Project Two

November 6, 2020

1 Requirements

Each group is required to submit a final report together with source code for your project, stating in details the derivation of pricing method (if not discussed in class), the choice of numerical algorithm, test results and analysis of results.

You can choose one of the topics for your project: Pricing Spread Option or Implied Binomial Tree and Barrier Option

Deadline: Nov 30, 2020

2 Pricing Spread Option

Let $S_1(t)$ and $S_2(t)$ be the prices of two stocks such that

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sigma_i dW_i(t), \qquad i = 1, 2 \tag{1}$$

and

$$dW_1(t)dW_2(t) = \rho dt \tag{2}$$

where $W_1(t)$ and $W_2(t)$ are two Brownian motions with instantaneous correlation ρ under the risk neutral measure $\mathbb{P}$. Assume that σ_1, σ_2 and ρ are constant. A spread call option with expiry T and strike K has the payoff

$$\max\{S_1(T) - S_2(T) - K, 0\} \tag{3}$$

Spread option is often priced using Kirk's approximation (see Kirk (1995)) for small K

$$C(S_1, S_2, T) = S_1\mathcal{N}(d_1) - (S_2 + Ke^{-rT})\mathcal{N}(d_2) \tag{4}$$

where $\mathcal{N}$ is the distribution function of a standard normal distribution

$$d_{1,2} = \frac{\ln\left(S_1/(S_2 + Ke^{-rT})\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \tag{5}$$

and

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2\left(\frac{S_2}{S_2 + Ke^{-rT}}\right)^2 - 2\rho\sigma_1\sigma_2\left(\frac{S_2}{S_2 + Ke^{-rT}}\right)} \tag{6}$$

In this project, you shall

- Derive Kirk's approximation formula.

  Hint: Let $Y(t) = S_2(t) + Ke^{-r(T-t)}$. For $K << S_2$, assume $Y(t)$ also follows log-normal. Derive the SDE for $Y(t)$ and then apply the formula for exchange option we derived in class.

- Compare Kirk's approximation with Monte Carlo simulation for different scenarios of stock prices, volatilities and correlations. For example, you can investigate the price difference between Kirk's approximation and MC simulation for the following scenarios

  1. Price: $30, 50, 70, 90, 100, 110, 130, 150, 200$
  2. Volatility: $10\%, 20\%, 50\%, 70\%, 90\%$
  3. Correlation: $-0.9, -0.5, 0.2, 0, 0.2, 0.5, 0.9$
  4. Strike: $10\%, 50\%, 100\%, 200\%$ of stock price,
  5. Combinations of different price, volatility and correlation levels.

  Note that these values are just for your references. You may choose any levels and combinations you would like to.

  Comment on the accuracy of Kirk's approximation based on the results you have obtained.

3 Implied Binomial Tree and Barrier Option

In this project, you will

1. implement an implied binomial tree model; and
2. **(optional)** use it to price a Barrier option.

For your reference, I outline the steps for building the model and pricing barrier options below.

1. Gather Market Data

   Choose a date for which you want to run your model and gather market data for this date. Two types of market data are needed: a) prices for the underlying and b) prices (or implied volatilities) for options.

   To build the model, you will need an implied volatility surface $\Sigma(T, K)$. In the market, for a fixed expiry T, there are usually only a finite number of options available for different strikes. That is, we only have values for $\Sigma(T, K_i)$ for $1 \le i \le n$. We need to interpolate and extrapolate the volatility curve $\Sigma(T, \cdot)$ for all strikes. A popular choice is to use cubic spline $\Sigma(T, K)$ as a function of K.

   For the time dimension, we are faced with similar issue. There are usually only have a limited number of expiries $T_1, \ldots, T_m$. Hence we need interpolation in time dimension to get implied volatility for any expiry T. One possible choice is to interpolate in variance. For example, $T_1 < T < T_2$ and we know implied volatility σ_1 and σ_2 for expiries T_1 and T_2 respectively. Since the variance for T_1 and T_2 are $\sigma_1^2 T_1$ and $\sigma_2^2 T_2$, using linear interpolation in variance, we have

$$\sigma^2 T = \frac{T_2 - T}{T_2 - T_1}\sigma_1^2 T_1 + \frac{T - T_1}{T_2 - T_1}\sigma_2^2 T_2 \tag{7}$$

   from which we can solve σ.

   In some markets, implied volatility surface are more commonly quoted in terms of expiry and delta, i.e., $\Sigma(\Delta, T)$. In this case, the strike dimension interpolation (extrapolation) can be carried out in delta.

2. Build Implied Binomial Tree

   With market prices for underlyings and an implied volatility surface, we can obtain prices for all European options. We can the follow the algorithm we discussed in class to build an implied binomial tree.

3. Price Barrier Option

   Once the implied binomial tree has been built, pricing barrier options becomes straightforward. The pricing algorithm is the same as a normal binomial tree.