let g be any function. Then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \cdot dx.$$

Ex:- ① A be a square of area a in $[0,1]^2$



Suppose n points are sampled uniformly and independenty at random from $[0,1]^2$.

What is the expected number of points inside A?

② Suppose $P_1, \dots P_n$ are n points in $[0,1]^2$

$\xi_1, \dots, \xi_n$ be n- points distributed independ. according to Gaussian with $\mu=0$ & s.dev σ. Consider $P_1+\xi_1, P_2+\xi_2, \dots, P_n+\xi_n$.

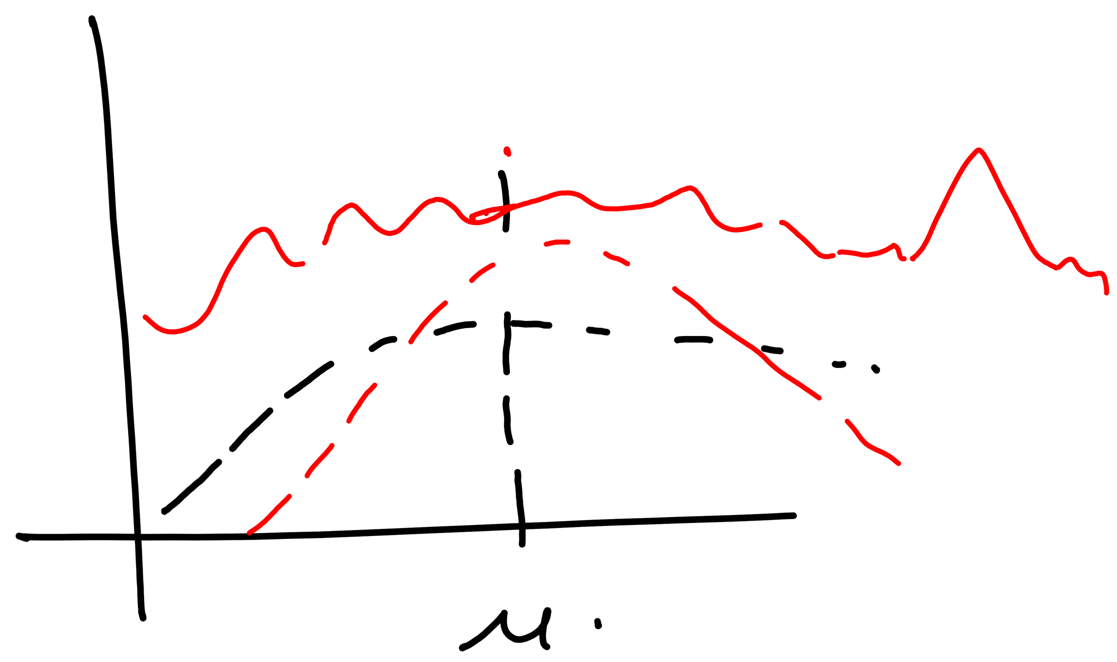
Compute: the expected # of points that lie outside $[0,1]^2$.

Lemma [Linearity of Expectation] Let X & Y be two random variables

then $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

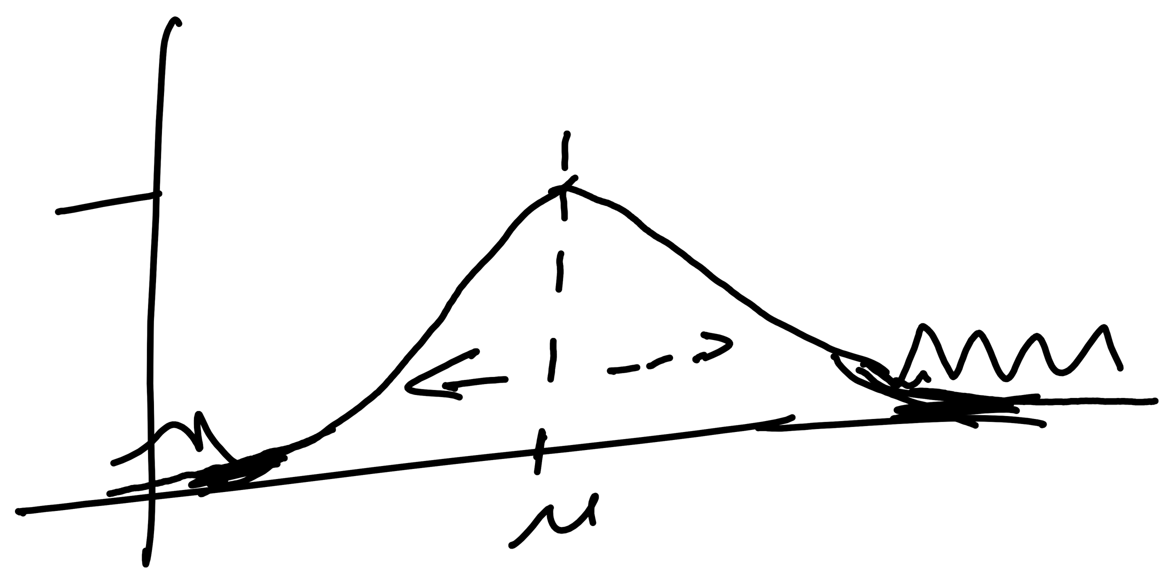
Defn:- [Variance].

$$\mathrm{Var}(X) = \mathbb{E}\left[(X-\mathbb{E}[X])^2\right]$$



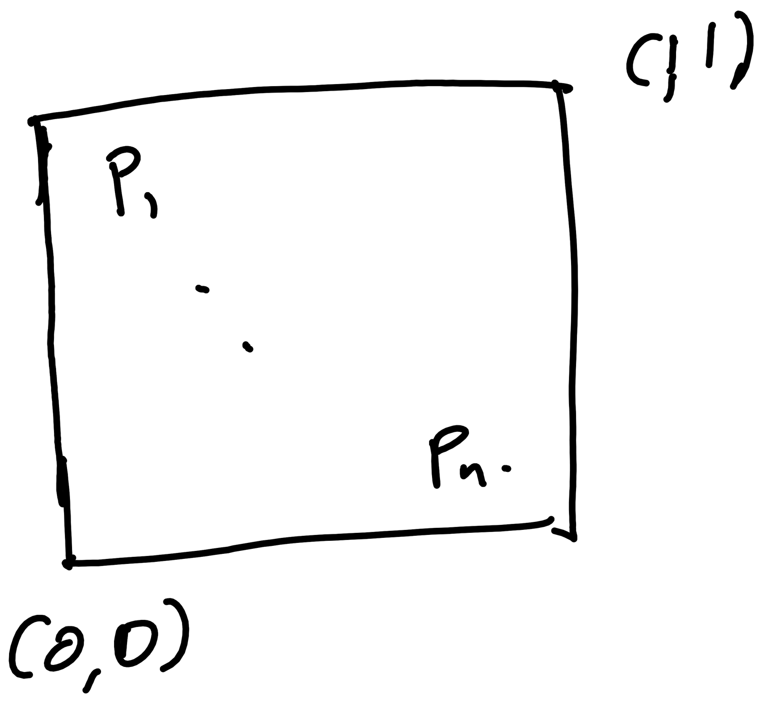
$$Y = X - \mathbb{E}[X]$$

Obs:- $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$



Ex:- ① Let $p_1, \dots p_n \in [0,1]^2$.

$T = \{p_1, \dots p_n\}$.



Show that
$$TSP(T) = O(\sqrt{n}).$$

② Show that the above bound is tight.

Lecture 8 — Smoothed Analysis of 2-OPT

ref: Notes by Manthey.

Interval Lemma

Lemma 1: Let $\phi > 0$ & X be a r.v. with PDF $f: \mathbb{R} \to [0, \phi]$, i.e. $\forall x\ f(x) \le \phi$

$\forall \epsilon > 0$ & t

$$\Pr[X \in (t, t+\epsilon)] \le \epsilon\phi.$$

$$\hookrightarrow \int_t^{t+\epsilon} f(x)\,dx \le \int \phi\,dx = \phi\epsilon.$$

Lemma 2: X - Gaussian with mean μ & std. dev σ

$\forall \epsilon > 0,\ t$

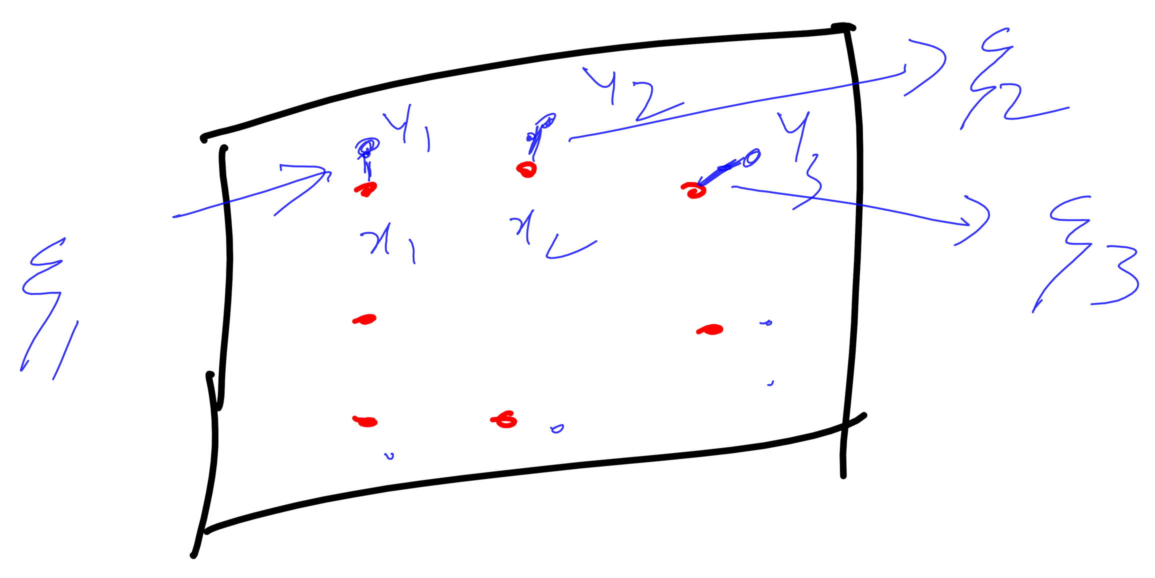
$$\Pr[X \in (t, t+\epsilon)] \le \frac{\epsilon}{2\sigma}$$

2-OPT initial tour T.

$\to$ sequence of 2-opt exchanges.

Let $x_1, \dots x_n$ be n-points in $[0,1]^2$ - fixed by adversary.

$\xi_1, \dots \xi_n$ - be independent Gaussian r.v. with $\mu = 0$ & std. dev σ.



$y_i = x_i + \xi_i$

$Y = \{y_1, \dots y_n\}.$

Assumptions

$\to$ We consider Squared Euclidean distances

i.e. $d((a_1, b_1)(a_2, b_2))$

$= (a_1 - a_2)^2 + (b_1 - b_2)^2.$

$\rightarrow \sigma \leq \frac{1}{2\sqrt{n \log n}}$ — in fact the arg works for general σ, but a bit more involved.

Main idea

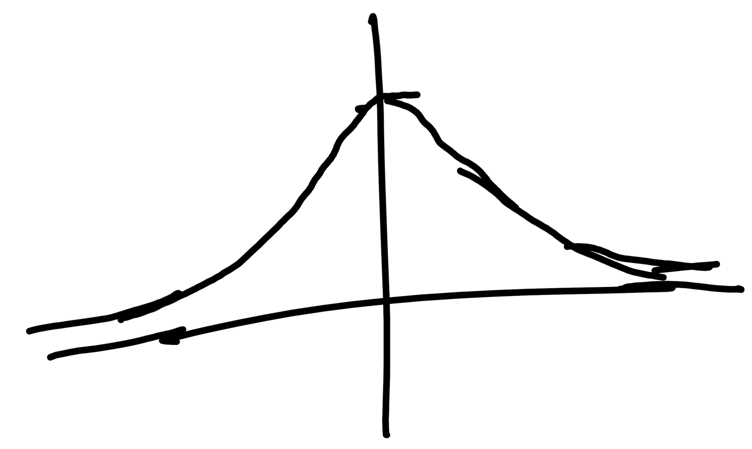
Let L_{init} be the cost of initial tour T.

Δ_{min} be the minimum gain among all 2-opt exchanges. (n^4)

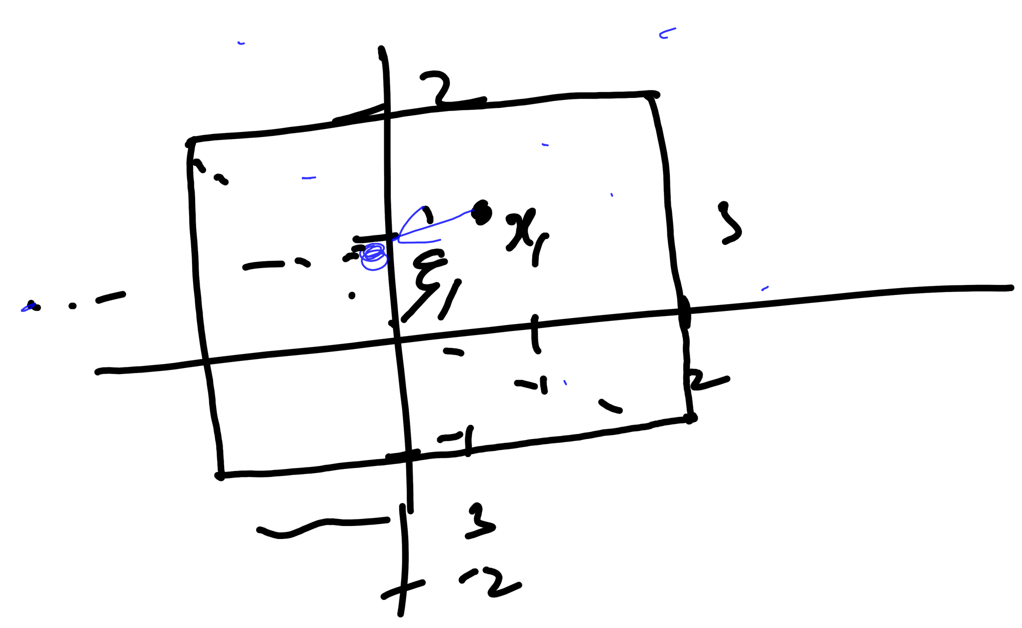
Then bound on the running time $\leq \frac{L_{init}}{\Delta_{min}}$.

Goal:- Obtain a good upper bound on L_{init}

" " Lower bound for Δ_{min}.



Lemma:- $Pr[L_{init} > 18n] \leq \frac{1}{n!}$



Pf:- If $Y \subseteq [-1,2]^2$

then $L_{init} \leq 18n$.

Then enough to estimate $Pr[Y \notin [-1,2]^2]$.

ie $Pr[L_{init} > 18n] \leq Pr[Y \notin [-1,2]^2]$

$Y \notin [-1,2]^2 \Rightarrow \exists\, i$ s.t. $y_i = x_i + \xi_i \notin [-1,2]^2$.

ie $\max\{x_i^1 + \xi_i^1,\ x_i^2 + \xi_i^2\} \geq 1$.

ie $\max\{|\xi_i^1|, |\xi_i^2|\} \geq 1$.

Lemma [Tail bound for Gaussian]. $X \sim N(\mu, \sigma^2)$

then $\forall t$ $\Pr[X \geq \mu + \sigma t] = \Pr[X \leq \mu - \sigma t]$

$$\leq \frac{1}{t \cdot \sqrt{2\pi}} \cdot e^{-t^2/2}.$$

set $t = \frac{1}{\sigma}$.

$$\Pr[\xi_i^j \geq 0 + \sigma \cdot \tfrac{1}{\sigma}] \leq \frac{\sigma}{\sqrt{2\pi}} \cdot e^{-\sigma^2/2}.$$

$$\sigma \leq \frac{1}{2\sqrt{n \log n}} \Rightarrow \Pr[\xi_i^j \geq 1] \leq \left(\frac{1}{n!}\right)^2.$$

Applying union bound over $i \in \{1 \ldots n\}$ & $j \in \{1, 2\}$.

we get $\Pr[Y \notin [-1, 2]^2] \leq \frac{1}{n!}$

To Lower bound Δ_{min}.