

CS6100

Lecture 9 + 10 + 11 + 12

Knapsack Problem

n-items labelled $1, \dots, n$. $I = \{1, \dots, n\}$.
 $p_1, \dots, p_n \in \mathbb{R}_{\geq 0}$ p_i - profit for i th item.
 $w_1, \dots, w_n \in \mathbb{R}_{\geq 0}$ w_i - weight of i th item.

W - capacity.

Goal: obtain $x_1, \dots, x_n \in \{0, 1\}^n$ s.t

maximize

$$p_1 x_1 + \dots + p_n x_n$$

subject to

$$w_1 x_1 + \dots + w_n x_n \leq W$$

$$x_1, \dots, x_n \in \{0, 1\}^n$$

$$(p_1, \dots, p_n) \binom{x_1}{x_n}$$

$$x = (x_1, \dots, x_n) \in \{0, 1\}^n$$

$$p = (p_1, \dots, p_n) \in \mathbb{R}_{\geq 0}^n$$

$$w = (w_1, \dots, w_n) \in \mathbb{R}_{\geq 0}^n$$

$$\max \quad p \cdot x^T$$

s.t

$$w x^T \leq W$$

$$x \in \{0, 1\}^n$$

Pareto Optimality

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0011

Let $x \in \mathbb{R}_{\geq 0}^n$

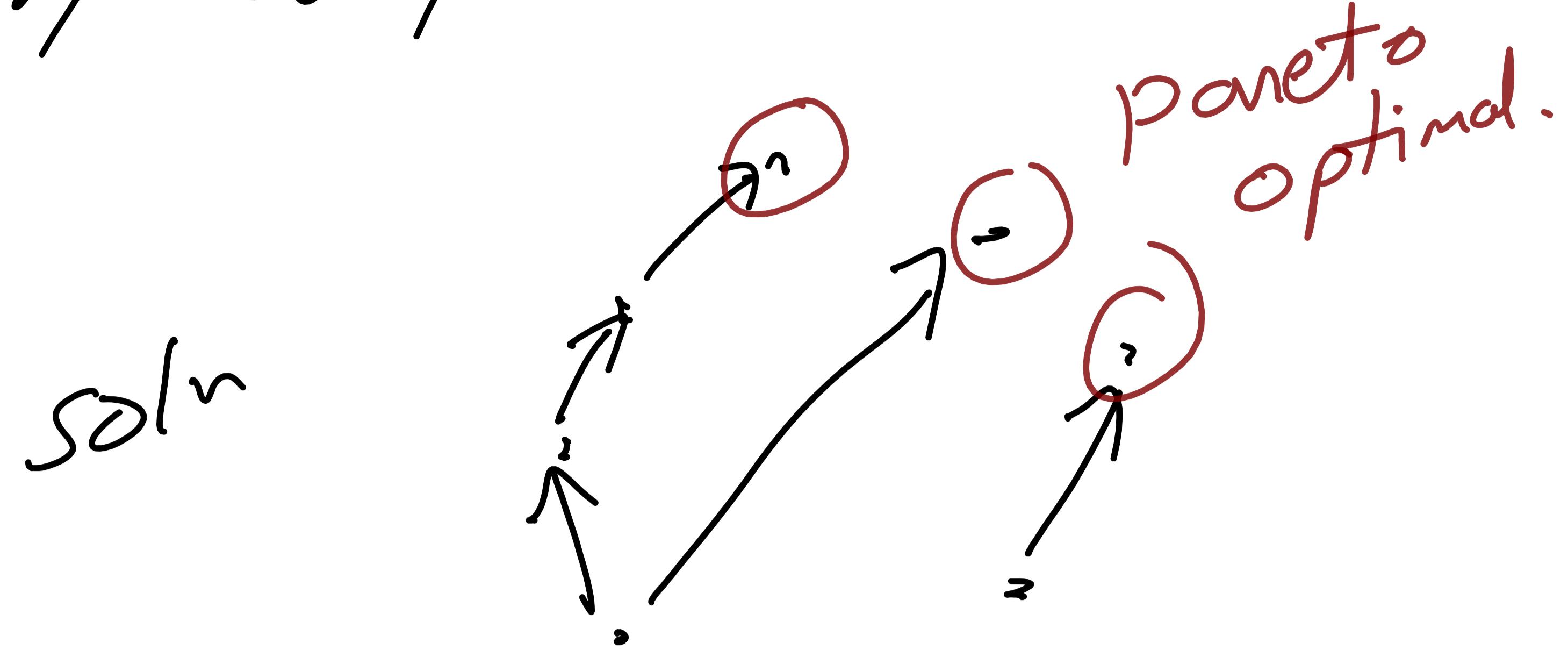
We say object solution y dominates x if

$$\textcircled{1} \quad p y^T \geq p x^T \text{ i.e. } y \text{ has at least as much profit as } x$$

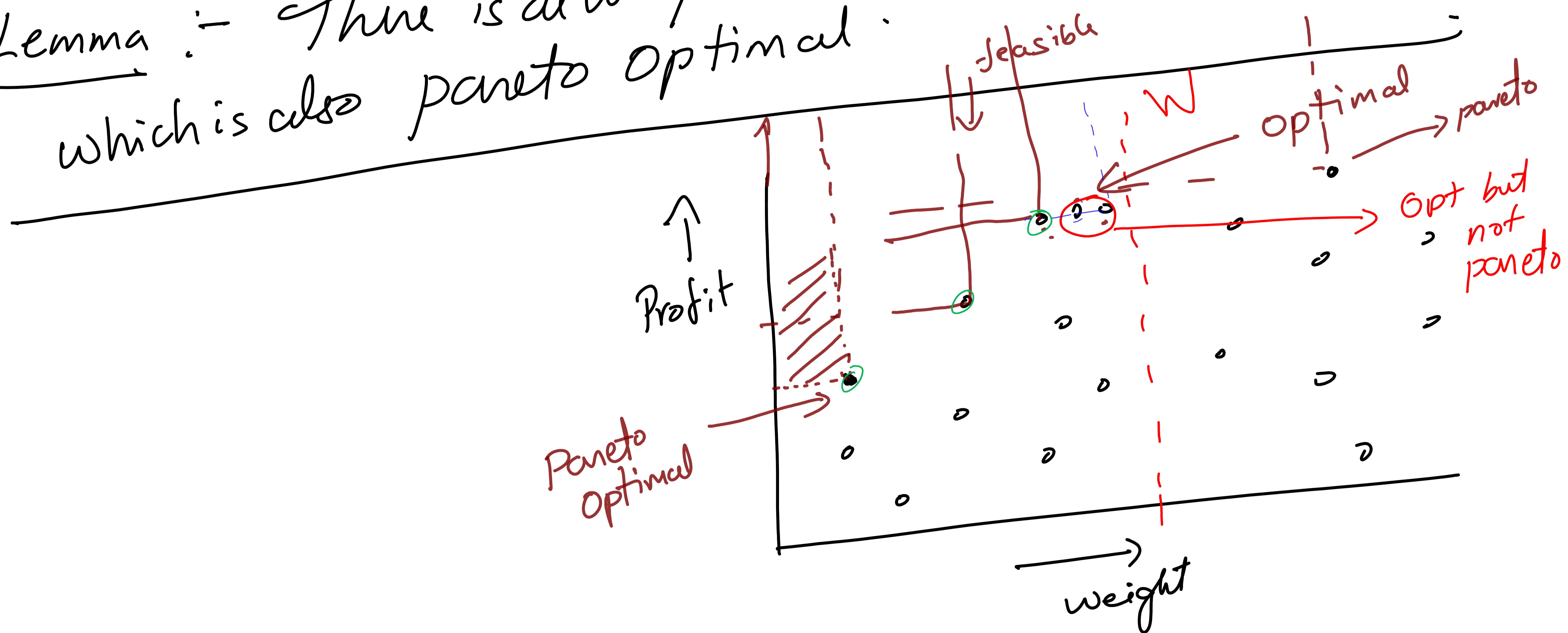
$$\textcircled{2} \quad w y^T \leq w x^T$$

and at least one of these is a strict inequality.

Defn: A solution y is said to be "paveto optimal" if it is not dominated by any other solution z .



Lemma :- There is always an optimal solution which is also paveto optimal.



Pf: Let x be an optimal solution.

Suppose x is not paveto, then there must be y that dominates x .

$$\text{ie } p_y^T \geq p_x^T \text{ & } w_y^T \leq w_x^T \text{ one & the incg strict.}$$

$$\text{Since } x \text{ is opt, } p_y^T = p_x^T \\ \therefore w_y^T < w_x^T$$

repeat the argument with y , y is not paveto. Since there are only finitely many solutions, repeating the argument gives an opt soln that is also paveto optimal.

for a given Knapsack instance (I, P, W, w)

Let P denote the set of all pareto optimal solutions.

Obs:- if P is known, then an optimal solution can be computed in time $\underline{O(|P|)}$.
Then we immediately have an algo for Knapsack with time $O(|P|) +$ time for computing P .

Question:-

- ① How large can $|P|$ be? YES
(Nemhauser & Ullmann)
- ② Can P be computed efficiently? ie in time $\text{poly}(n, |P|)$

Next:-

- ① Describe Nemhauser-Ullman.
- ② Show that $|P|$ is $\text{poly}(n)$ when instances are perturbed

Task:- To construct P .

$$I = \{1, \dots, n\}$$

$$I_i = \{1, \dots, i\}$$

$$S_i = \{x \in \{0, 1\}^n \mid x_j = 0 \quad \forall j > i\}$$

$$P_i = \{x \mid \begin{array}{l} x \text{ is a} \\ \text{pareto optimal} \\ \text{soln for } I_i \end{array}\}$$

$$x \in S_{i-1}$$

x^{+i} is a solution obtained

by adding i to x ie

$$x_j^{+i} = \begin{cases} x_j & j < i \\ 1 & j = i \\ 0 & \text{o.w.} \end{cases}$$

for any $T \subseteq \{0, 1\}^n$

$$T^{+i} = \{y \in \{0, 1\}^n \mid \exists x \in T \quad y = x^{+i}\}.$$

Lemma : $\forall i \in \{1, \dots\}, P_i \subseteq P_{i-1} \cup P_{i-1}^{+i}$

Pf:- Consider $x \in P_i$ ie x is a Pareto optimal soln for $I_i = \{1, \dots, i\}$.

Case 1:- $x_i = 0$

Claim: $x \in P_{i-1}$. Suppose not.

i.e. $\exists y \in P_{i-1}$ that dominates x .

i.e. $y \in S_i$ i.e. y is a candidate soln for I_i

\therefore by defn y is optimal for I_i

$\therefore x$ cannot be Pareto optimal for I_i

$\therefore x$ dominates it \rightarrow contradiction to $x \in P_i$

as y dominates it

Case 2:- $x_i = 1$. Claim: $x \in P_{i-1}^{+i}$

$x = \underbrace{x_1, \dots, x_i}_{\text{s.t. } x_i = 1}, \underbrace{x_{i+1}, \dots, x_n}_{\in S_i} \in S_i$

$\therefore \exists y \in S_{i-1}$ s.t. $x = y^{+i}$

\therefore y is Pareto optimal for I_{i-1} .

Enough to show $y \in P_{i-1}$

Suppose not say there is $z \in S_{i-1}$ s.t.

z dominates y

$$\begin{aligned} p_z^T &\geq p_y^T && \text{one extra strict.} \\ w_z^T &\leq w_y^T && \text{strict.} \end{aligned}$$

$x' = z^{+i}$

then $p_{x'}^T \geq p_x^T$ & with some α less strict

$w_{x'}^T \leq w_x^T$ as $p_z^T + p_i \geq p_y^T + p_i$

$\therefore \exists x' \in S_i$ that dominates x

$\therefore x \notin P_i$ — contradiction
 $\therefore y \in P_{i-1}$ where $x = y^{+i}$

$\therefore x \in P_{i-1} \cup P_{i-1}^{+i}$

Algo- $P_1 \quad P_2 \quad P_3 \quad \dots \quad - \quad P_n = P$

Lecture - II
Nemhauser - Ullmann Algorithm

1. $P_0 = \{0^n\}$.

2. For $i = 1$ to n do

$$Q_i = P_{i-1} \cup P_{i-1}^{+i}$$

3. $Q_i = \{x \in Q_i \mid \nexists y \in Q_i : y \text{ dominates } x\}$
 i.e. there is no y in Q_i that dominates x .

4. $P_i = \{x \in Q_i \mid \nexists p \in P_{i-1} \text{ such that } p \geq x^T / w_{x^T} \leq w\}$.

5. Return $x^* = \arg \max_{x \in P_n} \{ p \geq x^T / w_{x^T} \leq w\}$.

Time $\sum_{i=1}^n |P_i|^2$.

Thm: The N-U algorithm solves the knapsack problem and can be implemented in time $O\left(\sum_{i=0}^{n-1} |P_i|\right)$ — ref: Röglin's notes.

Limitation:- One can construct instances when

$|P|$ can be exponential.

input: $I = \{1, \dots, n\}$.
 $p = (p_1, \dots, p_n)$
 $w = (w_1, \dots, w_n)$.

Perturbation model

Choose $p = (p_1, \dots, p_n)$ -arbitrarily.
draw w_1, \dots, w_n independently s.t w_i is drawn according to $f_i: [0, 1] \rightarrow [0, \phi]$. (ie $\forall x \in [0, 1], f(x) \leq \phi$)

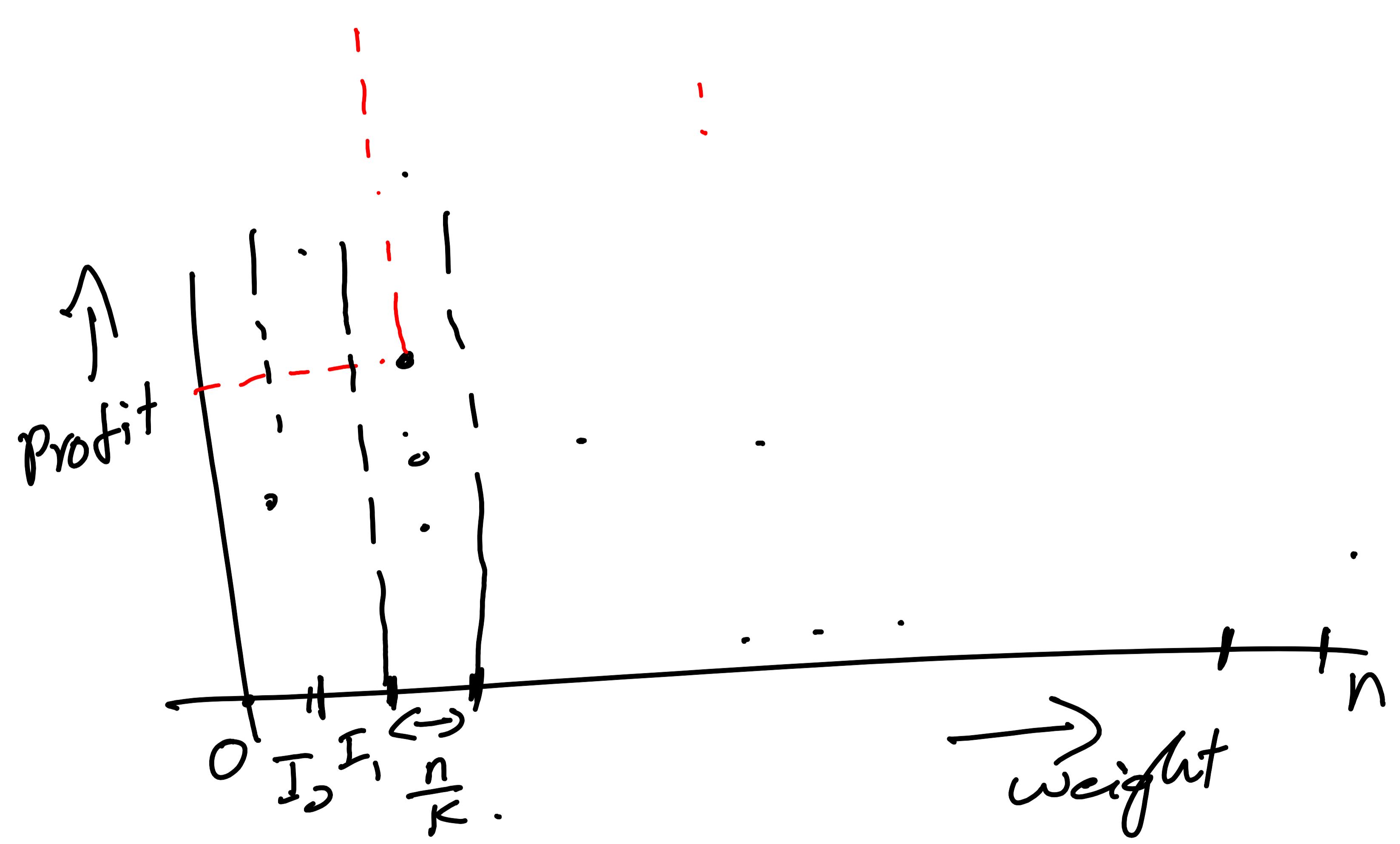
Thm Let profits p_1, \dots, p_n be chosen arbitrarily and the weights w_1, \dots, w_n s.t w_i is drawn independently according to $f_i: [0, 1] \rightarrow [0, \phi]$ where ϕ is a parameter. Then the expected # solutions $x \in [0, 1]^n$ where pareto optimal is bounded from above by $n^2\phi + 1$

Proof idea:-

$$0 \leq w_i \leq 1$$

$$\text{And } x^T w \leq n$$

$K \in \mathbb{N}$ - be a parameter.



partition $[0, n]$ into K -equal sized intervals

Say $I_0, I_1, I_2, \dots, I_{K-1}$

where

$$I_j = \left[\frac{n_j}{K}, \frac{n_{j+1}}{K} \right]$$

Let $\underline{x}^k = \cup \{j \mid \exists x \in P, w x^T \in I_j\}$
 ie # of intervals with at least one
 pareto optimal solution.

If K is large enough

$$\text{d}\mu(P) = \underline{x}^k$$

$$\mathbb{E}[|P|] = \mathbb{E}[x^k]$$

Define the event

$F_{ik}: \exists x \neq y \in \{0, 1\}^n$ with

$$|w x^T - w y^T| \leq \frac{n}{K}$$

(ie $x \neq y$ lie in some I_j)

Lemma:- $\Pr[F_{ik}] \leq 2^{2n+1} \cdot \frac{n\phi}{K} \quad \forall k \in \mathbb{N}$.

Let $x \neq y \in \{0, 1\}^n$ be fixed. Let i be s.t

Pf:-

Let $x \neq y \in \{0, 1\}^n$

wlog $x_i = 0$ & $y_i = 1$

$x_i \neq y_i$ all weights w_j except w_i are revealed.

Suppose

$$w x^T - w y^T = \alpha - w_i$$

for some α .

$$\Pr[|w x^T - w y^T| \leq \frac{n}{K}] = \Pr[|\alpha - w_i| \leq \frac{n}{K}]$$

$$\therefore \Pr[w_i \in [\alpha - \frac{n}{K}, \alpha + \frac{n}{K}]] \leq \frac{2n}{K} \cdot \phi$$

$$= \Pr[w_i \in [\alpha - \frac{n}{K}, \alpha + \frac{n}{K}]]$$

$$\int_{\alpha - \frac{n}{K}}^{\alpha + \frac{n}{K}} f_i(x) dx \\ = 2n \phi$$

By taking union bound over all pairs $x \neq y$

we have $\Pr[F_{1c}] \leq \underline{2} \frac{n\phi}{k}^{2n+1}$

Lemma 2: $\forall 1 < i < n \quad \mathbb{E}[x^i] \leq n^2\phi + 1.$

Assume Lemma 2

Proof of Thm

$$\mathbb{E}[|P|] = \sum_{i=1}^{2^n} i \cdot \Pr[|P|=i] \quad \text{by defn of } \mathbb{E}$$

$$= \sum_{i=1}^{2^n} i \cdot (\Pr[|P|=i \cap F_{1c}] \xrightarrow{\text{AND}} + \Pr[|P|=i \cap \bar{F}_{1c}])$$

$$\boxed{\Pr[A] = \Pr[A \cap B] + \Pr[A \cap \bar{B}]}$$

$$= \sum_{i=1}^{2^n} i \cdot \Pr[|P|=i \cap F_{1c}] + \sum_{i=1}^{2^n} i \cdot \Pr[|P|=i \cap \bar{F}_{1c}]$$

$$\Pr[A \cap B] \\ = \Pr[B] \cdot \Pr[A|B]$$

$$= \sum_{i=1}^{2^n} i \cdot \Pr[F_{1c}] \cdot \Pr[|P|=i | F_{1c}] + \sum_{i=1}^{2^n} i \cdot \Pr[|P|=i \cap \bar{F}_{1c}]$$

no interval has more than one solution.
ie $|P| = x^i$.

$$= \Pr[F_{1c}] \cdot \sum_{i=1}^{2^n} i \cdot \Pr[|P|=i | F_{1c}] + \sum_{i=1}^{2^n} i \cdot \Pr[x^i = i]$$

$$\begin{aligned}
 &= R_v[F_{1\kappa}] \cdot \sum_{i=1}^{2^n} i \cdot P_v[|P|=i] F_{ik} + E[X^k] \\
 &\leq 2^{2n+1} \frac{n\phi}{\kappa} [2^{2n}] + n^2\phi + 1 \\
 &\leq 2^{4n+1} \frac{n\phi}{\kappa} + n^2\phi + 1 \\
 \text{ie } E[|P|] &\leq 2^{4n+1} \frac{n\phi}{\kappa} + n^2\phi + 1 \quad \forall n \in \mathbb{N}. \rightarrow \textcircled{*} \\
 \therefore E[|P|] &\leq n^2\phi + 1 \quad \text{as } \textcircled{*} \text{ holds for all } \kappa \in \mathbb{N}.
 \end{aligned}$$

Lecture-12

To prove that $E[X^k] \leq n^2\phi + 1$

$$X^k = \left\{ ; \mid \exists x \in P \text{ s.t } x \in \overline{I}_i \right\} + 1$$

Define $X_i = \begin{cases} 1 & \text{if interval } \overline{I}_i \text{ contains at least one point opt i.e } \overline{I}_i \cap P \neq \emptyset \\ 0 & \text{o.w.} \end{cases}$

$\overline{I}_0, \overline{I}_1, \dots, \overline{I}_{k-1}$

$$\begin{aligned}
 X^k &= X_0 + \dots + X_{k-1} + 1 \\
 &= 1 + \left(\sum_{i=0}^{k-1} X_i \right)
 \end{aligned}$$

$$\begin{aligned}
 E[X^k] &= E[1 + \sum_{i=0}^{k-1} X_i] = 1 + E[\sum_{i=0}^{k-1} X_i] \\
 &= 1 + \sum_{i=0}^{k-1} E[X_i]
 \end{aligned}$$

Now

$$\mathbb{E}[x_i] = \frac{0 \cdot \Pr[x_i=0] + 1 \cdot \Pr[x_i=1]}{\Pr[\exists x \in P \text{ s.t. } w^T x \in I_i]}$$

Lemma 3: $\forall t \geq 0 \text{ & } \varepsilon > 0$

$$\Pr[\exists x \in P \text{ s.t. } w^T x \in [t, t+\varepsilon]] \leq n\phi\varepsilon.$$

by Lemma 3. we have $\mathbb{E}[x_i] \leq \frac{n^2\phi}{1\kappa}$

$$\therefore \mathbb{E}[x] \leq 1 + \frac{n^2\phi}{1\kappa} = \underline{1+n^2\phi}$$

To prove Lemma 3

x^* : most profitable
soln with $w^T x^* \leq t$

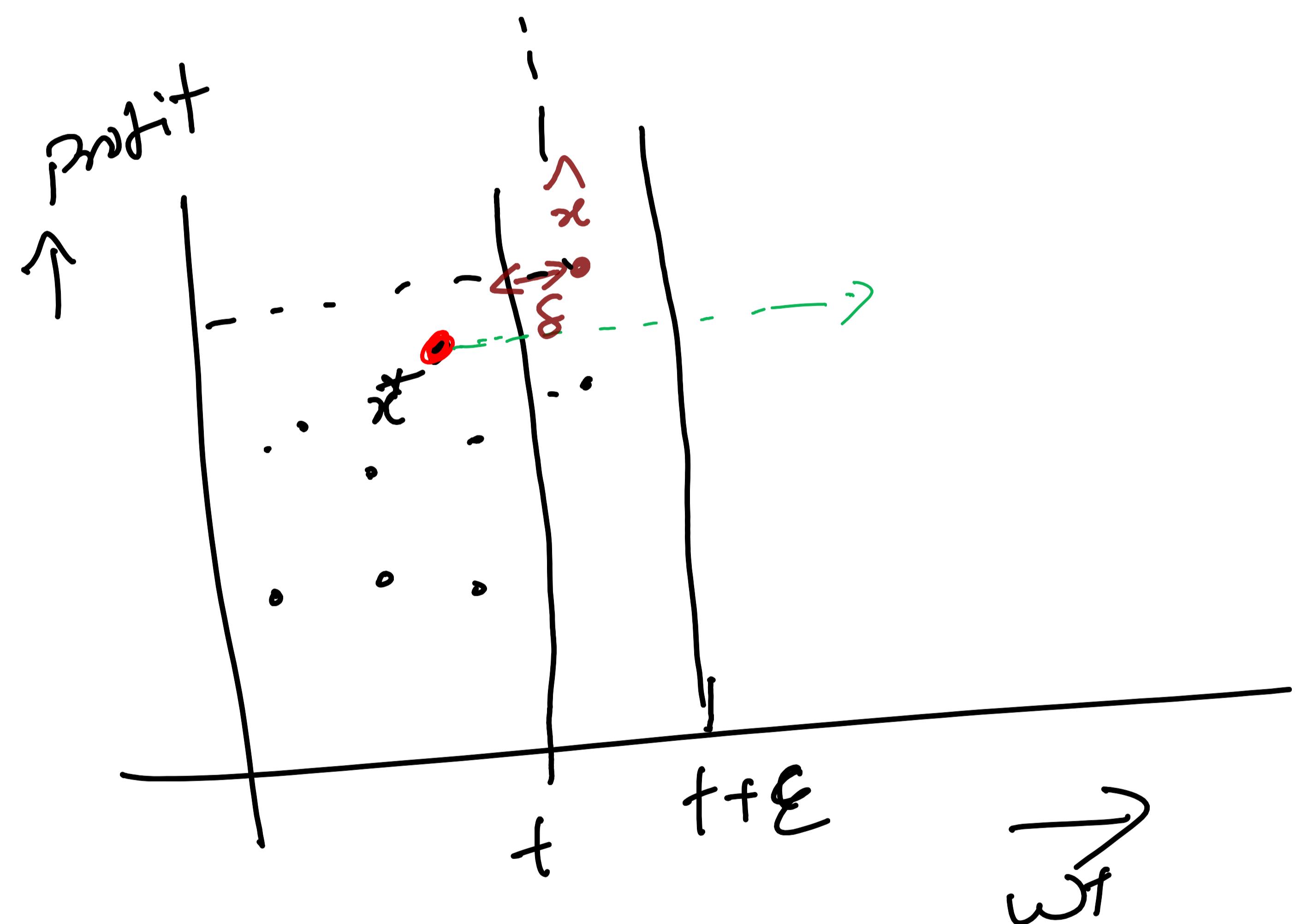
Define $\pi(t)$ van $\Lambda(t)$

s.t. $\pi(t) \leq \varepsilon \iff \exists x \in P: w^T x \in [t, t+\varepsilon]$

$$x^* = \arg \max_x \left\{ p x^T \mid x \in \{0,1\}^n \text{ & } w^T x \leq t \right\}.$$

$\begin{aligned} & \arg \min_x f(x) \\ & \text{= value of } x \\ & \text{s.t. } f(x) \text{ is min.} \end{aligned}$

we call x^* as the winner.



$\forall t \geq 0$, x^* - exists.

A solution x is a loser if $Px^T > Px^{*T}$

i.e. $wx^T > t$.

Let \hat{x} be the loser with least weight
 $\{wx^T \mid x \in \{0,1\}^n, Px^T > Px^{*T}\}$.

$$\hat{x} = \arg \min_x \{wx^T \mid x \in \{0,1\}^n, Px^T > Px^{*T}\}$$

It is possible that \hat{x} is not defined.
in such a case, we set $\hat{x} = \perp$. (bottom)

Define $\Lambda(t) = \begin{cases} w\hat{x}^T + & \text{if } \hat{x} \neq \perp \\ \perp & \text{otherwise} \end{cases}$

$$\boxed{\exists x \in P \quad wx^T \in [t, t+\varepsilon] \iff \Lambda(t) \leq \varepsilon}$$

$$\text{i.e. } F[x_i] = \Pr[\exists x \in P \quad wx^T \in [t, t+\varepsilon]]$$

$$= \Pr[\Lambda(t) \leq \varepsilon]$$

$$\sum \hat{x}_i w_i - t$$

$$\sum \hat{x}_i w_i - t \leq \varepsilon$$

To estimate $\Pr[\Lambda(t) \leq \varepsilon]$

Define for $i \in \{1, \dots, n\}$, $b \in \{0, 1\}$.

$$S^{x_i = b} = \{x \in \{0, 1\}^n \mid x_i = b\}.$$

$$x^{*,i} = \arg \max \{px^T \mid x \in S^{x_i=0}, wx^T \leq t\}.$$

$$\text{Then } x^* = \max_i x^{*,i}$$

i.e. $x^{*,i}$ is the winner among all soln that exclude item i .

$$\text{Defin } \hat{x}^i = \arg \min \{wx^T \mid x \in S^{x_i=1} \text{ s.t. } px^T > px^{*,i}\}$$

$$\lambda^i(t) = \begin{cases} w_{\hat{x}^i}^T - \epsilon, & \text{if } \hat{x}^i \neq \perp \\ 0 - \omega, & \text{if } \hat{x}^i = \perp \end{cases}$$

Lemma 4:- For every choice of profits & weights either $\lambda^i(t) \geq \perp$ or there is an $i \in \{1, \dots, n\}$ s.t $\lambda^i(t) = \lambda^i(t)$. [Proof later]

$$\text{Lemmas: } \forall i \in \{1, \dots, n\} \text{ & } \epsilon > 0$$

$$\Pr[\lambda^i(t) \leq \epsilon] \leq e^{-\phi}.$$

$$\Pr[\exists i, \lambda^i(t) \leq \epsilon] \leq n e^{-\phi}$$

Corollary:-

Proof [Lemma 5]

$w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n$

Suppose all weights except w_i are fixed.
 ie weights of all solutions in $S^{x_i=0}$ are fixed
 and hence x^{*i} is also fixed.

$w x^{*i}$ is fixed.

\hat{x}^i is also fixed as $w_i \geq 0$
 \hat{x}^i is the least weight solution among all x
 s.t. $p x^T \geq p x^{*i T}$

in fact the set

$$L = \left\{ x \in S^{x_i=1} \mid p x^T > p \underline{x^{*i}} \right\}$$

also fixed (as p_1, \dots, p_n are not random).

\hat{x}^i is the least weight element in L .

$$\therefore \text{the event } \lambda^i(t) \leq \epsilon = \lambda^i(t) \in [0, \epsilon]$$

equivalent to $\underline{w \hat{x}^i T + \epsilon} \in [0, \epsilon]$ where \hat{x}^i is fixed.

ie there is a $x \in S^t$

$$w_i \in (\kappa, \kappa + \epsilon)$$

$$\therefore \Pr [\lambda^i(t) \leq \epsilon] = \Pr [w_i \in (\kappa, \kappa + \epsilon)]$$

$$\leq \int_{\kappa}^{\kappa+\epsilon} f_i(x) dx \leq \int_{\kappa}^{\kappa+\epsilon} \phi(x) dx \leq \phi \epsilon.$$

