Introduction to Vacuum Gauge Electrodynamics

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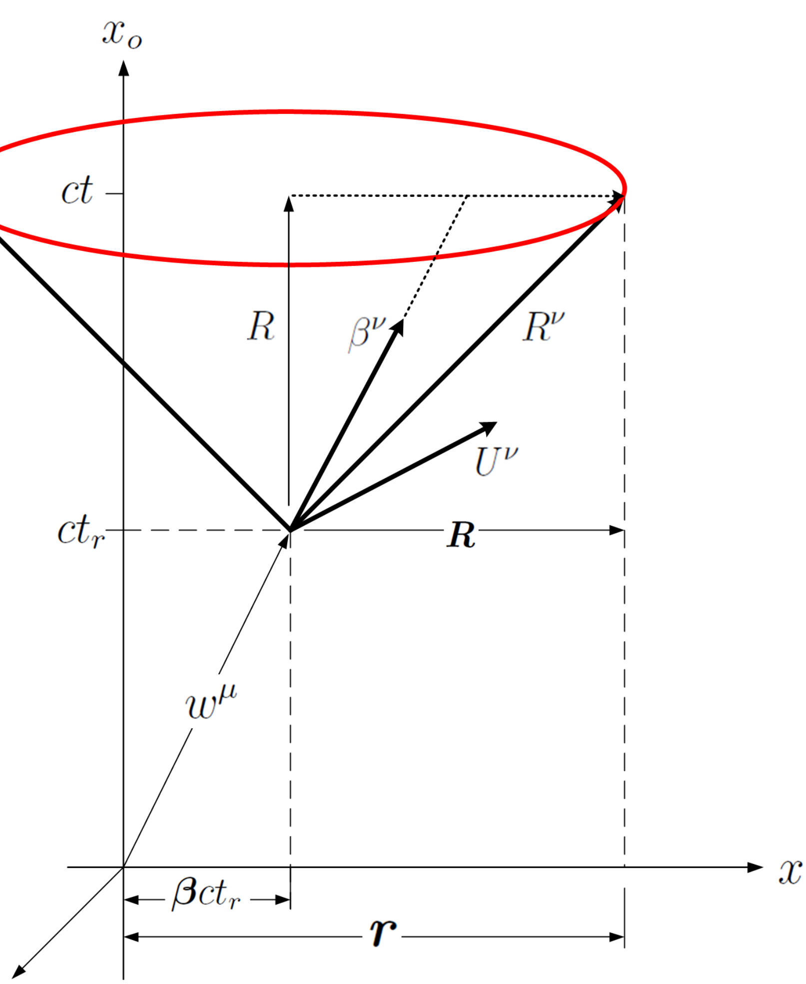

Abstract

The electrodynamics of a classical electron is reformulated based on an application of the vacuum gauge condition. The emergence of the spacelike gauge field allows the velocity and acceleration fields of the particle to managed as independent theories determined from associated vacuum gauge potentials.

Contents

1 Vacuum Gauge Velocity Potentials **4**

1.1 Differential Equations for Velocity Potentials 4

1.2 Vacuum Gauge Condition 5

1.3 Velocity Potentials and Velocity Fields 6

1.4 Covariant Integral 8

1.5 Lorentz Transformation of Potentials and Fields 10

2 Vacuum Gauge Acceleration Potentials **11**

2.1 Differential Equation for Acceleration Potentials 11

2.2 Acceleration Potentials and Acceleration Fields 12

2.3 Total Potentials: The Vacuum Gauge Universe 13

A Derivatives of the Null Vector **16**

B Electromagnetic Lagrangian and Stress Tensor **17**

List of Figures

1 Vacuum gauge velocity potentials 7

2 Integration of the vacuum gauge condition 9

1 Vacuum Gauge Velocity Potentials

A theory of the classical electron, written in terms of vacuum gauge potentials, can be approached from several different vantage points–each having its own merits. No particle accelerations are assumed anywhere in this section, which will be shown to be a fundamental requirement for the development of the theory.

1.1 Differential Equations for Velocity Potentials

In the flat spacetime of special relativity the metric tensor is defined by the signature $(+ - --)$. An inertial electron moves at velocity $\boldsymbol{\beta}$ and is characterized by the point-like four-current density J_e^ν. The associated fields satisfy the ***Maxwell-Lorentz equations***

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J_e^\nu \tag{1.1}$$

The name is appropriate since the original Maxwell theory of the electromagnetic field was a macroscopic theory, which Lorentz first applied to a charged particle. A theory of potentials can be developed by writing field strength tensor in terms of the well-known formula

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{1.2}$$

Inserting into (1.1) will then produce the source equations

$$\Box^2 A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J_e^\nu \tag{1.3}$$

In the conventional theory, gauge freedom allows for the Lorentz gauge condition $\partial_\mu A^\mu = 0$ which simplifies (1.3) as a set of source wave equations for the potentials. Instead, it will prove useful to address the problem of (1.3) from a different perspective. Suppose that the potential formulation is only to be applied—as already stipulated—to non-accelerated motions of the particle. For this, choose the covariant gauge condition

$$\partial_\nu A^\nu = \frac{e}{\rho^2} \tag{1.4}$$

Inserting into (1.3) gives

$$\Box^2 A^\nu - \partial^\nu \left[\frac{e}{\rho^2} \right] = \frac{4\pi}{c} J_e^\nu \tag{1.5}$$

One way to view this equation is to re-define the second term on the left as a contribution to the source term on the right.

$$\Box^2 A^\nu = \frac{4\pi}{c} \left[J_e^\nu + J_s^\nu \right] \tag{1.6}$$

Here, J_s^ν may be refered to as a ***velocity current*** defined by writing the spacelike vector

$$U^\nu = \frac{R^\nu}{\rho} - \beta^\nu \tag{1.7}$$

so that

$$J_s^\nu \equiv \frac{ec}{2\pi}\frac{U^\nu}{\rho^3} \tag{1.8}$$

The solution to (1.6) is facilitated by inserting $A^\nu = A_t^\nu + A_s^\nu$ to produce the decoupled pair

$$\Box^2 A_t^\nu = \frac{4\pi}{c} J_e^\nu \tag{1.9a}$$

$$\Box^2 A_s^\nu = \frac{4\pi}{c} J_s^\nu \tag{1.9b}$$

1.2 Vacuum Gauge Condition

A more extensive discussion of the gauge condition in (1.4) follows by first giving it a formal definition in terms of the fields:

$$\boxed{|\partial_\nu A^\nu| \equiv \sqrt{\mathrm{E}^2 - \mathrm{B}^2}} \tag{1.10}$$

The gauge introduced by this formula may be referred to as the ***vacuum gauge***. While its primary purpose is not understood yet, its definition above demonstrates an intimate link between $\partial_\nu A^\nu$—more specifically A^ν itself—and the classical theory of the electromagnetic field since one may also write $\partial_\nu A^\nu = \pm\sqrt{8\pi\mathcal{L}_{em}}$.

The covariant nature of the vacuum gauge gives it similar properties to the Lorentz gauge. For example, the restricted gauge transformation defined by

$$A^\nu \to A^\nu - \partial^\nu \Lambda \qquad \text{where} \qquad \Box^2\Lambda = 0 \tag{1.11}$$

will preserve the vacuum gauge condition. This means that—like the Lorentz gauge—the vacuum gauge represents an entire class of potentials. For the special case of electromagnetic waves, the left side of (1.10) is zero and the vacuum gauge is identical to the Lorentz gauge.

When the gauges are not identical, it is possible to implement a specific gauge transformation which connects them. If A_t^ν are the Lorentz gauge potentials satisfying $\partial_\nu A_t^\nu = 0$ then the vacuum gauge potentials follow from

$$A^\nu = A_t^\nu + \partial^\nu \varphi \tag{1.12}$$

Applying the divergence operator to both sides of this equation and inserting the vacuum gauge condition shows that the gauge field φ must satisfy

$$\Box^2 \varphi = \frac{e}{\rho^2} \tag{1.13}$$

One approach to solving this equation is by implicit differentiation. Applying the operator ∂_ν to

$$\partial^\nu \varphi = \frac{\partial \varphi}{\partial \rho} \partial^\nu \rho \tag{1.14}$$

and remembering that accelerations are not being considered results in a source equation which may be written

$$L[\,\varphi\,] = -\frac{\partial}{\partial \rho}\left[\rho^2 \frac{\partial}{\partial \rho}\right]\varphi = e \tag{1.15}$$

The operator on the left is self-adjoint and admits a general solution which includes two linearly independent solutions to the homogeneous equation:

$$\varphi = C_1 \varphi_1 + C_2 \varphi_2 + \varphi_p \tag{1.16}$$

For the record, $\varphi_1 = 1$ and $\varphi_2 = e/\rho$, but neither of these contributions are necessary for the design of the new theory. Instead the particular solution

$$\varphi(\boldsymbol{r}, t) = -e \ln \rho \tag{1.17}$$

is sufficient to determine the potentials. A solution is also available by considering a first order equation in the scalar field ψ defined through the relation

$$\varphi(\rho) = \int^\rho \psi(\rho')\, d\rho' \tag{1.18}$$

The associated Green function for the first order problem is determined from

$$-\frac{\partial}{\partial \rho}\left[\rho^2\, G(\rho, s)\right] = \delta(\rho - s) \tag{1.19}$$

and seems to provide a neater solution.

1.3 Velocity Potentials and Velocity Fields

One way to solve equations (1.9) is in the rest frame followed by a Lorentz transformation to moving frame coordinates. However, it is a simple matter to recognize (1.9a) as the differential equation solved by the Liénard-Wiechert potentials. Moreover, equation (1.9b) can be solved by applying the operator ∂^ν to the scalar field φ in equation (1.17). Results are

$$A_t^\nu = \frac{e}{\rho} \beta^\nu \tag{1.20a}$$

$$A_s^\nu = \frac{e}{\rho} U^\nu \tag{1.20b}$$

These are complimentary timelike and spacelike potentials (un-identical twins) which may be added together immediately using (1.7) to determine the vacuum gauge velocity potentials given by

$$\boxed{A^\nu = \frac{eR^\nu}{\rho^2}} \tag{1.21}$$

A graphical depiction of the potentials is shown in Figure 1. For constant velocity

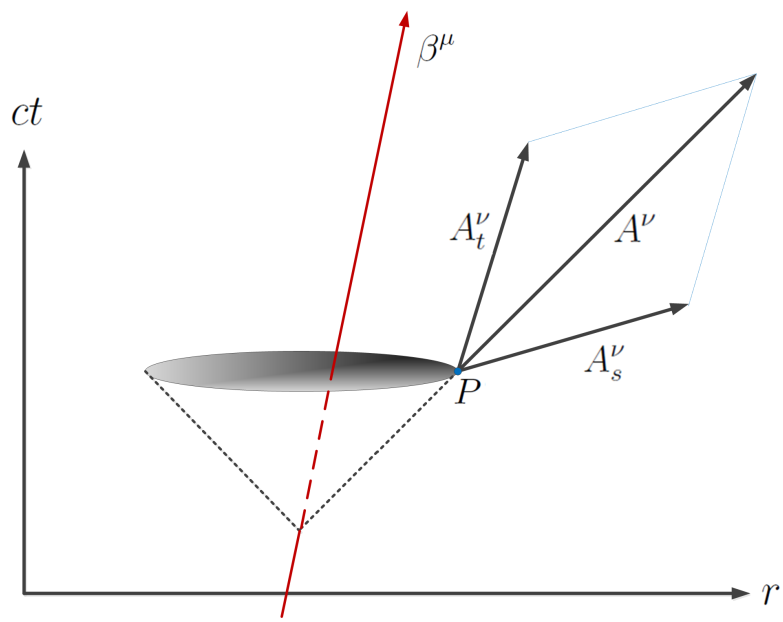

Figure 1: *Vacuum gauge velocity potentials at a point P. The light cone traces the potentials back to the world line of the charge at the retarded time.*

motion the effect of the vacuum gauge condition is to generate the gauge field A_s^ν which rotates the Lorentz gauge (Liénard-Wiechert) potentials onto the surface of the light cone. In Minkowski space both A_t^ν and A_s^ν have equal lengths except for an overall sign. They are mutually orthogonal timelike and spacelike components of the vacuum gauge velocity potentials.

Vacuum gauge potentials can be used to verify the correct form of the velocity portion of the field strength tensor. Technically, this is not necessary since it is known that the velocity potentials differ from the Liénard-Wiechert potentials by a gauge transformation. However, this is not the whole story and it will be mandatory to perform the calculation:

$$\begin{aligned} F_v^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\ &= \frac{e}{\rho^2}\left(\partial^\mu R^\nu - \partial^\nu R^\mu\right) - \frac{2e}{\rho^3}\left(R^\nu \partial^\mu \rho - R^\mu \partial^\nu \rho\right) \end{aligned} \tag{1.22}$$

But implicit derivatives (which include the possibility of particle accelerations) are

$$\partial^\mu R^\nu = \mathrm{g}^{\mu\nu} - \frac{1}{\rho} R^\mu \beta^\nu \tag{1.23a}$$

$$\partial^\mu \rho = \beta^\mu - \frac{R^\mu}{\rho} + \frac{a^\lambda R_\lambda}{\rho} R^\mu \tag{1.23b}$$

and these can be inserted above to yield

$$F_v^{\mu\nu} = \frac{e}{\rho^3}\left(R^\mu \beta^\nu - R^\nu \beta^\mu\right) \tag{1.24}$$

Of particular importance is how the calculation of the field strength tensor completely rejects all terms associated with the acceleration of the particle. This property is also visible in the calculation of the vacuum gauge condition where (1.23b) will be required along with $\partial_\nu R^\nu = 3$ to determine

$$\partial_\nu A^\nu = \frac{e}{\rho^2} \tag{1.25}$$

The implication here is that the Maxwell-Lorentz field of the classical electron will require independent theories of the particles' velocity and acceleration fields—a requirement imposed by the vacuum gauge condition.

1.4 Covariant Integral

An important application of Gauss' law begins with the integral of the vacuum gauge condition

$$I = \int_{\mathcal{V}} \partial_\nu A^\nu \, d^4x \tag{1.26}$$

where $\mathcal{V}$ represents the four-volume of the light cone shown in Figure 2. Since the integrand and the volume element are both Lorentz scalars, this means that I is a scalar invariant. It is most easily evaluated in the rest frame where

$$I = \int_0^{c\tau} \left[\int_0^{c\tau'} \frac{e}{r^2} r^2 dr \, d\Omega \right] c d\tau' = 2\pi e c^2 \tau^2 \tag{1.27}$$

Since this result is proportional to the interval, in a frame moving with velocity $\boldsymbol{\beta}$ it generalizes to

$$x^\mu x_\mu = \frac{1}{2\pi e} \int_{\mathcal{V}} \partial_\nu A^\nu d^4x \tag{1.28}$$

where $x^\mu = (ct, \boldsymbol{\beta} ct)$ is the coordinate vector of the particle.

Another approach to evaluate I is to apply Gauss' law and integrate over the surface of the light cone. First note that the volume integral can be written

$$I = \int_{\mathcal{V}} \partial_\nu A_t^\nu d^4x + \int_{\mathcal{V}} \partial_\nu A_s^\nu d^4x \tag{1.29}$$

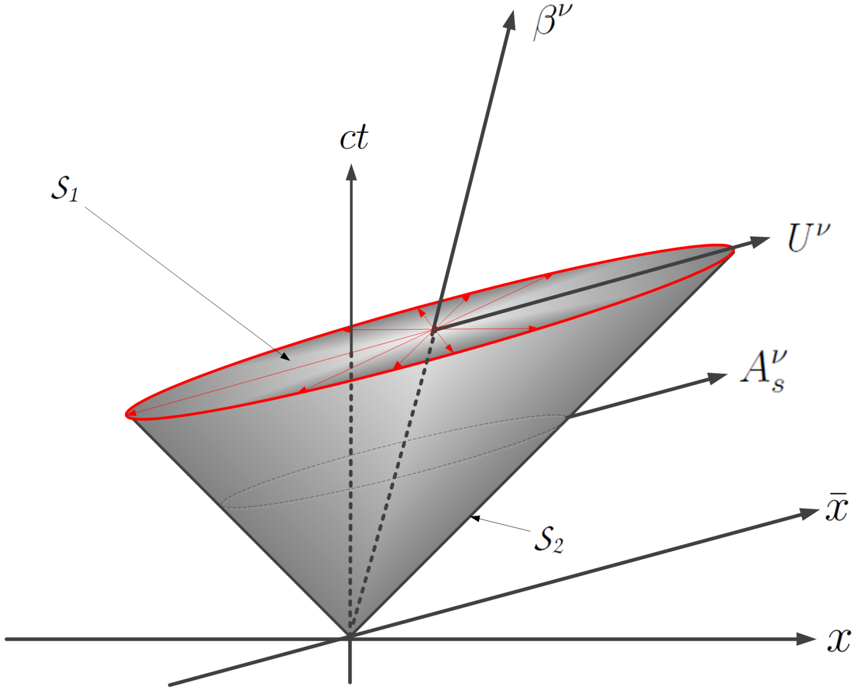

Figure 2: *The gauge field A_s^ν can be integrated over the hyper-surfaces enclosing the four-volume of the light cone. There is no contribution from the integral over the hyper-ellipse.*

The first integral is zero and this implies that the surface integrals of the Liénard-Wiechert potentials enclosing the volume $\mathcal{S}$ add to zero:

$$\int_{\mathcal{S}} A_t^\nu d_\nu \mathcal{S} = 0 \tag{1.30}$$

More specifically, non-zero integrals over each of two hypersurfaces are

$$\int_{\mathcal{S}_1} A_t^\nu \beta_\nu \, d^3\sigma + \int_{\mathcal{S}_2} A_t^\nu R_\nu \, d^2\omega = 0 \tag{1.31}$$

As always, these integrals are most easily evaluated in the rest frame having values $\pm 2\pi e c^2 \tau^2$.

All that remains is to apply Gauss' law to the second integral in equation (1.29), but as indicating by figure 2, the gauge field only contributes along the light cone producing the result

$$I = \int_{\mathcal{S}_2} A_s^\nu R_\nu \, d^2\omega \tag{1.32}$$

1.5 Lorentz Transformation of Potentials and Fields

The most general transformation equations for the velocity potentials mirror those of the coordinate transformation:

$$A' = \gamma(A + \boldsymbol{\beta} \cdot \boldsymbol{A}) \tag{1.33a}$$

$$\boldsymbol{A}' = \boldsymbol{A} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{A})\boldsymbol{\beta} + \gamma A \boldsymbol{\beta} \tag{1.33b}$$

However, velocity potentials are composed of timelike and spacelike components—each of which may be transformed by itself. Component transformations from the rest frame are particularly simple.

$$A'_t = \gamma A_t \qquad \boldsymbol{A}'_t = \gamma \boldsymbol{\beta} A_t \tag{1.34a}$$

$$A'_s = \gamma \boldsymbol{\beta} \cdot \boldsymbol{A}_s \qquad \boldsymbol{A}'_s = \boldsymbol{A}_s + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{A}_s)\boldsymbol{\beta} \tag{1.34b}$$

The norms of both four-vectors are easily shown to be Lorentz scalars. Moreover, their norms are the same to within sign which will enforce a zero norm for the velocity potential

$$A'^{\nu} = A'^{\nu}_t + A'^{\nu}_s \tag{1.35}$$

The importance of the spacelike potentials emerge when the velocity field strength tensor is written

$$eF'^{\mu\nu}_v = [A'^{\mu}_s,\, A'^{\nu}_t] \tag{1.36}$$

The anti-symmetric tensor loses its traditional identity as a fundamental object—being replaced by an interaction among four-potentials. The second order transformation law for $F'^{\mu\nu}_v$ is then a combination of first order transformations

$$[A'^{\mu}_s,\, A'^{\nu}_t] = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \left[A^{\alpha}_s,\, A^{\beta}_t\right] \tag{1.37}$$

Transformation Law for Velocity Fields: As an example of the use of the gauge field, velocity electric and magnetic field vectors in a moving frame are

$$\mathbf{E}'_v = \frac{\gamma}{\rho}\left(\boldsymbol{A}'_s - A'_s \boldsymbol{\beta}\right) \qquad \mathbf{B}'_v = \frac{\gamma}{\rho}\left(\boldsymbol{\beta} \times \boldsymbol{A}'_s\right) \tag{1.38}$$

Now suppose the transformation of the gauge field in equation (1.34b) is inserted into (1.38). In terms of rest frame potentials one finds

$$\mathbf{E}'_v = \frac{\gamma}{\rho}\left[\boldsymbol{A}_s - \frac{\gamma}{\gamma + 1}(\boldsymbol{\beta} \cdot \boldsymbol{A}_s)\boldsymbol{\beta}\right] \tag{1.39a}$$

$$\mathbf{B}'_v = \frac{\gamma}{\rho}(\boldsymbol{\beta} \times \boldsymbol{A}_s) \tag{1.39b}$$

In the rest frame, $\boldsymbol{A}_s/\rho = \mathbf{E}_v$ which derives the transformation law for the electric and magnetic field vectors of the particle

$$\mathbf{E}'_v = \gamma\mathbf{E}_v - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{E}_v)\boldsymbol{\beta} \tag{1.40a}$$

$$\mathbf{B}'_v = \gamma\boldsymbol{\beta}\times\mathbf{E}_v \tag{1.40b}$$

2 Vacuum Gauge Acceleration Potentials

As already discussed, the rejection of particle accelerations by the velocity potentials imply that the acceleration potentials will satisfy their own independent differential equation. Once derived, its solution and subsequent calculation of the acceleration field strength tensor $F_a^{\mu\nu}$ are straight-forward.

2.1 Differential Equation for Acceleration Potentials

The potential formulation of the Maxwell-Lorentz equations is

$$\Box^2 A^\nu - \partial^\nu\partial_\mu A^\mu = \frac{4\pi}{c}J_e^\nu \tag{2.1}$$

As a trial solution, A^ν can be inserted into the left side of this equation, but now including the possibility of particle accelerations. Excluding the location of the charge, the calculation shows that

$$\Box^2 A^\nu = \frac{2e}{\rho^3}U^\nu \tag{2.2a}$$

$$\partial^\nu\partial_\mu A^\mu = \frac{2e}{\rho^3}U^\nu - \frac{2e}{\rho^4}(a^\lambda R_\lambda)R^\nu \tag{2.2b}$$

The trial solution therefore differs from the actual solution by a single term involving the acceleration of the particle. It is convenient to write (2.2b) as

$$\partial^\nu\partial_\mu A^\mu = \frac{4\pi}{c}[J_s^\nu + J_a^\nu] \tag{2.3}$$

which defines the ***acceleration current***

$$J_a^\nu \equiv -\frac{ec}{2\pi\rho^4}(a^\lambda R_\lambda)R^\nu \tag{2.4}$$

Now assume that the potentials during accelerated motions can be written as

$$A^\nu \longrightarrow A^\nu + A_a^\nu \tag{2.5}$$

Inserting this into (2.1) along with the simultaneous appearance of J_a^ν shows that

$$\Box^2 A^\nu + \Box^2 A_a^\nu - \partial^\nu \partial_\mu A^\mu - \partial^\nu \partial_\mu A_a^\mu = \frac{4\pi}{c} J_e^\nu + \frac{4\pi}{c} J_a^\nu \tag{2.6}$$

Velocity and acceleration terms can now be de-coupled resulting in two independent equations which obey

$$\Box^2 A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J_e^\nu \tag{2.7a}$$

$$\Box^2 A_a^\nu - \partial^\nu \partial_\mu A_a^\mu = \frac{4\pi}{c} J_a^\nu \tag{2.7b}$$

While the first equation has already been investigated, the mathematical form of the acceleration equation implies that the acceleration fields will satisfy their own set of Maxwell-like equations

$$\partial_\mu F_a^{\mu\nu} = \frac{4\pi}{c} J_a^\nu \tag{2.8}$$

Moreover, J_a^ν is a null vector and will satisfy the continuity equation $\partial_\nu J_a^\nu = 0$ as it must since $F_a^{\mu\nu}$ is anti-symmetric. This is easily proved by noting that

$$\partial_\nu (a^\lambda R_\lambda) R^\nu = a^\lambda R_\lambda \tag{2.9}$$

The fact that J_a^ν points in the direction of R^ν gives it a physical interpretation as a massless $1/r^2$ electrical current which radiates from the instantaneous retarded position of the charge. This means that every point in space containing non-zero fields $\mathbf{E}_a$ and $\mathbf{B}_a$ can be associated with a local vacuum current density at that same point. Now write

$$e J_a^\nu \equiv -\frac{c}{2\pi} (a^\lambda A_\lambda) A^\nu \tag{2.10}$$

In other words, except for an overall constant, J_a^ν is nothing more than the projection of the particle four-acceleration along the direction of the velocity potentials.

2.2 Acceleration Potentials and Acceleration Fields

It may be possible to solve the differential equation in (2.7b) by a brute force calculation, but it is easier to determine its solution by referring directly to the gauge transformation in equation (1.12). Including particle accelerations this is

$$A^\nu = A_t^\nu + \partial^\nu \varphi \tag{2.11a}$$

$$= A_t^\nu - \frac{e}{\rho} \left[\beta^\nu - \frac{R^\nu}{\rho} (1 - a^\lambda R_\lambda) \right] \tag{2.11b}$$

The gauge transformation eliminates the Liénard-Wiechert potentials altogether and replaces them with null potentials. In addition, both the velocity and acceleration potentials are easily recognizeable in the brackets. The acceleration potentials and the acceleration current are then

$$\boxed{A_a^\nu = -\frac{e}{\rho^2}(a^\lambda R_\lambda)R^\nu} \qquad \boxed{J_a^\nu \equiv -\frac{ec}{2\pi\rho^4}(a^\lambda R_\lambda)R^\nu} \tag{2.12}$$

As with the velocity theory, the vacuum gauge is requiring radial null potentials such that $A_\nu^a A_a^\nu = 0$. Note also that A_a^ν does not diminish for large R.

To determine the acceleration field strength tensor first write

$$A_a^\nu = -(a^\lambda R_\lambda)A^\nu \tag{2.13}$$

This can be differentiated to produce

$$F_a^{\mu\nu} = -\partial^\mu(a^\lambda R_\lambda)A^\nu + \partial^\nu(a^\lambda R_\lambda)A^\mu - (a^\lambda R_\lambda)F_v^{\mu\nu} \tag{2.14}$$

A rigorous calculation of $\partial^\mu(a^\lambda R_\lambda)$ requires some computational stamina but leads to the simple result

$$\partial^\nu(a^\lambda R_\lambda) = a^\nu + \frac{\dot{a}^\lambda R_\lambda}{\rho}R^\nu \tag{2.15}$$

where $\dot{a}^\lambda = \partial a^\lambda/\partial\tau$ and where τ is the proper time. Now define the scalar field[1] $\xi \equiv a^\lambda R_\lambda/\rho$ and easily determine the final form of the field strength tensor:

$$\begin{aligned} F_a^{\mu\nu} &= \frac{e}{\rho^2}\left[R^\mu a^\nu - R^\nu a^\mu - \xi(R^\mu\beta^\nu - R^\nu\beta^\mu)\right] \\ &= [A^\mu, a^\nu] - \xi[A^\mu, \beta^\nu] \end{aligned} \tag{2.16}$$

2.3 Total Potentials: The Vacuum Gauge Universe

Combining the velocity potentials in (1.21) with the acceleration potentials in (2.12), the general vacuum gauge solution for arbitrary motions of a charged particle can be written

$$\boxed{A^\nu(\boldsymbol{r}, t) = \frac{e(1 - a^\lambda R_\lambda)}{\rho^2}R^\nu} \tag{2.17}$$

Based on previous discussion, the mathematical structure of this formula also allows the potentials to be divided as

$$A^\nu = A_t^\nu + A_s^\nu + A_a^\nu \tag{2.18}$$

where

$$A_t^\nu = \frac{e}{\rho}\beta^\nu \qquad A_s^\nu = \frac{e}{\rho}U^\nu \qquad A_a^\nu = -\frac{e}{\rho}\xi R^\nu \tag{2.19}$$

[1] This definition of ξ is similar to Rohrlich's definition $a_u \equiv a^\nu U_\nu$. The new definition was chosen to avoid confusion with the covariant four-acceleration.

Each of these potentials can be associated with its own vector current density, and each current density points in the same direction as its associated potentials. Two of the currents are conserved but one is not:

$$\partial_\nu J^\nu_e = 0 \qquad \partial_\nu J^\nu_s = -\frac{ec}{2\pi\rho^4} \qquad \partial_\nu J^\nu_a = 0 \tag{2.20}$$

Finally, it is important to compare the magnitudes of the velocity and acceleration potentials which become equal when $a^\lambda R_\lambda \sim c^2$. Unless accelerations are extremely large, it can be assumed that acceleration potentials represent only a small correction.

Multi-Particle System: In the constant velocity theory, the scalar field φ produces the gauge transformation between the Lieńard-Wiechert and vacuum gauge potentials. But the scalar field can also be shown to produce the entire theory of vacuum gauge electrodynamics. Begin with $\varphi = -e \ln \rho$ having first and second derivatives for general accelerated motions given by

$$\partial^\nu \varphi = A^\nu + A^\nu_a - A^\nu_t \tag{2.21a}$$

$$\partial^\mu \partial^\nu \varphi = \partial^\mu A^\nu + \partial^\mu A^\nu_a - \partial^\mu A^\nu_t \tag{2.21b}$$

Now form the object

$$\partial_\mu \left[\partial^\mu \partial^\nu \varphi - \partial^\nu \partial^\mu \varphi\right] = 0 \tag{2.22}$$

This set of operations can be written in terms of the potentials as

$$\Box^2 A^\nu + \Box^2 A^\nu_a - \Box^2 A^\nu_t - \partial^\nu \partial_\mu A^\mu - \partial^\nu \partial_\mu A^\mu_a = 0 \tag{2.23}$$

But, as has already been shown, the fourth term generates the acceleration current J^ν_a and this allows the velocity and acceleration terms to be separated. Moreover, the point charge four-current J^ν_e replaces the wave operator acting on the Liénard-Wiechert potentials. What remains is independent equations of motion given by

$$\Box^2 A^\nu - \partial^\nu \partial_\mu A^\mu - \frac{4\pi}{c} J^\nu_e = 0 \tag{2.24a}$$

$$\Box^2 A^\nu_a - \partial^\nu \partial_\mu A^\mu_a - \frac{4\pi}{c} J^\nu_a = 0 \tag{2.24b}$$

This calculation effectively summarizes the entire theory of vacuum gauge potentials for a charged particle.

Now suppose an electrically neutral universe explodes at time $c\tau = 0$ producing a collection of $N/2$ electrons and $N/2$ positrons of individual charge q_i. For all future times

$$\sum_{i=1}^{N} q_i = 0 \tag{2.25}$$

Each particle can also be described by a position vector

$$\boldsymbol{w}_i(c\tau) \quad \text{where} \quad \boldsymbol{w}_i(0) = 0 \qquad i = 1, ..., N \tag{2.26}$$

and a retarded position coordinate

$$\boldsymbol{R}_i = \boldsymbol{r} - \boldsymbol{w}_i(c\tau) \qquad i = 1, ..., N \tag{2.27}$$

Other constraints may also be imposed. If the initial universe exists in a cosmological rest frame, then for every particle created with initial velocity $\boldsymbol{\beta}_i(0) = \dot{\boldsymbol{w}}_i(0)$ there exists an associated anti-particle with velocity $\boldsymbol{\beta}_j(0) = -\boldsymbol{\beta}_i(0)$. Moreover, under the assumption of a cosmological principle, then the number density of particles created in any given direction must be approximately constant. The individual $\boldsymbol{\beta}_i(0)$ might otherwise be determined from a random number generator.

In accordance with vacuum gauge theory, now suppose each created particle is characterized by a scalar field

$$\varphi_i = -q_i \ln \rho_i \tag{2.28}$$

Summing over all electrons and positrons in the universe produces the resultant field

$$\varphi_{unv}(\boldsymbol{r}, t) = \sum_{i=1}^{N} \varphi_i = \ln \prod_{i=1}^{N} (\rho_i)^{-q_i} \tag{2.29}$$

Based on the previous calculation, it is a simple matter to derive a set of Maxwell-Lorentz source equations for potentials A^{μ}_{unv} which satisfy

$$\Box^2 A^{\nu}_{unv} - \partial^{\nu}\partial_{\mu} A^{\mu}_{unv} = \frac{4\pi}{c}(J^{\nu}_{e})_{unv} \tag{2.30}$$

where $(J^{\nu}_{e})_{unv}$ is a sum of point charge current densities at various locations. The universe generated by this calculation requires the fields $\varphi_{unv}(\boldsymbol{r}, t)$ and $A^{\nu}_{unv}(\boldsymbol{r}, t)$ to permeate all points of spacetime not associated with the source infinity at the location of a particle. This is the vacuum gauge universe.

A Derivatives of the Null Vector

The covariant derivative of R^ν is

$$\partial^\mu R^\nu = \mathrm{g}^{\mu\nu} - \frac{R^\mu \beta^\nu}{\rho} \tag{A.1}$$

The trace of the resulting matrix gives the 4-divergence $\partial_\nu R^\nu = 3$. In terms of individual components–and with the inclusion of a sign–a useful construction is:

$$-\partial^\mu R^\nu = \begin{bmatrix} -\dfrac{\partial R}{\partial ct} & -\dfrac{\partial \boldsymbol{R}}{\partial ct} \\ \boldsymbol{\nabla} R & \boldsymbol{\nabla}\boldsymbol{R} \end{bmatrix} \tag{A.2}$$

where individual components are given by

$$\frac{\partial R}{\partial ct} = 1 - \frac{\gamma R}{\rho} \qquad\qquad \frac{\partial \boldsymbol{R}}{\partial ct} = \frac{-\gamma R \boldsymbol{\beta}}{\rho} \tag{A.3}$$

$$\boldsymbol{\nabla} R = \frac{\gamma \boldsymbol{R}}{\rho} \qquad\qquad \boldsymbol{\nabla}\boldsymbol{R} = \mathbf{1} + \frac{\gamma \boldsymbol{R}\boldsymbol{\beta}}{\rho} \tag{A.4}$$

The determinant of (A.2) can be written $\det[\partial^\mu R^\nu] = 0$. The divergence and curl of $\boldsymbol{R}$ may be written

$$\boldsymbol{\nabla} \cdot \boldsymbol{R} = 3 + \frac{\gamma \boldsymbol{R} \cdot \boldsymbol{\beta}}{\rho} = Tr[\boldsymbol{\nabla}\boldsymbol{R}] \tag{A.5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{R} = \frac{\gamma}{\rho} \boldsymbol{R} \times \boldsymbol{\beta} \tag{A.6}$$

Let $\boldsymbol{w}(ct_r)$ be the retarded position of a charged particle at time ct_r. The light cone condition is defined by

$$\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{w}(ct_r) \qquad\qquad R \equiv ct - ct_r \tag{A.7}$$

Derivatives of the retarded time with respect to present time coordinates are

$$\frac{\partial ct_r}{\partial ct} = \frac{\gamma R}{\rho} \tag{A.8}$$

$$\boldsymbol{\nabla} ct_r = \frac{-\gamma \boldsymbol{R}}{\rho} \tag{A.9}$$

B Electromagnetic Lagrangian and Stress Tensor

The separation of velocity and acceleration fields generated by the vacuum gauge condition allows for the possibility of constructing a classical Lagrangian for a charged particle as a function of independent field quantities. In consideration of notational simplicity it is useful to make the temporary replacements

$$A^\nu \to \mathcal{V}^\nu \qquad \partial^\mu A^\nu \to \mathcal{V}^{\mu\nu} \tag{B.1a}$$

$$A_a^\nu \to \mathcal{A}^\nu \qquad \partial^\mu A_a^\nu \to \mathcal{A}^{\mu\nu} \tag{B.1b}$$

Hamilton's principle for the new Lagrangian will then be

$$\delta\mathcal{S} = \delta \int \mathcal{L}_{vg}[\mathcal{V}^{\mu\nu}, \mathcal{A}^{\mu\nu}, \mathcal{V}^\nu,\, \mathcal{A}^\nu,\, x^\nu]d^4x \tag{B.2}$$

The functional form of $\mathcal{L}_{vg}$ can be derived beginning with the classical electromagnetic Lagrangian

$$\mathcal{L}_{em} = -\frac{1}{16\pi}(\partial_\mu A_\nu - \partial_\mu A_\nu)(\partial^\mu A^\nu - \partial^\mu A^\nu) - \frac{1}{c} J_e^\nu A_\nu \tag{B.3}$$

The field quantities in (B.1) can be readily inserted but this must be accompanied by the insertion of the acceleration 4-current density through the replacement $J_e^\nu \to J_e^\nu + J_a^\nu$. The resulting Lagrangian is

$$\begin{aligned}\mathcal{L}_{vg} = -\frac{1}{8\pi}[&\mathcal{V}^{\mu\nu}\mathcal{V}_{\mu\nu} - \mathcal{V}^{\nu\mu}\mathcal{V}_{\mu\nu} + 2\mathcal{A}^{\mu\nu}\mathcal{V}_{\mu\nu} - 2\mathcal{A}^{\nu\mu}\mathcal{V}_{\mu\nu} + \mathcal{A}^{\mu\nu}\mathcal{A}_{\mu\nu} - \mathcal{A}^{\nu\mu}\mathcal{A}_{\mu\nu}] \\ &- \frac{1}{c}J_e^\nu\,\mathcal{V}_\nu - \frac{1}{c}J_e^\nu\,\mathcal{A}_\nu - \frac{1}{c}J_a^\nu\,\mathcal{V}_\nu - \frac{1}{c}J_a^\nu\,\mathcal{A}_\nu\end{aligned} \tag{B.4}$$

The functional value of the free field Lagrangian is independent of particle accelerations and essentially identical to the conventional theory. Although acceleration interaction terms are required by the vacuum gauge, they also make no contribution to functional value. Euler-Lagrange equations for the independent field quantities follow from Hamilton's principle leading to equations of motion given by

$$\partial_\mu \frac{\partial \mathcal{L}_{vg}}{\partial \mathcal{V}_{\mu\nu}} - \frac{\partial \mathcal{L}_{vg}}{\partial \mathcal{V}_\nu} = 0 \tag{B.5a}$$

$$\partial_\mu \frac{\partial \mathcal{L}_{vg}}{\partial \mathcal{A}_{\mu\nu}} - \frac{\partial \mathcal{L}_{vg}}{\partial \mathcal{A}_\nu} = 0 \tag{B.5b}$$

Not surprisingly, each set of Lagrange equations produces identical equations of motion

$$\partial_\mu F_v^{\mu\nu} = \frac{4\pi}{c} J_e^\nu \tag{B.6a}$$

$$\partial_\mu F_a^{\mu\nu} = \frac{4\pi}{c} J_a^\nu \tag{B.6b}$$

Now suppose $\mathcal{L}_{vg}$ is used to derive the canonical stress tensor from

$$T^{\mu\nu}_{vg} = \frac{\partial \mathcal{L}_{vg}}{\partial \mathcal{V}_{\mu\lambda}} \mathcal{V}^{\nu}{}_{\lambda} + \frac{\partial \mathcal{L}_{vg}}{\partial \mathcal{A}_{\mu\lambda}} \mathcal{A}^{\nu}{}_{\lambda} - \mathrm{g}^{\mu\nu}\mathcal{L}_{vg} \tag{B.7}$$

It is convenient to write this equation as

$$T^{\mu\nu}_{vg} = \Theta^{\mu\nu}_{vg} + T^{\mu\nu}_{D} \tag{B.8}$$

where

$$\Theta^{\mu\nu}_{vg} = \frac{\partial \mathcal{L}_{vg}}{\partial \mathcal{V}_{\mu\lambda}}(\mathcal{V}^{\nu}{}_{\lambda} - \mathcal{V}_{\lambda}{}^{\nu}) + \frac{\partial \mathcal{L}_{vg}}{\partial \mathcal{A}_{\mu\lambda}}(\mathcal{A}^{\nu}{}_{\lambda} - \mathcal{A}_{\lambda}{}^{\nu}) - \mathrm{g}^{\mu\nu}\mathcal{L}_{vg} \tag{B.9}$$

$$T^{\mu\nu}_{D} = \frac{\partial \mathcal{L}_{vg}}{\partial \mathcal{V}_{\mu\lambda}} \mathcal{V}_{\lambda}{}^{\nu} + \frac{\partial \mathcal{L}_{vg}}{\partial \mathcal{A}_{\mu\lambda}} \mathcal{A}_{\lambda}{}^{\nu} \tag{B.10}$$

The tensor $\Theta^{\mu\nu}_{vg}$ is the most general, symmetric and traceless, gauge invariant tensor. Identifying $F^{\mu\nu}$ as the total field strength tensor leads to

$$\Theta^{\mu\nu}_{vg} \equiv \frac{1}{4\pi}\left[F^{\mu\lambda}F_{\lambda}{}^{\nu} + \frac{1}{4}\mathrm{g}^{\mu\nu}F_{\alpha\lambda}F^{\alpha\lambda}\right] \tag{B.11}$$

The remainder $T^{\mu\nu}_{D}$, which is neither symmetric nor gauge invariant, follows as

$$T^{\mu\nu}_{D} = -\frac{1}{4\pi}F^{\mu\lambda}\partial_{\lambda}A^{\nu} \tag{B.12}$$

This term can be removed[2] from (B.8) with the observation that $\partial_{\mu}T^{\mu\nu}_{D} = 0$. The equivalence of the conventional and vacuum gauge electromagnetic theories for a charged particle in arbitrary motion is therefore

$$\Theta^{\mu\nu}_{vg} \equiv \Theta^{\mu\nu}_{em} = \Theta^{\mu\nu} \tag{B.13}$$

For reference, the specific form of $\Theta^{\mu\nu}$ can be written as the sum of three parts with individual components given by

$$\Theta^{\mu\nu}_{1} = \frac{e^2}{4\pi\rho^4}\left[\frac{1}{\rho}(R^{\mu}\beta^{\nu} + R^{\nu}\beta^{\mu}) - \frac{1}{\rho^2}R^{\mu}R^{\nu} - \frac{1}{2}\mathrm{g}^{\mu\nu}\right] \tag{B.14a}$$

$$\Theta^{\mu\nu}_{2} = \frac{e^2}{4\pi\rho^4}\left[a^{\mu}R^{\nu} + R^{\mu}a^{\nu} - \xi(R^{\mu}\beta^{\nu} + R^{\nu}\beta^{\mu}) + \frac{2\xi}{\rho}R^{\mu}R^{\nu}\right] \tag{B.14b}$$

$$\Theta^{\mu\nu}_{3} = -\frac{e^2}{4\pi\rho^4}(\xi^2 + a_{\lambda}a^{\lambda})R^{\mu}R^{\nu} \tag{B.14c}$$

[2] For details on symmetrizing the canonical stress tensor, see Jackson, *Classical Electrodynamics, Second Edition*, section 12.10; Landau and Lifshitz, *Classical Theory of Fields*, Section 94.

Each part is characterized by its dependence on the radial coordinate.

While details of the derivation of (B.14) show no contribution from the acceleration current density, which is something of a spectator in the calculation, a divergence calculation must include J_a^ν as a legimate source. In the vacuum gauge $\Theta_1^{\mu\nu}$ is the velocity term so that the acceleration current will be linked to the divergence of $\Theta_2^{\mu\nu}$:

$$\partial_\mu \Theta_1^{\mu\nu} = \frac{1}{c} F^{\lambda\nu} J_{e\,\lambda} \tag{B.15a}$$

$$\partial_\mu \Theta_2^{\mu\nu} = \frac{1}{c} F^{\lambda\nu} J_\nu^a \tag{B.15b}$$

$$\partial_\mu \Theta_3^{\mu\nu} = 0 \tag{B.15c}$$