11. WORST CASE GROWTH RATES

11.1. Introduction

Previous chapters focused on the behavior of the *stochastic* versions of problems in Euclidean combinatorial optimization. The notion of boundary functionals has been central to our analysis. Boundary functionals, with their intrinsic superadditivity, are the key to proving general umbrella theorems for power-weighted versions of Euclidean functionals. They are also the basis for finding rates of convergence and for developing ergodic theorems.

In this chapter we will see that boundary functionals are valuable tools in a setting that involves no probabilistic assumptions. Boundary functionals turn out to be valuable in describing growth rates of the worst case values of many of the classic problems in combinatorial optimization and operations research. It is somewhat surprising that the asymptotics for the worst case lengths match those found in the probabilistic setting.

Throughout we focus on the following *worst case* counterparts of the TSP, MST, and minimal matching functionals. We let $|V|$ denote the cardinality of the set V.

Definition 11.1.

(i) The largest possible length of any minimal traveling salesman tour (worst case tour) with pth power weighted edges formed from n points in $[0,1]^d$ is

$$\tau^p(n) := \max_{V \subset [0,1]^d, |V|=n} T^p(V). \tag{11.1}$$

(ii) The largest possible length of any minimum spanning tree with pth power weighted edges formed from n points in $[0,1]^d$ is

$$\mu^p(n) := \max_{V \subset [0,1]^d, |V|=n} M^p(V). \tag{11.2}$$

(iii) The largest possible length of any minimal matching with pth power weighted edges formed from n points in $[0,1]^d$ is

$$\sigma^p(n) := \max_{V \subset [0,1]^d, |V|=n} S^p(V). \tag{11.3}$$

The functions $\tau^p(n)$, $\mu^p(n)$, and $\sigma^p(n)$ represent the worst case values of the TSP, MST, and minimal matching functionals, respectively. The worst case values of the boundary TSP, MST, and minimal matching functionals are defined similarly and are denoted by $\tau_B^p(n)$, $\mu_B^p(n)$, and $\sigma_B^p(n)$, respectively.

The points in V need not be distinct. Thus the worst case functions (11.1), (11.2) and (11.3) are continuous functions on the compact set formed from the product of n copies of $[0,1]^d$. These functions therefore exist and are monotone increasing, that is $\tau^p(n) \le \tau^p(n+1)$ and similarly for μ^p and σ^p.

The bulk of this chapter provides an approach which yields asymptotics for the worst case functions (11.1) - (11.3). This approach also provides the asymptotics for the worst case versions of other Euclidean functionals, including the rectilinear Steiner tree, treated by Snyder (1992). We will not pursue this, but limit the discussion to the worst case versions (11.1) - (11.3) of the archetypical functions.

Our main result gives growth rates for the functions (11.1) - (11.3). These growth rates are identical to those in the basic limit Theorems 4.1 and 4.3 and the umbrella Theorem 7.1.

Theorem 11.2. *Let $1 \le p < d$. Let $\rho^p(n)$ denote either of the three worst case functions given by Definition 11.1. Then*

$$\lim_{n\to\infty} \rho^p(n)/n^{(d-p)/d} = \beta(\rho^p, d), \tag{11.4}$$

where $\beta(\rho^p, d)$ is a positive constant depending only on ρ^p and d.

Although worst case functions have received considerable attention, only Snyder (1987) and Steele and Snyder (1989) treat their asymptotics. We will not try to determine the values of the constant $\beta(\rho^p, d)$ for the different choices of ρ.

11.2. Superadditivity

The method for proving (11.4) uses a strategy that is by now familiar: use the intrinsic superadditivity of the worst case boundary functions ρ^p_B to easily establish that ρ^p_B satisfies the asymptotics (11.4) and then conclude the proof of (11.4) by checking that $\rho^p_B(n)$ is within $o(n^{(d-p)/d})$ of $\rho^p(n)$.

To see that (11.4) holds for the worst case boundary function ρ^p_B, we first show that ρ^p_B satisfies *monotonicity, boundedness, and superadditivity.* To check monotonicity, we have already seen that ρ is monotone and for the same reasons ρ^p_B is also monotone, i.e.,

$$\rho^p_B(n) \le \rho^p_B(n+1). \tag{11.5}$$

Recalling the growth bounds (3.7) of our archetypical functionals, we have

$$\rho^p_B(n) \le C_2 n^{(d-p)/d}. \tag{11.6}$$

We now check superadditivity, which says that for all $1 \le p < d$ and positive integers n and k we have

$$n^{d-p}\rho_B^p(k) \le \rho_B^p(kn^d). \tag{11.7}$$

Let us check that (11.7) holds when ρ is the worst case MST function. We proceed as follows. Subdivide $[0,1]^d$ into n^d subcubes $Q_1, ..., Q_{n^d}$ of edge length $1/n$. In each subcube put k points in such a way that the worst case value of $n^{-p}\mu_B^p(k)$ is achieved. The union of these points over all n^d subcubes gives a set F of cardinality kn^d. By the definition of the worst case MST, we have clearly $M_B^p(F) \le \mu_B^p(kn^d)$. It only remains to show that $M_B^p(F) \ge n^{d-p}\mu_B^p(k)$.

To see this last inequality, let G be the tree which realizes the boundary functional $M_B^p(F)$ on F. The restriction of G to each subcube Q_i, $1 \le i \le n^d$, generates a boundary rooted tree on $F \cap Q_i$. Every such tree has a length which is at least as large as $n^{-p}\mu_B(k)$, the length of the boundary rooted MST on $F \cap Q_i$. This holds for all $1 \le i \le n^d$, which shows that $n^{d-p}\mu_B^p(k) \le M_B^p(F)$, as desired.

This simple argument shows that the worst case MST boundary function is superadditive. Modifications of this argument show that the worst case TSP and minimal matching boundary functionals are superadditive as well.

Conditions (11.5)-(11.7) are rather strong and it is not surprising that together they imply the existence of the limit

$$\lim_{n\to\infty} \rho_B^p(n)/n^{(d-p)/d} = \beta(\rho_B^p, d). \tag{11.8}$$

To show (11.8), define for fixed $1 \le p < d$ the function $\phi(n) := \rho_B^p(n)/n^{(d-p)/d}$. Set $\beta := \limsup_{n\to\infty} \phi(n)$ and note that $\beta < \infty$ by (11.6). We want to show that $\liminf_{n\to\infty} \phi(n) \ge \beta$.

To see this, note that condition (11.7) tells us that for all positive integers n and k we have

$$\phi(n^d k) \ge \phi(k). \tag{11.9}$$

Now given $\epsilon > 0$, find $k_o := k_o(\epsilon)$ such that $\phi(k_o) \ge \beta - \epsilon$. Thus for all $n \ge 1$ and $k \ge k_o$, (11.9) gives

$$\phi(n^d k) \ge \beta - \epsilon.$$

We now use an interpolation argument and the assumed monotonicity (11.5) to deduce that $\phi(j) \ge \beta - 2\epsilon$ for all j sufficiently large. Indeed, find n_o such that $(\beta - \epsilon)(\frac{n}{n+1})^{d-p} \ge \beta - 2\epsilon$ holds for all $n \ge n_o$. Given $j \ge n_o^d k_o$, find the unique $n \ge n_o$ such that $n^d k_o \le j < (n+1)^d k_o$. Then (11.5) implies

$$\phi(j) \ge \phi(n^d k_o)(\frac{n}{n+1})^{(d-p)} \ge \beta - 2\epsilon,$$

by definition of n_o. Thus we have shown $\liminf_{n\to\infty} \phi(n) \ge \beta - 2\epsilon$, as desired. Let ϵ tend to zero to complete the proof of (11.8).

11.3. Closeness

The relation (11.8) gives the asymptotics for the worst case boundary functions. It is relatively easy to show that (11.8) implies asymptotics for the worst case standard functionals. To obtain (11.4) from (11.8), it suffices to show

$$|\rho_B^p(n) - \rho^p(n)| = o(n^{(d-p)/d})$$

whenever ρ is the worst case TSP, MST, or minimal matching function. This relation resembles the closeness estimate Lemma 3.7, which says that the TSP, MST, and minimal matching functionals are pointwise close to their respective boundary functionals. Whenever ρ is the worst case TSP, MST, or minimal matching function we will show

$$|\rho_B^p(n) - \rho^p(n)| \leq C\left(n^{(d-p-1)/(d-1)} \vee \log n\right). \tag{11.10}$$

We prove (11.10) individually for the MST, TSP, and minimal matching functionals as follows.

(i) *The Worst Case MST Function.* Since $\mu_B^p(n) \leq \mu^p(n)$, it suffices to show

$$\mu^p(n) \leq \mu_B^p(n) + C(n^{(d-p-1)/(d-1)} \vee \log n). \tag{11.11}$$

Let n be given and let $F \subset [0,1]^d$ denote a set of size n which realizes the worst case MST function $\mu^p(n)$. Let T denote the spanning tree given by the MST boundary functional $M_B^p(F)$ on F. Lemma 3.8 tells us that the sum of the pth powers of the lengths of the edges rooted to the boundary by T is bounded by $C(n^{(d-p-1)/(d-1)} \vee \log n)$.

Let $\mathcal{M} \subset \partial[0,1]^d$ denote the points where the rooted edges in T meet the boundary. Consider the length $T^p(\mathcal{M})$ of the optimal TSP tour T' on $\mathcal{M}$; the edges in T' lie on $\partial[0,1]^d$. Using the edges in the rooted tree T as well as those in the tour T' and applying the triangle inequality $(x+y+z)^p \leq C(x^p + y^p + z^p)$ for $x, y,$ and z positive, we obtain a feasible spanning tree through F having a length of at most

$$M_B^p(F) + CT^p(\mathcal{M}) + C(n^{(d-p-1)/(d-1)} \vee \log n), \tag{11.12}$$

where the last term represents a bound on the pth powers of the lengths of the boundary rooted edges, some of which are needed *two times* in the construction of the feasible tree. Since the edges in T' lie on the (d-1)-dimensional boundary, we have $T^p(\mathcal{M}) \leq C(n^{(d-p-1)/(d-1)} \vee \log n)$ by Lemma 3.3. The proof of the estimate (11.11) is completed by noting that $\mu^p(n) = M^p(F)$ is bounded by the length of the suboptimal feasible tree given by (11.12) and then using $M_B^p(F) \leq \mu_B^p(n)$.

(ii) *The Worst Case TSP Function.* We need to show the estimate

$$\tau^p(n) \le \tau_B^p(n) + C(n^{(d-p-1)/(d-1)} \vee \log n). \tag{11.13}$$

Given n, let $F \subset [0,1]^d$ denote a set of size n which realizes the value of the worst case TSP function $\tau^p(n)$. Consider the boundary functional $T_B^p(F)$. Let T be the tour given by this functional and let $F' \subset F$ denote the subset of F which is rooted to the boundary by the tour T. Let $\mathcal{M} \subset \partial[0,1]^d$ denote the set of points where the edges meet the boundary. Let N denote the common cardinality of the sets F' and $\mathcal{M}$. The goal here is to use T to construct a feasible tour through F.

Consider the length $S^p(\mathcal{M})$ of the minimal matching on $\mathcal{M}$ with edges on $\partial[0,1]^d$. This matching generates tours $C_1, ..., C_R$ $(R \le N)$ on the union $F \cup \mathcal{M}$. Given tour C_i, $1 \le i \le R$, select a point $M_i \in \mathcal{M} \cap C_i$ and set $\mathcal{M}' := (M_1, ..., M_R)$. The triangle inequality, the estimate $S^p(\mathcal{M}) \le C(n^{(d-p-1)/(d-1)} \vee 1)$, and Lemma 3.8 (for the TSP) together tell us that we may add and delete edges from the tours $C_1, ..., C_R$ to generate tours $C_1', ..., C_R'$ on the smaller set $F \cup \mathcal{M}'$ at an extra cost bounded by $C(n^{(d-p-1)/(d-1)} \vee \log n)$. Moreover, the sum of the pth powers of the lengths of the edges with a vertex in $\mathcal{M}'$ is bounded by $C\left(n^{(d-p-1)/(d-1)} \vee \log n\right)$.

Finally, consider the optimal tour of length $T^p(\mathcal{M}')$ with edges which lie on the boundary of $[0,1]^d$. By Lemma 3.3, $T^p(\mathcal{M}') \le C(n^{(d-p-1)/(d-1)} \vee 1)$.

The above construction, which is achieved at a cost of at most $T_B^p(F) + C(n^{(d-p-1)/(d-1)} \vee \log n)$, generates a connected graph G through $F \cup \mathcal{M}'$ consisting of tours $C_1', ..., C_R'$ through $F \cup \mathcal{M}'$ as well as single tour through $\mathcal{M}'$ with length at most $C(n^{(d-p-1)/(d-1)} \vee 1)$. Since the sum of the pth powers of the lengths of the edges in G with a vertex in $\mathcal{M}'$ is bounded by $C(n^{(d-p-1)/(d-1)} \vee \log n)$, the triangle inequality and an obvious patching argument imply that we may construct a tour through F at an extra cost of at most $C(n^{(d-p-1)/(d-1)} \vee \log n)$. We have thus shown

$$\tau^p(n) \le T_B^p(F) + C(n^{(d-p-1)/(d-1)} \vee \log n),$$

Since $T_B^p(F) \le \tau_B(n)$, (11.13) follows as desired.

(iii) *The Worst Case Minimal Matching Function.* We may show

$$\sigma^p(n) \le \sigma_B^p(n) + C(n^{(d-p-1)/(d-1)} \vee \log n) \tag{11.14}$$

by following the arguments used to prove the analogous estimate (11.11). To prove (11.14), let $F \subset [0,1]^d$ denote a set of size n which realizes the worst case minimal matching $\sigma^p(n)$. Let T denote the graph described by $S_B^p(F)$ and let $\mathcal{M} \subset \partial[0,1]^d$ the set of points where the edges in T meet the boundary. Let $F' \subset F$ be the set of points in F which are matched to the boundary. Consider the length $S^p(\mathcal{M})$ of the minimal matching on $\mathcal{M}$ with edges lying on $\partial[0,1]^d$. By Lemma 3.3, the edges given by $S^p(\mathcal{M})$ have a total length of at most $C(n^{(d-p-1)/(d-1)} \vee 1)$. Using these edges we may construct a natural pairing of points in F' which by the triangle inequality and Lemma 3.8 applied to $S_B^p(F)$, is achieved at an extra cost of at most $C(n^{(d-p-1)/(d-1)} \vee \log n)$. This produces a feasible matching of F and shows that

$$\sigma^p(n) \leq S_B^p(F) + +C(n^{(d-p-1)/(d-1)} \vee \log n).$$

Since $S_B^p(F) \leq \sigma_B^p(n)$, the proof of (11.14) is complete.

We have thus established that the worst case versions of the TSP, MST, and minimal matching functions are close to their worst case boundary versions in the sense that the estimate (11.10) is satisfied. Since the worst case boundary versions satisfy the asymptotics (11.4) the proof of Theorem 11.2 is complete.

11.4. Concluding Remarks

1. It is unclear whether the present method yields rates of convergence with an error term which is more precise than $C(n^{(d-p-1)/(d-1)} \vee \log n)$. Using superadditivity of the boundary functional together with (11.10) it is straightforward to obtain the upper bound

$$\tau^p(n) \leq \beta(\tau^p, d)n^{(d-p)/d} + C(n^{(d-p-1)/(d-1)} \vee \log n)$$

with similar estimates for $\mu^p(n)$ and $\sigma^p(n)$. It is unclear whether this can be developed into a two-sided inequality.

2. It is also unclear whether Theorem 11.2, which holds for power weighted edges with power p satisfying $1 \leq p < d$, can be modified to treat powers p lying in the ranges $0 < p < 1$ and $d \leq p < \infty$. In Yukich (1996a) it was claimed that Theorem 11.2 holds for all $0 < p < 1$, but this remains to be shown.

3. The worst case functions all satisfy the "smoothness" condition

$$|\rho^p(n) - \rho^p(n+k)| \leq Ck^{(d-p)/d},$$

where ρ denotes either the TSP, MST, or minimal matching function. It is not clear whether this smoothness property can be put to good use.

4. Steele and Snyder (1989) and Snyder (1987) were the first to investigate the asymptotics of the worst case functions (11.1) and (11.2). Instead of using boundary functionals they obtain asymptotics for $\tau^1(n)$ and $\mu^1(n)$ by showing that these functions have a slow incremental rate of growth and an approximate recursiveness which is akin to a superadditivity condition with no error term. They essentially show that if ρ denotes the worst case version of either the TSP or MST, then $\rho^p(n+1) \leq \rho^p(n) + Cn^{-p/d}$. From this recursion we immediately deduce that ρ satisfies the smoothness condition mentioned in Remark 3. Theorem 11.2 extends upon Steele and Snyder (1989) by treating the general case of power-weighted edges (i.e., $p > 1$).

5. There has been considerable work on estimating the constants $\beta(\rho^p, d)$ for various choices of the functional ρ. Steele and Snyder (1989) provides historical background on this subject and forms the basis for the remarks here. In dimension 2, Fejes-Tóth (1940) showed the lower bound $\tau(n) \geq (1-\epsilon)(4/3)^{1/4}n^{1/2}$ for all $n \geq n_o(\epsilon)$. Still in dimension 2, Verblunsky (1951) showed that $\tau(n) \leq (2.8n)^{1/2} + 3.15$ and later this was improved by Few (1955) to $\tau(n) \leq (2n)^{1/2} + 1.75$. For general $d \geq 2$, Few (1955) showed that $\tau(n) \leq d(2(d-1))^{(1-d)/2d} n^{(d-1)/d} + O(n^{1-2/d})$.

More recently, Supowit, Reingold, and Plaisted (1983) showed the lower bound $\tau(n) \geq (4/3)^{1/4}n^{1/2}$ for all $n \geq 1$. Finally, Karloff (1987) showed that in dimension 2 one has $\tau(n) < 0.984(2n)^{1/2} + 11$.

6. Snyder and Steele (1990) treat the asymptotics for the worst case version of the greedy matching heuristic. It remains to be seen whether the methods of this chapter deliver asymptotics for this and other heuristics.

7. Most of this chapter is based on Yukich (1996a).