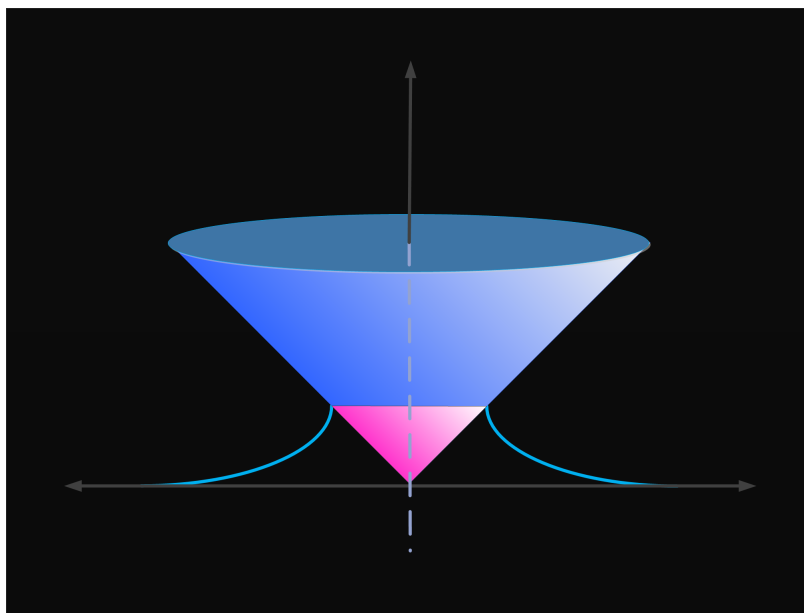


The Radius of the Electron

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Abstract

Causality requires generalized Coulomb fields of the classical electron to be carriers of momentum and energy. The inclusion of this energy requires the use of vacuum gauge potentials having a physical interpretation as deformations of the vacuum in the neighborhood of the charge. Inherent similarities with the classical theory of continuous media serve as a guide for the development of a *Vacuum Lagrangian*, reproducing all the results of conventional classical electron theory, with the additional benefit of eliminating problems of infinite self-energy and particle stability. The vacuum theory accomodates particle accelerations with the inclusion of an acceleration stress tensor, appearing as a perturbation to the velocity theory, and correctly generating additional acceleration terms of the symmetric stress tensor.

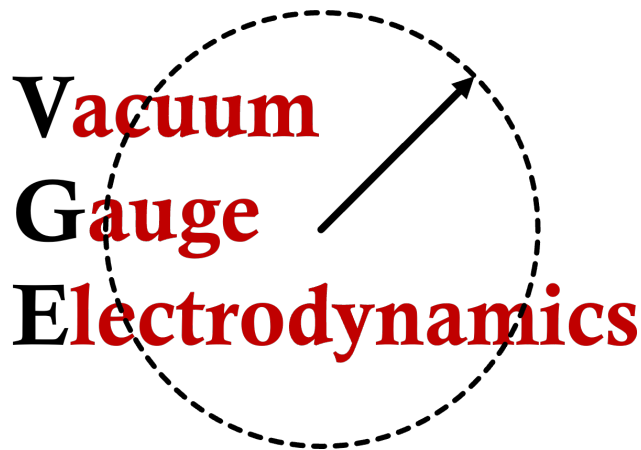
Links to youtube videos:

<https://youtu.be/ymgGwbXY50g> Working with VG potentials

<https://youtu.be/y6dK2URX8v8> Repeated Root

<https://youtu.be/YK9H41oblBI> Inertial Vacuum Power

<https://youtu.be/Vo06WXd5Xqc> Dirac Spinor



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1 Theory of the Classical Vacuum

1.1 Vacuum Gauge Potentials

For a successful model of a classical electron, consider a theory where the generalized Coulomb fields are re-interpreted as momentum flux fields. This can be accomplished by introducing a fundamental surface charge density $\sigma_e < 0$ and writing

$$\boldsymbol{\pi}_E = \sigma_e \mathbf{E}_v \qquad \boldsymbol{\pi}_B = \sigma_e \mathbf{B}_v \qquad (1.1)$$

Assume these fields are still capable of exerting forces on test particles, but now possessing the property of radiation fields. A theory of the electron based on (1.1) is most easily constructed with an application of the vacuum gauge condition:

$$|\partial_\nu A^\nu| \equiv \sqrt{E^2 - B^2} \qquad (1.2)$$

Here, the right side of the equation is assumed to be a positive root which mandates the use of absolute value signs on the left to account for the negative charge¹. In general, potentials satisfying the vacuum gauge condition are not unique, but if the additional requirement $A^\nu A_\nu = 0$ is imposed, this leads to the set of lightlike vacuum gauge potentials

$$A^\nu = \left[\frac{eR}{\rho^2}, \frac{e\mathbf{R}}{\rho^2} \right] \qquad (1.3)$$

For constant velocity motion the field strength tensor follows by writing the well known relation $F_v^{\mu\nu} = [\partial^\mu, A^\nu]$.

An important property of the vacuum gauge condition can be understood by appending to the potentials in (1.3) a complex exponential

$$e^{i\omega t_r/\gamma} \quad \text{where} \quad t_r = t - R/c + \tau_e \qquad (1.4)$$

Here t_r is the retarded time which has been written with a phase defined by $c\tau_e = r_e$. The potentials are now travelling spherical waves and determine the velocity field strength tensor given by

$$\begin{aligned} F_v^{\mu\nu} &= [\partial^\mu A^\nu - \partial^\nu A^\mu] \cdot e^{i\omega t_r/\gamma} + [A^\nu \partial^\mu - A^\mu \partial^\nu] \cdot e^{i\omega t_r/\gamma} \\ &= F_M^{\mu\nu} \cdot e^{i\omega t_r/\gamma} \end{aligned} \qquad (1.5)$$

In this equation the subscript M indicates the well-known Maxwell field strength tensor—now flanked by the same complex exponential applied to the potentials. The implication here is that the fields and the potentials still satisfy the vacuum gauge condition (1.2).

¹The theory described by (1.1) can be easily generalized to include the positron. Instead, for this investigation, we will assume the positron has not been discovered yet.

1.2 Momentum Flux in Electron Velocity Fields

The wave-like character of the vacuum gauge potentials and the resulting field strength tensor will require the fields to be carriers of both momentum and energy. It is easiest to begin in the rest frame with scalar and vector potentials given by

$$A = \frac{e}{r} e^{i\omega(t-r/c+\tau_e)} \quad \mathbf{A} = \frac{e}{r} \hat{\mathbf{r}} e^{i\omega(t-r/c+\tau_e)} \quad (1.6)$$

The vector potential has a longitudinal polarization and both potentials are travelling spherical waves having a structure similar to pressure and velocity waves in continuum mechanics. When combined, they produce an average energy flux vector given by

$$\mathbf{S}_E = \frac{c}{2a_e} A^* \mathbf{A} \quad a_e = 4\pi r_e^2 \quad (1.7)$$

The charge of the electron is $e = \sigma_e a_e$. In terms of the classical Coulomb field, energy- and momentum-flux fields follow by writing

$$\mathbf{S}_E = \frac{1}{2} \boldsymbol{\pi}_E c \quad \text{where} \quad \boldsymbol{\pi}_E = \sigma_e \mathbf{E} \quad (1.8)$$

Beginning with (1.8), a moving frame anti-symmetric flux tensor follows by applying a Lorentz transformation to the anti-symmetric rest frame tensor given by

$$\pi^{\mu\nu} = \begin{bmatrix} 0 & -\boldsymbol{\pi}_E \\ \boldsymbol{\pi}_E & \hat{0} \end{bmatrix} \quad (1.9)$$

The resulting tensor may be written $\pi^{\mu\nu} = \sigma_e F_v^{\mu\nu}$ and features an additional magnetic flux $\boldsymbol{\pi}_B = \hat{\mathbf{n}} \times \boldsymbol{\pi}_E$ circling around the particle at speed c .

A classical electron endowed with radiating Coulomb fields will be associated with *inertial vacuum power*. Of the many recipes for calculating this power, the simplest approach is to multiply the vector potential by c and interpret the resulting vector field as an electric current flowing through the particle radius. Evaluating the voltage and the current at the particle radius will then generate vacuum power by the well-known formula

$$P_{in} = \frac{1}{2} V^* I \quad \text{where} \quad \begin{cases} V = \frac{e}{r_e} e^{i\omega t} \\ I = \frac{ec}{r_e} \hat{\mathbf{r}} e^{i\omega t} \end{cases} \quad (1.10)$$

This result is striking and suggests that the even the quantum of electromagnetism possesses properties of an electric circuit. Note too that the voltage and current can be combined to produce a radiation resistance. This qualifies the classical electron as a fundamental radiating antenna.

Now a theory of the classical electron, constructed from a surface charge density σ_e , will determine the mass-energy from the classical relation $mc^2 = e^2/2r_e$. This allows the mass-energy of the particle to be written

$$mc^2 = P_{in} \cdot \tau_e \quad (1.11)$$

If inertial vacuum power radiates from a point at the center of the electron, this equation determines the mass-energy of the particle as the total amount of field energy inside the particle radius before being emitted into the vacuum. However, even with this neat and orderly interpretation of particle mass, the value of inertial vacuum power is disturbing at more than 10^{10} watts. This has all the makings of an astrophysical “vacuum catastrophe” leading to the suggestion that mass-energy equivalence may not be valid outside the particle radius.

Quantum Theory of the Radiation Field: Although the theory under development here is purely classical, it will be useful to briefly discuss a quantum mechanical view of the Coulomb field as a continuous stream of radiated particles with well defined energy and momentum. For this, introduce Plancks constant and write a formula for the frequency with which energy ε is dispensed to the vacuum through the particle radius:

$$\omega_D = \frac{P_{in} \cdot \tau_e}{\hbar} = \frac{mc^2}{\hbar} \quad (1.12)$$

This is the invariant Dirac frequency implying that inertial vacuum power can also be written

$$P_{in} = \varepsilon \cdot \omega_D \quad \text{where} \quad \varepsilon = \frac{\hbar}{\tau_e} \quad (1.13)$$

If ε is shared among two particles emitted back-to-back, then momentum is conserved and the energy violating decay law is

$$p^\nu \longrightarrow p^\nu + \varepsilon_1^\nu + \varepsilon_2^\nu \quad (1.14)$$

In general, energies of individual particles will show Doppler shifts in a moving frame, but the rest frame energy of both emitted particles is

$$\varepsilon_1 = \frac{1}{2}pc = \varepsilon_2 \quad \text{where} \quad p = \hbar/r_e \quad (1.15)$$

Accordingly, the rest frame emitted wavelength is $\lambda = 2\pi r_e$ and determining emitted energies of approximately 130 MeV.

1.3 Covariant Theory of Radiating Velocity Fields

In an arbitrary inertial frame, vacuum gauge velocity potentials may be written most simply as

$$A^\nu = \frac{eR^\nu}{\rho^2} \quad (1.16)$$

An important property of these potentials is their natural decomposition in terms of mutually orthogonal time-like and space-like components:

$$A^\nu = A_t^\nu + A_s^\nu \quad \text{where} \quad \begin{cases} A_t^\nu = \frac{e}{\rho} \beta^\nu \\ A_s^\nu = \frac{e}{\rho} U^\nu \end{cases}$$

The norms of both four-vectors are the same to within an overall sign and their similar structure promotes a designation as complimentary un-identical twins. For reference, the spacetime diagram of Figure 1 is included to illustrate the rotation of the Liénard–Wiechert potentials onto the light cone by the gauge field A_s^ν . Compatibility with Einstein’s energy-momentum relation can be exhibited by evaluating complimentary components at the electron radius $\rho = r_e$ and then forming the vector $S^\nu = p^\nu + mcU^\nu$. A contraction with lowered indices then implies $p^\nu p_\nu = m^2 c^2$. Now construct

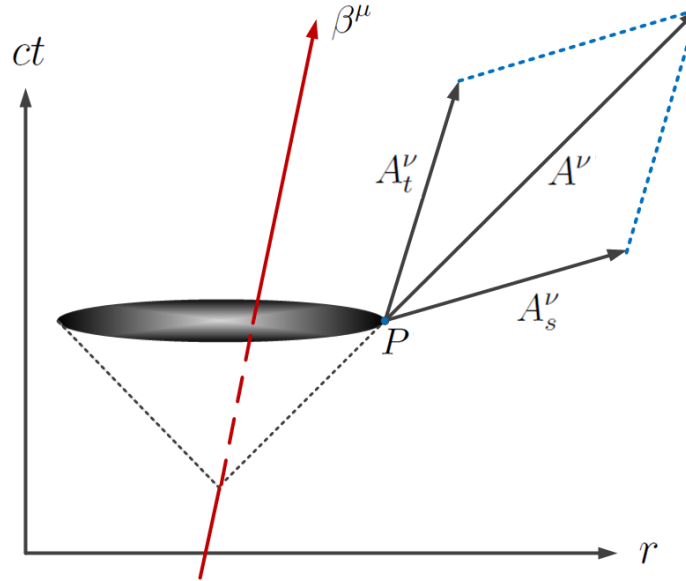


Figure 1: *Minkowski space diagram showing the addition of components of vacuum gauge potentials*

a momentum flux field strength tensor by appending complex exponentials to the time-like and space-like potentials

$$\pi^{\mu\nu} = \frac{1}{a_e} [A_s^{*\mu}, A_t^\nu] \quad \text{where} \quad \begin{cases} A_t^\nu = \frac{e}{\rho} \beta^\nu \cdot e^{i\omega t_r/\gamma} \\ A_s^\nu = \frac{e U^\nu}{\rho} \cdot e^{i\omega t_r/\gamma} \end{cases} \quad (1.17)$$

In the moving frame it is reasonable to derive the invariant formula for inertial vacuum power similar to the rest frame calculation. The potentials still qualify as a scalar voltage and vector current, but both receive “Doppler shifts” in the forward direction by special relativity. In this case the power integral is

$$P_{in} = \frac{c}{2a_e} \oint \mathbf{A} \mathbf{A} \cdot \hat{\mathbf{n}} R^2 (1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta}) d\Omega = \frac{mc^2}{\tau_e} \quad (1.18)$$

For the remainder of this investigation momentum flux in electron velocity fields will be assumed without any necessity to justify its presence in terms of the average momentum associated with the propagation of vacuum waves.

1.4 Vacuum Stress and Strain Tensors

The nature of vacuum gauge potentials as radial null-potentials is an indication of their ability to describe the physics of a single electron. In fact, the vacuum gauge can be used to develop a theory of the vacuum by a simple re-definition of the velocity potentials as a *vacuum displacement vector* or *vacuum dilatation vector*:

$$A^\nu \equiv 4\pi\sigma_e \mathcal{U}^\nu \quad \text{where} \quad \mathcal{U}^\nu = \frac{r_e^2}{\rho^2} R^\nu \quad (1.19)$$

The four-vector \mathcal{U}^ν is purely geometrical and describes a small orifice created by deforming the “medium” around the instantaneous retarded position of the particle out to a radius r_e —hereafter referred to as the classical radius of the electron.

Second rank tensors required for the new theory can be established beginning with the third rank tensor $\Psi^{\alpha\mu}_\nu$ composed of the four components of the dilatation along with two occurrences of the fourth rank, totally anti-symmetric, Levi-Civita symbol:

$$\Psi^{\alpha\mu}_\nu \equiv \frac{1}{2} \epsilon^{\alpha\mu\sigma\tau} \epsilon_{\sigma\tau\nu\kappa} \mathcal{U}^\kappa \quad (1.20)$$

Now consider a divergence operation on either of the last two indices. The inclusion of the metric is appropriate here to raise the last index with the result

$$\partial_\alpha \Psi^{\alpha\mu\nu} = \partial_\alpha [g^{\nu\lambda} \Psi^{\alpha\mu}_\lambda] = \partial^\nu \mathcal{U}^\mu - g^{\mu\nu} \partial_\alpha \mathcal{U}^\alpha \quad (1.21)$$

The first term on the right in this equation can be established as a unitless vacuum strain tensor:

$$\boxed{\eta^{\mu\nu} = \eta^{\dagger\nu\mu} \equiv \partial^\mu \mathcal{U}^\nu} \quad \text{vacuum strain tensor} \quad (1.22)$$

A concise representation of a vacuum stress tensor may then be determined from the entire right side of (1.21) with the inclusion of the modulus $\mu_e = 4\pi\sigma_e^2$:

$$\boxed{\Delta^{\dagger\mu\nu} \equiv \mu_e [\eta^{\nu\mu} - g^{\mu\nu} \eta]} \quad \text{vacuum stress tensor} \quad (1.23)$$

The peculiar inversion of the indices on the previous two equations justifies the inclusion of the superscript (\dagger) transpose operator in the definitions. For the stress tensor however, this and the un-transposed version $\Delta^{\mu\nu}$ will both be useful in the four-space theory of the electron.

Equation (1.23) defines a medium which can be characterized through comparisons with the 3-space theory of elastic media. For the 3-space theory, the medium is tacitly assumed to be embedded in a Galilean spacetime featuring independent temporal and spatial dimensions. For the vacuum theory under development here, equation (1.23) describes a deformable medium residing in four-dimensional Minkowski spacetime. While an elastic medium also comes complete with a unique undeformed ground state energy (or mass) density, this is an impossible requirement for the medium described by (1.23) which can be shown to possess both a restoring and admissibility property. Admissibility implies a negative force constant which, unlike Hooke's law, will allow the medium to create itself indefinitely. This medium can only move at the speed of light, and its specific density at any spacetime event will depend on the overall distribution of charged electrons, positrons, and other particles. Referring to this medium as the *vacuum*, a precise definition can be given as:

Vacuum— A self-generating deformable medium, embedded in Minkowski spacetime, and characterized by restoring and admissibility force constants having equal magnitudes.

The vacuum will be shown to form stable electrons (and positrons) continuously propagating electromagnetic momentum and energy via (1.1) during constant velocity motion, with additional (and more familiar) energy flux radiated during particle accelerations.

Explicit forms for the stress tensor may be determined by differentiating equation (1.19) or by differentiating timelike and spacelike components separately, and combining them together. The spacelike portion is symmetric and may be written in terms of differential operations on the scalar field φ . The timelike portion is easier to calculate and provides a single off-diagonal term to the final result:

$$\Delta^{\mu\nu} = \mu_e \frac{r_e^2}{\rho^2} [2U^\mu U^\nu - \beta^\mu \beta^\nu + U^\mu \beta^\nu] \quad (1.24)$$

Curiously, this tensor does not contain the metric as might be implied by its definition. Instead, the strain tensor carries the metric which is subtracted away.

Vacuum Dilatation: An important correspondence between the theory of elastic media and the theory presented here is determined by

$$\partial_\nu u^\nu = \partial_\nu u_s^\nu = \square^2 \varphi \quad \longleftrightarrow \quad \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{u}_l = -\nabla^2 \varphi \quad (1.25)$$

According to the 3-space theory, the divergence on the right measures the net distortion, or volumetric dilatation, of the medium. This suggests that the unitless

four-divergence

$$\partial_\nu \mathcal{U}^\nu = \frac{r_e^2}{\rho^2} \quad (1.26)$$

is really a ***vacuum dilatation*** gauge condition—a measure of the amount by which the surrounding vacuum is deformed by the presence of the particle. This is an inherently positive quantity which has the effect of compressing the vacuum around the charge. As already indicated this is interpreted to mean that the dilatation is creating a microscopic spherical orifice (void) in the vacuum having a classical radius r_e . This is a very desirable feature of any classical theory of the electron to address—among other things—the well known problem of the divergent self-energy. Moreover, a particle radius in the context of an appropriately chosen gauge condition necessarily avoids the introduction of form factors and other senseless constructs based on macroscopic classical notions. The theory of vacuum dilatation can be compared with the Liénard-Wiechert potentials where the application of the Lorenz gauge condition $\partial_\nu A_t^\nu = 0$ generates a dilatation of zero and is unmistakably associated with a point-particle theory.

1.5 Differential Operations on the Vacuum Stress Tensor

The divergence operator applied to the second index of the vacuum stress tensor is an identity

$$\partial_\nu \Delta^{\mu\nu} = 0 \quad (1.27)$$

This can be proved from the definition in equation (1.23), or by applying ∂_ν directly to (1.24) where the divergence of each individual term can be shown to vanish. Applying the divergence operator to the first index is more complex and can be shown to be proportional to the acceleration four-current density J_a^ν when the position of the point-source is excluded. However, as with all terms involving particle accelerations, vacuum gauge electrodynamics will assign J_a^ν to an independent vacuum theory of acceleration strain. Withholding particle acceleration for now, a divergence on the first index follows from

$$\partial_\mu \Delta^{\mu\nu} = \partial_\mu \left[\frac{1}{2}(\Delta^{\mu\nu} + \Delta^{\nu\mu}) + \frac{1}{2}(\Delta^{\mu\nu} - \Delta^{\nu\mu}) \right] = \frac{1}{2}\partial_\mu \Delta^{\mu\nu} + \frac{1}{2}\sigma_e \partial_\mu F_v^{\mu\nu} \quad (1.28)$$

From this it can only be concluded that

$$\partial_\mu \Delta^{\mu\nu} = 4\pi\sigma_e \frac{1}{c} J_e^\nu \quad (1.29)$$

where J_e^ν is the conventional point charge source current of the electromagnetic theory. This equation differs from the potential formulation of Maxwell's equations by a constant σ_e , yet derived from a completely different premise. The four equations are essentially the inhomogeneous Maxwell equations except that they identify two gauge

invariant forms of momentum flux $\boldsymbol{\pi}_E$ and $\boldsymbol{\pi}_B$. As the homogeneous equations follow automatically from differential operations on vacuum gauge potentials, one finds

$$\boldsymbol{\nabla} \cdot \boldsymbol{\pi}_E = 4\pi\rho_e \quad \boldsymbol{\nabla} \times \boldsymbol{\pi}_B = 4\pi\mathbf{f}_e + \frac{1}{c} \frac{\partial \boldsymbol{\pi}_E}{\partial t} \quad (1.30a)$$

$$\boldsymbol{\nabla} \times \boldsymbol{\pi}_E = -\frac{1}{c} \frac{\partial \boldsymbol{\pi}_B}{\partial t} \quad \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_B = 0 \quad (1.30b)$$

and it may be observed that the source $f_e^\nu = (\rho_e, \mathbf{f}_e)$ is now a point force density. An energy flux version of these equations follows by multiplying the above by an additional factor of $c/2$. In addition to momentum and energy flux in the fields, conventional classical electrodynamics also requires additional energy flux defined by the Poynting vector. This is still valid for the vacuum theory except the form of the equation must be slightly modified to read

$$\mathbf{S} = \frac{c}{\mu_e} \boldsymbol{\pi}_E \times \boldsymbol{\pi}_B \quad (1.31)$$

1.6 Covariant Integration of the Vacuum Stress Tensor

The radiation rate four-vector can be determined by integrations over both timelike and spacelike surfaces. First, consider the timelike 3-surface element illustrated in Figure 2 given by

$$d\sigma_\nu^t = U_\nu R^2 d\Omega d\tau \quad (1.32)$$

This leads to the flux integral²

$$\mathcal{E}^\mu(\tau) = \oint_{\rho=r_e} \frac{1}{2} \Delta^{\dagger\mu\nu} d\sigma_\nu^t = -\frac{1}{2} \dot{\varrho} c \tau \beta^\mu \cdot \vartheta(\tau) \quad (1.34)$$

Of interest here is the conspicuous appearance of the minus sign. Moreover, there is a temptation to include an integration constant and write the total energy four-vector as

$$\mathcal{E}^\mu(\tau) = [mc^2 - \frac{1}{2} \dot{\varrho} c \tau] \beta^\nu \cdot \vartheta(\tau) = \mathcal{H}(\tau) \beta^\mu \cdot \vartheta(\tau) \quad (1.35)$$

where \mathcal{H} is the invariant Hamiltonian for the system.

Vacuum power also results by choosing the spacelike hyper-surface pointing along the direction of the 4-velocity given by

$$d\sigma_\nu^s = \beta_\nu \rho^2 d\rho d\Omega' \quad (1.36)$$

²This integral is easy to evaluate as the moving frame solid angle integral in an arbitrary reference frame is

$$\int_\Omega U^\mu d\Omega' = 0 \quad (1.33)$$

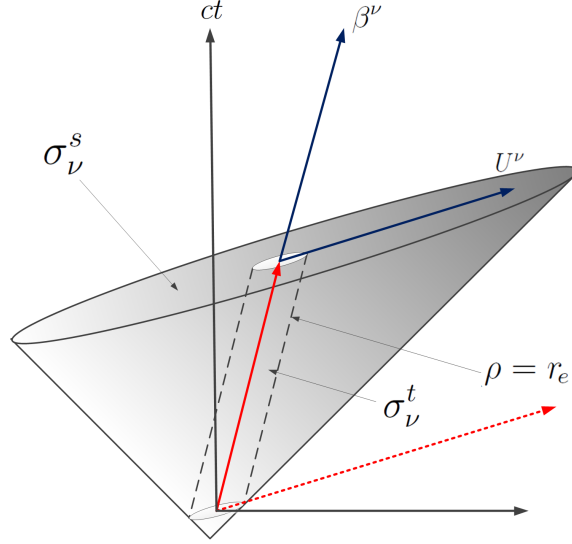


Figure 2: *Illustrating timelike and spacelike surfaces of integration for the vacuum stress tensor.*

Once again, $\mathcal{E}^\mu(\tau)$ is determined from the integral

$$\begin{aligned}\mathcal{E}^\mu(\tau) &= \frac{1}{2} \int \Delta^{\dagger\mu\nu} d\sigma_\nu^s \\ &= -\frac{1}{2} c \mu_e r_e^2 \beta^\mu \int \vartheta(\tau - \rho/c + \tau_e) d\rho d\Omega'\end{aligned}\quad (1.37)$$

To perform the last integral it is necessary to require the lower limit $\rho = r_e$, which is mandated by the vacuum dilatation. The upper integration limit can be anything since it is under control of the causality step. Adding an integration constant as before yields exactly the final result produced by the timelike integral.

2 Lagrangian Formulation of the Vacuum Gauge Electron

The correspondence between a radiation-based theory of the classical electron and the theory of elastic continua can be used as a guide for the development of a highly specialized Lagrangian formulation. A strictly inflexible requirement for a successful Lagrangian is the re-production of all the usual gauge invariant results of the conventional Maxwell-Lorentz electron theory. In particular, the vacuum Lagrangian \mathcal{L}_{vac} must generate the symmetric stress tensor of the conventional theory. In addition, stresses associated with radiated vacuum energy must also be generated.

2.1 Derivation of the Vacuum Lagrangian

A specific form for \mathcal{L}_{vac} could easily be postulated based on definitions (1.22) and (1.23), and knowledge of continuum mechanics, but a more suitable approach is to begin with the classical electromagnetic Lagrangian interacting with the source term

$$\mathcal{L}_{em} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - \frac{1}{c}J_e^\nu A_\nu \quad (2.1)$$

But stress and strain tensors can be inserted through the relations

$$\sigma_e F^{\mu\nu} = \Delta^{\mu\nu} - \Delta^{\nu\mu} = \mu_e(\eta^{\mu\nu} - \eta^{\nu\mu}) \quad (2.2)$$

Re-arranging constants in the interaction term leads to

$$\begin{aligned} \mathcal{L}_{em} &= -\frac{1}{4}(\Delta^{\mu\nu} - \Delta^{\nu\mu})(\eta_{\mu\nu} - \eta_{\nu\mu}) - \frac{1}{c}J_e^\nu A_\nu \\ &= -\frac{1}{2}\Delta^{\mu\nu}\eta_{\mu\nu} + \frac{1}{2}\Delta^{\nu\mu}\eta_{\mu\nu} - 4\pi f_e^\nu \mathcal{U}_\nu = \mathcal{L}_o + \mathcal{L}_x + \mathcal{L}_{int} \end{aligned} \quad (2.3)$$

It is easy to show that the term \mathcal{L}_x produces no equations of motion and removing it from the theory is a tempting possibility. But if equivalence between the electromagnetic Lagrangian and the vacuum theory is a fundamental requirement, then it is more appropriate to include \mathcal{L}_x along with \mathcal{L}_o and define the equivalent two component **Vacuum Lagrangian** in terms of the field quantities \mathcal{U}^ν and $\eta^{\mu\nu}$:

$$\mathcal{L}_{vac} = -\frac{1}{2}\Delta^{\mu\nu}\eta_{\mu\nu} - 4\pi f_e^\nu \mathcal{U}_\nu \quad (2.4a)$$

$$\mathcal{L}_x = \frac{1}{2}\Delta^{\nu\mu}\eta_{\mu\nu} \quad (2.4b)$$

The legitimacy of the vacuum Lagrangian can be immediately verified by applying the Euler-Lagrange equations resulting in the momentum flux version of the Maxwell-Lorentz theory of (1.30)

$$\partial_\mu \Delta^{\mu\nu} = 4\pi f_e^\nu \quad (2.5a)$$

$$\partial_\mu \Delta^{\nu\mu} = 0 \quad (2.5b)$$

It seems reasonable to consider the vacuum Lagrangian and its equation of motion as a symmetry property of classical electron theory. If required, the conventional theory can be instantly recovered simply by subtracting equations (2.5).

Lagrangian for a Moving Vacuum: The second term of the vacuum stress tensor is proportional to the metric and shows three positive definite space-space diagonal components. These terms represent the admissibility property of the stress tensor allowing the vacuum continuum to self-generate. The Lagrangian in equation (2.4a) can only accommodate this feature with an additional term. It must be linear in field

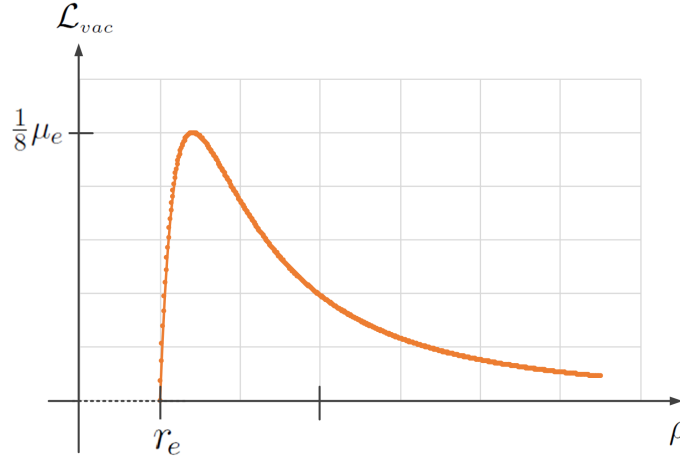


Figure 3: Plot of \mathcal{L}_{vac} versus the invariant scalar ρ . The value of the Lagrangian minimizes at the boundary $\rho = r_e$ where it vanishes. Inside the boundary assume \mathcal{L}_{vac} is not defined.

quantities and, like the term \mathcal{L}_x , can have no effect on the equations of motion. A simple and meaningful possibility is to observe that the vacuum Lagrangian possesses an internal symmetry under the variation

$$\Delta^{\mu\nu} \longrightarrow \Delta^{\mu\nu} - \mu_e g^{\mu\nu} \quad (2.6)$$

The resulting interacting velocity Lagrangian is then

$$\mathcal{L}_{vac} \equiv -\frac{1}{2}\Delta^{\mu\nu}\eta_{\mu\nu} + \frac{1}{2}\mu_e\eta - 4\pi f_e^\nu \mathcal{U}_\nu \quad (2.7)$$

The linear term should be properly referred to as a vacuum dilatation producing dilatation stresses which—as already indicated—function to propagate field energy away from the source. Its presence is reminiscent of a term added to a Lagrangian in point particle mechanics to represent forces of constraint. The analogy has some grey area but μ_e assumes the role of a Lagrange multiplier. For convenience, a plot of $\mathcal{L}_{vac}(\rho)$ is included in Figure 3.

2.2 Symmetric and Total Stress Tensors

The Noether current generated from invariance under the infinitesimal translation group $x^\mu \rightarrow x^\mu + \epsilon^\mu$ is the *canonical stress tensor*

$$T_{vac}^{\mu\nu} = \frac{\partial \mathcal{L}_{vac}}{\partial \eta_{\mu\lambda}} \eta^\nu{}_\lambda - g^{\mu\nu} \mathcal{L}_{vac} \quad (2.8)$$

To facilitate the construction of an appropriate stress tensor based on (2.7) it is beneficial to begin by considering only terms quadratic in field quantites for which

$$T_{vac}^{\mu\nu} = \frac{1}{2}g^{\mu\nu}\Delta^{\alpha\lambda}\eta_{\alpha\lambda} - \Delta^{\mu\lambda}\eta^\nu{}_\lambda \quad (2.9)$$

By design, the conservation law implied by Noether's theorem is $\partial_\mu T_{vac}^{\mu\nu} = 0$. It can be verified explicitly for the vacuum theory by considering only the second term on the right in (2.9) and differentiating:

$$\partial_\mu(\Delta^{\mu\lambda}\eta^\nu{}_\lambda) = \Delta^{\mu\lambda} \cdot \partial_\mu\eta^\nu{}_\lambda = \Delta^{\mu\lambda} \cdot \partial^\nu\eta_{\mu\lambda} = \frac{1}{2}\partial^\nu(\Delta^{\mu\lambda}\eta_{\mu\lambda}) \quad (2.10)$$

To determine the symmetric stress tensor from (2.8) it is only necessary to write out two individual pieces

$$\Theta_{vac}^{\mu\nu} = \mu_e \left[\frac{1}{2}g^{\mu\nu}\eta^2 - \eta^{\mu\lambda}\eta^\nu{}_\lambda \right] \quad (2.11a)$$

$$T_D^{\mu\nu} = \mu_e\eta\eta^{\nu\mu} \quad (2.11b)$$

As with the conventional theory, $T_D^{\mu\nu}$ has a non-symmetric component and can be eliminated by showing that $\partial_\mu T_D^{\mu\nu} = 0$ which leaves only the symmetric tensor. If symmetrizing this way is deemed unsatisfactory, it is reasonable to include contributions to $T_D^{\mu\nu}$ resulting from the excluded Lagrangian \mathcal{L}_x . In this case

$$\begin{aligned} T_D^{\mu\nu} &\longrightarrow \mu_e\eta\eta^{\nu\mu} + \Delta^{\lambda\mu}\eta^\nu{}_\lambda - \frac{1}{2}g^{\mu\nu}\Delta^{\lambda\alpha}\eta_{\alpha\lambda} \\ &= \mu_e \left[\eta^{\lambda\mu}\eta^\nu{}_\lambda - g^{\mu\nu}\eta^2 \right] = \frac{1}{4\pi}F^{\lambda\mu}\partial_\lambda A^\nu \end{aligned} \quad (2.12)$$

The term on the far right lacks gauge invariance and is the same leftover term that must be removed in the conventional theory.

A divergence operation on the symmetric stress tensor is somewhat different than the conventional theory and follows by re-inserting the vacuum tensor so that

$$\partial_\mu\Theta_{vac}^{\mu\nu} = \mu_e \left[\frac{1}{2}\partial^\nu\eta^2 - \partial_\mu\eta\eta^{\nu\mu} \right] - \partial_\mu\Delta^{\mu\lambda}\eta^\nu{}_\lambda - \Delta^{\mu\lambda} \cdot \partial_\mu\eta^\nu{}_\lambda = -4\pi f_e^\lambda\eta^\nu{}_\lambda \quad (2.13)$$

The source term here is composed of the point force density f_e^λ contracted with the strain tensor. Nevertheless, in the language of conventional electrodynamics, it is exactly the Lorentz force density associated with a point electron interacting with its own fields. This implies an equivalence relation linking the electromagnetic and vacuum gauge theories

$$\Theta_{vac}^{\mu\nu} \equiv \Theta_{em}^{\mu\nu} \quad (2.14)$$

This equivalence will be extended to include particle accelerations in section 3.

Applying equation (2.8) to the propagation term of the Lagrangian is rather trivial here, providing an additional stress

$$\Lambda^{\mu\nu} \equiv \frac{1}{2}\Delta^{\dagger\mu\nu} \quad (2.15)$$

which stands on its own without need of any symmetrization. Placing this term next to the symmetric stress tensor then defines the **Total Vacuum Stress Tensor**:

$$\mathcal{T}^{\mu\nu} \equiv \mu_e \left[\frac{1}{2} g^{\mu\nu} \eta^2 - \eta^{\mu\lambda} \eta^\nu{}_\lambda \right] + \Lambda^{\mu\nu} \quad (2.16)$$

Since $\partial_\mu \Lambda^{\mu\nu} = 0$, the differential law for the total vacuum stress tensor is no different than the conventional theory

$$\partial_\mu \mathcal{T}^{\mu\nu} = -4\pi f_e^\lambda \eta^\nu{}_\lambda \quad (2.17)$$

A useful representation of $\Lambda^{\mu\nu}$ follows by grouping the components as³

$$\begin{aligned} U_\Lambda &\equiv -\frac{1}{2} \mu_e \nabla \cdot \mathbf{u} & \frac{1}{c} \mathbf{S}_V &\equiv -\frac{1}{2} \mu_e \nabla u \\ \frac{1}{c} \mathbf{S}_A &\equiv \frac{1}{2} \mu_e \frac{\partial \mathbf{u}}{\partial ct} & \hat{\mathbf{T}}_\Lambda &\equiv \frac{\mu_e}{2} [-\nabla \mathbf{u} + \mathbf{1} \partial_\lambda u^\lambda] \end{aligned} \quad (2.18)$$

Individual elements are the canonical work-energy density U_Λ , six “components” of the canonical energy flux vectors \mathbf{S}_V and \mathbf{S}_A , and the canonical stress-strain tensor $\hat{\mathbf{T}}_\Lambda$. Now add a well known representation of the symmetric stress tensor and write the total stress tensor as

$$\mathcal{T}^{\mu\nu} = \begin{bmatrix} U & \frac{1}{c} \mathbf{S} \\ \frac{1}{c} \mathbf{S} & -\hat{\mathbf{T}} \end{bmatrix} + \begin{bmatrix} U_\Lambda & \frac{1}{c} \mathbf{S}_V \\ \frac{1}{c} \mathbf{S}_A & \hat{\mathbf{T}}_\Lambda \end{bmatrix} \quad (2.19)$$

As a reminder all elements of the symmetric stress tensor are derived from vacuum gauge potentials. For example, energy flux in the vacuum theory should properly be determined from the formula

$$\mathbf{S} = \mu_e \nabla u^\nu \frac{\partial u_\nu}{\partial t}$$

although this quantity has already been constructed previously from the flux fields.

2.3 Particle Stability in the Rest Frame

The most straight forward application of the formalism in (2.19) is in the rest frame where the presence of the vacuum tensor provides a simple solution for a stable particle of radius r_e . Most of the energy flux components are zero and (2.19) reduces to

$$\mathcal{T}^{\mu\nu} = \begin{bmatrix} U & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + \begin{bmatrix} U_\Lambda & \frac{1}{c} \mathbf{S}_V \\ 0 & \hat{\mathbf{T}}_\Lambda \end{bmatrix} \quad (2.20)$$

³The subscript V appended to the energy flux \mathbf{S}_V has been borrowed from the conventional theory and only serves as a means of distinction from the energy flux \mathbf{S}_A .

Using $(c\tau, \mathbf{r})$ as rest frame coordinates, the total stress-energy at some time τ may be written as a combination of the light sphere and the infinite space tensor $\mathcal{T}_M^{\mu\nu}$. The integral to be evaluated takes the form

$$\mathcal{E}^{\mu\nu}(\tau) = \int \mathcal{T}_M^{\mu\nu} \cdot \vartheta(\tau - r/c + \tau_e) d^3r \quad (2.21)$$

Components of $\mathcal{T}^{\mu\nu}$ are determined entirely by the vacuum dilatation vector, which requires that the radial integration have a lower limit r_e . As indicated by the illustration in Figure 4, this occurs at time $c\tau = 0$. Also shown is the upper integration limit chosen to be the finite time $c\tau + r_e$. A calculation of all 16 integrals given by (2.21) is

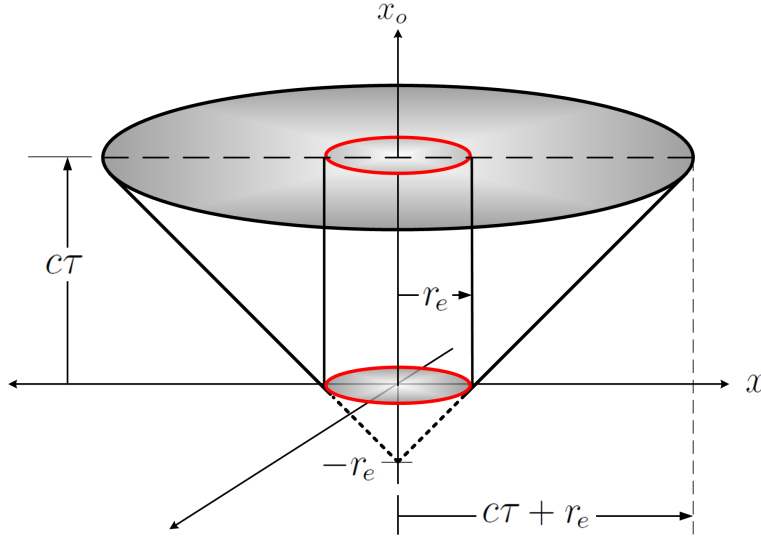


Figure 4: Spacetime diagram indicating the expansion of the radial step $\vartheta(\tau - r/c + \tau_e)$ in the rest frame. The radius r_e divides the causal light cone into interior and exterior four-volumes.

not necessary since several of the components share similar characteristics. Essential details of the integrations fit appropriately into 3 groups:

1. Total Energy:

$$\mathcal{T}^{\tau\tau} = \frac{1}{2}\mu_e(\partial_\nu \mathcal{U}^\nu)^2 - \frac{1}{2}\mu_e \nabla \cdot \mathbf{u} = \frac{1}{2}\mu_e \left[\frac{r_e^4}{r^4} - \frac{r_e^2}{r^2} \right] \cdot \vartheta$$

The energy density is zero at the electron radius. Integrals of each term over all space are complicated by the presence of the causality sphere and may be performed with different levels of mathematical rigor. Results generate the total invariant energy, or total particle Hamiltonian

$$\mathcal{H} = mc^2 \cdot \vartheta(\tau) - \frac{1}{2}\dot{\rho}c\tau \quad (2.22)$$

In this equation $\dot{\rho} = 4\pi\mu_e r_e^2 = \mu_e a_e > 0$ is the total scalar momentum radiated per unit time.

2. Space-Space Terms:

Diagonal pressure terms are all similar. The component \mathcal{T}^{xx} may be written

$$\begin{aligned}\mathcal{T}^{xx} &= -\frac{1}{2}\mu_e(\partial_\nu \mathcal{U}^\nu)^2 - \mu_e \frac{\partial \mathcal{U}^\nu}{\partial x} \frac{\partial \mathcal{U}_\nu}{\partial x} + \frac{1}{2}\mu_e \left[\partial_\alpha \mathcal{U}^\alpha - \frac{\partial \mathcal{U}_x}{\partial x} \right] \\ &= \frac{1}{2}\mu_e \left[\frac{r_e^4(-x^2 + y^2 + z^2)}{r^6} + \frac{r_e^2}{r^2} + \frac{r_e^2(x^2 - y^2 - z^2)}{r^4} \right] \cdot \vartheta\end{aligned}$$

The second line shows clearly how the outward pressure from the symmetric stress tensor is removed by the vacuum strain at $r = r_e$. In the language of electrodynamics, this outward pressure is interpreted as resulting from the repulsive force of the elementary charge on itself. In the new language, a restoring force resulting from the deformed vacuum exerts an equal and opposite force providing for a stable particle. Angular integrations are not trivial here but final results for all three space-space diagonal terms are identical. We list the xx-components as:

$$\mathcal{E}_{part}^{xx} = \frac{1}{3}mc^2 \cdot \vartheta(\tau) \qquad \mathcal{E}_{vac}^{xx} = \frac{1}{3}\dot{\rho}c\tau \qquad (2.23)$$

Like the diagonal terms, the off-diagonal terms are all similar to the \mathcal{T}^{xy} component

$$\mathcal{T}^{xy} = -\mu_e \frac{\partial \mathcal{U}^\nu}{\partial x} \frac{\partial \mathcal{U}_\nu}{\partial y} - \frac{1}{2}\mu_e \frac{\partial \mathcal{U}_y}{\partial x} = \mu_e \left[-\frac{r_e^4 xy}{r^6} + \frac{r_e^2 xy}{r^4} \right] \cdot \vartheta$$

Here again the vacuum strain removes the symmetric (or electromagnetic) stress at $r = r_e$. Solid angle integrals then require all off-diagonal pressure terms to be zero.

3. Energy Flux:

In the rest frame, there is no contribution from the time-space and space-time components of the symmetric stress tensor. Contributions to the total energy come only from the vacuum stress tensor and are determined by

$$\frac{1}{c}\mathbf{S}_V = -\frac{1}{2}\mu_e\nabla\mathcal{U} = -\frac{1}{2}\sigma_e\mathbf{E}$$

$$\frac{1}{c}\mathbf{S}_A = \frac{1}{2}\mu_e\frac{\partial\mathbf{u}}{\partial c\tau} = \mathbf{0}$$

The vector \mathbf{S}_V has been written in terms of the rest frame electric field vector. Since this vector is purely radial, the 3-space integral is zero. The vector \mathbf{S}_A represents canonical energy flux based on explicit changes of the vacuum deformation with time and they vanish for the stationary electron.

To summarize, calculations in items 1–3 show that all terms of $\mathcal{T}^{\mu\nu}$ vanish at the radius r_e , except for the presence of residual energy flux propagating away from the source. Appropriate spatial integrations for all terms can be collected together in a matrix format as the total energy tensor:

$$\begin{aligned} \mathcal{E}_{total}^{\mu\nu} &\equiv \mathcal{E}_{part}^{\mu\nu} + \mathcal{E}_{vac}^{\mu\nu} \\ &= \begin{bmatrix} mc^2 & 0 & 0 & 0 \\ 0 & \frac{1}{3}mc^2 & 0 & 0 \\ 0 & 0 & \frac{1}{3}mc^2 & 0 \\ 0 & 0 & 0 & \frac{1}{3}mc^2 \end{bmatrix} \cdot \vartheta(\tau) + \begin{bmatrix} -\frac{1}{2}\dot{\varrho}c\tau & 0 & 0 & 0 \\ 0 & \frac{1}{3}\dot{\varrho}c\tau & 0 & 0 \\ 0 & 0 & \frac{1}{3}\dot{\varrho}c\tau & 0 \\ 0 & 0 & 0 & \frac{1}{3}\dot{\varrho}c\tau \end{bmatrix} \end{aligned} \quad (2.25)$$

The term $\mathcal{E}_{part}^{\mu\nu}$ is a traceless mass-energy tensor just like the symmetric stress tensor from which it derives. Its components are representative of the energy associated with the locally compressed vacuum. In contrast, the term $\mathcal{E}_{vac}^{\mu\nu}$ represents an unstable continuum. In the language of deformable media, it refers to the total canonical stress imparted to the vacuum in a time $c\tau$ by the canonical momentum. The energy component is the total work done on the vacuum which is negative and assumed to result from the admissibility force constant.

2.4 Stability in a Moving Frame: Dirac Electron

Stability relative to an arbitrary frame can be addressed by first defining the unitless quantities

$$\mathcal{G}_1^{\mu\nu} \equiv 2\beta^\mu\beta^\nu - 2U^\mu U^\nu - g^{\mu\nu} \quad (2.26a)$$

$$\mathcal{G}_2^{\mu\nu} \equiv \beta^\mu\beta^\nu + \beta^\mu U^\nu - g^{\mu\nu} = -\partial^\nu R^\mu \quad (2.26b)$$

The symmetric stress tensor and the vacuum tensor are then

$$\Theta_1^{\mu\nu} = \frac{1}{2}\mu_e\eta^2\mathcal{G}_1^{\mu\nu} \quad \Lambda^{\mu\nu} = -\frac{1}{2}\mu_e\eta[\mathcal{G}_1^{\mu\nu} - \mathcal{G}_2^{\mu\nu}] \quad (2.27)$$

and the total stress may be written

$$\mathcal{T}^{\mu\nu} = -(\mathcal{L}_o + \mathcal{L}_\Lambda)\mathcal{G}_1^{\mu\nu} + \mathcal{L}_\Lambda\mathcal{G}_2^{\mu\nu} \quad (2.28)$$

Stability is determined by the first term which vanishes at $\rho = r_e$. What remains only has meaning at the electron radius but its importance is paramount as the particle radius is synonymous with the vacuum boundary. Define this stress by

$$\mathcal{E}_{rad}^{\mu\nu} \equiv \mathcal{L}_\Lambda\mathcal{G}_2^{\mu\nu} \Big|_{\rho=r_e} = -\frac{1}{2}\mu_e\partial^\nu R^\mu \quad (2.29)$$

It may be readily integrated over the boundary to determine the four-vector vacuum radiation rate. Using \mathcal{V}_e as the 3-volume inside the electron radius, the scalar contraction is an easily recognizable quantity

$$\mathcal{E}_{rad} = -\frac{mc^2}{\mathcal{V}_e} \quad (2.30)$$

More importantly however, $\mathcal{E}_{rad}^{\mu\nu}$ can be used as a vehicle to quantize the classical particle. This follows by writing it as a composition of the timelike and spacelike four-vectors

$$\mathcal{E}_{rad}^{\mu\nu} = \frac{mc^2}{4\pi r_e^3} [U^\nu\beta^\mu + \beta^\mu\beta^\nu - \beta^\lambda(\beta_\lambda + U_\lambda)g^{\mu\nu}] \quad (2.31)$$

Terms multiplying the metric are superfluous but are necessary to quantize the particle while enforcing the overall structure of the tensor as a quantity $X^{\mu\nu} - Xg^{\mu\nu}$. Now replace the four-velocity with the four-momentum using $p^\nu = mc\beta^\nu$ and make the quantum mechanical substitution

$$\boxed{U^\nu \longrightarrow -\gamma^\nu} \quad (2.32)$$

where γ^ν are the Dirac matrices. The result is a new tensor

$$\mathcal{E}_{Dirac}^{\mu\nu} = \frac{c}{4\pi r_e^3} \left[-\gamma^\nu p^\mu + \frac{1}{mc} p^\mu p^\nu - mc g^{\mu\nu} + \gamma^\lambda p_\lambda g^{\mu\nu} \right] \quad (2.33)$$

The term inside the brackets contains the Dirac energy-momentum tensor which becomes apparent by selectively substituting $p^\nu = i\hbar\partial^\nu$, then operating on the left and right with Dirac spinor fields

$$\frac{1}{mc} p^\mu p^\nu \bar{\psi}\psi - \mathsf{T}_{Dirac}^{\mu\nu} = 0 \quad (2.34)$$

The Dirac energy-momentum tensor is

$$\mathsf{T}_{Dirac}^{\mu\nu} = \bar{\psi}[i\hbar\gamma^\mu\partial^\nu - g^{\mu\nu}(i\hbar\gamma^\lambda\partial_\lambda - mc)]\psi \quad (2.35)$$

but one can also derive this result from the Dirac Lagrangian determined by the contraction

$$\mathcal{L}_{Dirac} = \bar{\psi} \left[\frac{1}{c} \mathcal{E}_{Dirac} \cdot \mathcal{V}_e \right] \psi \quad (2.36)$$

Applying the Lagrange equations yields

$$\mathsf{T}_{Dirac}^{\mu\nu} = \frac{\partial \mathcal{L}_{Dirac}}{\partial(\partial_\mu \psi)} \cdot \partial^\nu \psi - g^{\mu\nu} \mathcal{L}_{Dirac} \quad (2.37)$$

The simplicity by which Dirac electron theory has emerged from the vacuum gauge electron is impressive. Of importance is the fact that $\mathcal{E}_{Dirac}^{\mu\nu}$ vanishes when operated on by a set of Dirac spinors. This can be interpreted to mean that stress associated with the flow of the vacuum field cannot be known to the quantum mechanical particle. One may consider a classical radiating particle or quantum-mechanical particle—but not both at the same time. This fact is also indicated more directly by the quantization procedure in (2.32) where the spacelike vector can be traced directly to the presence of the gauge field A_s^ν . In the classical theory, the gauge field propagates vacuum energy; but when the gauge field is removed in favor Dirac matrices, field energy yields to the quantum mechanical theory.

Total Energy Tensor in a Moving Frame: Lorentz transformation of $\mathcal{E}^{\mu\nu}$ in equation (2.25) proceeds by transforming individual portions of the tensor separately. In the rest frame, components of the traceless part are

$$\mathcal{K}_{part}^{\mu\nu} = mc^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \vartheta(\tau) \quad \mathcal{P}_{part}^{\mu\nu} = \frac{1}{3}mc \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \vartheta(\tau) \quad (2.38)$$

Applying the general Lorentz transformation $\mathcal{X}^{\mu\nu} = L^\mu_\alpha \mathcal{X}^{\alpha\beta} L^\nu_\beta$ to both portions and combining derives the moving particle energy tensor

$$\mathcal{E}_{part}^{\mu\nu} = [mc^2\beta^\mu\beta^\nu + \frac{1}{3}mc^2(\beta^\mu\beta^\nu - g^{\mu\nu})] \cdot \vartheta(\tau) \quad (2.39)$$

A scalar contraction with the metric may be implemented which—upon removal of a spurious factor—leads to the Einstein formula

$$\boxed{p^\nu p_\nu = m^2 c^2} \quad (2.40)$$

A similar breakdown can be arranged for the vacuum terms

$$\mathcal{K}_{vac}^{\mu\nu} = -\frac{1}{2}\dot{\varrho}c\tau \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{P}_{vac}^{\mu\nu} = \frac{1}{3}\dot{\varrho}c\tau \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.41)$$

Combining transformed quantities again shows that

$$\mathcal{E}_{vac}^{\mu\nu} = -\frac{1}{2}\dot{\varrho}c\tau\beta^\mu\beta^\nu + \frac{1}{3}\dot{\varrho}c\tau(\beta^\mu\beta^\nu - g^{\mu\nu}) \quad (2.42)$$

Contracting both terms with the metric tensor as before then derives the general scalar relation

$$\mathcal{K} = \frac{1}{2}\mathcal{P}c \quad (2.43)$$

where both \mathcal{K} and \mathcal{P} are seen to be scalar invariants. Combining vacuum and particle terms leads to the symmetric total energy tensor

$$\begin{aligned} \mathcal{E}_{total}^{\mu\nu} = [mc^2\beta^\mu\beta^\nu + \frac{1}{3}mc^2(\beta^\mu\beta^\nu - g^{\mu\nu})] \cdot \vartheta(\tau) \\ - \frac{1}{2}\dot{\varrho}c\tau\beta^\mu\beta^\nu + \frac{1}{3}\dot{\varrho}c\tau(\beta^\mu\beta^\nu - g^{\mu\nu}) \end{aligned} \quad (2.44)$$

A single contraction with the four-velocity derives the total particle energy 4-vector

$$\mathcal{E}_{total}^\nu = [mc^2 \cdot \vartheta(\tau) - \frac{1}{2}\dot{\varrho}c\tau]\beta^\nu = \mathcal{H}\beta^\nu \quad (2.45)$$

3 Accelerated Motions of the Electron

It is well known that an accelerating charge emits transverse electromagnetic waves over a range of frequencies. For the vacuum gauge particle these waves must be re-interpreted as transverse distortions of the vacuum travelling at light speed and independent of radiation associated with the velocity theory. A description of these variations will naturally require the inclusion of an acceleration strain tensor $\epsilon^{\mu\nu}$. One essential property of $\epsilon^{\mu\nu}$ is immediately available by requiring that it may not be a source of additional vacuum dilatation so that $\epsilon = 0$ only. This requirement may be used to classify classical radiation theory as an observable phenomena in contrast with the theory of dilatation radiation which is assumed to be unobservable.⁴

Naturally, $\epsilon^{\mu\nu}$ is expected to be functionally dependent on the four-acceleration of the particle, and as a preliminary calculation it will be useful to define the quantity a_\perp^ν orthogonal to the spacelike vector U^ν

$$a_\perp^\nu \equiv a^\nu + (a^\lambda U_\lambda) U^\nu \quad (3.1)$$

⁴This section relies heavily on various contractions of the strain tensors $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu}$. These contractions have been evaluated explicitly in separate tables in Appendix B serving as a useful calculational tool.

A link between a_\perp^ν and the velocity theory can be immediately established through sets of eigenvalue equations

$$\Delta^{\mu\nu} a_\nu^\perp = 0 \quad \longleftrightarrow \quad \eta^{\mu\nu} a_\nu^\perp = \eta a_\perp^\mu \quad (3.2a)$$

$$\Delta^{\mu\nu} a_\mu^\perp = 0 \quad \longleftrightarrow \quad \eta^{\mu\nu} a_\mu^\perp = \eta a_\perp^\nu \quad (3.2b)$$

In general, interactions of a_\perp^ν with other first and second rank tensors is quite limited. Evidence for this follows from orthogonality with the four-vectors R^ν and β^ν in addition to its defined orthogonality with U^ν .

3.1 Theory of Acceleration Strain

Beginning with the theory of potentials, suppose an acceleration strain tensor can be determined by defining $\varepsilon^{\mu\nu} \equiv \partial^\mu A_a^\nu$. Calculating ε is an extensive exercise in the use of implicit differentiation, but an application of the chain rule shows that

$$\varepsilon^{\mu\nu} = -a^\lambda R_\lambda (\partial^\mu A^\nu) - \partial^\mu (a^\lambda R_\lambda) \cdot A^\nu \quad (3.3)$$

Unfortunately, a serious problem with this equation lies in the fact that it leads to a non-zero vacuum dilatation $\varepsilon = -2(a^\lambda R_\lambda) \partial_\nu A^\nu$, with each of the separate terms on the right side of (3.3) making equal contributions. This violates the premise that acceleration strain may not dilate the vacuum.

A clue to resolving this problem is available from the well known anti-symmetry of the acceleration field strength tensor determined from $[\partial^\mu, A_a^\nu]$. This suggests a strategy to ‘cleanse’ $\varepsilon^{\mu\nu}$ of unwanted terms leaving an appropriate tensor $\epsilon^{\mu\nu}$ with the required property $\epsilon = 0$. As an initial (and final) guess, suppose all symmetric terms are separated from $\varepsilon^{\mu\nu}$ by

$$\varepsilon^{\mu\nu} = s^{\mu\nu} - \epsilon^{\nu\mu} \quad (3.4)$$

The anti-symmetric combination eliminates the symmetric term and leads to the equation

$$\partial_\mu [\epsilon^{\mu\nu} - \epsilon^{\nu\mu}] = \frac{4\pi}{c} J_a^\nu \quad (3.5)$$

The task of removing the symmetric terms depends critically on the expansion of the acceleration four-vector in terms of a_\perp^ν with the suprisingly simple result

$$\boxed{\epsilon^{\mu\nu} \equiv A^\mu a_\perp^\nu} \quad (3.6)$$

There are many interesting properties of this bi-linear quantity including $\epsilon = 0$ and vanishing determinants of all individual two-by-two minors which implies $\det \epsilon^{\mu\nu} = 0$. Divergences on each index of $\epsilon^{\mu\nu}$ are

$$\partial_\mu \epsilon^{\nu\mu} = \frac{4\pi}{c} (\tilde{J}_a^\nu - J_a^\nu) \quad \partial_\mu \epsilon^{\mu\nu} = \frac{4\pi}{c} \tilde{J}_a^\nu \quad (3.7)$$

New Definitions for the Vacuum Theory: The previous calculations of strain from the theory of potentials can be reconciled with the vacuum theory by a rearrangement of constants and the introduction of the elastic modulus μ_e . Table 1 gives a summary of the new definitions. The most striking feature of the table is

Names	Column 1	Column 2
Stress/Strain	$\Delta_a^{\mu\nu} = \mu_e \epsilon^{\mu\nu}$	$\epsilon^{\mu\nu} = \mathbf{u}^\mu a_\perp^\nu / c^2$
Force current densities	$\tilde{f}_a^\nu = \frac{\mu_e \eta}{4\pi c^2} a_\perp^\nu$	$f_a^\nu = -\frac{\mu_e \eta}{2\pi c^2 \rho^2} a^\lambda R_\lambda R^\nu$
Equations of motion.	$\partial_\mu \Delta_a^{\mu\nu} = 4\pi \tilde{f}_a^\nu$	$\partial_\mu \Delta_a^{\nu\mu} = 4\pi [\tilde{f}_a^\nu - f_a^\nu]$

Table 1: Showing “normalized” definitions of acceleration stress and strain tensors along with associated force-densities and equations of motion.

the emergence of momentum flux associated with transverse shear stresses on the medium resulting from particle accelerations. This is consistent with the appearance of transverse momentum associated with shear waves in elastic media, and it means that Maxwell acceleration fields determined from

$$\partial_\mu [\Delta_a^{\mu\nu} - \Delta_a^{\nu\mu}] = 4\pi f_a^\nu \quad (3.8)$$

must now be re-intrepreted a momentum flux fields, similar to the fields in the velocity theory. According to vacuum gauge electrodynamics, all electromagnetic fields therefore transport momentum and Maxwell’s equations for particle accelerations are

$$\nabla \cdot \boldsymbol{\pi}_{Ea} = 4\pi \rho_a \quad \nabla \times \boldsymbol{\pi}_{Ba} = 4\pi \mathbf{f}_a + \frac{1}{c} \frac{\partial \boldsymbol{\pi}_{Ea}}{\partial t} \quad (3.9a)$$

$$\nabla \times \boldsymbol{\pi}_{Ea} = -\frac{1}{c} \frac{\partial \boldsymbol{\pi}_{Ba}}{\partial t} \quad \nabla \cdot \boldsymbol{\pi}_{Ba} = 0 \quad (3.9b)$$

Formula’s for acceleration flux vectors as functions of the dilatation vector may be written

$$\boldsymbol{\pi}_{Ea} = \mu_e [\mathbf{u} a_\perp - \mathbf{u} a_\perp^0] \quad (3.10a)$$

$$\boldsymbol{\pi}_{Ba} = \mu_e \mathbf{u} \times \mathbf{a}_\perp \quad (3.10b)$$

Acceleration flux fields themselves are not to be associated with radiated flux as can be verified by dotting equations (3.10) with $\hat{\mathbf{n}}$. However, a generalization of the

Poynting vector in (1.31) inclusive of acceleration flux fields reads

$$\mathbf{S} = \frac{c}{\mu_e} (\boldsymbol{\pi}_E + \boldsymbol{\pi}_{Ea}) \times (\boldsymbol{\pi}_B + \boldsymbol{\pi}_{Ba}) \quad (3.11)$$

This is exactly the Poynting vector of the conventional theory written in terms of a new set of fundamental quantities.

3.2 Generalized Vacuum Lagrangian and Stress Tensor

With appropriate definitions of acceleration stress and strain, a generalized vacuum Lagrangian may be determined beginning with the explicit form of the velocity Lagrangian:

$$\mathcal{L}_{vac} = -\frac{1}{2}\mu_e [\partial^\mu \mathcal{U}^\nu \partial_\mu \mathcal{U}_\nu - (\partial_\nu \mathcal{U}^\nu)^2] - 4\pi f_e^\nu \mathcal{U}_\nu \quad (3.12)$$

Under conditions where accelerated motions occur assume that $\partial^\mu \mathcal{U}^\nu$ receives a small anti-symmetric perturbation, first order in the quantity a_\perp^ν , with the appearance of an acceleration four-current:

$$\partial^\mu \mathcal{U}^\nu \longrightarrow \partial^\mu \mathcal{U}^\nu - \mathcal{U}^\mu a_\perp^\nu + \mathcal{U}^\nu a_\perp^\mu \quad f_e^\nu \longrightarrow f_e^\nu - f_a^\nu \quad (3.13)$$

Keeping only first order corrections in a_\perp^ν the modified velocity theory can be written

$$\begin{aligned} \mathcal{L}_{vac} = & -\frac{1}{2}\mu_e [\partial^\mu \mathcal{U}^\nu \partial_\mu \mathcal{U}_\nu - (\partial_\nu \mathcal{U}^\nu)^2] - 4\pi f_e^\nu \mathcal{U}_\nu \\ & + \mu_e [\partial^\mu \mathcal{U}^\nu \mathcal{U}_\mu a_\nu^\perp - \partial^\mu \mathcal{U}^\nu \mathcal{U}_\nu a_\mu^\perp] + 4\pi f_a^\nu \mathcal{U}_\nu \end{aligned} \quad (3.14)$$

All additional terms resulting from particle accelerations do not make contributions to the functional value of the Lagrangian, while derivatives with respect to the field quantity and its derivative are

$$\frac{\partial \mathcal{L}_{vac}}{\partial (\partial_\mu \mathcal{U}_\lambda)} = -\mu_e [\partial^\mu \mathcal{U}^\lambda - g^{\mu\lambda} \partial_\nu \mathcal{U}^\nu - \mathcal{U}^\mu a_\perp^\lambda + \mathcal{U}^\lambda a_\perp^\mu] \quad (3.15)$$

$$\frac{\partial \mathcal{L}_{vac}}{\partial \mathcal{U}_\lambda} = \mu_e [\partial^\lambda \mathcal{U}^\mu - \partial^\mu \mathcal{U}^\lambda] a_\mu^\perp - 4\pi [f_e^\lambda - f_a^\lambda] \quad (3.16)$$

Appealing to equation (3.2), the second equation produces all the correct current densities necessary to write

$$\mu_e \partial_\mu [\partial^\mu \mathcal{U}^\lambda - g^{\mu\lambda} \partial_\nu \mathcal{U}^\nu] = 4\pi f_e^\lambda = \partial_\mu \Delta^{\mu\lambda} \quad (3.17a)$$

$$\mu_e \partial_\mu [\mathcal{U}^\mu a_\perp^\lambda] = 4\pi \tilde{f}_a^\lambda = \partial_\mu \Delta_a^{\mu\lambda} \quad (3.17b)$$

$$\mu_e \partial_\mu [\mathcal{U}^\lambda a_\perp^\mu] = 4\pi [\tilde{f}_a^\lambda - f_a^\lambda] = \partial_\mu \Delta_a^{\lambda\mu} \quad (3.17c)$$

Unfortunately, the generalized Lagrangian is not suitable for calculating an associated stress tensor from an equation like (2.8). The problem lies in the fact that acceleration stresses in the vacuum gauge are to be treated as small corrections to the velocity theory, and this same approach must be applied to the symmetric stress tensor. The velocity portion of the symmetric stress tensor has already been derived in section 2.2:

$$\Theta_{vac}^{\mu\nu} = \mu_e \left[\frac{1}{2} g^{\mu\nu} \eta^2 - \eta^{\mu\lambda} \eta^\nu{}_\lambda \right] \quad (3.18)$$

The inclusion of particle accelerations follows immediately from the generalized strain

$$\zeta^{\mu\nu} \equiv \eta^{\mu\nu} - \epsilon^{\mu\nu} \quad (3.19)$$

The resulting stress tensor is

$$\Theta_{vac}^{\mu\nu} = \mu_e \left[\frac{1}{2} g^{\mu\nu} \eta^2 - \eta^{\mu\lambda} \eta^\nu{}_\lambda + \eta^{\mu\lambda} \epsilon^\nu{}_\lambda - \epsilon^{\mu\lambda} \epsilon^\nu{}_\lambda + \epsilon^{\mu\lambda} \eta^\nu{}_\lambda \right] \quad (3.20)$$

which is exactly the symmetric stress tensor of the electromagnetic theory. One could also include the conjugate acceleration strain and write the variation

$$\eta^{\mu\nu} \longrightarrow \eta^{\mu\nu} - \epsilon^{\mu\nu} + \epsilon^{\nu\mu} \quad (3.21)$$

but all extra terms in $\Theta_{vac}^{\mu\nu}$ are zero and equation (3.20) remains unchanged. For a more concise representation of the symmetric stress tensor define

$$\mathcal{R}^{\mu\nu} \equiv \mu_e [\zeta^{\mu\lambda} \zeta^\nu{}_\lambda] \quad (3.22)$$

The total stress tensor including the radiation term is then

$$\boxed{\mathcal{T}^{\mu\nu} = \left[\frac{1}{4} g^{\mu\nu} \mathcal{R} - \mathcal{R}^{\mu\nu} \right] + \Lambda^{\mu\nu}} \quad (3.23)$$

With the inclusion of particle accelerations, $\mathcal{T}^{\mu\nu}$ above can be used to determine the action integral \mathcal{S} while simultaneously showing that accelerations do not contribute to the action. For this use the four-volume element $d^4V = \rho^2 d\rho d\Omega' cd\tau$ and write

$$\mathcal{S} = \frac{1}{c} \int \beta_\mu \mathcal{T}^{\mu\nu} \beta_\nu d^4V = \frac{1}{c} \int \mathcal{L}_{vac} d^4V \quad (3.24)$$

The full integration will show that the radiation term alone will depend on the square of the proper time. Instead, it is more appropriate to withhold the proper time integral and determine the particle Hamiltonian from

$$\frac{d\mathcal{S}}{d\tau} = -\mathcal{H} \quad (3.25)$$

For a final calculation it will be necessary to adequately address the problem of the divergence of $\mathcal{T}^{\mu\nu}$ in vacuum gauge electrodynamics. If acceleration terms are withheld from a divergence operation on $\Theta_1^{\mu\nu}$, then an acceleration term must be included for the generalized tensor

$$\partial_\mu \Theta_{vac}^{\mu\nu} = -4\pi [f_e^\lambda \eta^\nu{}_\lambda + \eta f_a^\nu] \quad (3.26)$$

On the other hand, the propagation term can have no generalization and must remain as part of the velocity theory only.

A Derivatives of the Null Vector

The covariant derivative of R^ν is

$$\partial^\mu R^\nu = g^{\mu\nu} - \frac{R^\mu \beta^\nu}{\rho} \quad (\text{A.1})$$

The trace of the resulting matrix gives the 4-divergence $\partial_\nu R^\nu = 3$. In terms of individual components—and with the inclusion of a sign—a useful construction is:

$$-\partial^\mu R^\nu = \begin{bmatrix} -\frac{\partial R}{\partial ct} & -\frac{\partial \mathbf{R}}{\partial ct} \\ \nabla R & \nabla \mathbf{R} \end{bmatrix} \quad (\text{A.2})$$

where individual components are given by

$$\frac{\partial R}{\partial ct} = 1 - \frac{\gamma R}{\rho} \quad \frac{\partial \mathbf{R}}{\partial ct} = \frac{-\gamma R \boldsymbol{\beta}}{\rho} \quad (\text{A.3})$$

$$\nabla R = \frac{\gamma \mathbf{R}}{\rho} \quad \nabla \mathbf{R} = \mathbf{1} + \frac{\gamma \mathbf{R} \boldsymbol{\beta}}{\rho} \quad (\text{A.4})$$

The determinant of (A.2) can be written $\det[\partial^\mu R^\nu] = 0$. The divergence and curl of \mathbf{R} may be written

$$\nabla \cdot \mathbf{R} = 3 + \frac{\gamma \mathbf{R} \cdot \boldsymbol{\beta}}{\rho} = \text{Tr}[\nabla \mathbf{R}] \quad (\text{A.5})$$

$$\nabla \times \mathbf{R} = \frac{\gamma}{\rho} \mathbf{R} \times \boldsymbol{\beta} \quad (\text{A.6})$$

Let $\mathbf{w}(ct_r)$ be the retarded position of a charged particle at time ct_r . The light cone condition is defined by

$$\mathbf{R} \equiv \mathbf{r} - \mathbf{w}(ct_r) \quad R \equiv ct - ct_r \quad (\text{A.7})$$

Derivatives of the retarded time with respect to present time coordinates are

$$\frac{\partial ct_r}{\partial ct} = \frac{\gamma R}{\rho} \quad (\text{A.8})$$

$$\nabla ct_r = \frac{-\gamma \mathbf{R}}{\rho} \quad (\text{A.9})$$

B Contractions of Vacuum Strain Tensors

$$\eta^{\mu\nu} = \eta \left[g^{\mu\nu} - \frac{R^\mu \beta^\nu}{\rho} - \frac{2R^\nu \beta^\mu}{\rho} + \frac{2R^\mu R^\nu}{\rho^2} \right] \quad (\text{B.1})$$

$$\epsilon^{\mu\nu} = \eta \left[R^\mu a^\nu - \chi \frac{R^\mu \beta^\nu}{\rho} + \chi \frac{R^\mu R^\nu}{\rho^2} \right] \quad (\text{B.2})$$

$\eta^{\mu\lambda} \eta_{\lambda}{}^{\nu} = \eta^2 \left[g^{\mu\nu} - \frac{R^\mu \beta^\nu}{\rho} \right]$	$\eta^{\mu\lambda} \eta^{\nu}{}_{\lambda} = \eta^2 \left[g^{\mu\nu} - \frac{1}{\rho} (R^\mu \beta^\nu + R^\nu \beta^\mu) + \frac{R^\mu R^\nu}{\rho^2} \right]$
$\eta^{\lambda\mu} \eta^{\nu}{}_{\lambda} = \eta^2 \left[g^{\mu\nu} - \frac{\beta^\mu R^\nu}{\rho} \right]$	$\eta^{\lambda\mu} \eta_{\lambda}{}^{\nu} = \eta^2 \left[g^{\mu\nu} - \frac{1}{\rho} (R^\mu \beta^\nu + \beta^\mu R^\nu) \right]$

Table 2: Contractions of $\eta^{\mu\nu}$.

$\eta^{\mu\lambda} \epsilon^{\nu}{}_{\lambda} = \eta \epsilon^{\nu\mu}$	$\epsilon^{\mu\lambda} \eta^{\nu}{}_{\lambda} = \eta \epsilon^{\mu\nu}$	$\epsilon^{\mu\lambda} \epsilon^{\nu}{}_{\lambda} = -\Theta_3^{\mu\nu}$
$\eta^{\mu\lambda} \epsilon_{\lambda}{}^{\nu} = 0$	$\epsilon^{\mu\lambda} \eta_{\lambda}{}^{\nu} = \eta \epsilon^{\mu\nu}$	$\epsilon^{\mu\lambda} \epsilon_{\lambda}{}^{\nu} = 0$
$\eta^{\lambda\mu} \epsilon^{\nu}{}_{\lambda} = \eta \epsilon^{\nu\mu}$	$\epsilon^{\lambda\mu} \eta^{\nu}{}_{\lambda} = 0$	$\epsilon^{\lambda\mu} \epsilon^{\nu}{}_{\lambda} = 0$
$\eta^{\lambda\mu} \epsilon_{\lambda}{}^{\nu} = -\eta \epsilon^{\mu\nu}$	$\epsilon^{\lambda\mu} \eta_{\lambda}{}^{\nu} = -\eta \epsilon^{\nu\mu}$	$\epsilon^{\lambda\mu} \epsilon_{\lambda}{}^{\nu} = 0$

Table 3: Contractions of $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu}$.