Black-Scholes formula: applications

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Forward contract

Let us consider a financial product with payoff at $T : \chi = f(S_T)$ $\mathcal{F}_T-$measurable, and suppose that we are in the framework of BS model. Let us suppose that we are now at instant $t < T$.

Definition

A forward contract with maturity T on χ, concluded at instant $t < T$, is a contract in which the buyer will pay a price K (forward price) at date T and will receive χ at this date.

Nothing is paid nor received at date t, but the forward price K that will be paid at T is determined at date t (at the signature of the forward contract).

Notation

The forward price K determined at instant t will be denoted by $f(t; T, \chi)$.
We will compute it now.

Forward price

Clearly, the amount received at T by the buyer is

$$Y = \chi - K$$

and the amount received at date t is equal to 0 (no cash-flow at t=0 by definition of the forward contract).

By using the fundamental thm on pricing in an arbitrage free and complete market, we directly get :

$$E_{\mathbb{Q}}\left[e^{-r(T-t)}(\chi - K) \mid \mathcal{F}_t\right] = 0$$

which implies

$$E_{\mathbb{Q}}[\chi \mid \mathcal{F}_t] = E_{\mathbb{Q}}[K \mid \mathcal{F}_t] = f(t; T, \chi)$$

since K is $\mathcal{F}_t-$measurable (K is fixed at t)

Forward price

We hence have the following result :

Proposition

The forward price $f(t; T, \chi)$ determined at instant t, on the financial product with payoff χ $\mathcal{F}_T$-measurable is :

$$f(t; T, \chi) = E_{\mathbb{Q}}[\chi \mid \mathcal{F}_t]$$

In particular, if $\chi = S_T$, then

$$f(t; T, S_T) = e^{r(T-t)} S_t.$$

This last inequality simply comes from the fact that $(e^{-rt} S_t)$ is a martingale under $\mathbb{Q}$.

Call on forward

We now consider a European call option with maturity T and strike K, on a forward contract with underlying S_t and maturity $T_1 > T$.

The call holds more specifically on the forward price with maturity T_1, which is worth at t :

$$F(t, T_1; S) = F(t, T_1) = e^{r(T_1 - t)} S_t$$

The underlying is hence a forward price associated to a given maturity date T_1.

At date T, the payoff of the call is :

$$(F(T, T_1) - K)_+ = \left(e^{r(T_1 - T)} S_T - K\right)_+$$

$$= e^{-r(T-t)} \left(e^{r(T_1 - t)} S_T - e^{r(T-t)} K\right)_+$$

Call on forward

The call price is hence equal to

$$e^{-r(T-t)}C(t, e^{r(T_1-t)}S_t, K' = e^{r(T-t)}K, T)$$

i.e. :

$$e^{-r(T-t)}\left[e^{r(T_1-t)}S_t\Phi(d_1) - K\Phi(d_2)\right]$$

$$= e^{-r(T-t)}\left[F_t\Phi(d_1) - K\Phi(d_2)\right]$$

$$d_1 = \frac{\ln \frac{F_t}{e^{r(T-t)}K} + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Call on forward : Black formula

Proposition (Black's formula)

The price at t of a European call with maturity T and strike K, on the forward price of maturity $T_1 > T$ on the underlying S_t, is given by :

$$e^{-r(T-t)}\left[F(t, T_1)\,\phi(d_1) - K\phi(d_2)\right]$$

with

$$d_1 = \frac{\ln(F/K)+\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Exchange option

Let us consider a market composed of a risk free asset (risk free rate r) and of two risky assets of price $S_1(t)$ and $S_2(t)$ satisfying the assumptions of Black-Scholes' model :

$$\begin{aligned} d\beta_t &= r\beta_t d\beta_t \\ dS_1(t) &= \alpha_1 S_1(t)dt + \sigma_1 S_1(t)dW_t^1 \\ dS_2(t) &= \alpha_2 S_2(t)dt + \sigma_2 S_2(t)dW_t^2 \end{aligned}$$

with corr $\left(W_t^1, W_t^2\right) = 0$ (independence between both stocks S_1 et S_2).

We consider an option allowing to exchange stock 2 against stock 1 at instant T ("exchange option") :

payoff of the option :

$$\left(S_1\left(T\right) - S_2\left(T\right)\right)_+$$

Exchange option : pricing by replication

A first method consists to build a self-financing replicating portfolio, of value

$$V(t) = V(t, S_1, S_2) = a_1(t)S_1(t) + a_2(t)S_2(t) + b(t)\beta_t$$

like in the case of the European call, and apply the multidimensional Itô lemma.

We can then see than we arrive to the following PDE :

$$\partial_t V + rs_1\partial_1 V + rs_2\partial_2 V + \frac{1}{2}s_1^2\sigma_1^2\partial_{11}^2 V + \frac{1}{2}s_2^2\sigma_2^2\partial_{22}^2 V = rV$$

with terminal condition $V(T, s_1, s_2) = (s_1 - s_2)^+$ for all $(s_1, s_2) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Exchange option : pricing by replication

The key of the remaining part is to see that the terminal condition is homogeneous with degree 1 in the following sense :

$$H(s_1, s_2) = (s_1 - s_2)^+ = s_2 \left(\frac{s_1}{s_2} - 1 \right)^+ = s_2 H\left(\frac{s_1}{s_2}, 1 \right).$$

One can show that this property is kept in the solution of the PDE : by introducing the change of variable

$$V(t, s_1, s_2) = s_2 F\left(t, \frac{s_1}{s_2} \right)$$

we get a new PDE on the unknown function $F(t, z)$, more simple than the preceding PDE (one dimension less), and which has a solution.

Exchange option : pricing by replication

Indeed, the PDE for the new unknown function $F(t,z)$ becomes :

$$\begin{cases} \partial_t F(t,z) + \frac{1}{2} z^2 \partial^2_{zz} F(t,z) \left(\sigma_1^2 + \sigma_2^2\right) = 0 \\ F(T,z) = (z-1)_+ \end{cases}$$

which is exactly the Black-Scholes PDE :

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF = 0, \quad F(T,S) = (S-K)_+$$

avec :

$$\begin{aligned} "r" &= 0 \\ "K" &= 1 \\ "\sigma" &= \sqrt{\sigma_1^2 + \sigma_2^2} \end{aligned}$$

Exchange option : pricing by replication

The solution F is hence given by the following formula :

$$F(t,z) = z\Phi(d_1(z)) - \Phi(d_2(z))$$

$$d_1 = \frac{\ln z + \frac{\sigma_1^2+\sigma_2^2}{2}(T-t)}{\sqrt{(\sigma_1^2+\sigma_2^2)(T-t)}}, \quad d_2 = d_1 - \sqrt{\sigma_1^2+\sigma_2^2}\sqrt{T-t}$$

The price at t of the exchange option is hence :

$$V(t,s_1,s_2) = s_2 F\left(t, \frac{s_1}{s_2}\right) = s_1\Phi\left(d_1\left(\frac{s_1}{s_2}\right)\right) - s_2\Phi\left(d_2\left(\frac{s_1}{s_2}\right)\right)$$

This formula is known as the so called “Margrabe formula”.

Exchange option

Theorem (Margrabe formula)

Let two risky assets $S_1(t)$ and $S_2(t)$ satisfying the assumptions of the Black-Scholes model :

$$dS_1(t) = \alpha_1 S_1(t)dt + S_1(t)\sigma_1 dW_1(t)$$
$$dS_2(t) = \alpha_2 S_2(t)dt + S_2(t)\sigma_2 dW_2(t)$$

with $W_1(t)$ and $W_2(t)$ independent, and let r be the risk free rate. Then the price at t of an exchange option of maturity T, of payoff : $(S_2 - S_1)^+$ is given by :

$$S_1\Phi(d_1) - S_2\Phi(d_2)$$

$$d_1 = \frac{\ln\frac{S_1}{S_2} + \frac{\sigma_1^2+\sigma_2^2}{2}(T-t)}{\sqrt{(\sigma_1^2+\sigma_2^2)(T-t)}} \quad d_2 = d_1 - \sqrt{(\sigma_1^2+\sigma_2^2)(T-t)}$$

One can generalize this formula to the case where $corr(W_1(t), W_2(t)) = \rho$.

Exchange option : pricing by change of numeraire

Another method consists to apply the technique of change of numeraire.

The departure point consists to see that the price of the exchange option is equal to :

$$E_{\mathbb{Q}}[e^{-r(T-t)}\left(S_1\left(T\right)-S_2\left(T\right)\right)_+|\mathcal{F}_t]=E_{\mathbb{Q}}\left[e^{-r(T-t)}S_2\left(T\right)\left(\frac{S_1\left(T\right)}{S_2\left(T\right)}-1\right)_+|\mathcal{F}_t\right]$$

The idea is to change measure : we will pass to a new equivalent measure under which the prices expressed w.r.t. the second asset S_2 are martingales.

Numeraire

A **numeraire** is an asset exchanged on the market having a strictly positive value and not paying any dividend.
Example : $\beta_t = e^{rt}$ the saving/bank account.

Once we have a numeraire, we can express the prices of any financial product w.r.t. that numeraire :
If N_t is (the price at t of) a numeraire, and $P(t)$ is the price of a financial product, the its price expressed w.r.t. that numeraire is : $P(t)/N(t)$.

Example with the bank account : discounted prices $P(t)e^{-rt} = \frac{P(t)}{\beta_t}$ correspond actually to prices expressed w.r.t. the particular choice of numeraire of the bank account : $N(t) = \beta_t$.

We know that under the risk neutral measure $\mathbb{Q}$, the discounted prices of assets are martingales. Measure $\mathbb{Q}$ is actually the martingale measure associated to the numeraire choice β_t.

We will show that one can associate a martingale measure to any numeraire choice.

Change of numeraire : change of measure

Generally, when we pass to an equivalent measure (e.g. from $\mathbb{Q}_1$ to $\mathbb{Q}_2$), the measure change can be done with the Radon-Nikodym derivative :

$$Z = \frac{d\mathbb{Q}^2}{d\mathbb{Q}^1}$$

such that $\mathbb{E}_{\mathbb{Q}^2}[Y] = \mathbb{E}_{\mathbb{Q}^1}[ZY]$ for any random variable Y.

If we work in a probability space with filtration $(\mathcal{F}_t)_{t\in[0,T]}$, we can define a stochastic process (Z_t) from the R-N derivative :

$$Z_t := E_{\mathbb{Q}^1}[Z|\mathcal{F}_t]$$

By construction, Z_t is a $\mathbb{Q}_1$-martingale adapted to this filtration, and

$$Z_0 = \mathbb{E}_{\mathbb{Q}_1}[Z|\mathcal{F}_0] = \mathbb{E}_{\mathbb{Q}_1}[Z] = 1, \quad Z_T = \mathbb{E}_{\mathbb{Q}_1}[Z|\mathcal{F}_T] = \mathbb{E}_{\mathbb{Q}_1}[Z|\mathcal{F}] = Z.$$

Change of numeraire : change of measure

We have the first following result :

Lemma

If Y is a random variable $\mathcal{F}_t$–measurable, then :

$$E_{\mathbb{Q}^2}[Y] = E_{\mathbb{Q}^1}[Z_t Y]$$

Indeed :

$$E_{\mathbb{Q}^2}[Y] = E_{\mathbb{Q}^1}[ZY] = E_{\mathbb{Q}^1}[E_{\mathbb{Q}^1}[ZY|\mathcal{F}_t]] = E_{\mathbb{Q}^1}[YE_{\mathbb{Q}^1}[Z|\mathcal{F}_t]] = E_{\mathbb{Q}^1}[YZ_t]$$

This lemma means in other words that :

$$\left.\frac{d\mathbb{Q}^2}{d\mathbb{Q}^1}\right|_{\mathcal{F}_t} = Z_t$$

Change of numeraire : change of measure

We will see that we can use the process Z_t to change measure in the **conditional expectations** :

Lemma

If Y is $\mathcal{F}_T$ measurable, then :

$$E_{\mathbb{Q}^2}[Y|\mathcal{F}_t] = E_{\mathbb{Q}^1}\left[Y\frac{Z_T}{Z_t}|\mathcal{F}_t\right] = \frac{1}{Z_t}E_{\mathbb{Q}^1}[YZ_T|\mathcal{F}_t] \qquad (*)$$

Proof :

By definition of the conditional expectation, $E_{\mathbb{Q}^2}[Y|\mathcal{F}_t]$ is the rando variable U –$\mathcal{F}_t$ measurable such that for any $A \in \mathcal{F}_t$, we have :

$$E_{\mathbb{Q}^2}[U\mathbb{I}_A] = E_{\mathbb{Q}^2}[Y\mathbb{I}_A] \qquad (**)$$

Let us show that $U = E_{\mathbb{Q}^1}[Y\frac{Z_T}{Z_t}|\mathcal{F}_t]$ well satisfies this condition.

Change of numeraire : change of measure

Let $A \in \mathcal{F}_t$. Then $E_{\mathbb{Q}^2}[U\mathbb{I}_A]$ is equal to :

$$E_{\mathbb{Q}^2}\left[\underbrace{\mathbb{I}_A}_{\in\mathcal{F}_t}\underbrace{E_{\mathbb{Q}^1}\left[Y\frac{Z_T}{Z_t}|\mathcal{F}_t\right]}_{\in\mathcal{F}_t}\right] = E_{\mathbb{Q}^1}\left[Z_t\mathbb{I}_A E_{\mathbb{Q}^1}\left[Y\frac{Z_T}{Z_t}|\mathcal{F}_t\right]\right]$$

by the preceding result. Since $1/Z_t$ is $\mathcal{F}_t$ measurable, we can put is out of the conditional expectation :

$$= E_{\mathbb{Q}^1}\left[Z_t\mathbb{I}_A\frac{1}{Z_t}E_{\mathbb{Q}^1}[YZ_T|\mathcal{F}_t]\right] = E_{\mathbb{Q}^1}[\mathbb{I}_A E_{\mathbb{Q}^1}[YZ_T|\mathcal{F}_t]]$$

By definition of the conditional expectation $E_{\mathbb{Q}^1}[YZ_T|\mathcal{F}_t]$, this is equal to :

$$E_{\mathbb{Q}^1}[YZ_T\mathbb{I}_A] = E_{\mathbb{Q}^2}[Y\mathbb{I}_A]$$

once again by applying the first lemma with now $t = T$, which is well the right-and side of (**). $\square$

Change of numeraire and martingale measure

When we have a numeraire $N(t)$, it is possible to associate to it a measure $\mathbb{Q}^N \sim \mathbb{Q}$ (and hence $\mathbb{Q}^N \sim \mathbb{P}$) such that the prices of financial products expressed with respect to that new numeraire are martingales under $\mathbb{Q}^N$:

$$\mathbb{E}_{\mathbb{Q}^N}\left[\frac{S_T}{N_T}|\mathcal{F}_t\right] = \frac{S_t}{N_t}$$

Indeed, it suffices to define $\mathbb{Q}^N$ from $\mathbb{Q}$ as follows :

$$\frac{d\mathbb{Q}^N}{d\mathbb{Q}} = Z = \frac{M(0)N(T)}{N(0)M(T)}$$

(where M is the numeraire associated to $\mathbb{Q}$, $M(t) = \beta_t$ to fix ideas...).

The new measure $\mathbb{Q}^N$ has the property that **prices of assets expressed w.r.t. to numeraire $N(t)$ are martingales under $\mathbb{Q}^N$**.

Change of numeraire and martingale measure

To show that, it suffices to use the second lemma :

1) Let us first search what is the form of Z_t, the associated R-N process :

$$Z_t = \mathbb{E}_{\mathbb{Q}}\left[\frac{M(0)N(T)}{N(0)M(T)}|\mathcal{F}_t\right] = \frac{M(0)N(t)}{N(0)M(t)}$$

since $N(t)/M(t)$ is a $\mathbb{Q}$-martingale.

Change of numeraire and martingale measure

2) Let us now use the second lemma (change of measure in conditional expectations) :

$$\mathbb{E}_{\mathbb{Q}^N}\left[\frac{S(T)}{N(T)}|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{S(T)}{N(T)}\frac{Z_T}{Z_t}|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{S(T)}{N(T)}\frac{\frac{M(0)N(T)}{N(0)M(T)}}{\frac{M(0)N(t)}{N(0)M(t)}}|\mathcal{F}_t\right]$$

$$= \mathbb{E}_{\mathbb{Q}}\left[\frac{S(T)}{M(T)}\frac{M(t)}{N(t)}|\mathcal{F}_t\right] = \frac{M(t)}{N(t)}\mathbb{E}_{\mathbb{Q}}\left[\frac{S(T)}{M(T)}|\mathcal{F}_t\right]$$

$$= \frac{M(t)}{N(t)}\frac{S(t)}{M(t)} = \frac{S(t)}{N(t)}$$

and hence $\frac{S(t)}{N(t)}$ is a martingale under $\mathbb{Q}^N$. $\square$

Change of numeraire and martingale measure

What we have seen can be summarized in the following result :

Theorem (Change of numeraire)

Let $\mathbb{Q}$ be a martingale measure associated to numeraire $M(t)$, and let $N(t)$ be another numeraire. Then there exists a measure $\mathbb{Q}^N \sim \mathbb{Q}$ such that prices of assets expressed in that numeraire, $S(t)/N(t)$, are martingales under $\mathbb{Q}^N$. This measure can be obtained from $\mathbb{Q}$ as :

$$\frac{d\mathbb{Q}^N}{d\mathbb{Q}} = \frac{M(0)N(T)}{N(0)M(T)}$$

and for all random variable Y–$\mathcal{F}_T$ measurable,

$$\mathbb{E}_{\mathbb{Q}^N}[Y|\mathcal{F}_t] = \frac{M(t)}{N(t)}\mathbb{E}_{\mathbb{Q}}\left[Y\frac{N(T)}{M(T)}|\mathcal{F}_t\right]$$

Think about the case $\mathbb{Q}$ = sirk neutral measure, associated to numeraire $\beta_t = e^{rt}$ of the saving account.

Price of the exchange option by change of numeraire technique

[1] :

$$E_{\mathbb{Q}}\left[e^{-r(T-t)}S_2\left(T\right)\left(\frac{S_1\left(T\right)}{S_2\left(T\right)}-1\right)_+|\mathcal{F}_t\right]$$

This can be re-written as :

$$M(t)\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{S_1\left(T\right)}{S_2\left(T\right)}-1\right)_+\frac{N(T)}{M(T)}|\mathcal{F}_t\right]$$

if we take $M(t)=\beta_t$ the saving account , and $N(t)=S_2(t)$.

We will switch to martingale measure $\mathbb{Q}_2$ associated to numeraire $N(t)=S_2(t)$.

1. cf. Girsanov bi-dimensional

Price of the exchange option by change of numeraire technique

Indeed, the price of the option can be re-written by using this new measure thanks to the thm seen previously :

$$M(t)\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{S_1(T)}{S_2(T)}-1\right)_+\frac{N(T)}{M(T)}|\mathcal{F}_t\right]=N(t)\mathbb{E}_{\mathbb{Q}_2}\left[\left(\frac{S_1(T)}{S_2(T)}-1\right)_+|\mathcal{F}_t\right]$$

To compute this last expectation, we still miss :

- the RN derivative $\frac{d\mathbb{Q}_2}{d\mathbb{Q}}$, in order to identify $\mathbb{Q}_2$
- the dynamics of $Y(t)=\frac{S_1(t)}{S_2(t)}$ under $\mathbb{Q}_2$

Price of the exchange option by change of numeraire technique

The R-N derivative :

$$\frac{d\mathbb{Q}_2}{d\mathbb{Q}} = Z = Z_T = \frac{S_2(T)\beta_0}{S_2(0)\beta_T} = \frac{S_2(T)e^{-rT}}{S_2(0)} = e^{-\frac{\sigma_2^2}{2}T+\sigma_2 W_2(T)}$$

where $W_2(t)$ is a standard B.M. under $\mathbb{Q}$.

If we define $Y(t) = \frac{S_1(t)}{S_2(t)}$, then by applying the Itô lemma to $Y(t) = f(t, S_1(t), S_2(t))$, we get :

$$dY(t) = \frac{dS_1}{S_2} - \frac{S_1}{S_2^2}dS_2 + \frac{1}{2}S_1\frac{2S_2^2}{S_2^3}\sigma_2^2 dt$$

$$= ... = Y(t)[\sigma_2^2 dt + \sigma_1 dW_1(t) - \sigma_2 dW_2(t)]$$

$$= Y(t)[\sigma_1 dW_1(t) - \sigma_2 d(W_2(t) - \sigma_2 t)]$$

Price of the exchange option by change of numeraire technique

By Girsanov thm applied to $Z_T = e^{-\sigma_2^2 T/2 + \sigma_2 W_T}$, we know that

$$W_2^*(t) = W_2(t) - \sigma_2 t$$

is a standard B.M. under $\mathbb{Q}_2$. Hence under this new measure, the dynamics of $Y(t)$ can be written :

$$dY(t) = Y(t)[\sigma_1 dW_1(t) - \sigma_2 dW_2^*(t)]$$

Moreover, we can see that $W_1(t)$ is still a standard B.M. under $\mathbb{Q}_2$, independent from $W_2(t)$ [2].

We recover in particular the fact that Y is a martingale under $\mathbb{Q}_2$ (...).

2. cf. multidimensional version of Girsanov thm

Price of the exchange option by change of numeraire technique

On the other hand :

$$d(lnY) = \frac{1}{Y}dY - \frac{1}{2}\frac{1}{Y^2}dY.dY = \sigma_1 dW_1(t) - \sigma_2 dW_2^*(t) - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)dt$$

and hence :

$$\frac{Y(T)}{Y(t)} = e^{\sigma_1(W_1(T) - W_1(t)) - \sigma_2(W_2^*(T) - W_2^*(t)) - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)(T-t)}$$

where W_1, W_2^* are two independent standard M.B. under $\mathbb{Q}_2$, which implies :

$$Y(T)/Y(t) \sim LN(a, b)$$

with

$$a = -\frac{1}{2}(\sigma_1^2 + \sigma_2^2)(T-t), \quad b = \sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}$$

Price of the exchange option by change of numeraire technique

In consequence, the rpice of the exchange option appears as :

$$S_2(t)\mathbb{E}_{\mathbb{Q}_2}[(Y(T)-1)^+|\mathcal{F}_t]$$

with $\frac{Y(T)}{Y(t)} \sim LN(a,b)$ for a and b like in the Black-Scholes model with σ replaced by $\sigma = \sqrt{(\sigma_1^2+\sigma_2^2)}$.

We are hence exactly in the case of the Black-Scholes formula with :

$$r=0, K=1, \sigma = \sqrt{(\sigma_1^2+\sigma_2^2)}$$

and we get the Margrabe formula.

Change of numeraire : remark

We can re-use the general results on measure and numeraire change in the case of interest rates models. An equivalent martingale measure that plays an important role will be the forward measure :

In the case of options on zero-coupons (or on options on coupon bonds or on swaps), we will often make the change of numeraire associated to a particular zero-coupon. This is what we will call the*forward measure.*

If we take as numeraire the fixed leg of a swap, we will then talk about forward-swap measure.

FX options - Garman-Kohlagen formula

Assumptions :

- We suppose that there are 2 currencies : the domestic currency and a foreign currency
- in each currency, we have a risk free rate :
  - r_d : domestic risk free rate
  - r_f : foreign risk free rate
- We suppose that the market is perfect
- The exchange rate between the foreign and the domestic currencies follows a geometric Brownian motion :

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

where $S(t)$ = value at instant t of one foreign currency unit expressed in the domestic currency.

FX option - Garman-Kohlagen formula

We are interested in the price of a European call option of strike K on that foreign currency :

K represents in practice the maximum exchange rate at which we would accept to buy foreign currencies at a fixed future date. If we want to hedge ourselves against situations where the FX rate becomes superior to K, then we buy such a call.

$$Payoff = (S(T) - K)^+$$

FX option - Garman-Kohlagen formula

LEt us consider the following strategy :

- At t, we borrow an amount $S(t)$ at the risk-free rate r_d in the domestic market, and we buy one foreign currency unit
- At t, we invest this currency unit at the risk free rate r_f of the foreign market
- At T, the risk free investment in the foreign market is become equal to $e^{r_f(T-t)}$ foreign currency units.
- We then take that money from the foreign bank account and we exchange it against domestic currencies, at the FX rate that prevails at $T : S(T)$. We hence receive an amount $S(T)e^{r_f(T-t)}$.
- At T, we also redeem the borrowing at the risk free rate : we need to pay $S(t)e^{r_d(T-t)}$ domestic currency units.

FX option - Garman-Kohlagen formula

This strategy does not require any money at the beginning, so it is a "zero-cost" strategy, and its payoff at T is :

$$S(T)e^{r_f(T-t)} - S(t)e^{r_d(T-t)}$$

Hence the price of a financial product delivering the same final payoff must also be zero if we assume that the market is arbitrage free. So if $\mathbb{Q}$ is a (the) risk-neutral measure of the domestic market, necessarily :

$$e^{-r_d(T-t)}E_{\mathbb{Q}}[S(T)e^{r_f(T-t)} - S(t)e^{r_d(T-t)}|\mathcal{F}_t] = 0$$

$$\Leftrightarrow E_{\mathbb{Q}}[S(T)e^{(r_f-r_d)(T-t)}|\mathcal{F}_t] = S_t$$

$$\Leftrightarrow E_{\mathbb{Q}}[S(T)e^{-(r_d-r_f)T}|\mathcal{F}_t] = S_t e^{-(r_d-r_f)t}$$

i.e. $S(t)e^{-(r_d-r_f)t}$ is a martingale under $\mathbb{Q}$.

FX option - Garman-Kohlagen formula

] Since $\mathbb{Q}$ is equivalent to $\mathbb{P}$, we know by Girsanov thm that the dynamics of $S(t)$ under $\mathbb{Q}$ is identical to that under $\mathbb{P}$ except that the drift can be different. The martingale condition above implies necessarily under $\mathbb{Q}$,

$$dS(t) = (r_d - r_f)S(t)dt + \sigma S(t)dW(t)$$

i.e. the same equation as in the Black-Scholes model with dividends, in which the dividend rate is in fact the risk free rate in the foreign currency r_f.

Once again, we can see that the martingale measure is unique since the market is complete (...)

FX option - Garman-Kohlagen formula

If we consider a European call on the exchange rate whose payoff at T is $(S(T)-K)^+$, then its price is given by

$$C = E_{\mathbb{Q}}[e^{-r_d(T-t)}(S(T)-K)^+|\mathcal{F}_t] = e^{-r_f(T-t)}E_{\mathbb{Q}}[e^{-(r_d-r_f)(T-t)}(S(T)-K)^+|\mathcal{F}_t]$$

By applying the Black-Scholes formula in which we replace r by $r_d - r_f$, and by multiplying the whole by $e^{-r_f(T-t)}$, we get :

$$C = e^{-r_f(T-t)}S(t)\Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2)$$

$$d_1 = \frac{ln(S_t/K) + (r_d - r_f + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

(Garman-Kohlagen formula, 1983)