Multipole Fields in the Vacuum Gauge

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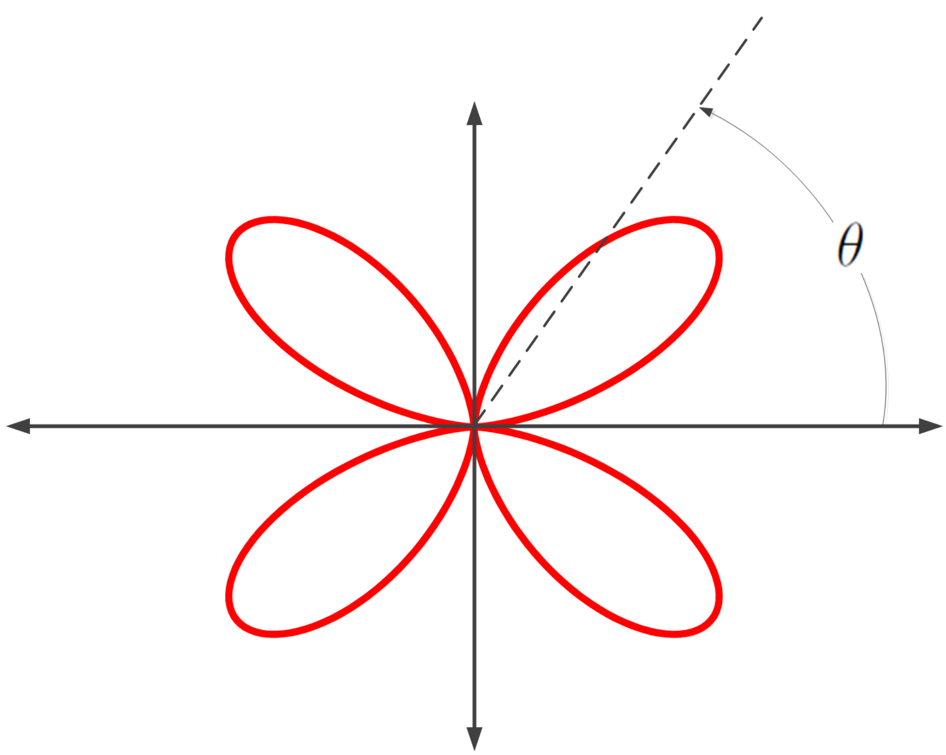

Abstract

The vacuum gauge requires velocity and acceleration fields of the electron to be considered as independent phenomena—and this remains valid for a description of multipole fields. For the velocity theory, all electric and magnetic multipoles can be written as linear combinations of vector multipole potentials. For the acceleration fields, the proportionality between the fields and potentials is equally beneficial allowing for simple calculations associated with multipole radiation.

Vacuum
Gauge
Electrodynamics

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1 Multipole Velocity Fields

In the vacuum gauge, Taylor expansions of both the scalar and vector fields are required. Of particular interest are expansions for individual particles, which is only possible in the vacuum gauge. Nevertheless, classical limit expansions for arbitrary distributions of charges and currents is still possible by summing over large numbers of particles.

1.1 Multipole Expansion in the Vacuum Gauge

A charged particle is displaced from the origin of a coordinate system by an amount $\boldsymbol{r}_o$ having arbitrary rectangular coordinates

$$\boldsymbol{r}_o = r_o(\sin\theta'\cos\phi'\hat{\boldsymbol{x}} + \sin\theta'\sin\phi'\hat{\boldsymbol{y}} + \cos\theta'\hat{\boldsymbol{z}}) \tag{1.1}$$

A graphic showing the coordinate system along with vectors $\boldsymbol{r}$ and $\boldsymbol{r}_o$ is shown in figure 1. The angle γ between the vectors is determined by the trigonometric identity

$$\cos\gamma = \sin\theta'\sin\theta\cos(\phi - \phi') + \cos\theta'\cos\theta \tag{1.2}$$

Now assume $r_o \ll r$ and define the small quantity

$$\begin{aligned}\boldsymbol{\varepsilon} &\equiv \frac{r_o}{r}\left[\cos\gamma\,\hat{\boldsymbol{r}} + \frac{\partial}{\partial\theta}\cos\gamma\,\hat{\boldsymbol{\theta}} + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\cos\gamma\,\hat{\boldsymbol{\phi}}\right] \\ &= \frac{r_o}{r}\left[\cos\gamma\,\hat{\boldsymbol{r}} + r\boldsymbol{\nabla}\cos\gamma\right]\end{aligned} \tag{1.3}$$

The well known scalar potential for this particle can be written as a sum over Legendre polynomials in $\cos\gamma$:

$$V(r,\theta,\phi) = e\sum_{l=0}^{\infty}\frac{r_o^l}{r^{l+1}}\mathcal{P}_l(\cos\gamma) \tag{1.4}$$

Equations for the first three terms in the series are:

$$V_0 = \frac{e}{r} \qquad V_1 = \frac{er_o}{r^2}\cos\gamma \qquad V_2 = \frac{er_o^2}{2r^3}[3\cos^2\gamma - 1] \tag{1.5}$$

Unfortunately, the expansion of the potentials in the vacuum gauge is not yet complete, and must include the vector potential also. The vector potential proves to be a more difficult expansion since the general displacement includes two instances of the small variable $\boldsymbol{\varepsilon}$:

$$\boldsymbol{A} = \frac{e}{r}\left[\frac{\hat{\boldsymbol{r}} - \boldsymbol{\varepsilon}}{1 - 2\hat{\boldsymbol{r}}\cdot\boldsymbol{\varepsilon} + \varepsilon^2}\right] \tag{1.6}$$

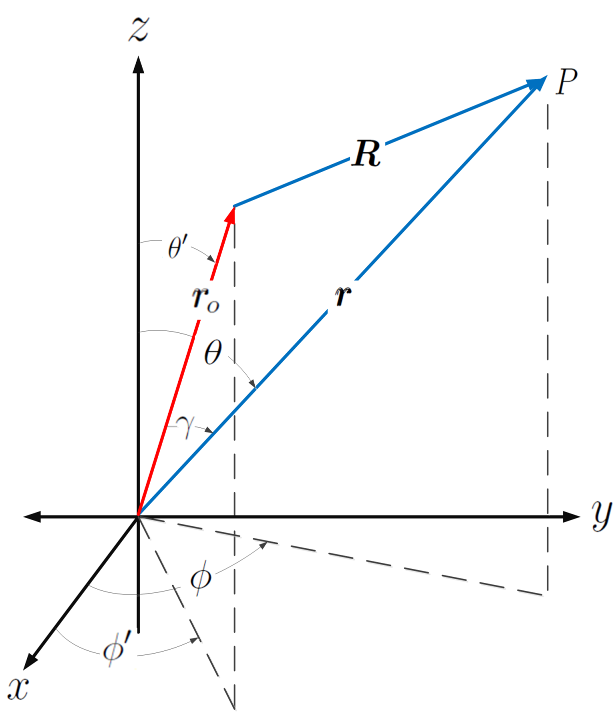

Figure 1: *Plot showing variables appropriate for a classical electron displaced from the origin of coordinates.*

A summed power series for $\boldsymbol{A}$ is overly complicated and it is easier to simply quote individual contributions to the sum. For our purposes the first three orders in the sum can be determined from

$$\boldsymbol{A} = \frac{e}{r}[\hat{\boldsymbol{r}} - \boldsymbol{\varepsilon}] \cdot \left[1 + 2\cos\gamma \cdot \varepsilon + (4\cos^2\gamma - 1) \cdot \varepsilon^2 + \ldots\right] \tag{1.7a}$$

with

$$\boldsymbol{A}_0 = \frac{e}{r}\hat{\boldsymbol{r}} \tag{1.7b}$$

$$\boldsymbol{A}_1 = \frac{2e}{r}(\cos\gamma)\varepsilon\,\hat{\boldsymbol{r}} - \frac{e}{r}\boldsymbol{\varepsilon} \tag{1.7c}$$

$$\boldsymbol{A}_2 = \frac{e}{r}(4\cos^2\gamma - 1)\varepsilon^2\,\hat{\boldsymbol{r}} - \frac{2e}{r}(\cos\gamma)\varepsilon\,\boldsymbol{\varepsilon} \tag{1.7d}$$

For a stationary charge there are no magnetic fields. This means each multipole vector potential must obey $\boldsymbol{\nabla} \times \boldsymbol{A}_n = 0$ as can be easily verified.

The importance of the vector potential arises when considering expansions of the electric field vector. One finds

$$\mathbf{E}_m = \frac{1}{r}\sum_{n=0}^{m}\mathcal{P}_{m-n}(\cos\gamma)\varepsilon^{m-n}\,\boldsymbol{A}_n \tag{1.8}$$

Now observe that a sum of equation (1.8) over all multipoles can be written

$$\boxed{e\mathbf{E} = \sum_{m,n=0}^{\infty} V_m \boldsymbol{A}_n} \tag{1.9}$$

According to this formula, the electric field is purely a function of vacuum gauge potentials, and each multipole l can be read by including all terms such that $l = m + n$. The three lowest order contributions are

$$e\mathbf{E}_0 = V_0\boldsymbol{A}_0 \tag{1.10a}$$

$$e\mathbf{E}_1 = V_1\boldsymbol{A}_0 + V_0\boldsymbol{A}_1 \tag{1.10b}$$

$$e\mathbf{E}_2 = V_2\boldsymbol{A}_0 + V_1\boldsymbol{A}_1 + V_0\boldsymbol{A}_2 \tag{1.10c}$$

Based on these results, a coordinate free form for the electric dipole field can be derived by inserting previously calculated forms of vector and scalar potentials. First write

$$e^{-1}[V_1\boldsymbol{A}_0 + V_0\boldsymbol{A}_1] = \frac{e}{r^2}\left[3\cos\gamma\,\varepsilon\,\hat{\boldsymbol{r}} - \boldsymbol{\varepsilon}\right] \tag{1.11}$$

But $\boldsymbol{\varepsilon}$ can be associated with the electric dipole moment by writing $\boldsymbol{p} = er\boldsymbol{\varepsilon}$ and this leads immediately to the familiar formula

$$\mathbf{E}_1 = \frac{1}{r^3}\left[3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}\right] \tag{1.12}$$

Now consider re-capturing $\mathbf{E}$ by summing the series:

$$\mathbf{E} = \sum_{m=0}^{\infty}\mathbf{E}_m = \frac{1}{r}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\mathcal{P}_{m-n}(\cos\gamma)\varepsilon^{m-n}\,\boldsymbol{A}_n \tag{1.13}$$

Inserting the series for $\boldsymbol{A}_m$ from (1.7) and using the following two relations,

$$(l+1)\mathcal{P}_l(\cos\gamma) = \sum_{n=0}^{l}\mathcal{P}_{l-n}(\cos\gamma)\cos n\gamma \tag{1.14}$$

$$\boldsymbol{\nabla}\mathcal{P}_l(\cos\gamma) = \sum_{n=1}^{l}\frac{1}{n}\mathcal{P}_{l-n}(\cos\gamma)\boldsymbol{\nabla}\cos n\gamma \tag{1.15}$$

derives the formula

$$\mathbf{E} = e \sum_{l=0}^{\infty} \frac{r_o^l}{r^{l+2}} \left[(l+1)\mathcal{P}_l(\cos\gamma)\hat{\boldsymbol{r}} - r\boldsymbol{\nabla}\mathcal{P}_l(\cos\gamma) \right] \tag{1.16}$$

Of course this equation can also be derived directly from equation (1.4) from a gradient operation.

1.2 Multipole Expansion along the Z-Axis:

A simplification occurs in the expansion of the vector potential when considering displacement from the origin along the z-axis. Equation (1.1) simplifies to $\boldsymbol{r}_o = z_o\hat{\boldsymbol{z}}$ while the value of $\boldsymbol{\varepsilon}$ in spherical-polar coordinates reads

$$\boldsymbol{\varepsilon} = \frac{z_o}{r}(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \tag{1.17}$$

In the cartesian coordinate system the multipole potentials are determined from

$$\boldsymbol{A}(\boldsymbol{r}) = \sum_{n=1}^{\infty} \frac{e z_o^{n-1}}{r^n} \left[\sin n\theta \cos\phi,\ \sin n\theta \sin\phi,\ \cos n\theta \right] \tag{1.18}$$

but an expansion in curvi-linear coordinates gives pure multi-angle contributions along each of two components:

$$\boldsymbol{A}(r,\theta) = \frac{e}{r} \sum_{n=0}^{\infty} \left(\frac{z_o}{r} \right)^n \left[\cos n\theta\,\hat{\boldsymbol{r}} + \sin n\theta\,\hat{\boldsymbol{\theta}} \right] \tag{1.19}$$

Each component of the vector potential is also a Fourier series in the polar angle. Fourier coefficients along each direction are identical but follow from separate integrals,

$$\varepsilon^n = \frac{2}{\pi} \int_0^{\pi} g_\varepsilon(\theta) \cos n\theta\, d\theta = \frac{2}{\pi} \int_0^{\pi} h_\varepsilon(\theta) \sin n\theta\, d\theta \tag{1.20}$$

where the two functions $g_\varepsilon(\theta)$ and $h_\varepsilon(\theta)$ are given by

$$g_\varepsilon(\theta) = \frac{1 - \varepsilon\cos\theta}{1 - 2\varepsilon\cos\theta + \varepsilon^2} \qquad\qquad h_\varepsilon(\theta) = \frac{\varepsilon\sin\theta}{1 - 2\varepsilon\cos\theta + \varepsilon^2} \tag{1.21}$$

Based on the construction of equation (1.19), it may also be reasonable to consider individual multipole potentials in term of the complex potential $\widetilde{A}$ defined by

$$\widetilde{A} \equiv \frac{e}{r} \sum_{n=0}^{\infty} \left(\frac{z_o}{r} e^{i\theta} \right)^n \tag{1.22}$$

Components of the vector potential are then determined from the real and imaginary parts of $\widetilde{A}$.

The divergence of $\boldsymbol{A}$ measures vacuum dilatation term by term:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{e}{r^2} \sum_{n=0}^{\infty} \left[\frac{z_o}{r} \right]^n \left[\frac{\sin(n+1)\theta}{\sin\theta} \right] \tag{1.23}$$

According to this formula the total dilatation receives contributions from multipoles of all orders. The dilatation from each multipole is a function of angle and can be either positive or negative. However, the total dilatation is positive and this can be shown by replacing $\sin(n+1)\theta$ with complex exponentials and summing the series to determine

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{e}{r^2(1 - 2\varepsilon\cos\theta + \varepsilon^2)} \tag{1.24}$$

Calculation of the quantity $\hat{\boldsymbol{r}} \times \boldsymbol{A}$ from equation (1.19) requires the nth term in the expansion to take the form

$$\hat{\boldsymbol{r}} \times \boldsymbol{A}_n = \left[\frac{e z_o^n}{r^{n+1}} \right] \sin n\theta \, \hat{\boldsymbol{\phi}} \tag{1.25}$$

While the $n = 0$ term vanishes, each of the remaining terms represents a Lorentz gauge vector potential satisfying $\boldsymbol{\nabla} \cdot \boldsymbol{A}_L = 0$, and capable of determining multipole magnetic fields. For example, the $n = 1$ term takes the form

$$\boldsymbol{A}_L \longrightarrow \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{r^2} \tag{1.26}$$

and generates a magnetic dipole field with a curl operation.

1.3 Ring of Charge

The formalism developed here can be applied in a many particle theory. Begin with an electron located in the xy-plane with position vector

$$\boldsymbol{r}_o = r_o(\cos\phi' \, \hat{\boldsymbol{x}} + \sin\phi' \, \hat{\boldsymbol{y}}) \tag{1.27}$$

In a spherical coordinate system this vector is

$$\boldsymbol{r}_o = r_o \left[\sin\theta\cos(\phi - \phi') \, \hat{\boldsymbol{r}} + \cos\theta\cos(\phi - \phi') \, \hat{\boldsymbol{\theta}} - \sin(\phi - \phi') \, \hat{\boldsymbol{\phi}} \right] \tag{1.28}$$

and the quantity $\cos\gamma$ simplifies to

$$\cos\gamma = \sin\theta\cos(\phi - \phi') \tag{1.29}$$

The lowest order contributions to the vector potential from equation (1.7) are:

$$\boldsymbol{A}_o = \frac{e}{r}\hat{\boldsymbol{r}} \tag{1.30a}$$

$$\boldsymbol{A}_1 = \frac{er_o}{r^2}\left[\sin\theta\cos(\phi-\phi')\,\hat{\boldsymbol{r}} - \cos\theta\cos(\phi-\phi')\,\hat{\boldsymbol{\theta}} + \sin(\phi-\phi')\,\hat{\boldsymbol{\phi}}\right] \tag{1.30b}$$

$$\begin{aligned}\boldsymbol{A}_2 = \frac{2er_o^2}{r^3}[(\sin^2\theta\cos^2(\phi-\phi')-1)\,\hat{\boldsymbol{r}} - \sin\theta\cos\theta\cos^2(\phi-\phi')\,\hat{\boldsymbol{\theta}}\\ + \sin\theta\cos(\phi-\phi')\sin(\phi-\phi')\,\hat{\boldsymbol{\phi}}]\end{aligned} \tag{1.30c}$$

As in the previous section, these potentials can be used via equations (1.10) to generate monopole, dipole and quadrupole fields for a particle in the x-y plane, but this is not the goal here.

Instead, we will use the vector potentials of (1.30) for the case where there are a large number N of electrons at radius r_o evenly distributed about the origin. The position of each charge may be a random variable but for the sake of simplicity it will be assumed that the n^{th} particle is located at angular coordinate $\phi' = 2\pi n/N$ as indicated by the graphic in figure 2.

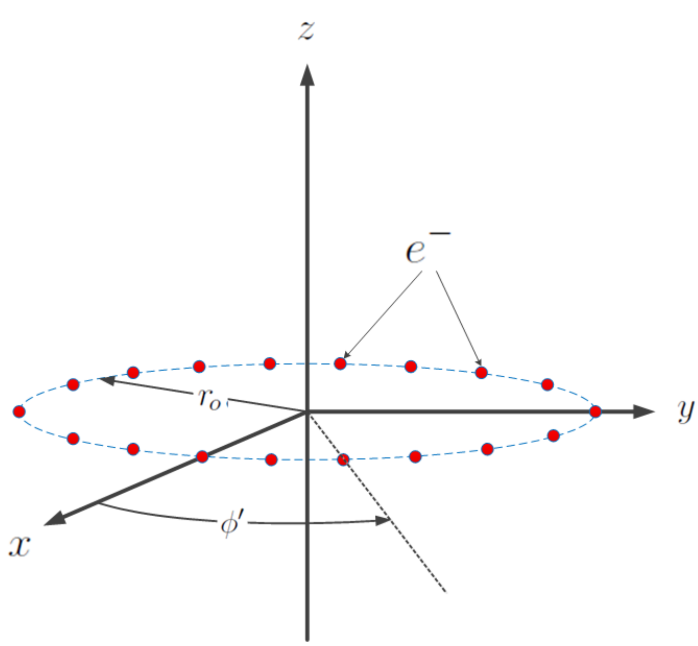

Figure 2: *Electrons distributed in a circle of radius r_o in the xy-plane.*

Now the fields of equation (1.10) can be determined by summing over individual particles. With

$$\phi_n \equiv \phi - \frac{2\pi n}{N} \tag{1.31}$$

the following sums are useful and simple to evaluate:

$$\sum_{n=1}^{N} \cos\phi_n = \sum_{n=1}^{N} \sin\phi_n = 0 \qquad \sum_{n=1}^{N} \cos^2\phi_n = \frac{N}{2} \tag{1.32}$$

The monopole field calculates to $\mathbf{E}_0 = [Ne/r^2]\hat{\boldsymbol{r}}$ while the dipole field vanishes. The quadrupole field is non-vanishing with individual portions given by

$$\frac{1}{e}\sum_{n=1}^{N} V_2 \boldsymbol{A}_0 = -\frac{Ner_o^2}{2r^4}\left[\frac{1}{2}(3\cos^2\theta - 1)\,\hat{\boldsymbol{r}}\right] \tag{1.33a}$$

$$\frac{1}{e}\sum_{n=1}^{N} V_1 \boldsymbol{A}_1 = +\frac{Ner_o^2}{2r^4}\left[\sin^2\theta\,\hat{\boldsymbol{r}} - \sin\theta\cos\theta\,\hat{\boldsymbol{\theta}}\right] \tag{1.33b}$$

$$\frac{1}{e}\sum_{n=1}^{N} V_0 \boldsymbol{A}_2 = -\frac{Ner_o^2}{2r^4}\left[2\cos^2\theta\,\hat{\boldsymbol{r}} + 2\sin\theta\cos\theta\,\hat{\boldsymbol{\theta}}\right] \tag{1.33c}$$

Including the monopole field and defining the total charge $Q = Ne$, an approximation for electric field vector of the ring of charge can be written

$$\mathbf{E}_0 + \mathbf{E}_2 = \frac{Q}{r^2}\,\hat{\boldsymbol{r}} - \frac{3Qr_o^2}{2r^4}\left[\frac{1}{2}(3\cos^2\theta - 1)\,\hat{\boldsymbol{r}} + \sin\theta\cos\theta\,\hat{\boldsymbol{\theta}}\right] \tag{1.34}$$

Of course this result is also available by calculating $-\boldsymbol{\nabla}V$ from the scalar potential

$$V(r,\theta) = \frac{Q}{r}\left[1 - \frac{r_o^2}{2r^2}\mathcal{P}_2(\cos\theta)\right] \tag{1.35}$$

but this calculation lacks the physical insights offered by the vacuum gauge.

Magnetic Dipole Field: The previous calculation can be extended further by setting individual electrons in motion around the ring. This is easy to do by introducing a time dependence in equation (1.27) for each of the N charges:

$$\boldsymbol{r}_n(t) = r_o\left[\cos\left(\omega t + \frac{2\pi n}{N}\right)\hat{\boldsymbol{x}} + \sin\left(\omega t + \frac{2\pi n}{N}\right)\hat{\boldsymbol{y}}\right] \tag{1.36}$$

If ϕ_n is re-defined by

$$\phi_n \longrightarrow \phi - \omega t - \frac{2\pi n}{N} \tag{1.37}$$

then values for the position and velocity of the n^{th} electron in the spherical-polar coordinate system can be written

$$\boldsymbol{r}_n(t) = r_o \left[\sin\theta\cos\phi_n\,\hat{\boldsymbol{r}} + \cos\theta\cos\phi_n\,\hat{\boldsymbol{\theta}} - \sin\phi_n\,\hat{\boldsymbol{\phi}}\right] \tag{1.38}$$

$$\boldsymbol{\beta}_n(t) = \frac{\omega r_o}{c}\left[\sin\theta\sin\phi_n\,\hat{\boldsymbol{r}} + \cos\theta\sin\phi_n\hat{\boldsymbol{\theta}} + \cos\phi_n\hat{\boldsymbol{\phi}}\right] \tag{1.39}$$

It has already been shown that the dipole electric field vector vanishes for the ring of charge and this holds true even if the charges are in motion. However, the dipole electric field from each charge still produces a contribution to a magnetic dipole field which can be calculated from

$$\mathbf{B}_1 = \sum_{n=1}^{N}\boldsymbol{\beta}_n \times \mathbf{E}_1 = \frac{1}{e}\sum_{n=1}^{N}\left[V_1\,\boldsymbol{\beta}_n \times \boldsymbol{A}_0 + V_0\,\boldsymbol{\beta}_n \times \boldsymbol{A}_1\right] \tag{1.40}$$

Performing the summations leads to

$$\mathbf{B}_1 = \left[\frac{Ne\omega r_o^2}{2r^3c}\left(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)\right] \tag{1.41}$$

Now define the magnitude of the current in the loop by $I = Ne\nu$ and the z-directed magnetic dipole moment

$$\boldsymbol{m} = I\pi r_o^2\hat{\boldsymbol{z}} \tag{1.42}$$

In coordinate free form the dipole magnetic flux field can then be written

$$\mathbf{B}_1 = \frac{1}{r^3c}\left[3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \boldsymbol{m}\right] \tag{1.43}$$

It might be suggested that the results of this section qualify as a semi-classical approach to a determination of electric and magnetic fields since the calculations are performed by populating the ring (a crystal lattice of copper, maybe) with individual quanta of the electromagnetic field. A more complete approach might include statistics and the law of averages over a large value of N which—for a one foot long copper wire—would be $N \sim 10^{23}$. With or without statistics however, it is important to observe that no integrations or differential operations are required to determine the fields. Instead, the vacuum gauge potentials take care of everything as they represent smallest possible division of the classical electromagnetic field.

2 Multipole Acceleration Fields

In the vacuum gauge, the proportionality between fields and potentials can be extended to include particle accelerations by writing a covariant expression for the acceleration field strength tensor as

$$F_a^{\mu\nu} = [A^\mu, a_\perp^\nu] \tag{2.1}$$

This leads to electric and magnetic field vectors[1]

$$\boldsymbol{\mathcal{E}} = \boldsymbol{A} a_{\perp}^{o} - V \boldsymbol{a}_{\perp} \tag{2.2a}$$

$$\boldsymbol{\mathcal{B}} = \boldsymbol{a}_{\perp} \times \boldsymbol{A} \tag{2.2b}$$

and it's easy to verify $\boldsymbol{\mathcal{B}} = \hat{\boldsymbol{n}} \times \boldsymbol{\mathcal{E}}$.

A general formula for radiated power follows by making substitutions into the Poynting vector. With help from the orthogonality relation $a_{\perp}^{\nu} R_{\nu} = 0$, the vacuum gauge version of the Poynting vector (including factors of c) is

$$\mathbf{S}_{a} = \frac{c}{4\pi} \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}} = \frac{1}{4\pi c^{3}} \left[\boldsymbol{a}_{\perp} \cdot \boldsymbol{a}_{\perp} - a_{\perp}^{o} (\boldsymbol{a}_{\perp} \cdot \hat{\boldsymbol{n}}) \right] V \boldsymbol{A} \tag{2.3}$$

This equation can be integrated over a closed surface to determine the Liénard generalization of the Larmor power formula. Unfortunately, the integration is difficult since components of $a_{\perp}^{\mu}$ are complicated functions of the velocity and acceleration in the relativistic limit. To derive Larmor power first write the acceleration four-vector

$$a^{\nu} = [\gamma^{4} \boldsymbol{\beta} \cdot \boldsymbol{a},\, \gamma^{2} \boldsymbol{a} + \gamma^{4} (\boldsymbol{\beta} \cdot \boldsymbol{a}) \boldsymbol{\beta}] \tag{2.4}$$

The perpendicular component of the four-acceleration is

$$a_{\perp}^{\nu} = a^{\nu} - \frac{a^{\mu} R_{\mu}}{\rho} \beta^{\nu} \tag{2.5}$$

In the low velocity limit, approximations through first order in $\boldsymbol{\beta}$ are

$$a_{\perp}^{o} = \boldsymbol{a} \cdot \hat{\boldsymbol{n}} + (\boldsymbol{a} \cdot \hat{\boldsymbol{n}}) \boldsymbol{\beta} \cdot \hat{\boldsymbol{n}} \qquad \boldsymbol{a}_{\perp} = \boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{n}}) \boldsymbol{\beta} \tag{2.6}$$

Inserting only the lowest order of these expressions into equation (2.3) will verify the Larmor power formula from the closed integral

$$P_{larmor} = \frac{1}{4\pi c^{3}} \oint \left[\boldsymbol{a} \cdot \boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{n}})^{2} \right] V \boldsymbol{A} \cdot d\boldsymbol{s} = \frac{2e^{2}a^{2}}{3c^{3}} \tag{2.7}$$

2.1 Electric Dipole Radiation

The simplest application of the new formalism is the determination of acceleration fields for an electron undergoing oscillations on the z-axis. As in the previous section, write

$$\boldsymbol{z}(t_{r}) = z_{o} \cos \omega t_{r} \hat{\boldsymbol{z}} \tag{2.8}$$

Two time derivatives and ignoring the retardation condition in the cosine produce components of $a_{\perp}^{\nu}$. According to equation (2.6), $\boldsymbol{a} = \boldsymbol{a}_{\perp}$ in lowest order so

$$\boldsymbol{a} = -\omega^{2} z_{o} \cos \omega (t - r/c) \hat{\boldsymbol{z}} \tag{2.9a}$$

$$a_{\perp}^{o} = \omega^{2} z_{o} \cos \omega (t - r/c) \cos \theta \tag{2.9b}$$

[1] In this section, use the symbols $(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}})$ to represent electric and magnetic field vectors associated with acceleration fields. This provides a clear distinction with velocity fields of the previous section.

Now include the rest frame (lowest order) vacuum gauge potentials

$$A^\nu = \left(\frac{e}{r}, \frac{e}{r}\hat{\boldsymbol{r}}\right) \tag{2.10}$$

Combining this information into (2.2) leads to explicit expressions for the fields

$$\boldsymbol{\mathcal{E}}_{e1} = -\frac{e\omega^2 z_o}{rc^2} \sin\theta \cos\omega(t - r/c)\hat{\boldsymbol{\theta}} \tag{2.11a}$$

$$\boldsymbol{\mathcal{B}}_{e1} = -\frac{e\omega^2 z_o}{rc^2} \sin\theta \cos\omega(t - r/c)\hat{\boldsymbol{\phi}} \tag{2.11b}$$

Defining the dipole moment $\boldsymbol{p} = ez_o\hat{\boldsymbol{z}}$ and including a complex phase will then allow the fields to be written

$$\boldsymbol{\mathcal{E}}_{e1} = k^2(\hat{\boldsymbol{r}} \times \boldsymbol{p} \times \hat{\boldsymbol{r}})\frac{e^{-i\omega(t-r/c)}}{r} \tag{2.12a}$$

$$\boldsymbol{\mathcal{B}}_{e1} = k^2(\hat{\boldsymbol{r}} \times \boldsymbol{p})\frac{e^{-i\omega(t-r/c)}}{r} \tag{2.12b}$$

The total power radiated follows from the Poynting vector. In terms of vacuum gauge potentials this is

$$\langle\, \mathbf{S}\, \rangle = \frac{c}{8\pi}k^4 z_o^2 \sin^2\theta \cdot A\boldsymbol{A} \tag{2.13}$$

Compare the derivation of the fields in (2.11) with conventional textbook calculations —For example Griffiths *Introduction to Electrodynamics*, third edition. The vacuum gauge solution is very precise and focuses directly on the motion of the source instead of relying heavily on the potentials which must be differentiated in conventional calculations.

2.2 Electric Quadrupole Radiation

Figure 3 shows two electrons separated by a small distance d on the z-axis. If the two particles are oscillating out of phase, the dipole moment will be zero allowing for the generation of electric quadrupole radiation. The positions of the particles as a function of time are given by

$$\boldsymbol{z}_{up}(t) = +\frac{d}{2}\left[1 + \sin\omega t_r\right]\hat{\boldsymbol{z}} \tag{2.14a}$$

$$\boldsymbol{z}_{dn}(t) = -\frac{d}{2}\left[1 + \sin\omega t_r\right]\hat{\boldsymbol{z}} \tag{2.14b}$$

To solve for the acceleration fields, begin with two time derivatives $\boldsymbol{z}_{up}(t)$ and $\boldsymbol{z}_{dn}(t)$:

$$\boldsymbol{a}_{up} = -\frac{d\omega^2}{2}\sin\omega\left(t - \tfrac{1}{c}|\boldsymbol{r} - \boldsymbol{d}/2|\right)\hat{\boldsymbol{z}} \tag{2.15}$$

$$\boldsymbol{a}_{dn} = +\frac{d\omega^2}{2}\sin\omega\left(t - \tfrac{1}{c}|\boldsymbol{r} + \boldsymbol{d}/2|\right)\hat{\boldsymbol{z}} \tag{2.16}$$

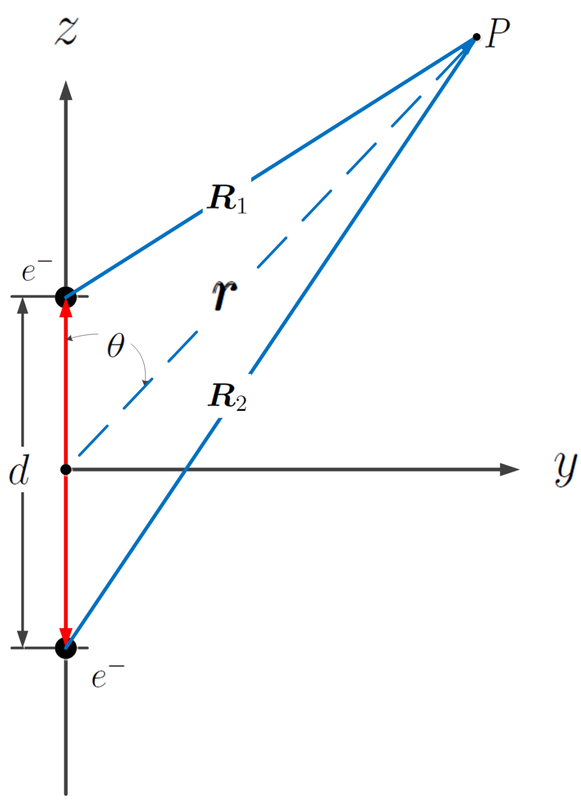

Figure 3: *Configuration of two oscillating electrons on the z-axis producing quadrupole radiation in the far field limit.*

Expanding inside the sine functions for the small quantity $\varepsilon = d/r$ gives the approximations

$$\boldsymbol{a}_{up} = +\frac{d\omega^2}{2}\left[\sin\omega(t-r/c) + \frac{\omega d}{2c}\cos\theta\cos\omega(t-r/c)\right]\hat{\boldsymbol{z}} \tag{2.17}$$

$$\boldsymbol{a}_{dn} = -\frac{d\omega^2}{2}\left[\sin\omega(t-r/c) - \frac{\omega d}{2c}\cos\theta\cos\omega(t-r/c)\right]\hat{\boldsymbol{z}} \tag{2.18}$$

To lowest order, the components $a_\perp^o$ for each particle may be written

$$a^o_{\perp\,up} = \boldsymbol{a}_{up}\cdot\hat{\boldsymbol{n}} = +\frac{d\omega^2}{2}\left[\sin\omega(t-r/c) + \frac{\omega d}{2c}\cos\theta\cos\omega(t-r/c)\right]\cos\theta \tag{2.19}$$

$$a^o_{\perp\,dn} = \boldsymbol{a}_{dn}\cdot\hat{\boldsymbol{n}} = -\frac{d\omega^2}{2}\left[\sin\omega(t-r/c) - \frac{\omega d}{2c}\cos\theta\cos\omega(t-r/c)\right]\cos\theta \tag{2.20}$$

The quadrupole electric field vector from equation (2.2) summed over two particles is then

$$\boldsymbol{\mathcal{E}}_{e2} = \boldsymbol{A}a^o_{\perp\,up} + \boldsymbol{A}a^o_{\perp\,dn} - A\,\boldsymbol{a}_{\perp up} - A\,\boldsymbol{a}_{\perp dn} \tag{2.21}$$

where A and $\boldsymbol{A}$ are (again) the lowest order potentials of equation (2.10). An explicit expression for the resulting field is

$$\boldsymbol{\mathcal{E}}_{e2} = \frac{e\omega^3 d^2}{2rc^3} \sin\theta \cos\theta \cos\omega(t - r/c)\, \hat{\boldsymbol{\theta}} \tag{2.22}$$

The magnetic field vector can be determined directly from the electric field vector by calculating the cross-product $\boldsymbol{\mathcal{B}}_{e2} = \hat{\boldsymbol{r}} \times \boldsymbol{\mathcal{E}}_{e2}$, but it may also be determined as

$$\boldsymbol{\mathcal{B}}_{e2} = [\boldsymbol{a}_{up} + \boldsymbol{a}_{dn}] \times \boldsymbol{A} \tag{2.23}$$

Quadrupole Tensor: It is important to write quadrupole electric and magnetic field strengths in terms of a quadrupole tensor. In this case, the charge is $2e$ and $z_o = d/2$. In terms of the quadrupole tensor, the fields can be written as the real part of

$$\boldsymbol{\mathcal{E}}_{e2} = \frac{k^3}{3} \left[\hat{\boldsymbol{r}} \times (\widehat{\boldsymbol{Q}} \cdot \hat{\boldsymbol{r}}) \times \hat{\boldsymbol{r}}\right] \frac{e^{i\omega(t-r/c)}}{r} \tag{2.24a}$$

$$\boldsymbol{\mathcal{B}}_{e2} = -\frac{k^3}{3} \left[\hat{\boldsymbol{r}} \times \widehat{\boldsymbol{Q}} \cdot \hat{\boldsymbol{r}}\right] \frac{e^{i\omega(t-r/c)}}{r} \tag{2.24b}$$

The time-averaged momentum flux can be calculated from the complex Poynting vector

$$\mathbf{S} = \frac{c}{8\pi}(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}}^*) \tag{2.25}$$

Power radiated per unit solid angle will then be

$$\frac{dP}{d\Omega} = \frac{ce^2 d^4 k^6}{32\pi} \sin^2\theta \cos^2\theta \tag{2.26}$$

2.3 Magnetic Dipole Radiation

Acceleration fields produced by an oscillating magnetic dipole can be calculated using the ring of charge in section 1.3 except that each of the N electrons in the ring must undergo oscillations about their specificed angle $2\pi n/N$. This will occur if an oscillating voltage is applied to a current loop. If α is a small angle in radians, then the position of each charge at the retarded time can be written

$$\begin{aligned} \boldsymbol{r}_n(t) &= r_o \cos(\phi_n + \alpha \cos\omega t_r)\, \hat{\boldsymbol{x}} + r_o \sin(\phi_n + \alpha\cos\omega t_r)\, \hat{\boldsymbol{y}} \\ &\approx r_o (\cos\phi_n - \alpha \sin\phi_n \cos\omega t_r)\, \hat{\boldsymbol{x}} + r_o(\sin\phi_n + \alpha\cos\phi_n \cos\omega t_r)\, \hat{\boldsymbol{y}} \end{aligned} \tag{2.27}$$

The magnitude of this vector to lowest order in α is just r_o while components of velocity and acceleration vectors are

$$\boldsymbol{v}_n = \alpha\omega r_o \sin\omega t_r \left[\sin\phi_n \hat{\boldsymbol{x}} - \cos\phi_n \hat{\boldsymbol{y}}\right] \tag{2.28a}$$

$$\boldsymbol{a}_n = \alpha\omega^2 r_o \cos\omega t_r \left[\sin\phi_n \hat{\boldsymbol{x}} - \cos\phi_n \hat{\boldsymbol{y}}\right] \tag{2.28b}$$

$$a_\perp^o = \boldsymbol{a}_n(t) \cdot \hat{\boldsymbol{r}} = \alpha\omega^2 r_o \cos\omega t_r \sin\theta \sin(\phi - \phi_n) \tag{2.28c}$$

Since the n^{th} particle is oscillating at the retarded time we expand the cosine term and approximate

$$\cos\omega[t - |\boldsymbol{r} - \boldsymbol{s}_n|/c] \approx \cos\omega(t - r/c) - \frac{\omega r_o}{c}\sin\theta\cos(\phi - \phi_n)\sin\omega(t - r/c) \tag{2.29}$$

This can be inserted into into equation (2.28b) but before this is done it will be prudent to write the equation in terms of spherical polar coordinates. Here we find

$$\sin\phi_n \hat{\boldsymbol{x}} - \cos\phi_n \hat{\boldsymbol{y}} = \sin\theta\sin(\phi_n - \phi)\hat{\boldsymbol{r}} + \cos\theta\sin(\phi_n - \phi)\hat{\boldsymbol{\theta}} - \cos(\phi_n - \phi)\hat{\boldsymbol{\phi}} \tag{2.30}$$

With the previous two substitutions, $\boldsymbol{a}_n$ is composed of six terms. Following sums over N particles, all terms vanish except for one in the direction of $\hat{\boldsymbol{\phi}}$. More specifically we can write

$$\sum_{n=1}^{N} a_\perp^o = 0 \qquad\qquad \sum_{n=1}^{N} \boldsymbol{a}_n = \frac{N\alpha\omega^3 r_o^2}{2c}\sin\theta\sin\omega(t - r/c)\hat{\boldsymbol{\phi}} \tag{2.31}$$

Electric and magnetic field vectors from equations (2.2) may then be written

$$\boldsymbol{\mathcal{E}}_{m1} = -A\sum_{n=1}^{N} \boldsymbol{a}_n = -\frac{Ne\alpha\omega^3 r_o^2}{2c^3}\left[\frac{\sin\theta}{r}\right]\sin\omega(t - r/c)\,\hat{\boldsymbol{\phi}} \tag{2.32a}$$

$$\boldsymbol{\mathcal{B}}_{m1} = -\boldsymbol{A} \times \sum_{n=1}^{N} \boldsymbol{a}_n = \frac{Ne\alpha\omega^3 r_o^2}{2c^3}\left[\frac{\sin\theta}{r}\right]\sin\omega(t - r/c)\,\hat{\boldsymbol{\theta}} \tag{2.32b}$$

To complete the calculation define the current and magnetic dipole moment by

$$I(t_r) = Ne\alpha\nu\sin\omega t_r \qquad\qquad \boldsymbol{m}(t_r) = I(t_r)\pi r_o^2 \hat{\boldsymbol{z}} \tag{2.33}$$

This allows the dipole fields to be written more concisely as

$$\boldsymbol{\mathcal{E}}_{m1} = -\frac{m_o\,\omega^2}{c}\left[\frac{\sin\theta}{r}\right]\sin\omega(t - r/c)\,\hat{\boldsymbol{\phi}} \tag{2.34a}$$

$$\boldsymbol{\mathcal{B}}_{m1} = +\frac{m_o\,\omega^2}{c}\left[\frac{\sin\theta}{r}\right]\sin\omega(t - r/c)\,\hat{\boldsymbol{\theta}} \tag{2.34b}$$

The average energy flux from this current configuration is

$$\langle\, \mathbf{S}\, \rangle = \frac{1}{4\pi c^3}\langle\, \boldsymbol{a}\cdot\boldsymbol{a}\rangle A\boldsymbol{A} = \frac{m_o^2\omega^4}{8\pi c^5}\left[\frac{\sin^2\theta}{r^2}\right]\hat{\boldsymbol{r}} \tag{2.35}$$

Once again, the vacuum gauge reigns supreme in its ability to provide simple, straightforward calculations of complex quantities. The only potentials required for calculations of all electric and magnetic acceleration fields are those of equation (2.10). All logical steps are purely algebraic and require no knowledge of vector calculus.