Inertial Vacuum Power

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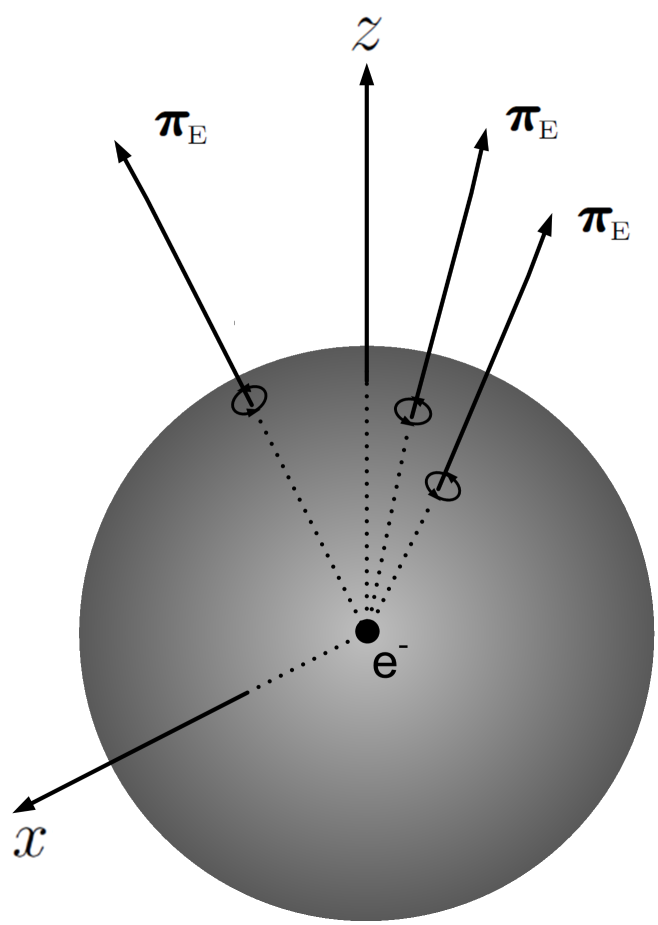
March 17, 2024



Abstract

Electromagnetic flux can be introduced into the velocity fields of the classical electron. The inherent violation of energy conservation relies heavily on the use of vacuum gauge potentials in multiple scenarios for the calculation of inertial vacuum power.

Links to youtube videos:
https://youtu.be/PHEeNRdzCX4
https://youtu.be/gk8QoH8dPXE

Vacuum
Gauge
Electrodynamics

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1 Vacuum Radiation in Electron Coulomb Fields

A model for a theory of the classical electron can be constructed under the assumption that particle Coulomb fields are carriers of radiated momentum flux. This is most easily accomplished by introducing an elementary constant surface charge density $\sigma_e < 0$ and writing:

$$\boldsymbol{\pi}_{\text{E}} = \sigma_e \mathbf{E}_v \qquad \boldsymbol{\pi}_{\text{B}} = \sigma_e \mathbf{B}_v \tag{1.1}$$

Assume the fields are radiating at the speed of light and possess an associated energy flux similar to particle acceleration fields. Of course this is a flagrant violation of energy conservation, but it can be assumed that the existence of momentum flux does not inhibit the ability of the fields to exert forces on other charges. This means that the introduction of momentum flux is a symmetry operation which leaves the fields invariant.

An interesting consequence of (1.1) is that electrons and positrons will be associated with identical radiation fields, and both will satisfy the same set of (re-defined) Maxwell-Lorentz equations:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{\text{E}} = 4\pi f_o \qquad \boldsymbol{\nabla} \times \boldsymbol{\pi}_{\text{B}} = \frac{4\pi}{c}\boldsymbol{f} + \frac{1}{c}\frac{\partial \boldsymbol{\pi}_{\text{E}}}{\partial t} \tag{1.2a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\pi}_{\text{E}} = -\frac{1}{c}\frac{\partial \boldsymbol{\pi}_{\text{B}}}{\partial t} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{\text{B}} = 0 \tag{1.2b}$$

The sources here are still those of a point particle but now re-interpreted as force densities. Discrete symmetry operations can be performed[1] showing C- and P- invariance, but violations of T-symmetry occur in the two source equations. However, this is not in conflict with the CPT-theorem as the question of how fast a Coulomb field radiates has nothing to do with the electromagnetic force it exerts.

Radiating Coulomb fields still allow for momentum and energy flux attributed to Poynting's theorem, but require a slightly new formulation of the Poynting vector—inclusive of a fundamental pressure constant[2]

$$\mathbf{S} = \frac{c}{\mu_e}\boldsymbol{\pi}_{\text{E}} \times \boldsymbol{\pi}_{\text{B}} \tag{1.4}$$

For constant velocity motion, $\mathbf{S}$ does not qualify as radiation as it lacks the r^{-2} character. However, during particle accelerations the Poynting vector generalizes to

$$\mathbf{S} = \frac{c}{\mu_e}\left[(\boldsymbol{\pi}_{\text{E}} + \boldsymbol{\pi}_{\text{E}a}) \times (\boldsymbol{\pi}_{\text{B}} + \boldsymbol{\pi}_{\text{B}a})\right] \tag{1.5}$$

[1]See Appendix for symmetry operations performed on momentum flux fields.

[2]The pressure constant given by

$$\mu_e = 4\pi\sigma_e^2 \tag{1.3}$$

is fundamental to classical electron theory as it represents the self-induced outward pressure attributed to the particles' own surface charge density.

This means that the inclusion of momentum flux in velocity fields will also require momentum flux in acceleration fields. But momentum in acceleration fields is not associated with radiation. This follows since acceleration fields oscillate perpendicular to the direction of motion and average to zero as the wave propagates.

It is not difficult to include radiating velocity fields into the general scheme of theoretical physics since their presence amounts to only a slight modification of the 1905 postulates. The postulate of relativity remains the same but the constancy of light postulate must be altered to read:

> ***–Postulate 2: Electromagnetic radiation fields propagate at speed c independent of the motion of the source.***

This is mainly just a generalization of the Einstein postulate to include all possible forms of radiation, but it's also a unifying statement which renders the descriptive title, '*static field*' as an unsubstantiated figment of the imagination.

While the introduction of momentum in particle velocity fields might seem like a good idea in its own right, the motivation for equation (1.1) can be traced to observations of dark forms of energy known to exist in the universe. Certainly, there are an overwhelming abundance of questions regarding properties of radiated field energy described by (1.1), an (unfortunately) only a small portion will be addressed on in the rest of this investigation, and in the Appendix.

1.1 Vacuum Gauge Condition

Insertion of momentum and energy in electron velocity fields is facilitated by the application of the covariant vacuum gauge condition.

$$\partial_\nu A^\nu = \sqrt{\mathrm{E}^2 - \mathrm{B}^2} \tag{1.6}$$

The solution to this partial differential equation is most easily addressed in the rest frame which determines the spacelike field

$$\boldsymbol{A}_s = \frac{e}{r}\hat{\boldsymbol{r}} \tag{1.7}$$

This is a covariant four-vector with no time component in the rest frame. A Lorentz transformation in combination with theory of retarded potentials will then identify the general moving frame vector

$$A_s^\nu = \frac{e}{\rho}U^\nu = \frac{eR^\nu}{\rho^2} - \frac{e\beta^\nu}{\rho} \tag{1.8}$$

Now observe that any four-vector field with a zero divergence can be added to A_s^ν without violating the vacuum gauge condition. Since the last term on the right side of (1.8) are the divergence free Lienard-Wiechert potentials, this means the light-like potentials (labeled without a subscript)

$$\boxed{A^\nu = \frac{eR^\nu}{\rho^2}} \tag{1.9}$$

are a legitimate set of vacuum gauge potentials still satisfying the vacuum gauge condition.

Complex Vacuum Gauge Potentials: An important property of the vacuum gauge condition can be understood by appending to the potentials in (1.9) a complex exponential

$$A^\nu = \frac{eR^\nu}{\rho^2} \cdot \mathrm{e}^{i\omega t_r/\gamma} \quad \text{where} \quad t_r = t - R/c + \tau_e \tag{1.10}$$

Here t_r is the retarded time which which has been written with a phase defined by $c\tau_e = r_e$. The potentials are now travelling spherical waves and determine the velocity field strength tensor given by

$$\begin{aligned} F_v^{\mu\nu} &= [\partial^\mu A^\nu - \partial^\nu A^\mu] \cdot \mathrm{e}^{i\omega t_r/\gamma} + [A^\nu \partial^\mu - A^\mu \partial^\nu] \cdot \mathrm{e}^{i\omega t_r/\gamma} \\ &= F_M^{\mu\nu} \cdot \mathrm{e}^{i\omega t_r/\gamma} \end{aligned} \tag{1.11}$$

In this equation the subscript M indicates the well-known Maxwell field strength tensor—now flanked by the same complex exponential applied to the potentials. The implication here is that the fields and the potentials still satisfy the vacuum gauge condition (1.6). In fact, the modulated vacuum gauge condition proceeds from

$$\begin{aligned} \partial_\nu \left[A^\nu \cdot \mathrm{e}^{i\omega t_r/\gamma}\right] &= \partial_\nu A^\nu \cdot \mathrm{e}^{i\omega t_r/\gamma} + A^\nu \cdot \partial_\nu \mathrm{e}^{i\omega t_r/\gamma} \\ &= \frac{e}{\rho^2} \cdot \mathrm{e}^{i\omega t_r/\gamma} \end{aligned} \tag{1.12}$$

In short, the vacuum gauge condition remains valid regardless of whether the travelling wave property of the fields and potentials is included or not.

1.2 Momentum Flux with Vacuum Gauge Potentials

The wave-like character of vacuum gauge potentials and the resulting field strength tensor will require the fields to be carriers of both momentum and energy. It is easiest to begin in the rest frame with scalar and vector potentials given by

$$V = \frac{e}{r}\mathrm{e}^{i\omega(\tau - r/c + \tau_e)} \qquad\qquad \boldsymbol{A} = \frac{e}{r}\hat{\boldsymbol{r}}\,\mathrm{e}^{i\omega(\tau - r/c + \tau_e)} \tag{1.13}$$

The vector potential has a longitudinal polarization and both potentials are travelling spherical waves having a structure similar to pressure and velocity waves in continuum mechanics. When combined, they produce an average energy flux vector given by

$$\langle \mathbf{S}_{E} \rangle = \frac{c}{2\mathrm{a}_e} V^* \boldsymbol{A} \qquad \mathrm{a}_e = 4\pi r_e^2 \tag{1.14}$$

The charge of the electron is $e = \sigma_e \mathrm{a}_e$. In terms of the classical Coulomb field, energy- and momentum-flux fields follow by writing

$$\langle \mathbf{S}_{E} \rangle = \frac{1}{2} \boldsymbol{\pi}_{E} c \quad \text{where} \quad \boldsymbol{\pi}_{E} = \sigma_e \mathbf{E} \tag{1.15}$$

Now construct a moving frame anti-symmetric flux tensor by applying a Lorentz transformation to the anti-symmetric rest frame tensor given by

$$\pi^{\mu\nu} = \begin{bmatrix} 0 & -\boldsymbol{\pi}_{\mathrm{E}} \\ \boldsymbol{\pi}_{\mathrm{E}} & \widehat{0} \end{bmatrix} \tag{1.16}$$

The result may be written $\pi^{\mu\nu} = \sigma_e F_v^{\mu\nu}$ and features an additional magnetic flux $\boldsymbol{\pi}_{\mathrm{B}} = \hat{\boldsymbol{n}} \times \boldsymbol{\pi}_{\mathrm{E}}$ circling around the particle at speed c.

Momentum flux in the moving frame can also be established by writing the potentials as the sum of timelike and spacelike components:

$$A^\nu = A_t^\nu + A_s^\nu \qquad \begin{cases} A_t^\nu = \dfrac{e}{\rho} \beta^\nu \cdot \mathrm{e}^{i\omega t_r/\gamma} \\ \\ A_s^\nu = \dfrac{e}{\rho} U^\nu \cdot \mathrm{e}^{i\omega t_r/\gamma} \end{cases} \tag{1.17}$$

In this case the momentum flux tensor takes the form

$$\pi^{\mu\nu} = \frac{1}{a_e} [A_s^{*\mu},\, A_t^\nu] \tag{1.18}$$

An important covariant flux integral over a timelike surface is

$$dE^\nu = \oint_{R=r_e} \left[\tfrac{1}{2} \pi^{\mu\nu} c\right] U_\mu R^2 d\Omega d\tau = P_{in} \beta^\nu d\tau \tag{1.19}$$

Dividing both sides by $d\tau$ determines the four-vector radiation rate per unit proper time comparable to a similar formula for the particle's acceleration fields. The invariant radiation rate is then determined as

$$P_{in} = \frac{e^2 c}{2 r_e^2} = \frac{mc^2}{\tau_e} = 1.73 \times 10^{10}\,\text{Watts} \tag{1.20}$$

This value of this power is extremely large—on the order of ten billion Watts. A more reasonable calculation is to determine the r^{-2} energy density—although this value is still relatively large. Regardless, the inherent violation of energy conservation in our theory leads directly to the question of whether mass-energy equivalence can be applied to Coulomb fields. Based on (1.20) the answer seems to be, no.

1.3 Vacuum Power from Compression Waves

Another interpretation of the light-like vacuum gauge potentials can be realized beginning with an isotropic velocity vector[3]:

$$\boldsymbol{v}(\tau) = \Re e\left[c\,\hat{\boldsymbol{r}} \cdot \mathrm{e}^{i\omega_e \tau}\right] \tag{1.21}$$

But $\boldsymbol{v}(\tau)$ can be viewed as the boundary of a more general velocity field at the radius r_e. In addition, the norm of the velocity field defines a scalar pressure field $cp = \mu_e|\boldsymbol{v}|$, and both may be written as the set:

$$p = \frac{\mu_e r_e}{r} \cdot \mathrm{e}^{i\omega_e(\tau - r/c + \tau_e)} \qquad\qquad \boldsymbol{v} = \frac{c r_e}{r}\hat{\boldsymbol{r}} \cdot \mathrm{e}^{i\omega_e(\tau - r/c + \tau_e)} \tag{1.22}$$

In the rest frame, these fields can be linked to a linearized Euler equation and source continuity equation, while showing properties similar to sound waves emitted by a spherically vibrating source. In an arbitrary inertial reference frame, equations (1.22) can be awarded the status as covariant four-vectors in the form of a timelike pressure field and a spacelike velocity field:

$$p^\nu = \frac{\mu_e r_e}{\rho}\beta^\nu \cdot \mathrm{e}^{i\omega_e z} \qquad\qquad v^\nu = \frac{c r_e}{\rho} U^\nu \cdot \mathrm{e}^{i\omega_e z} \tag{1.23}$$

In terms of these fields, the electromagnetic flux tensor is

$$\pi^{\mu\nu} = \frac{1}{c}[v^{*\mu},\, p^\nu] \tag{1.24}$$

Pressure and velocity four-vectors may also be combined to form a related null pressure field

$$c\mathcal{P}^\nu \equiv cp^\nu + \mu_e v^\nu \equiv [c\mathcal{P}, \mu_e \boldsymbol{\mathcal{V}}] \quad \text{where} \quad c\mathcal{P} = \mu_e|\boldsymbol{\mathcal{V}}| \tag{1.25}$$

Naturally, this four-vector is proportional to the vacuum gauge velocity potentials and features moving frame scalar and vector components given by

$$\mathcal{P} = \mu_e r_e \frac{R}{\rho^2} \cdot e^{i\omega_e z} \qquad\qquad \boldsymbol{\mathcal{V}} = c r_e \frac{\boldsymbol{R}}{\rho^2} \cdot e^{i\omega_e z} \tag{1.26}$$

The field $\mathcal{P}$ radiates spherical compression waves similar to acoustic phenomena. A formula for radiated vacuum power is still equation (1.20) but radiated energy flux averaged over time will be written

$$\langle\, \mathbf{S}_v \,\rangle = \frac{1}{2}\mathcal{P}^* \boldsymbol{\mathcal{V}} \tag{1.27}$$

This result and others like it are testament to the ubiquitous nature of the vacuum fields radiated by the classical electron. With two sets of defined pressure fields, one can ask what exactly is being compressed. This question can be answered by asserting the existence of a medium sitting on top of Minkowski space in which the classical electron appears as an excited state.

[3] Calculations in this section closely follow the theory of sound waves in chapter 9 (section 49) Fetter and Walecka, Theoretical Mechanics of Particles and Continua, 1980.

2 Classical Electron as a Radiating Antenna

2.1 Physics at the Particle Radius

A series LCR resonant circuit is illustrated in the left side of Figure 2.1. By choosing the capacitor and inductor appropriately, the current can oscillate at the resonant frequency. This creates a time-dependent magnetic field in the neighborhood of the circuit allowing it to function as a radiating antenna. In general, the effect is negligible—adding a small radiation resistance $\mathcal{R}$ to the circuit, which is typically not part of any discussion about the circuit. But if $\mathcal{R}$ is known, the power radiated by the circuit can be determined by writing

$$P_{LCR} = I_{rms}^2 \mathcal{R} \tag{2.1}$$

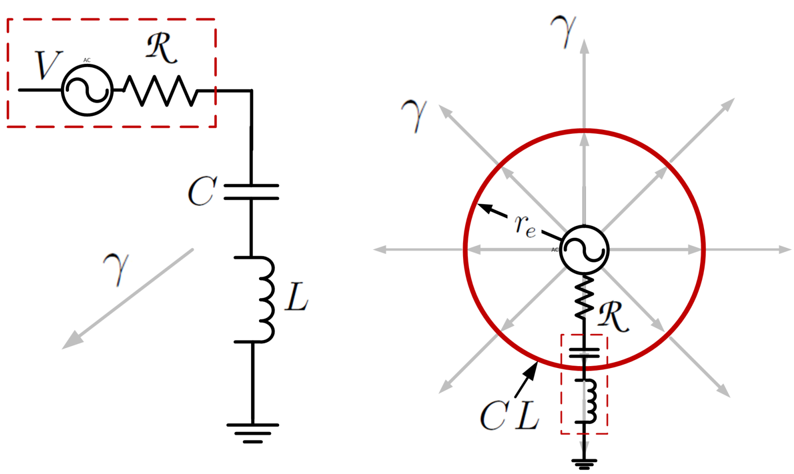

Figure 1: *Left: Radiative property of an LCR circuit. Right: Effective circuit diagram for a rest frame classical electron emitting electric vacuum current at high frequency.*

Now suppose a classical electron is described with the set of vacuum gauge potentials. In the rest frame, they are travelling spherical waves given by

$$V = \frac{e}{r} \cdot \mathrm{e}^{i\omega_e(\tau - r/c + \tau_e)} \qquad \boldsymbol{A} = \frac{e}{r}\hat{\boldsymbol{r}} \cdot \mathrm{e}^{i\omega_e(\tau - r/c + \tau_e)} \tag{2.2}$$

Evaluating at the electron radius, the scalar and vector components become simple oscillators. In the SI system of units they are

$$A^\nu \bigg|_{r_e} = \frac{1}{4\pi\epsilon_o}\left[\frac{e}{r_e}, \frac{e}{r_e}\hat{\boldsymbol{r}}\right] \cdot \mathrm{e}^{i\omega_e \tau} \equiv (\tilde{V}, \tilde{\boldsymbol{A}}) \tag{2.3}$$

As the electron is an emitter of electromagnetic radiation, we expect that it can be associated with its own radiation resistance $\mathcal{R}$. This can be derived by associating the vector potential with a vacuum current radiating perpendicular to the particle radius

$$\tilde{\boldsymbol{I}} = 4\pi\epsilon_o c \cdot i\tilde{\boldsymbol{A}} = \frac{ec}{r_e}\hat{\boldsymbol{r}} \cdot i\mathrm{e}^{i\omega_e\tau} \qquad \text{and} \qquad \|\tilde{\boldsymbol{I}}\| \sim 3.4 \times 10^4 \,\text{amps} \tag{2.4}$$

Now use $i\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{I}}\mathcal{R}$ to deterimine an almost trivial calculation of radiation resistance

$$\mathcal{R} = \frac{1}{4\pi}\sqrt{\frac{\mu_o}{\epsilon_o}} \sim 30\,\Omega \tag{2.5}$$

The complex charge at the particle radius can be read directly from the scalar potential. Moreover, a time derivative of the charge will generate the magnitude of the current flowing through the radius. A differential equation related the two is given by

$$\frac{d\tilde{I}}{d\tau} + \omega_e^2\tilde{q}(\tau) = 0 \quad \text{where} \quad \begin{cases} \tilde{q}(\tau) = e \cdot \mathrm{e}^{i\omega_e\tau} \\ \\ \tilde{I} = \dfrac{d\tilde{q}}{d\tau} = ie\omega_e \cdot \mathrm{e}^{i\omega_e\tau} \end{cases} \tag{2.6}$$

Comparing the scalar and vector forms of the current shows that the radiated frequency is given by the simple expression

$$\omega_e = \frac{1}{\tau_e} \tag{2.7}$$

With calculations of charge and current, it is useful to refer to the diagram on the right side of Figure 2.1. The central portion of the diagram may be thought of as an input AC power supply matched to a feedpoint at the boundary defined by the electron radius. This endows the electron with the property of a microscopic radiating antenna emitting a current of vacuum energy into the spacetime. Calculations of radiated vacuum power are

$$P_{in} = \frac{1}{2}\tilde{I}^*\tilde{I}\mathcal{R} = \frac{1}{2\mathcal{R}}\tilde{V}^*\tilde{V} = 1.73 \times 10^{10}\,\text{Watts} \tag{2.8}$$

and are fully consistent with previous calculations of P_{in}. Now, if the radiating antenna theory is correct, then the antenna must have the property of a resonant LC circuit. Derivations of L and C follow by writing

$$\tilde{V} \rightarrow \tilde{V}_C = \frac{\tilde{q}}{C} \qquad\qquad C = 4\pi\epsilon_o r_e = 1.57 \times 10^{-25}\,\text{F} \tag{2.9a}$$

$$\tilde{V}_L = L\frac{d\tilde{I}}{d\tau} = -\tilde{V}_C \qquad\qquad L = \frac{1}{4\pi}\mu_o r_e = 1.41 \times 10^{-22}\,\text{H} \tag{2.9b}$$

Particle impedance will certainly be given by the familiar formula

$$Z = \mathcal{R} + i\left[-\frac{1}{\omega_e C} + \omega_e L\right] \tag{2.10}$$

Insertion of known values for each circuit element and the frequency ω_e determine a reactance of zero leaving the simple formula $Z = \mathcal{R}$. Moreover, combinations of L and C also provide independent derivations of the frequency and radiation resistance

$$\omega_e = \frac{1}{\sqrt{LC}} \qquad \text{and} \qquad \mathcal{R} = \sqrt{\frac{L}{C}} \tag{2.11}$$

We note that calculations in this section rely on logic entirely opposite that of the classic resonant oscillator circuit problem where known values of the circuit elements are given in advance. Instead, the problem of the classical radius relies on a working knowledge of vacuum gauge potentials.

This view of the classical electron as a resonant radiator of field energy is impressive. One may now consider calculations of energy stored in the capacitive and inductive elements associated with the radius. Like all calculations in elementary circuit theory, maximum stored energies follow by writing

$$u_{max} = \frac{e^2}{8\pi\epsilon_o r_e} = \frac{1}{2}C\tilde{V}_C^*\tilde{V}_C = \frac{1}{2}L\tilde{I}^*\tilde{I} = mc^2 \tag{2.12}$$

But LC-circuits continously transfer energy between the capacitive and inductive elements which differ in phase by the phase angle $\phi = \pi$. This property of the radius follows by integrating equation (2.6) to capture the constant energy formula

$$u = u_L \cdot \sin^2 \omega_e \tau + u_C \cdot \cos^2 \omega_e \tau = mc^2 \tag{2.13}$$

To summarize, we have shown how a classical electron with radiating Coulomb fields possesses the property of a perpetual oscillator that cannot be destroyed except (possibly) by a collision with a positron. One may ask where charge of the electron went. We assert that it is rightfully determined by calculating the norm

$$e = -\sqrt{\tilde{q}^*\tilde{q}} \tag{2.14}$$

In other words, oscillations of electric charge and the implied radiation of vacuum energy is not verifiable by any local experiment. It might be suggested that vacuum radiation belongs to the realm of metaphysics—having no means of verification—but this may not be true if our theory correctly predicts dark forms of energy known to exist throughout the universe.

Moving Frame Solution: It is important to interpret all the previous rest frame equations in an arbitrary moving frame where the light-like vacuum gauge retarded potentials are concisely written as

$$A^\nu = \frac{eR^\nu}{\rho^2} \cdot \mathrm{e}^{i\omega_e t_r/\gamma} \tag{2.15}$$

Moving frame generalizations of scalar and vector potentials in equation (2.2) are

$$V = \frac{eR}{\rho^2} \cdot \mathrm{e}^{i\omega_e t_r/\gamma} \qquad\qquad \boldsymbol{A} = \frac{e\boldsymbol{R}}{\rho^2} \cdot \mathrm{e}^{i\omega_e t_r/\gamma} \tag{2.16}$$

and this establishes the invariant radiation resistance

$$\mathcal{R} = \frac{1}{4\pi\epsilon_o} \frac{V}{\|ic\boldsymbol{A}\|} \tag{2.17}$$

Now suppose that the potentials are evaluated at the classical electron radius defined by $R = r_e$. This is an equal time radius which is still a 3-sphere in the moving frame and producing the boundary

$$A^\nu \Big|_{r_e} = \frac{1}{\gamma^2(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2} \left[\frac{e}{r_e}, \frac{e}{r_e}\hat{\boldsymbol{n}}\right] \cdot \mathrm{e}^{i\omega_e t/\gamma} \tag{2.18}$$

According to this formula, relativity has “Doppler shifted” the spherical symmetry of the potentials at the radius. This requires radiated vacuum power to be determined by an integration over solid angle.

$$P_{in} = \frac{1}{4\pi} \oint \left[\frac{1}{2} V^* I\right] d\Omega(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}) \tag{2.19}$$

Next, it will be necessary to discuss the capacitance and inductance calculated in equation (2.9) in the rest frame. Both are derived from fundamental constants implying that both are relativistic invariants along with the frequency ω_e. This means that the impedance Z is also a relativistic invariant. In the moving frame, calculation of energy in the capacitive element is

$$u_C = \frac{1}{4\pi} \oint \left[\frac{1}{2} C \cdot [\mathrm{Re}V]^2\right] d\Omega(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}) = mc^2 \cdot \cos^2\omega_e t/\gamma \tag{2.20}$$

and a similar formula follows for the inductance. In the moving frame, the energy stored in the inductor and the capacitor is moving with the particle at speed v. This implies that both have momentum as well as energy which can be included by writing the four-vectors

$$p_C^\nu = u_C\beta^\nu \qquad p_L^\nu = u_L\beta^\nu \tag{2.21}$$

2.2 Source Current for the Vacuum Field

Calculations in the previous section were all performed following the introduction of vacuum gauge potentials. Nowhere in the discussion was any information regarding the electromagnetic fields produced by the potentials. For an elementary electron, equation (1.11) indicates that the (velocity) fields are determined in the moving frame as

$$F_v^{\mu\nu} = F_M^{\mu\nu} \cdot \mathrm{e}^{i\omega t_r/\gamma} \tag{2.22}$$

Verify Maxwell's equations by calculating the four-current density from a divergence operation

$$\begin{aligned}\partial_\mu F_v^{\mu\nu} &= \partial_\mu \left[F_{_M}^{\mu\nu} \cdot \mathrm{e}^{i\omega_e t_r/\gamma}\right] = \partial_\mu F_{_M}^{\mu\nu} \cdot \mathrm{e}^{i\omega_e t_r/\gamma} + F_{_M}^{\mu\nu} \cdot \partial_\mu \mathrm{e}^{i\omega_e t_r/\gamma} \\ &= \frac{4\pi}{c}\left[J_e^\nu \cdot \mathrm{e}^{i\omega_e t_r/\gamma} + J_{_N}^\nu\right] \end{aligned} \tag{2.23}$$

Here J_e^ν is the conventional point four-current density associated with the classical particle which has become a point oscillator. This is consistent with the rest frame voltage source in Figure 2.1. Equally important is the additional null source-current given by

$$J_{_N}^\nu = -\frac{i\omega_e e}{4\pi\rho^3} R^\nu \cdot \mathrm{e}^{i\omega_e z} \tag{2.24}$$

One finds that $\partial_\nu J_{_N}^\nu = 0$ even though it doesn't appear to be a conserved current. It's properties are most easily addressed in the rest frame where the charge density and the electric current density take the form radiating spherical waves—out of phase with the electric field—and determined by:

$$\varrho_{_N} = -\frac{ie\omega_e}{4\pi c r^2} \cdot \mathrm{e}^{i\omega_e(\tau - r/c + \tau_e)} \qquad\qquad \mathbf{J}_{_N} = -\frac{ie\omega_e}{4\pi r^2} \cdot \mathrm{e}^{i\omega_e(\tau - r/c + \tau_e)} \hat{\boldsymbol{r}} \tag{2.25}$$

As the three-vector current radiates isotropically through the vacuum boundary, it cannot be associated with any magnetic field. This can by proved by inserting $\mathbf{J}_{_N}$ into the Biot-Savart law to determine a magnetic field of zero. In addition, Ampère's law with the Maxwell correction in the rest frame is

$$\mathbf{0} = \boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}_{_N} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{2.26}$$

Without a magnetic field, $\mathbf{J}_{_N}$ follows from a time derivative of the electric field vector. For reference, the front page diagram shows the magnetic flux lines generated by the current density, which curl around the electric flux field and symmetrically cancel themselves out.

Vacuum Current in the Rest Frame: The source four-current density can be used to determine a wealth of information about the fields of the classical electron. As in the previous section, this is most easily understood in the rest frame where the total charge $q(\tau)$ follows from the indefinite volume integral

$$\tilde{q}(\tau) = \int \varrho_{_N} r^2 dr d\Omega = e \cdot \mathrm{e}^{i\omega_e z} \tag{2.27}$$

In a similar manner, the current density can be used to determine a formula for the total radiated vacuum current encompassing all directions from the source:

$$\tilde{I} = \oint \mathbf{J}_{_N} \cdot d\boldsymbol{a}_\perp = ie\omega_e \cdot \mathrm{e}^{i\omega_e z} = \frac{d\tilde{q}(\tau)}{d\tau} \tag{2.28}$$

Now suppose the current density is written $\mathbf{J}_N = \varrho_N c\hat{\boldsymbol{r}}$. A resistance follows (almost trivially) by writing

$$\mathcal{R} = \frac{1}{4\pi\epsilon_o}\frac{\varrho_N}{\|\mathbf{J}_N\|} = \frac{1}{4\pi}\sqrt{\frac{\mu_o}{\epsilon_o}} \tag{2.29}$$

While this is identical to the radiation resistance calculated in the previous section, now $\mathcal{R}$ is defined in terms of a scalar charge density and vector vacuum current in the space surrounding the particle. It seems reasonable to combine the resistance and current to form ohms law deriving the complex voltage

$$\tilde{V}_{\mathcal{R}} = \tilde{\mathrm{I}} \cdot \mathcal{R} = \frac{ie\omega_e}{4\pi\epsilon_o c} \cdot \mathrm{e}^{i\omega_e z} \tag{2.30}$$

This determines radiated vacuum power from

$$P_{in} = \frac{1}{2}\tilde{V}_{\mathcal{R}}^{*}\tilde{\mathrm{I}} = \frac{e^2\omega_e^2}{8\pi\epsilon_o c} \tag{2.31}$$

Now the resistance $\mathcal{R}$ will determine a resistivity of the vacuum for the diverging current source. This follows from the electric field and the volume current density (rest frame only):

$$\|\mathbf{E}\| = \rho_{res}\|\mathbf{J}_N\| \qquad\qquad \rho_{res} = \frac{1}{\epsilon_o\omega_e} \tag{2.32}$$

Radiating vacuum current must also be associated with capacitance and inductance. In the previous section, both were constants determined by the particle radius and well known vacuum constants (μ_o, ϵ_o). Here, we consider a vacuum impedance per unit solid angle in the space surrounding the classical particle

$$\frac{d\mathcal{Z}}{d\Omega} = \frac{d\mathcal{R}}{d\Omega} + i\left[\frac{dX_C}{d\Omega} + \frac{dX_L}{d\Omega}\right] \tag{2.33}$$

Each of the circuit elements have identical values to within a sign and may be written

$$\frac{d\mathcal{R}}{d\Omega} = \frac{1}{4\pi\epsilon_o c} \qquad\qquad \frac{dX_C}{d\Omega} = -\frac{1}{4\pi\epsilon_o c} \qquad\qquad \frac{dX_L}{d\Omega} = \frac{\mu_o c}{4\pi} \tag{2.34}$$

As the reactive elements cancel, the integrated impedance of the vacuum assumes the value

$$\mathcal{Z} = \sqrt{\frac{\mu_o}{\epsilon_o}} = 377\,\mathrm{ohms} \tag{2.35}$$

This is identical to the well-known free space result determined from the theory of electromagnetic plane waves.

Vacuum Current in a Moving Frame: As the null current is divergence free, in an arbitrary frame of reference the divergence theorem may be written

$$\int \partial_\nu J^\nu_{\scriptscriptstyle N}\, d^4\mathcal{V} = \oint J^\nu_{\scriptscriptstyle N}\, d^3\sigma_\nu = 0 \tag{2.36}$$

Since the left side of the equation must be zero, this implies that the right side can be split into separate integrals over spacelike and timelike hypersurfaces. Essentially, a flux integral of the current density through the radius for a time τ, is equal to the amount charge introduced to the vacuum in the corrseponding volume. Only one integral needs to be calculated and we choose the spacelike integral

$$d\tilde{q} = \oint_{r_e} J^\nu_{\scriptscriptstyle N} U_\nu R^2 d\Omega d\tau \tag{2.37}$$

In the moving frame this determines the vacuum current

$$\tilde{I} = \frac{d\tilde{q}}{d\tau} = \frac{iec}{r_e} \cdot \mathrm{e}^{i\omega_e \tau} \tag{2.38}$$

Comparison with the rest frame value indicates that the vacuum current is a Lorentz invariant. Since the resistance $\mathcal{R}$ is also a Lorentz invariant, then so is the voltage $\tilde{V}_{\mathcal{R}}$. Another interesting facet of the vacuum theory are expressions for both electric and magnetic velocity field vectors in terms of components of the null current. Defining

$$\alpha = -\frac{i\omega_e}{4\pi} \tag{2.39}$$

The moving frame fields can be written

$$\mathbf{E}_v = \frac{e\gamma(\boldsymbol{R} - R\boldsymbol{\beta})}{\rho^3} \qquad \alpha\mathbf{E}_v = \gamma\left(\mathbf{J}_{\scriptscriptstyle N} - \varrho_{\scriptscriptstyle N} c\boldsymbol{\beta}\right) \tag{2.40a}$$

$$\mathbf{B}_v = \frac{e\gamma\boldsymbol{\beta} \times \boldsymbol{R}}{\rho^3} \qquad \alpha\mathbf{B}_v = \boldsymbol{\beta} \times \mathbf{J}_{\scriptscriptstyle N} \tag{2.40b}$$

A Derivatives of the Null Vector

The covariant derivative of R^ν is

$$\partial^\mu R^\nu = \mathrm{g}^{\mu\nu} - \frac{R^\mu \beta^\nu}{\rho} \tag{A.1}$$

The trace of the resulting matrix gives the 4-divergence $\partial_\nu R^\nu = 3$. In terms of individual components–and with the inclusion of a sign–a useful construction is:

$$-\partial^\mu R^\nu = \begin{bmatrix} -\dfrac{\partial R}{\partial ct} & -\dfrac{\partial \boldsymbol{R}}{\partial ct} \\ \boldsymbol{\nabla} R & \boldsymbol{\nabla} \boldsymbol{R} \end{bmatrix} \tag{A.2}$$

where individual components are given by

$$\frac{\partial R}{\partial ct} = 1 - \frac{\gamma R}{\rho} \qquad\qquad \frac{\partial \boldsymbol{R}}{\partial ct} = \frac{-\gamma R \boldsymbol{\beta}}{\rho} \tag{A.3}$$

$$\boldsymbol{\nabla} R = \frac{\gamma \boldsymbol{R}}{\rho} \qquad\qquad \boldsymbol{\nabla} \boldsymbol{R} = \boldsymbol{1} + \frac{\gamma \boldsymbol{R}\boldsymbol{\beta}}{\rho} \tag{A.4}$$

The determinant of (A.2) can be written $\det[\partial^\mu R^\nu] = 0$. The divergence and curl of $\boldsymbol{R}$ may be written

$$\boldsymbol{\nabla} \cdot \boldsymbol{R} = 3 + \frac{\gamma \boldsymbol{R} \cdot \boldsymbol{\beta}}{\rho} = Tr[\boldsymbol{\nabla} \boldsymbol{R}] \tag{A.5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{R} = \frac{\gamma}{\rho} \boldsymbol{R} \times \boldsymbol{\beta} \tag{A.6}$$

Let $\boldsymbol{w}(ct_r)$ be the retarded position of a charged particle at time ct_r. The light cone condition is defined by

$$\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{w}(ct_r) \qquad\qquad R \equiv ct - ct_r \tag{A.7}$$

Derivatives of the retarded time with respect to present time coordinates are

$$\frac{\partial ct_r}{\partial ct} = \frac{\gamma R}{\rho} \tag{A.8}$$

$$\boldsymbol{\nabla} ct_r = \frac{-\gamma \boldsymbol{R}}{\rho} \tag{A.9}$$

B Lorentz Transformation of the Causality Sphere

An essential problem for a theory of radiating electromagnetic fields is the development of an appropriate transformation law for the causality step which necessarily includes the particle radius as a phase. In the rest frame the causality step is

$$\vartheta = \vartheta(c\tau - s + r_e) \tag{B.1}$$

A graphical depiction of the this function is provided in figure 2 showing a temporal expansion over a time interval $c\tau + r_e$. Unfortunately, the presence of r_e in the argu-

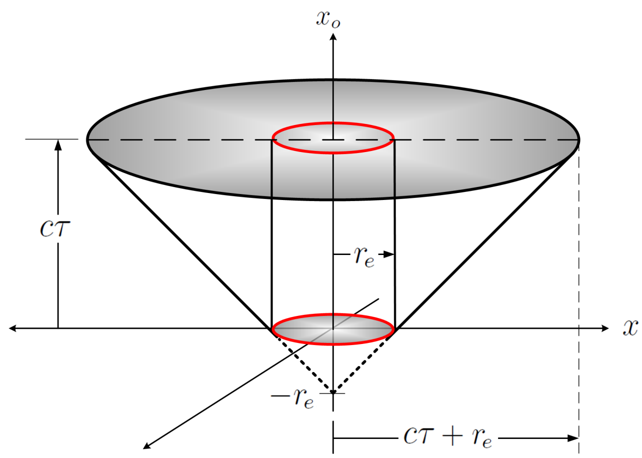

Figure 2: *Spacetime diagram indicating the expansion of the radial step $\vartheta(c\tau - s + r_e)$ in the rest frame. The inclusion of the phase r_e defines two regions inside and outside the particle radius.*

ment leads to difficulties when transforming to a moving frame. To understand why, insert the general homogeneous Lorentz transformation

$$c\tau = \gamma\,(ct - \boldsymbol{\beta}\cdot\boldsymbol{r}) \tag{B.2a}$$

$$\boldsymbol{s} = \boldsymbol{r} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{r})\boldsymbol{\beta} - \gamma ct\boldsymbol{\beta} \tag{B.2b}$$

showing that the step transforms as

$$L[\vartheta(c\tau - s + r_e)] = \vartheta(\gamma ct - \gamma\boldsymbol{\beta}\cdot\boldsymbol{r} - \rho + r_e) \tag{B.3}$$

The problem is to determine the collection of points which represents the boundary of the expanding step in the moving frame. Letting $\vartheta = \vartheta(Z)$, then the vanishing of Z implies the covariant condition

$$(x^\nu + r_e\beta^\nu)\cdot(x_\nu + r_e\beta_\nu) = 0 \tag{B.4}$$

This is the equation of a sphere with a radius $ct + \gamma r_e$ which expands about the point $-\gamma\boldsymbol{\beta} r_e$ so the moving frame step function is

$$L[\vartheta] = \vartheta(ct - \|\boldsymbol{r} - \gamma\boldsymbol{\beta} r_e\| + \gamma r_e) \tag{B.5}$$

In short, the homogeneous transformation is not a useful theoretical tool for keeping the mathematics inside the argument of the step simple.

Based on the previous result it is reasonable to enquire how the Lorentz transformation might be tailored to shift the origin of coordinates so that the causality step still expands from the spatial origin in the moving frame. This can be accomplished by defining proper frame and moving frame time coordinates as

$$c\hat{\tau} \equiv c\tau + r_e \qquad\qquad c\hat{t} \equiv ct + r_e \tag{B.6}$$

Replacing the time coordinates in (B.2) with the hatted coordinates renders a new set of equations

$$c\hat{\tau} = \gamma\left(c\hat{t} - \boldsymbol{\beta}\cdot\boldsymbol{r}\right) \tag{B.7a}$$

$$\boldsymbol{s} = \boldsymbol{r} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{r})\boldsymbol{\beta} - \gamma\boldsymbol{\beta} c\hat{t} \tag{B.7b}$$

This new set can be tested immediately to show that

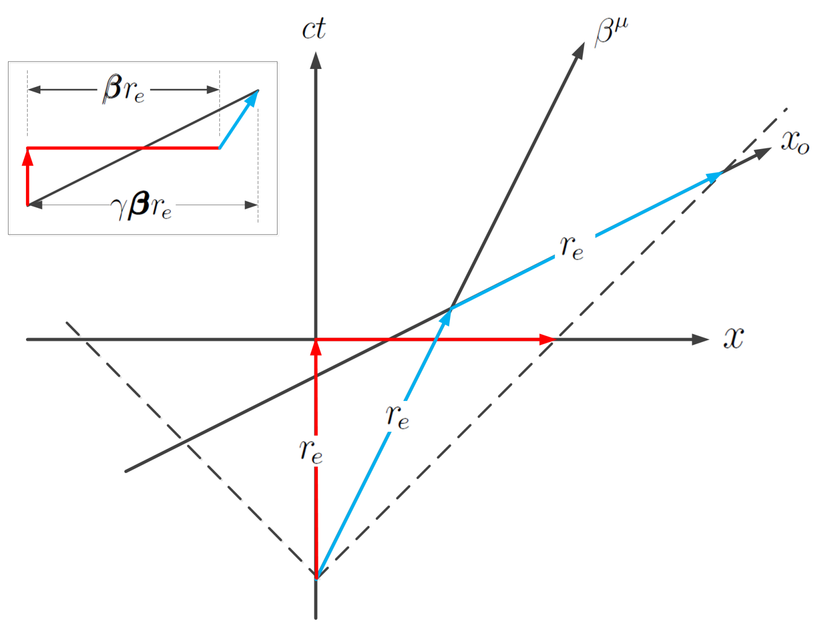

Figure 3: *Spacetime diagram based on equations (B.7). The transformation produces timelike and spacelike radius vectors (shown in red and blue) which are the same in both frames.*

$$L[\vartheta(c\tau - s + r_e)] = \vartheta(ct - r + r_e) \tag{B.8}$$

It is also important to consider equations (B.7) in terms of the retarded time ct_r and its associated vector $\boldsymbol{R}$. Relative to the moving frame, the position of the electron as a function of time is given by

$$w^\nu(ct_r) = (ct_r, \boldsymbol{\beta}c\hat{t}_r) \tag{B.9}$$

and the retardation condition is $R^\nu = x^\nu - w^\nu$. This substitution derives the following relations

$$c\tau + r_e = \rho + \frac{1}{\gamma}c\hat{t}_r \tag{B.10}$$

$$\boldsymbol{s} = \boldsymbol{R} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{R})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}R \tag{B.11}$$

Extraneous terms involving the radius drop out of the second equation which is important because it preserves the definition of the covariant scalar $\rho \equiv R^\nu\beta_\nu$. In terms of the new coordinates the causality sphere can be written

$$\vartheta = \vartheta[\gamma^{-1}(c\hat{t}_r)] \tag{B.12}$$

This is a sensible result which vanishes for retarded times less then r_e. Finally, equations (B.7) can be re-arranged to look something like a Poincaré transformation except the coordinate shifts are functions of particle velocity

$$\alpha_\mu \equiv [\gamma - 1, -\gamma\boldsymbol{\beta}]r_e \tag{B.13}$$

C Discrete Symmetries

Classical electromagnetism is known to be invariant under charge, parity, and time reversal operations. This is still true for the field strength tensor of equation (1.11). These fields—generated by changes to the source current—are high frequency outgoing travelling waves which are incoming waves upon application of the T-operator. For the classical particle however, some changes to discrete symmetries are necessary to address the problems of infinite self-energy and particle stability which are not relevant to the macroscopic theory. These changes imply the flux fields satisfying equations (1.2). In this case the radiation-based theory determines the following:

- C-Symmetry: The square of the charge appears in all equations. This seems to be equivalent to the statement that the theory is invariant under the operation of charge conjugation.
- P-Symmetry: $\boldsymbol{\pi}_{\mathrm{E}}$ remains odd under a parity operation since field momentum appears to move opposite to the original coordinate system. But $\boldsymbol{\pi}_{\mathrm{B}}$ must still remain an axial vector owing to the relation

  $$\boldsymbol{\pi}_{\mathrm{B}} = \boldsymbol{\beta} \times \boldsymbol{\pi}_{\mathrm{E}} \tag{C.1}$$

  Operations on equations (1.2) show that the momentum flux formulation remains invariant under parity.
- T-Symmetry: In the vacuum theory, electric flux fields radiate field energy. Clearly $\mathrm{T}[\boldsymbol{\pi}_{\mathrm{E}}] = -\boldsymbol{\pi}_{\mathrm{E}}$. Since the T- operation also changes the sign of $\boldsymbol{\beta}$ this requires $\mathrm{T}[\boldsymbol{\pi}_{\mathrm{B}}] = \boldsymbol{\pi}_{\mathrm{B}}$. Operations on equations (1.2) show that both source equations violate time reversal invariance.