Lecture-5

Analysis of 2-OPT

Project allocation

24 FEB Topics will be announced.

28 FEB Allocation

Office hours

After ~~all~~ lectures

Recall

Model for smoothed analysis

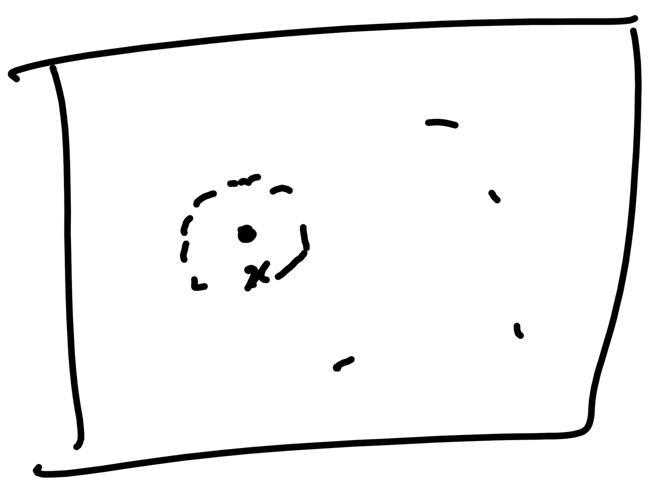
A - be an algorithm

x - be an i/p.

$A(x)$ the performance of the algorithm

π - a perturbation model & let $N(x)$ the neighborhood of

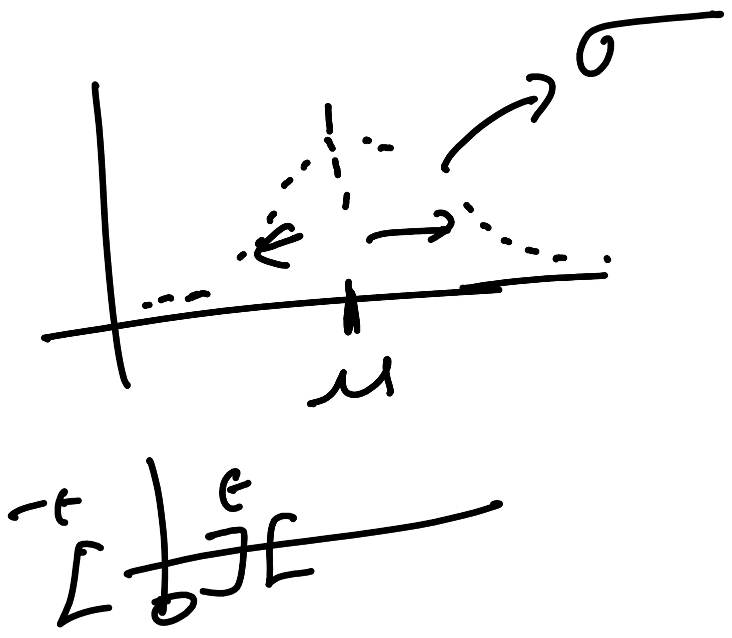
x.



$$\text{Smoothed } A(n) = \max_{x, |x|=n} \left\{ \underset{y \in N(x)}{E} [A(x)] \right\}$$

(original - Spielman & Teng)

Model 1:- Gaussian (Normal) Noise model.



x - $N(x)$ is defined by adding a Guassian distribution with μ & σ

$\underline{x} = (p_1, p_2, \cdots, p_n)$

$Z\ (z_1, z_2, \cdots z_n)$

$Z_i \sim$ independently distr $N(\mu, \sigma)$

Model2 :- density model.

parameter is ϕ

$p_1 \cdots p_n$

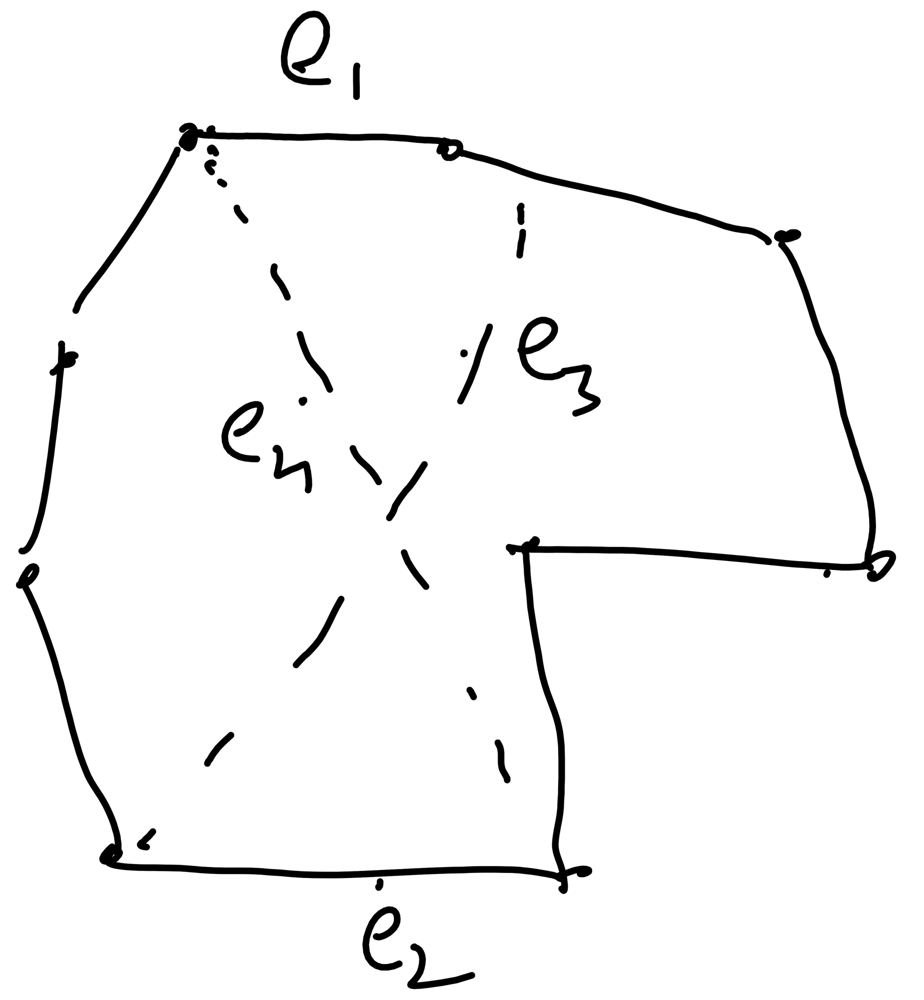
$f_1 \cdots d_n$

$f_i : [0,1]^2 \to [0, \phi]$

2-opt algorithm

i/p: n points $P_1 \cdots P_n$

T - some initial tour.



2-opt step.

Consider edge e_1, e_2

$e_1 = (v_1, v_2)$

$e_2 = (v_3, v_4)$

$cost(e_1) = d(v_1, v_2)$

Let T' be the tour obtained by replacing (v_1, v_2) & (v_3, v_4) with (v_1, v_3) (v_2, v_4)

$\Delta = d(e_1) + d(e_2) - d(e_3) - d(e_4)$

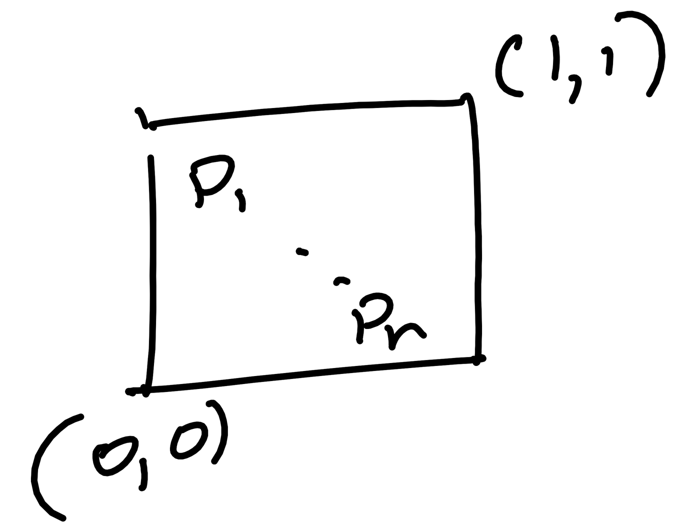
if $cost(T) > cost(T')$

replace T by T' & proceed.

Halt when no further improvement is possible.

Worst case :- Algo can run for $n!$ many iterations

Suppose $P_1, \cdots P_n \in [0,1]^2$



For any T, $cost(T) \leq \sqrt{2} \cdot n$.

Suppose Δ_{min} is the minimum improvement

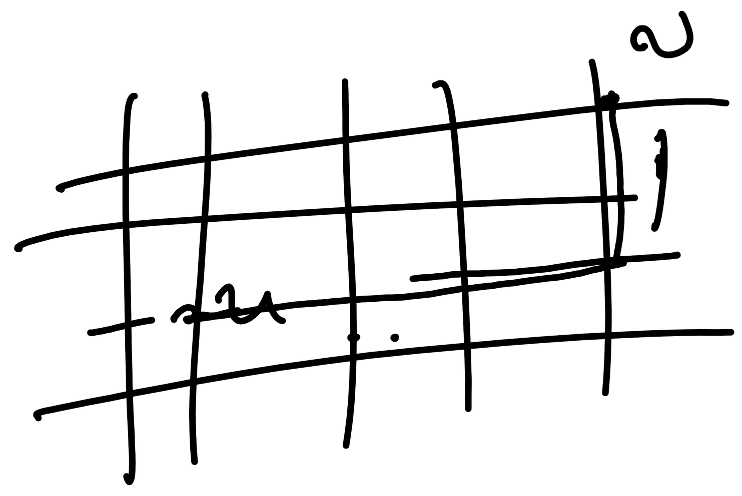
i.e. $cost(T) - cost(T') \geq \Delta_{min}$.

$\boxed{\text{Then upperbound on the runtime} \leq \frac{\sqrt{2}n}{\Delta_{min}}}$

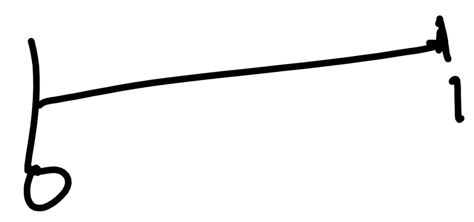
A simplified notion of distance.

$u = (x_1, y_1)$ be two points
$v = (x_2, y_2)$

the l_1-distance between u & v $\quad d_1(u,v) = |x_1 - x_2| + |y_1 - y_2|$.
(taxicab manhattan)



Suppose $P_1 \ldots P_n$ are uniformly distributed. $\quad P_1 = (x_1, y_1)$



S - be one 2-opt step.
(v_1, v_2, v_3, v_4)

$$\Delta(S) = d_1(v_1, v_2) + d_1(v_3, v_4) - d_1(v_1, v_3) - d_1(v_2, v_4)$$

$V = \{P_1, \ldots P_n\}$.

$$\Delta_{min} = \min_{S \subseteq V \times V \times V \times V} \{\Delta(S)\}.$$

$v_1 = (x_1, y_1)$
$v_2 = (x_2, y_2)$
v_3
$v_4 = (x_4, y_4)$

$x_1 - x_2$ or $x_2 - x_1$

$$\Delta(S) = |x_1 - x_2| + |y_1 - y_2| + |x_3 - x_4| + |y_3 - y_4| - |x_1 - x_3| + |y_1 - y_3| - |x_2 - x_4| + |y_2 - y_4|$$

Let us fix $\epsilon > 0$ as a threshold for Δ_{min}.
We want: $Pr[\Delta_{min} < \epsilon] \rightarrow (A)$

$\Delta_{min} < \epsilon$
$= \exists S \; \Delta(S) < \epsilon$.

— To estimate $Pr[\Delta(S) < \epsilon] \rightarrow (*)$

Once $(*)$ is done we can compute (A) using union bound.

we know $|x_1 - x_2| = x_1 - x_2$ or $x_2 - x_1$ depending on $x_1 \geq x_2$ or not

gives us 2^8 different possibities for $\Delta(S)$

eg: $\Delta(S) = x_1 - x_2 + y_1 - y_2 + x_3 - x_4 + y_3 - y_4 +$
$x_1 - x_3 + y_1 - y_3 + x_2 - x_4 + y_2 - y_4$

Fix one such possibility.

then $\Delta(S) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \beta_1 y_1 + \cdots + \beta_4 y_4$

$\alpha_1 x_1 + \gamma$

$Pr[\Delta(S) < \epsilon] \leq \epsilon$

t

$\frac{\epsilon}{t}$

$Pr[x_1 < \frac{\epsilon - \gamma}{\alpha_1}] \leq \epsilon$

$Pr[\exists$ is a possibility writ $\Delta(S) < \epsilon] \leq \#\gamma$ pass $\times\ \epsilon$
$= 2^8 \cdot \epsilon.$

$Pr[\Delta_{min} < \epsilon] = Pr[\exists S, \Delta(S) < \epsilon]$
$\leq \sum_S Pr[\Delta(S) < \epsilon]$
$\leq n^4 \cdot 2^8 \cdot \epsilon$

Choose an $\epsilon < \frac{1}{n^4 2^8}$

e.g. choose $\epsilon = \frac{1}{n^5}$

$Pr[\Delta_{min} < \frac{1}{n^5}] \leq n^4 \cdot 2^8 \cdot \frac{1}{n^5} \leq \frac{2^8}{n}.$

$1 - \frac{1}{n}$

$Pr[$2opt runs for $> 2 \cdot n^6] \leq \frac{2^8}{n}.$

$\mathbb{E}[\text{run time 2-opt}] \le 2 \cdot n^6.$