CS6100

Lecture 9 + 10 + 11 + 12 + 13

Knapsack Problem.

n- items labelled $1, \ldots, n$. $I = \{1, \ldots n\}$.

$p_1, \ldots p_n \in \mathbb{R}_{\geq 0}$ — p_i - profit for ith item.

$w_1, \ldots w_n \in \mathbb{R}_{\geq 0}$ — w_i - weight of ~~given~~ ith item.

W - capacity.

Goal: obtain $x_1, \ldots x_n \in \{0,1\}$ s.t

maximize $p_1 x_1 + \cdots + p_n x_n$

subject to $w_1 x_1 + \cdots + w_n x_n \leq W$.

$x_1, \ldots x_n \in \{0,1\}$.

$x = (x_1, \ldots x_n) \in \{0,1\}^n$.

$p = (p_1, \ldots p_n) \in \mathbb{R}^n_{\geq 0}$

$w = (w_1, \ldots w_n) \in \mathbb{R}^n_{\geq 0}$

max. $p \cdot x^T$ $\quad (p_1 \ldots p_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

s.t $w x^T \leq W$.

& $x \in \{0,1\}^n$

Pareto optimality.

0 1 0 0 1
0 0 1 1 1

Let x & $y \in \{0,1\}^n$

We say that solution y dominates x if.

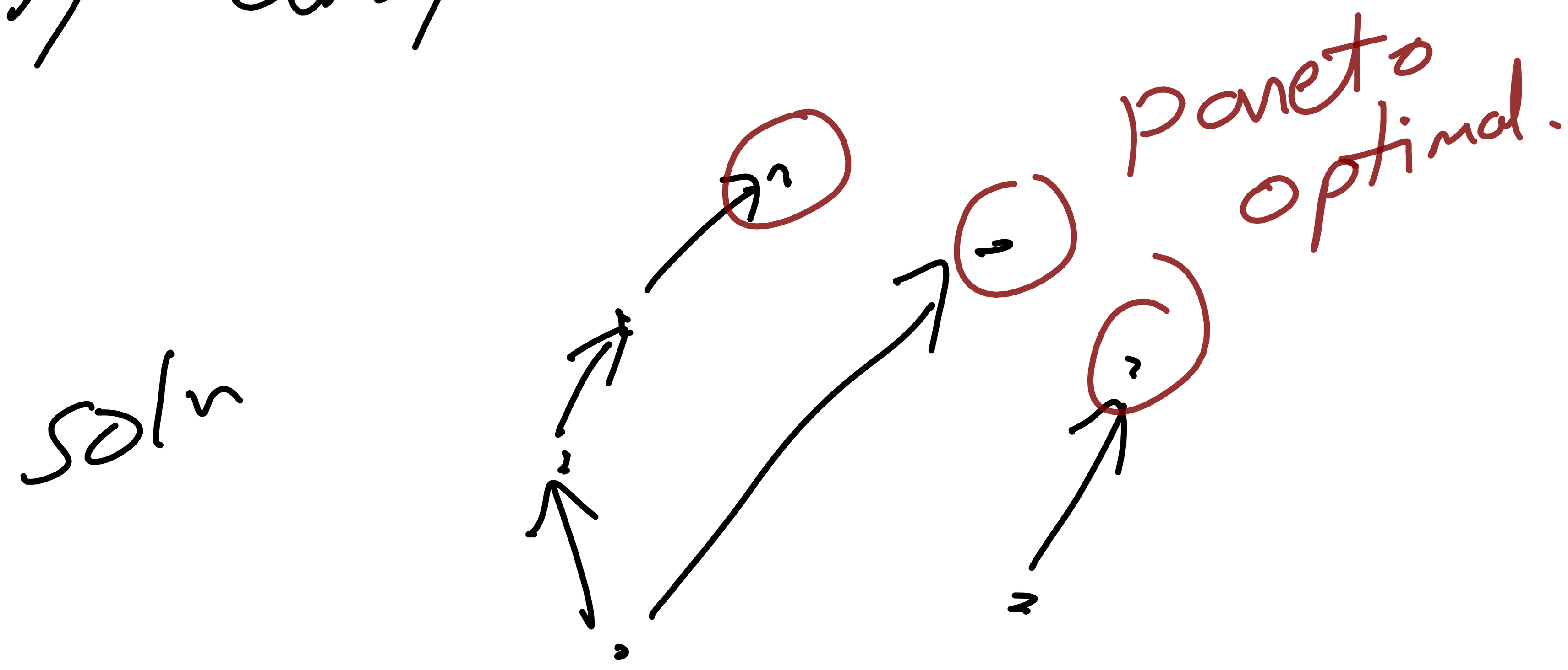
① $p y^T \geq p x^T$ (ie y has at least as much profit as x)

&

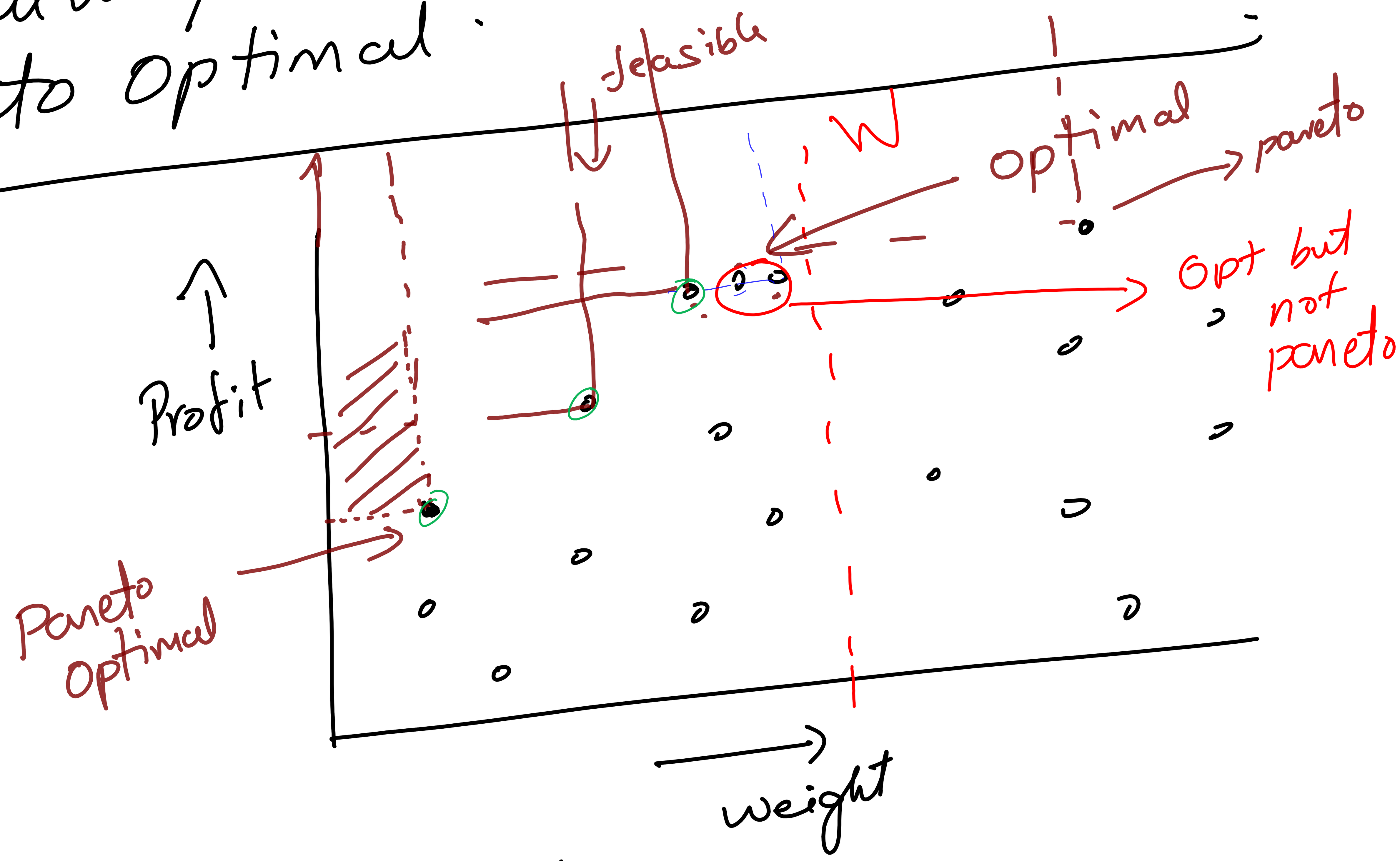
② $w y^T \leq w x^T$

and at least one of these is a strict inequality.

Defn: A solution y is said to be "pareto optimal" if it is not dominated by any other solution z.



Lemma :- There is always an optimal solution which is also pareto optimal.



Pf: Let x be an optimal solution.
Suppose x is not pareto, then there must be y that dominates x.

ie $\quad py^T \geq px^T$ & $wy^T \leq wx^T$ one of the ineq strict.

Since x is opt, $py^T = px^T$

$\therefore \quad wy^T < wx^T$

repeat the argument with y, if y is not pareto. Since there are only finitely many solution, repeating the argument gives an opt soln that is also pareto optimal.

for a given Knapsack instance (I, p, w, W)
Let P - denote the set of all pareto optimal solutions.

Obs:- If P is known, then an optimal solution can be computed in time $O(|P|)$.

Then we immediately have an algo for Knapsack with time $O(|P|)$ + time for computing P.

Question:-
① How large can P be?
② Can P be computed efficiently? ie in time $poly(n, |P|)$ ← YES (Nemhauser & Ullmann)

Next:-
① Describe Nemhauser-Ullman.
② Show that $|P|$ is $poly(n, \cdot)$ when instances are perturbed

Task:- To construct P.

$I = \{1 \ldots n\}$

$I_i = \{1 \ldots i\}$.

$S_i = \{ x \in \{0,1\}^n \mid x_j = 0 \ \forall j > i \}$

$P_i = \{ x \mid x \text{ is a pareto optimal soln for } I_i \}$

$x \in S_{i-1}$.

x^{+i} is a solution obtained by adding i to x ie

$$x_j^{+i} = \begin{cases} x_j & j < i \\ 1 & j = i \\ 0 & \text{o.w.} \end{cases}$$

for any $T \subseteq \{0,1\}^n$ $\quad T^{+i} = \{ y \in \{0,1\}^n \mid \exists x \in T \ \ y = x^{+i} \}$.

Lemma: $\forall\, i \in \{1, \dots n\}$, $P_i \subseteq P_{i-1} \cup P_{i-1}^{+i}$

pf:- Consider $x \in P_i$ ie x is a pareto optimal soln for $I_i := \{1 \dots i\}$.

$x = \underbrace{x_1 \dots x_n}_{1+\dots n}$ 0

Case 1:- $x_i = 0$ $\quad x \in P_i$

then Claim: $x \in P_{i-1}$. Suppose not. $\mid$ $y \in P_{i-1} \subseteq S_{i-1} \subseteq S_i$

ie $\exists\, y \in P_{i-1}$ that dominates x.

ie y is a candidate soln to I_i

$\therefore$ by definitn $y \in S_i$

$\therefore$ x cannot be pareto optimal for I_i as y dominates it $\longrightarrow$ contradiction to $x \in P_i$

$\therefore$ $x \in P_{i-1}$

Case 2:- $x_i = 1$. Claim: $x \in P_{i-1}^{+i}$ $\quad S_i = S_{i-1} \cup S_{i-1}^{+i}$

$x = \underbrace{x_1 \dots}\, x_i\, \overset{0}{x_{i+1}} \dots \overset{0}{x_n} \in S_i$

$\therefore \exists\, y \in S_{i-1}$ s.t $x = y^{+i}$

enough to show $y \in P_{i-1}$ ie y is pareto optimal for I_{i-1}.

Suppose not say there is a $z \in S_{i-1}$ s.t z dominates y

$pz^T \geq py^T$ one of them strict.
$wz^T \leq wy^T$. $\longrightarrow$ (*)

$x' = z^{+i}$

then $px'^T \geq px^T$ & $wx'^T \leq wx^T$ with one of them strict as $pz^T + p_i \geq py^T + p_i$ & $wz^T + w_i \leq wy^T + w_i$ from (*)

$\therefore \exists\, x' \in S_i$ that dominates x

$\therefore$ $x \notin P_i$ — contradiction

$\therefore$ $y \in P_{i-1}$ where $x = y^{+i}$

$\therefore$ $x \in P_{i-1} \cup P_{i-1}^{+i}$

Algo $P_1 \quad P_2 \quad P_3 \quad \cdot \quad - \quad P_n = P.$

Lecture - 11

Nemhauser - Ullmann Algorithm

1. $P_0 = \{0^n\}.$
2. For $i = 1$ to n do
3. $Q_i = P_{i-1} \cup P_{i-1}^{+i}$
4. $P_i = \{x \in Q_i \mid \nexists y \in Q_i \text{ } y \text{ dominates } x\}$

   ie there is no y in Q_i that dominates x.
5. Return $x^* = \arg\max_{x \in P_n} \{ px^T \mid wx^T \leq W\}.$

Time $\sum_{i=1}^{n} |P_i|^2.$

Thm: The N-U algorithm solves the knapsack problem and can be implemented in time $O\left(\sum_{i=0}^{n-1} |P_i|\right)$

— ref: Röglin's notes.

Limitation :- One can construct instances when $|P|$ can be exponential.

input: $I = \{1, \dots n\}$.

$p = (p_1, \dots p_n)$

$w = (w_1, \dots w_n)$.

Perturbation model

Choose $p = (p_1, \dots p_n)$ - arbitrarily.

draw $w_1 \dots w_n$ independently s.t w_i is drawn → parameter.

according to $f_i : [0,1] \to [0, \phi]$. (ie $\forall \alpha \in [0,1]$ $f(\alpha) \leq \phi$)

↳ without loss of generality (WLOG)

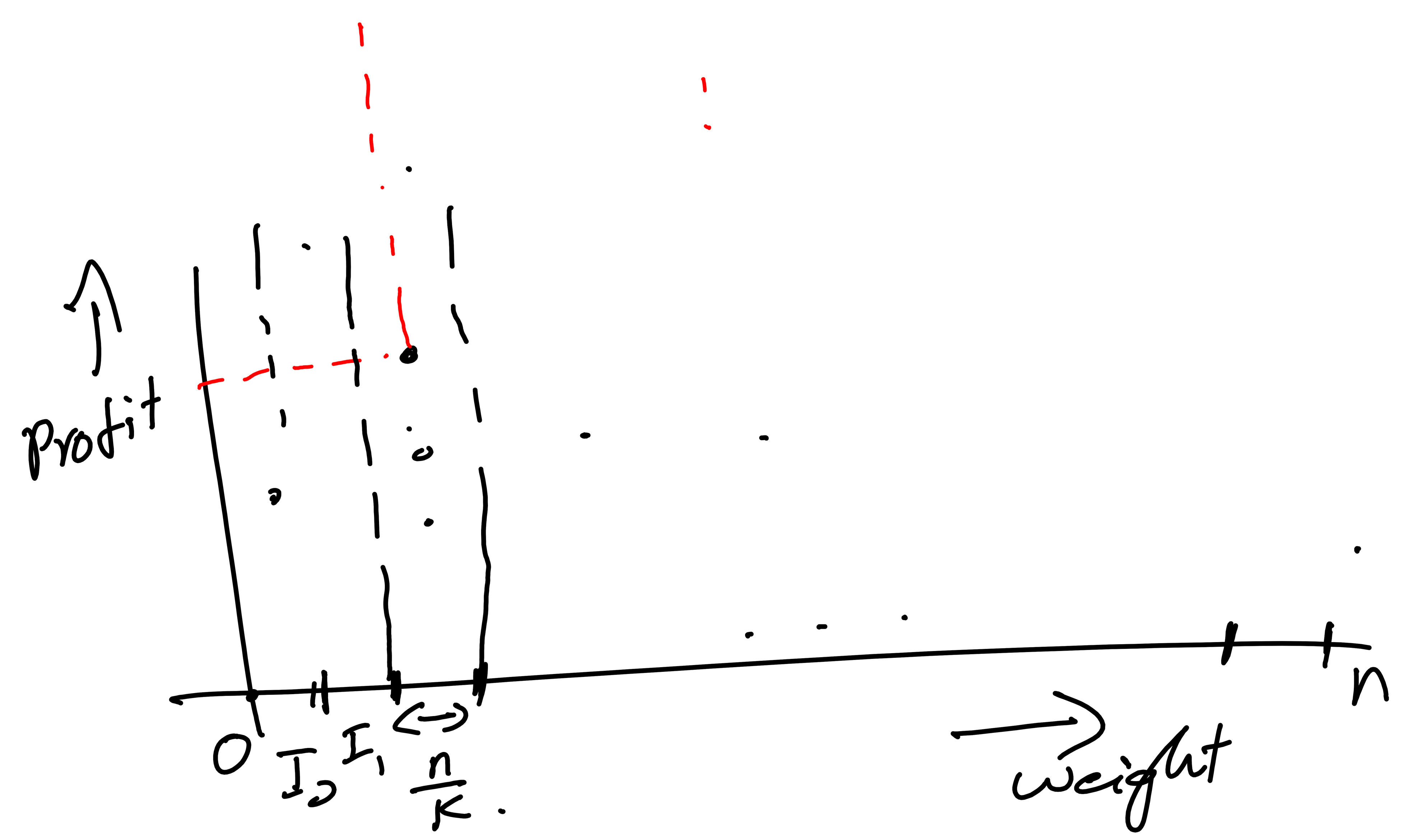
Thm Let profits $p_1, \dots, p_n$ be chosen arbitrarily and the weights $w_1 \dots w_n$ s.t w_i is drawn independently according to $f_i : [0,1] \to [0, \phi]$ where ϕ is a parameter. Then the expected # of solutions $x \in \{0,1\}^n$ there pareto optimal is bounded from above by $n^2\phi + 1$

Proof idea :-

$0 \leq w_i \leq 1$

$\forall x \quad wx^T \leq n$

$K \in \mathbb{N}$ - be a parameter.



Partition $[0,n]$ into K- equal sized intervals

Say $I_0, I_1, I_2, \dots, I_{K-1}$

where $I_j = \left[\frac{nj}{K}, \frac{n(j+1)}{K}\right]$

Let $X^K = 1 + \left|\{j \mid \exists x \in P, \ wx^T \in I_j\}\right|$

ie # of intervals with at least one pareto optimal solution.

If K is large enough

then $|P| = \cancel{X}^K$ then $E[|P|] = E[X^K]$

Define the event

F_K: $\exists\ x \neq y \in \{0,1\}^n$ with $|wx^T - wy^T| \leq \frac{n}{K}$

(ie x & y lie in some I_j)

<u>Lemma 1</u>:- $\Pr[F_K] \leq 2^{2n+1} \cdot \frac{n\phi}{K}$ $\forall K \in \mathbb{N}$.

<u>Pf</u>:- Let $x \neq y \in \{0,1\}^n$ be fixed. Let i be s.t

$x_i \neq y_i$ WLOG $x_i = 0$ & $y_i = 1$

Suppose all weights w_j except w_i are revealed.

then $wx^T - wy^T = \alpha - w_i$ for some α.

$\therefore \Pr[|wx^T - wy^T| \leq \frac{n}{K}] = \Pr[|\alpha - w_i| \leq \frac{n}{K}]$

$= \Pr\left[w_i \in \left[\alpha - \frac{n}{K}, \alpha + \frac{n}{K}\right]\right] \leq 2\frac{n}{K} \cdot \phi$

$$\int_{\alpha - n/K}^{\alpha + n/K} f_i(x)\,dx = 2nK\phi$$

By taking union bound over all pairs $x \neq y$

we have $\Pr[F_k] \leq \frac{2^{2n+1}}{\frac{n\phi}{k}}$

<u>Lemma2:-</u> $\forall k \in \mathbb{N}$ $\mathbb{E}[X^k] \leq n^2\phi + 1.$

<u>Assume Lemma2</u>

<u>Proof of Thm</u>

$$\mathbb{E}[|P|] = \sum_{i=1}^{2^n} i \cdot \Pr[|P| = i]$$ by defn of $\mathbb{E}$

$$= \sum_{i=1}^{2^n} i \cdot \Big(\Pr[|P| = i \wedge F_k] + \Pr[|P| = i \wedge \overline{F_k}] \Big)$$

AND

$\Pr[A] = \Pr[A \wedge B] + \Pr[A \wedge \overline{B}]$

$$= \sum_{i=1}^{2^n} i \cdot \Pr[|P| = i \wedge F_k] + \sum_{i=1}^{2^n} i \cdot \Pr[|P| = i \wedge \overline{F_k}]$$

$\Pr[A \wedge B] = \Pr[B] \cdot \Pr[A | B]$

$$= \sum_{i=1}^{2^n} i \cdot \Pr[F_k] \cdot \Pr[|P| = i \mid F_k] + \sum_{i=1}^{2^n} i \cdot \Pr[|P| = i \wedge \overline{F_k}]$$

no interval has more than one solution. ie $|P| = X^k$.

$$= \Pr[F_k] \cdot \sum_{i=1}^{2^n} i \cdot \Pr[|P| = i \mid F_k] + \sum_{i=1}^{2^n} i \cdot \Pr[X^k = i]$$

$$= \Pr[F_k] \cdot \sum_{i=1}^{2^n} i \cdot \Pr[|P| = i \mid F_k] + E[X^k]$$

$$\leq 2^{2n+1} \frac{n\phi}{k} \left[2^{2n} \right] + n^2\phi + 1$$

$$\leq 2^{4n+1} \cdot \frac{n\phi}{k} + n^2\phi + 1 \quad \forall n \in \mathbb{N}. \longrightarrow (*)$$

ie $E[|P|] \leq 2^{4n+1} \frac{n\phi}{k} + n^2\phi + 1$ holds for all $k \in \mathbb{N}$.

$\therefore E[|P|] \leq n^2\phi + 1$ as $(*)$ holds for all $k \in \mathbb{N}$.

Lecture-12

To prove that $E[X^k] \leq n^2\phi + 1$

$$X^k = \{ i \mid \exists x \in P \text{ s.t } x \in I_i \} + 1$$

Define $X_i = \begin{cases} 1 & \text{if interval } I_i \text{ contains at least one pareto opt ie } I_i \cap P \neq \phi \\ 0 & \text{o.w.} \end{cases}$

$I_0, I_1, \ldots I_{k-1}.$

$$X^k = X_0 + \cdots + X_{k-1} + 1$$
$$= 1 + \left(\sum_{i=0}^{k-1} X_i \right)$$

$$E[X^k] = E[1 + \sum X_i] = 1 + E[\sum X_i]$$
$$= 1 + \sum_{i=0}^{k-1} E[X_i]$$

Now $E[X_i] = \underline{0 \cdot Pr[X_i=0]} + 1 \cdot Pr[X_i = 1]$

$= Pr[\exists x \in P \text{ s.t. } wx^T \in I_i]$

$=$

Lemma3: $\forall t \geq 0$ & $\epsilon > 0$

$$Pr[\exists x \in P \ \ wx^T \in (t, t+\epsilon]] \leq n\phi\epsilon.$$

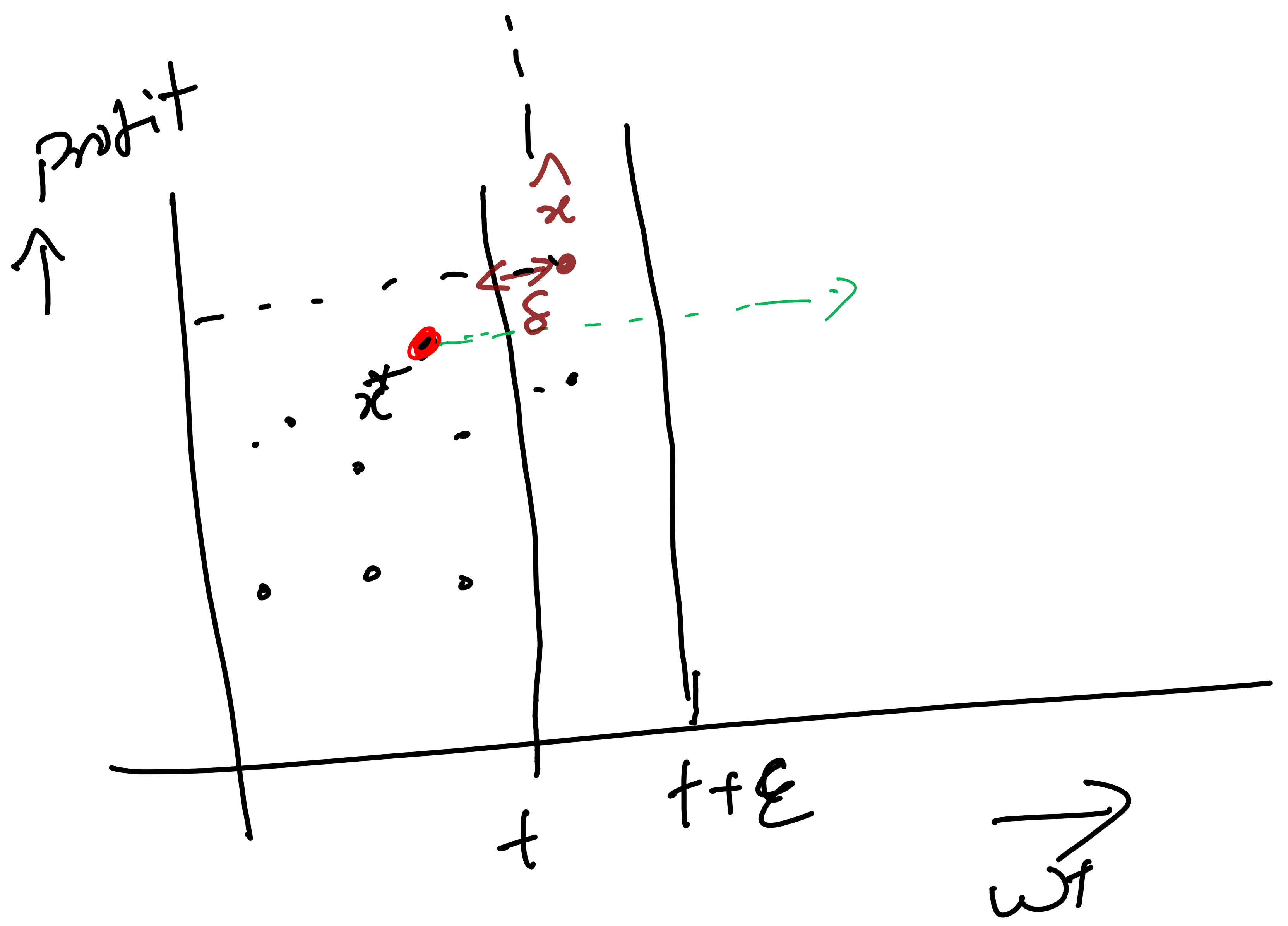
by Lemma3. we have $E[X_i] \leq \frac{n^2\phi}{k}$

$\therefore \ E[X^*] \leq 1 + \frac{n^2\phi}{k} \cdot k = 1 + n^2\phi$

$w_1 \cdots w_n$

To prove Lemma3

x^*: most ~~pro~~ profitable soln with $wx^T \leq t$



Define a random var $\Lambda(t)$

s.t $\Lambda(t) \leq \epsilon \iff \exists x \in P: wx^T \in (t, t+\epsilon]$

$$x^* = \arg\max_x \{ px^T \mid x \in \{0,1\}^n \ \& \ wx^T \leq t \}.$$

$\arg\min_x f(x)$
$=$ value of x s.t $f(x)$ is min.

We call x^* as the winner.

$\forall\, t \geq 0$, x^* - exists.

A solution x is a loser if $px^T > px^{*T}$

ie $wx^T > t$.

Let $\hat{x}$ be the loser with least weight

$$\hat{x} = \arg\min_x \{ wx^T \mid x \in \{0,1\}^n,\ px^T > px^{*T} \}.$$

It is possible that $\hat{x}$ is not defined.

in such a case, we set $\hat{x} = \perp$ (bottom).

Define

$$\Lambda(t) = \begin{cases} w\hat{x}^T - t & \text{if } \hat{x} \neq \perp \\ \perp & \text{o.w.} \end{cases}$$

$$\boxed{\exists x \in P \quad wx^T \in [t, t+\varepsilon] \iff \Lambda(t) \leq \varepsilon}$$

ie $E[X_i] = \Pr[\exists x \in P \quad wx^T \in [t, t+\varepsilon]]$

$= \underline{\underline{\Pr[\Lambda(t) \leq \varepsilon]}}$

$\underline{\underline{\sum \hat{x}_i w_i - t}}$

$\sum \hat{x}_i w_i - t \leq \varepsilon$

To estimate

$\underline{\Pr[\Lambda(t) \leq \varepsilon]}$

Define for $i \in \{1, \ldots n\}$, $b \in \{0,1\}$.

$$S^{x_i=b} = \{ x \in \{0,1\}^n \mid x_i = b \}.$$

$$x^{*,i} = \arg\max\{ p x^T \mid x \in S^{x_i=0},\ w x^T \le t \}.$$

Then $x^* = \max_i x^{*,i}$

ie $x^{*,i}$ is the winner among all soln that exclude item i.

Define $\hat{x}^i = \arg\min \{ w x^T \mid x \in S^{x_i=1}$ & $p x^T > p x^{*,i} \}$

$$\Lambda^i(t) = \begin{cases} w \hat{x}^{i\,T} - t & \text{if } \hat{x}^i \neq \perp \\ \perp & \text{o-w} \end{cases}$$

Lemma 4:- For every choice of profits & weights either $\Lambda(t) = \perp$ or there is an $i \in \{1, \dots, n\}$ s.t $\Lambda(t) = \Lambda^i(t)$. [Proof later]

Lemma 5:- $\forall i \in \{1..n\}$ & $\epsilon > 0$

$$\Pr[\Lambda^i(t) \le \epsilon] \le \epsilon \phi.$$

Corrollary:- $\Pr[\exists i,\ \Lambda^i(t) \le \epsilon] \le n \epsilon \phi$

Proof [Lemma 5]

$w_1, \dots w_{i-1}, w_{i+1} \dots w_n$

Suppose all weights except w_i are fixed.

ie weights of all solutions in $S^{x_i=0}$ are fixed

and hence x^{*i} is also fixed. $w x^{*i}$ is fixed.

$\hat{x}^i$: is also fixed as $w_i \geq 0$

$\hat{x}^i$ is the least weight solution among all x s.t $p x^T \geq p x^{*i\,T}$

in fact the set

$$L = \{ x \in S^{x_i=1} \mid p x^T > p x^{*i} \}$$ is

also fixed (as $p_1 \dots p_n$ are not random).

$\hat{x}^i$ is the least weight element in L.

$\therefore$ the event $\Lambda^i(t) \leq \epsilon = \Lambda^i(t) \in [0, \epsilon]$

equivalent to $w \hat{x}^{i\,T} - t \in (0, \epsilon]$ where $\hat{x}^i$ is fixed.

ie there is a k s.t

$w_i \in (k, k+\epsilon)$

$$\therefore \Pr[\Lambda^i(t) \leq \epsilon] = \Pr[w_i \in (k, k+\epsilon)] \text{ for some } k$$

$$\leq \int_k^{k+\epsilon} f_i(x)\,dx \leq \int_k^{k+\epsilon} \phi\,dx$$

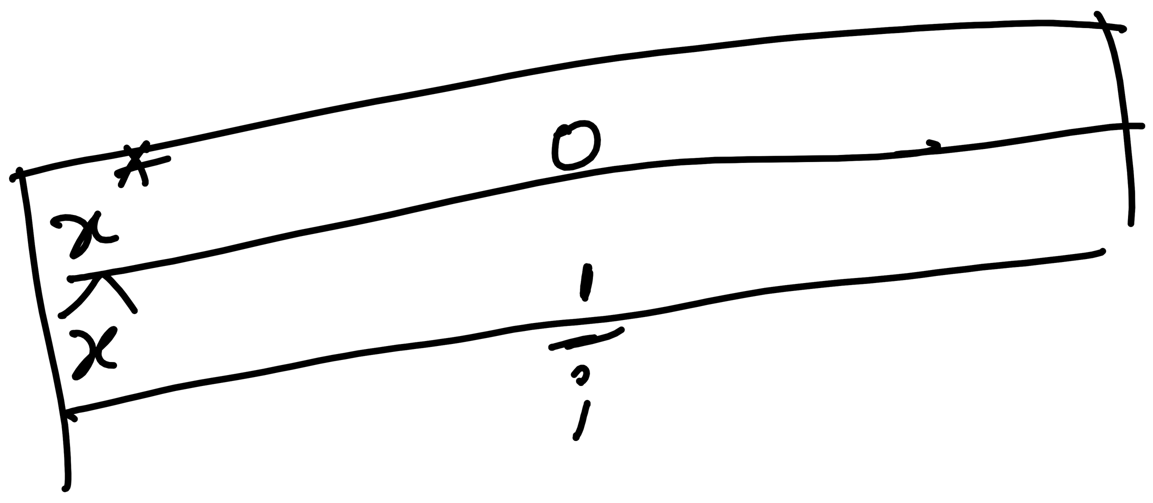
$$\leq \phi\epsilon.$$

Proof of Lemma 4

restatement:

For any choice of profits and weights, either $\Lambda(t) = \perp$ or $\exists\, i$ s.t. $\Lambda^i(t) = \Lambda(t)$.

Proof:- Suppose $\Lambda(t) = \perp$. There is a winner x^* and a loser $\hat{x}$.



As the weights are +ve $x^* \neq \hat{x}$:

We must have an i s.t. $x^*_i = 0$ & $\hat{x}_i = 1$ (i.e. i is not in x^* but in $\hat{x}$)

We claim that $\Lambda(t) = \Lambda^i(t)$ for this index i.

Since $x^*_i = 0$, we have $x^{*i} = x^*$

but since $\hat{x}_i = 1$, we must have $\hat{x} = \hat{x}^i$

$\Rightarrow \Lambda(t) = \Lambda^i(t)$

$\hat{x} = \arg\min_x px^T > px^{*T}$

$= \arg\min_x px > px^{*i}$

$\hat{x}^i = \arg\min_{x_i = 1,\, x \in S} px > px^{*i}$

Conclusion: # of Pareto optimal soln $= O(n^2 \phi)$ in expectation

$\therefore$ E time of N-U algo $= O(n^3 \phi)$

Lower bound

Thm Let $p_i = 2^i$ $\forall$ $i \in \{1, \dots n\}$. Let $w_1 - - - w_n$ chosen independently and uniformly from $[0,1]$.
Then the expected # of Pareto optimal solutions is $\Omega(n^2)$

— Our analysis is tight upto a constant factor. for $\phi = 1$