Applied Mathematics Letters 26 (2013) 904–907

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A simple derivation of Kirk's approximation for spread options

C.F. Lo

Institute of Theoretical Physics and Department of Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong

ARTICLE INFO

Article history:
Received 28 February 2013
Accepted 9 April 2013

Keywords:
Lognormal random variables
Black–Scholes equation
Spread options
Kirk's approximation
WKB approximation

ABSTRACT

Ever since Kirk proposed an approximate price formula for a European call spread option in 1995, Kirk's approximation has become the most widely used among the practitioners, especially in the energy markets. It is well known that Kirk's approximation extends from Margrabe's exchange option formula but no explicit derivation is available or has ever been published. In this paper we apply the idea of WKB method to provide a simple derivation of Kirk's approximation and discuss its validity.

1. Introduction

Spread options are options whose payoff is contingent upon the price difference (or the spread) of two underlying assets and form the simplest type of multi-asset options [1]. In spite of being extremely simple in nature, pricing spread options is a very challenging task and receives much attention in the literature. The main obstacle to exact analytic pricing lies in the lack of knowledge about the distribution of the spread which can assume negative values. One straightforward approach is to obtain the joint probability distribution of the two correlated underlyings and evaluate the expectation of the final payoff by means of numerical integration. However, spread option traders often prefer to use analytical approximations rather than numerical methods because of their computational ease and the availability of closed-form formulae for hedging ratios. Among various analytical approximations Kirk's approximation seems to be the most popular, especially in the energy markets [2]. It is well known that Kirk's approximation extends from Margrabe's exchange option formula [3] but, insofar as we know, no explicit derivation is available or has ever been published. In this paper we apply the idea of WKB method [4] to derive Kirk's approximation and discuss its validity.

2. Derivation of Kirk's approximation

The price of a European call spread option with the underlying assets S_1 and S_2 obeys the two-dimensional Black–Scholes equation

$$0 = \left\{ \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + rS_1 \frac{\partial}{\partial S_1} + rS_2 \frac{\partial}{\partial S_2} - r - \frac{\partial}{\partial \tau} \right\} P\left(S_1, S_2, \tau\right) \tag{1}$$

with the final payoff condition

$$P\left(S_1, S_2, 0\right) = \max\left(S_1 - S_2 - K, 0\right), \tag{2}$$

E-mail address: cflo@phy.cuhk.edu.hk.

0893-9659/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.aml.2013.04.004

where σ_1 and σ_2 are the volatilities of the two underlyings, ρ is the correlation between them, K is the strike price, r is the risk-free interest rate, and τ denotes the time-to-maturity. It is well known that no analytical solution is available in closed form and one needs to resort to numerical methods. In the following we try to apply the idea of WKB method to derive the price formula of Kirk's approximation.

To begin with, we re-write Black–Scholes equation as

$$0 = \left\{\frac{1}{2}\sigma_{\text{eff}}^2\left(S_2+Ke^{-r\tau}\right)^2\frac{\partial^2}{\partial S_2^2}+\rho\sigma_1\sigma_{\text{eff}}S_1\left(S_2+Ke^{-r\tau}\right)\frac{\partial^2}{\partial S_1\partial S_2}+\frac{1}{2}\sigma_1^2S_1^2\frac{\partial^2}{\partial S_1^2}\right.$$
$$\left.+\ rS_2\frac{\partial}{\partial S_2}+rS_1\frac{\partial}{\partial S_1}-r-\frac{\partial}{\partial\tau}\right\}P\left(S_1,S_2,\tau\right) \tag{3}$$

where

$$\sigma_{\text{eff}} = \sigma_2\left(\frac{S_2}{S_2+Ke^{-r\tau}}\right). \tag{4}$$

Proposition 1. *If σ_{eff} is replaced by a constant σ_{20}, then the solution is given by*

$$P_0\left(S_1,S_2,\tau\right) = S_1N\left(d_1\right)-\left(S_2+Ke^{-r\tau}\right)N\left(d_2\right) \tag{5}$$

where

$$d_1 = \frac{\ln\left(S_1\right)-\ln\left(S_2+Ke^{-r\tau}\right)}{\sigma_-\sqrt{\tau}}+\frac{1}{2}\sigma_-\sqrt{\tau} \tag{6}$$

$$d_2 = d_1-\sigma_-\sqrt{\tau} \tag{7}$$

$$\sigma_- = \sqrt{\sigma_1^2-2\rho\sigma_1\sigma_{20}+\sigma_{20}^2}. \tag{8}$$

Proof. In terms of the two new variables:

$$R_1 = \frac{S_1}{S_2+Ke^{-r\tau}}\quad\text{and}\quad R_2 = S_2+Ke^{-r\tau}. \tag{9}$$

Eq. (3) with σ_{eff} being replaced by a constant σ_{20} can be re-written in the following form

$$0 = \left\{\hat{L}-\frac{\partial}{\partial\tau}\right\}\frac{\partial P_0\left(R_1,R_2,\tau\right)}{\partial\tau}, \tag{10}$$

where

$$\hat{L} = \frac{1}{2}\sigma_-^2R_1^2\frac{\partial^2}{\partial R_1^2}+\left(\rho\sigma_1-\sigma_{20}\right)\sigma_{20}R_1R_2\frac{\partial^2}{\partial R_1\partial R_2}+\frac{1}{2}\sigma_{20}^2R_2^2\frac{\partial^2}{\partial R_2^2}-\left(\rho\sigma_1-\sigma_{20}\right)\sigma_{20}R_1\frac{\partial}{\partial R_1}+rR_2\frac{\partial}{\partial R_2}-r, \tag{11}$$

and the final payoff condition becomes

$$P_0\left(R_1,R_2,0\right) = R_2\max\left(R_1-1,0\right). \tag{12}$$

The formal solution of Eq. (10) is given by

$$P_0\left(R_1,R_2,\tau\right) = \exp\left\{\tau\hat{L}\right\}R_2\max\left(R_1-1,0\right) = R_2F\left(R_1,\tau\right) \tag{13}$$

where

$$F\left(R_1,\tau\right) = \exp\left\{\frac{1}{2}\tau\sigma_-^2R_1^2\frac{\partial^2}{\partial R_1^2}\right\}\max\left(R_1-1,0\right). \tag{14}$$

It is not difficult to see that $F\left(R_1,\tau\right)$ satisfies the partial differential equation

$$\frac{\partial F\left(R_1,\tau\right)}{\partial\tau} = \frac{1}{2}\sigma_-^2R_1^2\frac{\partial^2F\left(R_1,\tau\right)}{\partial R_1^2} \tag{15}$$

with the boundary condition $F\left(R_1,0\right) = \max\left(R_1-1,0\right)$. This equation is Black–Scholes equation for a future asset price R_1 and the solution $F\left(R_1,\tau\right)$ is given by

$$F\left(R_1,\tau\right) = R_1N\left(d_1\right)-N\left(d_2\right) \tag{16}$$

where $N(\cdot)$ denotes the cumulative normal distribution function, and

$$d_1 = \frac{\ln(R_1)}{\sigma_-\sqrt{\tau}} + \frac{1}{2}\sigma_-\sqrt{\tau} \tag{17}$$

$$d_2 = d_1 - \sigma_-\sqrt{\tau}. \tag{18}$$

As a result, we obtain

$$P_0(R_1, R_2, \tau) = R_1R_2N(d_1) - R_2N(d_2) \tag{19}$$

which is exactly the solution in Eq. (5). □

Since σ_{eff} is not a constant, it is obvious that the solution in Eq. (5) does not satisfy Eq. (3). Nevertheless, if σ_{eff} is a slowly-varying function of S_2 and τ, then, based upon the solution in Eq. (5), we can apply the idea of WKB method,[1] which is a powerful tool for obtaining a global approximation to the solution of a linear ordinary differential equation, to derive an accurate approximate solution to Eq. (3), which turns out to be identical to Kirk's approximation.

Proposition 2. *If σ_{eff} is a* slowly-varying function *of S_2 and τ, i.e.*

$$\frac{S_2}{\sigma_{\text{eff}}}\left|\frac{\partial\sigma_{\text{eff}}}{\partial S_2}\right| \ll 1 \tag{20}$$

and

$$\frac{1}{r\sigma_{\text{eff}}}\left|\frac{\partial\sigma_{\text{eff}}}{\partial \tau}\right| \ll 1, \tag{21}$$

then the solution $P(S_1, S_2, \tau)$ of Eq. (3) can be approximated by

$$P_{\text{eff}}(S_1, S_2, \tau) = S_1N(d_1) - \left(S_2 + Ke^{-r\tau}\right)N(d_2) \tag{22}$$

where

$$d_1 = \frac{\ln(S_1) - \ln\left(S_2 + Ke^{-r\tau}\right)}{\sigma_-\sqrt{\tau}} + \frac{1}{2}\sigma_-\sqrt{\tau} \tag{23}$$

$$d_2 = d_1 - \sigma_-\sqrt{\tau} \tag{24}$$

$$\sigma_- = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_{\text{eff}} + \sigma_{\text{eff}}^2} \tag{25}$$

which resembles the solution $P_0(S_1, S_2, \tau)$ in Eq. (5) very closely and is identical to Kirk's approximation.

Proof. First of all, it is not difficult to show that

$$\frac{S_2}{\sigma_{\text{eff}}}\left|\frac{\partial\sigma_{\text{eff}}}{\partial S_2}\right| = \left|\frac{Ke^{-r\tau}}{S_2 + Ke^{-r\tau}}\right| \ll 1 \tag{26}$$

provided $\left|Ke^{-r\tau}/S_2\right| \ll 1$. Similarly, we can also show that

$$\frac{\partial\sigma_{\text{eff}}}{\partial\tau} = rS_2\frac{\partial\sigma_{\text{eff}}}{\partial S_2} \tag{27}$$

[1] WKB method provides approximate solutions of differential equations of the form

$$\frac{d^2y(x)}{dx^2} + k(x)^2y(x) = 0,$$

provided that $k(x)$ is slowly varying, *i.e.*

$$\left|\frac{1}{k(x)}\frac{dk(x)}{dx}\right| \ll 1.$$

The completed approximate solution is given by

$$y(x) \approx \frac{1}{\sqrt{k(x)}}\exp\left\{\pm i\int k(x)dx\right\}.$$

It is obvious that the approximate solution will be reduced to the usual plane-wave solution if $k(x)$ is replaced by a constant. Details of the method can be found in Morse and Feshbach [4], Mathews and Walker [5] as well as Bender and Orszag [6].

which implies that

$$\frac{1}{r\sigma_{\text{eff}}}\left|\frac{\partial\sigma_{\text{eff}}}{\partial\tau}\right| \ll 1. \tag{28}$$

Then, substituting $P_{\text{eff}}(S_1, S_2, \tau)$ into the right-hand side (*R.H.S.*) of Eq. (3), we obtain, after simplification,

$$\begin{aligned} R.H.S. = {} & \frac{\sigma_2}{\sigma_-}\left\{\left(\frac{S_2}{S_2 + Ke^{-r\tau}}\right)\sigma_2 d_1 - \rho\sigma_1 d_2\right\}\Phi(d_1)\, S_1 S_2 \frac{\partial\sigma_-}{\partial S_2} \\ & + \frac{1}{2}\sigma_2^2\sqrt{\tau}\,\Phi(d_1)\, S_1 S_2^2 \left\{\frac{\partial^2\sigma_-}{\partial S_2^2} + \frac{d_1 d_2}{\sigma_-}\left(\frac{\partial\sigma_-}{\partial S_2}\right)^2\right\} \end{aligned} \tag{29}$$

where

$$\frac{\partial\sigma_-}{\partial S_2} = \left(\frac{\sigma_{\text{eff}} - \rho\sigma_1}{\sigma_-}\right)\frac{\partial\sigma_{\text{eff}}}{\partial S_2} \tag{30}$$

$$\frac{\partial^2\sigma_-}{\partial S_2^2} = \left(\frac{\sigma_{\text{eff}} - \rho\sigma_1}{\sigma_-}\right)\frac{\partial^2\sigma_{\text{eff}}}{\partial S_2^2} + \left(1-\rho^2\right)\frac{\sigma_1^2}{\sigma_-^3}\left(\frac{\partial\sigma_{\text{eff}}}{\partial S_2}\right)^2 \tag{31}$$

and $\Phi(\cdot)$ denotes the normal distribution function. Since σ_{eff} is a slowly-varying function of S_2 and τ as shown in Eqs. (20) and (21), it can be inferred that $R.H.S. \approx 0$ in Eq. (29) and $P_{\text{eff}}(S_1, S_2, \tau)$ can be a good approximate solution of Eq. (3). □

3. Conclusion

We have applied the idea of WKB method to provide a simple derivation of the price formula of Kirk's approximation for a European call spread option. According to our analysis, the validity of Kirk's approximation is dictated by the constraints stated in Eqs. (20) and (21), namely σ_{eff} is a slowly-varying function of S_2 and τ. Moreover, to further improve the accuracy of Kirk's approximation, we could apply the higher-order WKB approximation.

References

[1] R. Carmona, V. Durrleman, Pricing and hedging spread options, SIAM Review 45 (4) (2003) 627–685.
[2] E. Kirk, Correlation in the energy markets, in: Managing Energy Price Risk, Risk Publications, 1995.
[3] W. Margrabe, The value of an option to exchange one asset for another, Journal of Finance 33 (1978) 177–186.
[4] P. Morse, H. Feshbach, Methods of Theoretical Physics, McGraw-Hill, New York, 1953, p. 1092.
[5] J. Mathews, R.L. Walker, Mathematical Methods of Physics, Benjamin, New York, 1973, p. 27.
[6] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, Auckland, 1978, p. 484.