Dark Energy for the Schwarzschild Vacuum

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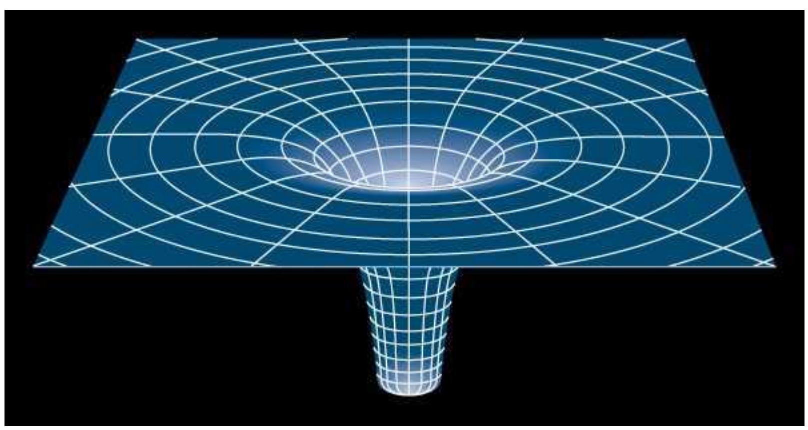

Abstract

In the weak field limit, dark energy is propagated through an otherwise static and spherically symmetric gravitational field. The inherent violation of energy conservation has negligible impact on local orbits of test bodies while generating a dark matter profile in the near neighborhood of a galaxy. The well-known problem of the Schwarzschild vacuum can accomodate the propagation of dark energy with the addition of a source dark stress tensor to the Einstein field equations. Solutions differ from the conventional problem by a small perturbation while producing radiation fields derived from travelling spherical curvature waves.

Links to youtube videos:
https://youtu.be/PHEeNRdzCX4
https://youtu.be/gk8QoH8dPXE

Vacuum
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1 Dark Radiation in the Weak Field Limit

1.1 Momentum Flux in the Gravitational Field

In the sun's core, radiated electromagnetic power and approximately 10^{38} radiated neutrinos per second are both traceable to p-p chain nuclear interactions. The total power output from these sources is documented to be $P_{core} \approx 3.9 \times 10^{26}$ Watts. In addition, the sun also ejects protons, electrons, and other charged particles which form the solar wind. Power output from this source is mainly the mass-energy of the particles and estimated to be $P_{sw} \approx 1.5 \times 10^{26}$ Watts. Combining these sources leads to a rough estimate of the total power radiated by the sun

$$P_\odot \approx P_{core} + P_{sw} \tag{1.1}$$

As the power spreads out in a (roughly isotropic) r^{-2} distribution about the sun, one might expect radiated particles inside the orbit of a planet will contribute to the total gravitational force on the planet, while radiated particles beyond will not. In terms of conservation of energy, the effective mass-energy of the sun is decreasing with time and includes a small—probably not measurable—time-dependent correction

$$M_\odot(t)c^2 = M_\odot c^2 - P_\odot t \tag{1.2}$$

Apart from the sun, a formula like this can probably be tailored to most gravitating bodies in the universe having a stable mass M, with the possibility that the power output might have the opposite sign.

Now suppose that, in addition to known radiated power, the sun emits a form of massless radiation in violation of the principle of energy conservation[1]. That is, the mass of the sun does not decrease as a result of this radiation as it would in (1.2). This statement can be immediately quantified by considering an isotropically radiated energy flux

$$\boldsymbol{S}_D(r) = \frac{\alpha M_\odot c^3}{4\pi r^2}\hat{\boldsymbol{r}} \tag{1.3}$$

where the unknown constant α has units of inverse-length. This flux must be justified in terms of the propagation of waves, and for this it is necessary to re-establish the Schwarzschild radius of the sun as the norm of a complex oscillator:

$$r_s = \sqrt{\boldsymbol{r}_s^* \cdot \boldsymbol{r}_s} \qquad \text{where} \qquad \boldsymbol{r}_s(t) = r_s\hat{\boldsymbol{r}} \cdot \mathrm{e}^{i\omega t} \tag{1.4}$$

A time derivative determines the velocity of the radius vector $\boldsymbol{v} = i\omega r_s\hat{\boldsymbol{r}} \cdot \mathrm{e}^{i\omega t}$ which is the boundary of radiating longitudinal velocity field

$$\boldsymbol{v}(r,t) = \frac{i\omega r_s^2}{r}\hat{\boldsymbol{r}} \cdot \mathrm{e}^{i\omega(t-r/c+\tau_s)} \tag{1.5}$$

[1]To justify energy violation in the classical theory, one must show that the inclusion of energy violation in elementary particle physics exists independently of particle interactions described by the standard model.

While the angular frequency is not yet known, the phase constant must be chosen to yield a simple oscillator at the Schwarzschild radius. Now construct a field of spherical compression waves from the magnitude of the velocity field

$$p(r,t) = \frac{p_o}{c}\|\boldsymbol{v}(r,t)\| \tag{1.6}$$

A familiar formula from continuum mechanics produces the required energy flux[2]

$$\boldsymbol{S}_{D}(r) = \frac{1}{2}p^{*}\boldsymbol{v} = \frac{1}{2c}\left[\frac{r_s^4}{r^2}\right]p_o\omega^2\,\hat{\boldsymbol{r}} \tag{1.8}$$

The two unknowns in this equation are p_o and ω, but both can be rationalized by common sense and their ability to reproduce the energy flux in equation (1.3):

$$\omega = \sqrt{\frac{\alpha}{r_s}}\,c \qquad\qquad p_o = \frac{M_{\odot}c^2}{2\pi r_s^3} \tag{1.9}$$

For the sun, the frequency $f \sim 10^{-5}$ Hz is available in Table 1 along with the calculated wavelength. The fundamental pressure constant is proportional to the energy density of the associated black hole and measuring approximately 10^{36} J/m^3.

The previous calculation of dark energy flux for the sun can be extended to include any spherically symmetric gravitating body of mass M. It's a trivial exercise to calculate a formula for dark power output:

$$P_{D} = \alpha M c^3 \tag{1.10}$$

A dark cuurent four-vector follows by dividing the energy flux by a factor of c and including its magnitude as the time component

$$\mathcal{P}_{D}^{\nu} = [p_{D}, \boldsymbol{p}_{D}] \qquad \text{where} \qquad \boldsymbol{p}_{D}(r) = \frac{\alpha M c^2}{4\pi r^2}\hat{\boldsymbol{r}} \tag{1.11}$$

The time-component may be referred to as a scalar **dark energy density** equal to the magnitude of the associated **dark momentum flux**. As the energy density is positive, one may presume an equivalent mass density by writing $p_{D} \equiv \rho_{D}c^2$. This implies that the total mass of a gravitating body inside some exterior radius r—excluding all other possible forms of radiation—will be

$$M(r) = M(1 + \alpha r) \tag{1.12}$$

[2]This derivation of $\boldsymbol{S}_{D}$ assumes pressure and velocity fields are strictly complex wave forms. A more physical approach begins with the radius vector

$$\boldsymbol{r}_s(t) = 2r_s\sin^2\left[\frac{\omega t}{2}\right]\hat{\boldsymbol{r}} \tag{1.7}$$

The average Schwarzschild radius r_s is used here instead of the norm.

This mass field can be inserted directly into the Newtonian gravitational force law leading to a re-fashioned field given by:

$$\mathbf{g}_{net} = -GM\left[\frac{1}{r^2} + \frac{\alpha}{r}\right]\hat{\boldsymbol{r}} = \mathbf{g} + \mathbf{g}_\alpha \tag{1.13}$$

The legitimacy of this field hinges on the requirement that α be a small number which labels it as a fundamental far field constant of the universe. In the far field limit the Newtonian term makes no contribution and velocities of large orbit test bodies become

$$v = \sqrt{\alpha GM} \tag{1.14}$$

It is important to observe that (1.13) is still a conservative field. While it may be perceived as a modification to Newton's law of gravitation, it is more appropriate to leave Newton's law intact and refer to $\mathbf{g}_{net}$ as a re-assessment of the distribution of mass-energy associated with gravitating bodies. A reasonable estimate for α follows from the stipulation that the total radiated mass density in the neighborhood of a galaxy is a good candidate for dark matter. A rough calculation is performed in the Appendix with the result

$$\alpha \approx 6.00 \times 10^{-21}\,\mathrm{m}^{-1} \tag{1.15}$$

All calculations in this investigation will assume this value when required.

1.2 Lagrangian and Stress Tensor for the Dark Field

Dark radiation theory of the previous section shows similarities to acoustic phenomena in continuous media. Even as no medium is specified, the theory successfully demonstrates the requirement for a small correction to the Newtonian law of gravity. However, another possibility for describing dark flux also exists as a theory of geometry, and this interpretation is more naturally accomodated by general relativity. For this reason, and for the purpose of introducing general covariance, it is sensible to discard the pressure and velocity fields in favor of the purely geometrical null four-vector defined by:

$$\phi^\mu = [\phi, \boldsymbol{\phi}] \qquad \text{where} \qquad \boldsymbol{\phi} = \frac{1}{r}\hat{\boldsymbol{r}} \tag{1.16}$$

For an arbitrary mass M, the gravitational field (less dark energy) may be determined from the bi-linear combination $\mathbf{g} = -GM \cdot \phi\boldsymbol{\phi}$, but the field and its magnitude also follow from the pair

$$\mathbf{g} = GM\left[\boldsymbol{\nabla}\phi + \frac{\partial \boldsymbol{\phi}}{\partial ct}\right] \qquad\qquad \|\mathbf{g}\| = GM\left[\frac{\partial \phi}{\partial ct} + \boldsymbol{\nabla}\cdot\boldsymbol{\phi}\right] \tag{1.17}$$

These equations can be written in a tensor notation as individual components on the left can be combined into a gravity null vector. For a time-independent ϕ^ν, this vector is

$$\mathrm{g}^\nu = (\|\mathbf{g}\|, -\mathbf{g}) = GM\,(\boldsymbol{\nabla}\cdot\boldsymbol{\phi}, -\boldsymbol{\nabla}\phi) \tag{1.18}$$

Now define the two second rank tensors

$$F^{\mu\nu} \equiv \begin{bmatrix} 0 & -\phi\boldsymbol{\phi} \\ \phi\boldsymbol{\phi} & 0 \end{bmatrix} \qquad\qquad S^{\mu\nu} \equiv \begin{bmatrix} \phi & -\boldsymbol{\phi} \\ \boldsymbol{\phi} & -\mathbf{1}\phi \end{bmatrix} \tag{1.19}$$

A gravitational field strength tensor follows from the combination

$$\mathcal{F}^{\mu\nu} = GM\left[F^{\mu\nu} + \alpha S^{\mu\nu}\right] \tag{1.20}$$

For this construction, the time-space and space-time components are exactly the slightly modified force law in (1.13) to within a possible sign. Moreover, a divergence operation will include both the ordinary matter source and the dark radiation source indicated by equation (1.11):

$$\partial_\mu \mathcal{F}^{\mu\nu} \equiv 4\pi G\left[\rho_m^\nu + \mathcal{P}_{\scriptscriptstyle D}^\nu/c^2\right] \tag{1.21}$$

To construct an appropriate Lagrangian for the dark field, use $z = t - r/c + \tau_s$ and extend the definition of ϕ^ν to an outgoing travelling spherical wave:

$$\phi^\nu \equiv \frac{1}{r}\mathrm{e}^{i\omega z} n^\nu \quad \text{where} \quad n^\nu = [1, \hat{\boldsymbol{r}}] \tag{1.22}$$

An appropriate Lagrangian follows from the binary combination

$$\mathcal{L}_{\scriptscriptstyle D} = -\frac{\epsilon}{2\kappa} S^{*\mu\nu} S_{\mu\nu} = -p_{\scriptscriptstyle D}(r) \tag{1.23}$$

and leading to the **dark stress tensor** (or dark tensor)

$$\begin{aligned} \mathcal{D}_{\mu\nu} &= -2\frac{\delta \mathcal{L}_{\scriptscriptstyle D}}{\delta \mathrm{g}^{\mu\nu}} + \mathrm{g}_{\mu\nu}\mathcal{L}_{\scriptscriptstyle D} \\ &= \frac{\epsilon}{\kappa}\left[S^*_{\mu\lambda} S_\nu{}^\lambda - \frac{1}{2}\mathrm{g}_{\mu\nu} S^{*\sigma\lambda} S_{\sigma\lambda}\right] \end{aligned} \tag{1.24}$$

An explicit form for the dark tensor may be obtained by writing $S^{\mu\nu}$ in terms of symmetric and anti-symmetric components $S^{\mu\nu} \equiv \phi^{\mu\nu} + \mathrm{g}^{\mu\nu}\phi$. The dark tensor is then

$$\kappa \mathcal{D}_{\mu\nu} = \epsilon \phi^*_{\mu\lambda} \phi_\nu{}^\lambda = \frac{\epsilon}{r^2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}_{tr} \tag{1.25}$$

where the subscript tr indicates truncation of all tensor components having angular directions. One may use $\mathcal{D}_{\mu\nu}$ immediately for the derivation of the momentum flux four-vector from the contraction $\mathcal{P}_{\scriptscriptstyle D}^\nu = -\mathcal{D}^{\mu\nu} n_\nu$.

A critical point involving the dark tensor stems from the observation that it contains no flux in the off-diagonal components $\mathcal{D}_{tr}$ and $\mathcal{D}_{rt}$. While the production of dark energy has not been discussed at length, one can speculate that general relativity might not be aware of the radial motion of the field. Certainly it can distinguish the two possible directions of a rotating spherical mass, but a radiating field like $\mathcal{D}_{\mu\nu}$ is a perfect spherical symmetry which appears static, and may not have observable effects except on cosmic scales.

2 Dark Radiation for the Schwarzschild Vacuum

The dark Lagrangian in equation (1.23) was derived assuming a flat spacetime, but is also valid in curved spacetimes. It can therefore be inserted into general relativity by including it as part of the Einstein-Hilbert action. If $\mathcal{R}$ is the Ricci scalar—and without the ordinary matter Lagrangian $\mathcal{L}_M$—the modified action for the space surrounding a spherically symmetric gravitating body is:

$$\mathcal{S} = \int \left[\frac{1}{2\kappa}\mathcal{R} + \mathcal{L}_D \right] \sqrt{-g}\, d^4x \tag{2.1}$$

From the smallness of the constant α, it is evident that $\mathcal{L}_D$ represents only a perturbative correction to the action. Regardless, the presence of $\mathcal{L}_D$ is the only requirement necessary to propagate dark energy through the Schwarzschild vacuum.

2.1 Schwarzschild Metric with Radiating Dark Energy

For a spherically symmetric and static spacetime, the Schwarzschild metric is given in terms of the line element

$$ds^2 = e^{2\Phi}dt^2 - e^{2\Lambda}dr^2 - r^2 d\Omega^2 \tag{2.2}$$

Without dark energy, there is no source and the solution to the field equations is determined solely by the Ricci tensor with $\mathcal{R}_{\mu\nu} = 0$. As discussed earlier however, the energy violating dark field does not introduce any explicit time-dependence to destroy the static nature of the problem. It is therefore safe to assume that the metric for the Schwarzschild vacuum, modified by the dark current, is still given by (2.2).

The explicit form of the dark stress tensor follows by appealing directly to equation (1.24) and replacing the flat metric with Schwarzschild metric. Written out in full, this is

$$\mathcal{D}_{\mu\nu} = \frac{\epsilon}{\kappa r^2} \begin{bmatrix} -\mathrm{e}^{2\Phi} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{2\Lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.3}$$

Now use the action given in equation (2.1) to determine the field equations for the "slightly modified" Schwarzschild vacuum:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu}\mathcal{R} = \kappa\mathcal{D}_{\mu\nu} \tag{2.4}$$

Assume the right side of this equation describes the external dark stresses for all spherically symmetric gravitating bodies—including black holes and neutron stars. Since the left side is the Einstein tensor which satisfies $G^{\mu\nu}_{\ \ ;\nu} = 0$, then it must be true that $\mathcal{D}^{\mu\nu}_{\ \ ;\nu} = 0$, which can be readily verified. For large values of r, inspection of the non-zero components indicate that the energy density is to be associated with a radial pressure having the opposite sign[3]. This suggests that the implied cosmic background of dark energy will possess the same property.

Explicit solutions to (2.4) follow by writing out components of the Ricci tensor explicitly in terms of Φ, Λ, and their derivatives:

$$\mathcal{R}_{\tau\tau} = e^{2\phi-2\Lambda}\left[\Phi'' + \Phi'^2 - \Lambda'\Phi' + 2\Phi'/r\right] \tag{2.5a}$$

$$\mathcal{R}_{rr} = -\Phi'' - \Phi'^2 + \Lambda'\Phi' + 2\Lambda'/r \tag{2.5b}$$

$$\mathcal{R}_{\theta\theta} = e^{-2\Lambda}\left[r\Lambda' - r\Phi' - 1\right] + 1 \tag{2.5c}$$

$$\mathcal{R}_{\phi\phi} = \mathcal{R}_{\theta\theta}\sin^2\theta \tag{2.5d}$$

Combining these using $\mathrm{g}^{\mu\nu}\mathcal{R}_{\mu\nu}$ derives

$$\mathcal{R} = 2\mathrm{e}^{-2\Lambda}\left[\Phi'' + \Phi'^2 - \Lambda'\Phi' + \frac{2\Phi'}{r} - \frac{2\Lambda'}{r} + \frac{1}{r^2}\right] - \frac{2}{r^2} \tag{2.5e}$$

Inserting into (2.4) leads to non-linear differential equations for the metric. The tt- and rr- components together imply $\Phi' + \Lambda' = 0$, and this relation may be used to render the single equation

$$\mathrm{e}^{-2\Lambda}\left[\frac{2\Lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} = -\frac{\epsilon}{r^2} \tag{2.6}$$

Components of the metric are therefore given by

$$\mathrm{e}^{2\Phi} = \left[1 - \frac{r_s}{r} + \epsilon\right] \qquad\qquad \mathrm{e}^{2\Lambda} = \left[1 - \frac{r_s}{r} + \epsilon\right]^{-1} \tag{2.7}$$

Individual components of the Ricci tensor (which would *all* be zero in the conventional model) are:

$$\mathcal{R}_{\tau\tau} = 0 \qquad\qquad \mathcal{R}_{rr} = 0 \tag{2.8a}$$

$$\mathcal{R}_{\theta\theta} = -\epsilon \qquad\qquad \mathcal{R}_{\phi\phi} = -\epsilon\sin^2\theta \tag{2.8b}$$

With known values of the Ricci tensor, it is easy to verify that the field equations $\mathrm{G}_{\mu\nu} = \kappa\mathcal{D}_{\mu\nu}$ are satisfied.

The most striking feature relating to the determination of the metric components in (2.7) is the notion that a gravitating body like the sun can radiate huge amounts of dark energy into the cosmos with only a vanishingly small perturbation to the metric. For the central Phoenix A black hole, one finds $\epsilon \approx 10^{-7}$, which would be

[3]The dark tensor $\mathcal{D}_{\mu\nu}$ does not seem to be derivable from the perfect fluid stress tensor which may be a sympton of energy violation.

the upper limit for ϵ if this black hole is the heaviest gravitating body in the universe described by the Schwarzschild metric. For very large values of r, (2.7) suggests that the radiation of dark energy precludes the existence of an absolutely flat spacetime anywhere in the universe. However, there might be a way for ϵ to be legitimately transformed away in this limit.

2.2 Properties of the Field Equations with Dark Radiation

Even without a solution to the field equations, they produce a wealth of useful information. The Ricci scalar is determined by evaluating the trace of both sides of (2.4):

$$\mathcal{R} = \frac{2\epsilon}{r^2} \tag{2.9}$$

This is a Gaussian-like curvature, which may also be referred to as a positive *dilatation* with $\mathcal{R} > 0$ everywhere. The positive sign is important as it implies that a small sphere surrounding a point in the spacetime has been compressed by the dilatation. As the dilatation is radial, this gives rise to the appearance of a theoretic hole in the spacetime through which the dark energy can flow. The notion of a singularity at the origin is removed by the dilatation in favor of a spacetime boundary with an interior excluded from the spacetime continuum.

Comparing the formula for the dark stress tensor derived in equation (1.24) with field equations of (2.4) leads to the conclusion that each term of the field equations has a representation as a bi-linear combination of $S_{\mu\nu}$:

$$\mathcal{R}_{\mu\nu} = \epsilon\, S^*_{\mu\lambda} S_\nu^{\ \lambda} \qquad\qquad \mathcal{R} = \epsilon\, S^*_{\mu\lambda} S^{\mu\lambda} \tag{2.10}$$

The equation on the right also implies a relation between the Ricci scalar and the dark Lagrangian $\mathcal{R} = -2\kappa\mathcal{L}_D$. But this is exactly the integrand of the Einstein-Hilbert action less the matter Lagrangian

$$\mathcal{S} = \int \left[\frac{1}{2\kappa}\mathcal{R} + \mathcal{L}_D\right] \sqrt{-g}\, d^4x = 0 \tag{2.11}$$

The value of the action is unchanged from the conventional Schwarzschild vacuum.

The trace reverse of field equations in (2.4) is

$$\mathcal{R}_{\mu\nu} = \kappa\left[\mathcal{D}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{D}\right] \tag{2.12}$$

but $\mathcal{D}_{\mu\nu}$ has been written explicitly in terms of the anti-symmetric $\phi_{\mu\nu}$ in equation (1.25). The implication is that the Ricci tensor is its own stress tensor associated with a Lagrangian given by

$$\mathcal{L}_{Ricci} = -\frac{\epsilon}{2}\phi^*_{\mu\nu}\phi^{\mu\nu} \tag{2.13}$$

2.3 Field Equations and Gaussian Curvature

An important operation on $S_{\mu\nu}$ in the Schwarzschild metric is its decomposition into the sum of two parts occupying independent subspaces:

$$S_{\mu\nu} \equiv \mathcal{A}_{\mu\nu} + \xi_{\mu\nu} \tag{2.14}$$

On the right side of this equation $\mathcal{A}_{\mu\nu}$ is associated with the four space/time components at the upper left, while $\xi_{\mu\nu}$ covers the remaining diagonal terms on the lower right. The merits of this decomposition become evident when inserting the right side of (2.14) into the Lagrangian density

$$\begin{aligned}\mathcal{L}_D = -\frac{\epsilon}{2\kappa}S^*_{\mu\nu}S^{\mu\nu} \\ = -\frac{\epsilon}{2\kappa}(\mathcal{A}^*_{\mu\nu} + \xi^*_{\mu\nu})(\mathcal{A}^{\mu\nu} + \xi^{\mu\nu}) \\ = -\frac{\epsilon}{2\kappa}\xi^*_{\mu\nu}\xi^{\mu\nu} = -p_D(r)\end{aligned} \tag{2.15}$$

Not only is the value of the Lagrangian invariant to the transformation, but the stress tensor for the dark field still follows in terms of the new field quantity

$$\mathcal{D}_{\mu\nu} = -2\frac{\delta\mathcal{L}_D}{\delta \mathrm{g}^{\mu\nu}} + \mathrm{g}_{\mu\nu}\mathcal{L}_D = \frac{\epsilon}{\kappa}\left[\xi^*_{\mu\lambda}\xi^{\lambda}_{\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu}\,\xi^*_{\sigma\lambda}\xi^{\sigma\lambda}\right] \tag{2.16}$$

A link exists between $\xi_{\mu\nu}$ and the differential geometry of a 2D sphere in R^3 since the two-by-two subspace spanned by $\xi_{\mu\nu}$ is exactly the second fundamental form derived for an oscillating complex spherical surface defined by $\boldsymbol{\sigma}(\theta,\phi) \equiv \boldsymbol{r}\cdot \mathrm{e}^{i\omega_s z^*}$. Specifically, one may write

$$\mathrm{II} = \begin{bmatrix} -r & 0 \\ 0 & -r\sin^2\theta \end{bmatrix} \cdot \mathrm{e}^{i\omega_s z^*} \tag{2.17}$$

As long as we remain in the subspace, it is then possible to determine a real Gaussian curvature from the ratio of determinants of first and second fundamental forms. This link is strong enough to define identical principal curvatures in the 4-space theory as components K_1 and K_2 of the mixed tensor $\xi_\mu^{\ \nu}$ which are propagating spherical waveforms:

$$K_1 = K_2 = \frac{1}{r}\mathrm{e}^{i\omega_s z^*} \equiv \phi \tag{2.18}$$

Each principal curvature satisfies the wave equation and the two may be combined as a Gaussian curvature $K = K_1^* K_2 = 1/r^2 = \phi^*\phi$. Principal curvatures may also be referenced by the Ricci tensor, the Ricci scalar, and the dark stress tensor. For the latter two they appear as sums over indivudual components, for example:

$$\mathcal{R} = \epsilon\,(K_1^*K_1 + K_2^*K_2) \tag{2.19}$$

In this interpretation, the Ricci scalar is a sum of independent polarizations of radiating curvature waves.

For some additional perspective, it can be shown that perturbations to the metric components and the Ricci tensor—brought about by the introduction of the dark tensor—can be traced to a single independent component of the Riemann curvature tensor, namely

$$\mathcal{R}^{\phi}_{\theta\phi\theta} = \frac{r_s}{r} - \epsilon \tag{2.20}$$

This component is exactly the single component which arises in the problem of a 2D spherical surface suggesting that the propagation of the dark field in the four-space Lorentzian manifold is the natural extension of the 2D problem.

2.4 Gravitational Redshift of Dark Radiation

In the radiation-based theory, dark energy is emitted at the speed of light through a vacuum portal created by the dilatation. The radius of the portal is not explicitly given but assumed to be the Schwarzschild radius. As with the photon, dark waves climbing through the deformed spacetime will undergo a gravitational redshift determined by the metric component g_{tt}.

$$1 + z = \sqrt{\frac{g_{tt}(rec)}{g_{tt}(src)}} = \sqrt{\frac{1 - \dfrac{r_s}{r_1} + \epsilon}{1 - \dfrac{r_s}{r_2} + \epsilon}} \tag{2.21}$$

This formula unambiguously sets the sign of α. At radius $r > r_s$, the frequency of the emitted wave will be

$$f(r) = \frac{\sqrt{\epsilon}}{\sqrt{1 - \dfrac{r_s}{r} + \epsilon}} f_o \quad \text{where} \quad f_o = \frac{c}{2\pi r_s} \tag{2.22}$$

Energy densities must be proportional to the square of the frequency which follows by calculating the energy component of the dark source tensor. The energy density and the integrated power formula are

$$\mathcal{D}^{tt} = -\frac{1}{\kappa r^2}\left[\frac{f}{f_o}\right]^2 \qquad\qquad P_D = \mathrm{e}^{2\Lambda}\alpha M c^3 \tag{2.23}$$

A plot of the redshift for the sun is available in Figure 1. From the plot it is evident that the frequency of the waves redshift by more than eight orders of magnitude before they reach the surface of the sun. Such large redshifts imply that radiation from large gravitating bodies is (ultimately) of extremely small frequencies and long wavelengths. For comparison, Table 1 shows values for emitted and redshifted frequencies from three well known sources based on the value of α given in equation (1.15). For most bodies like the sun and earth, nearly one hundred percent of the full redshift occurs before the waves radiate from the surface of the body.

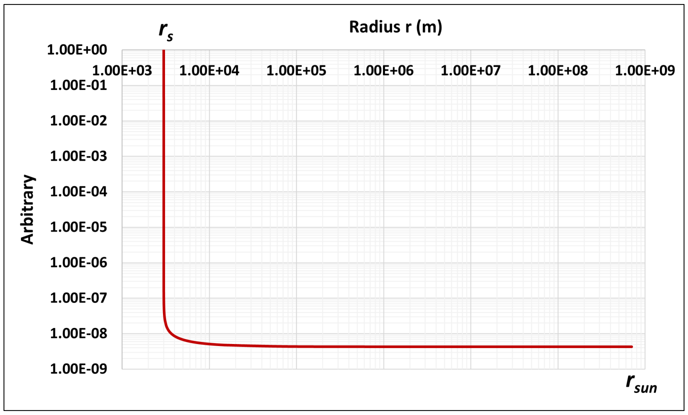

Figure 1: *Redshift of dark waves from Schwarzschild radius of the sun to the radius of the sun.*

Body	$f(r_s)$	$f(\infty)$	$\lambda(\infty)$
Earth	5.39 GHz	3.92×10^{-2} Hz	7.65×10^{9} m
Sun	16.1 kHz	6.78×10^{-5} Hz	4.42×10^{12} m
Sagittarius A*	3.90 mHz	3.34×10^{-8} Hz	8.98×10^{15} m

Table 1: *Gravitational redshifts of dark radiation emitted by three well known sources.*

In the radiation-based theory, heavenly bodies in the universe therefore function as cosmic engines capable of interacting with their environment through the perpetual manufacture of longitudinal waves. Larger objects radiate at lower frequencies in analogy with locomotives in a train yard generating acoustic waves of much lower frequencies than passing automobiles.

2.5 Calculations with the Tortoise Coordinate

The addition of dark flux to the problem of the Schwarzschild vacuum has the effect of slightly displacing the coordinate singularity but it still remains. In the conventional theory, the singularity is removed through the introduction of the tortoise coordinate r^* which leads to the Eddington-Finklestein coordinates. For the radiation-based theory, EF coordinates still exist but the tortoise coordinate must be modified to include the constant ϵ:

$$r^* \equiv \frac{1}{1+\epsilon}\left[r + \frac{r_s}{1+\epsilon}\ln\left|\frac{r(1+\epsilon)}{r_s} - 1\right|\right] \tag{2.24}$$

The factor $(1+\epsilon)^{-1}$ outside the large brackets is of primary importance since without it, the function r^* is nothing more than the conventional tortoise coordinate with a re-scaled Schwarzschild radius. Instead, the function r^* defined here leads to the derivative

$$\frac{\partial r^*}{\partial r} = \frac{1}{1 - \dfrac{r_s}{r} + \epsilon} \tag{2.25}$$

For outgoing waves only, the transformation law for the EF coordinates remains as $u = t - r^*$ but the new line element is

$$ds^2 = \left(1 - \frac{r_s}{r} + \epsilon\right) dt^2 + 2du dr - r^2 d\Omega^2 \tag{2.26}$$

The form of the dark stress tensor in the transformed metric is obvious

$$\mathcal{D}_{\mu\nu} = -p_D(r) \cdot \begin{bmatrix} 1 - r_s/r + \epsilon & 1 \\ 1 & 0 \end{bmatrix}_{tr} \tag{2.27}$$

The transformation $t' = t - (r^* - r)$ is also found in the literature generating a similar line element with another form of $\mathcal{D}_{\mu\nu}$. In either case the trace of the dark tensor remains the same as does the value of the Ricci tensor and the Ricci scalar.

Covariant and contravariant forms of the null vector in the Schwarzschild metric are

$$n_\nu = \left[1, -\mathrm{e}^{2\Lambda}, 0, 0\right] \qquad\qquad n^\nu = \left[\mathrm{e}^{-2\Phi}, 1, 0, 0\right] \tag{2.28}$$

This vector is a solution to the geodesic equation for outgoing light rays in the Schwarzschild metric and may be contracted with both $\mathcal{D}^{\mu\nu}$ and $S^{\mu\nu}$:

$$\mathcal{P}^\mu = -\mathcal{D}^{\mu\nu} n_\nu = p_D(r) n^\mu \tag{2.29a}$$

$$\phi^\mu = \frac{1}{2} S^{\mu\nu} n_\nu = \phi n^\mu \tag{2.29b}$$

For complex quantities like the four-potential, the Schwarzschild metric requires the argument of the complex exponential to be written in terms of the tortoise coordinate. Using $z^* \equiv t - r^*/c + \tau_s^*$, the definition of the four-potential will then generalize as

$$\phi^\nu = \phi n^\nu \cdot \mathrm{e}^{i\omega_s z^*} \quad \text{and} \quad \partial_\nu \phi^\nu = \frac{1}{r^2} \cdot \mathrm{e}^{i\omega_s z^*} \tag{2.30}$$

A somewhat more difficult problem is to calculate the divergence of $S^{\mu\nu}$ which assumes the explicit form

$$S_{\mu\nu} \equiv \frac{1}{r} \begin{bmatrix} \mathrm{e}^{2\Phi} & 1 & 0 & 0 \\ -1 & -\mathrm{e}^{2\Lambda} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{bmatrix} \mathrm{e}^{i\omega_s z^*} \tag{2.31}$$

The flat spacetime divergence was calculated previously in section (1.1) but is now complicated by the presence of additional Schwarzschild metric Christoffel symbols and the complex exponential:

$$S^{\mu\nu}{}_{;\,\mu} = S^{\mu\nu}{}_{,\,\mu} + S^{\lambda\nu}\,\Gamma^{\mu}_{\lambda\mu} + S^{\mu\lambda}\,\Gamma^{\nu}_{\lambda\mu} = \mathrm{e}^{-2\Lambda}\frac{1}{r}\phi^{\nu} \tag{2.32}$$

Another example of the use of the tortoise coordinate is the determination of the dark Lagrangian beginning with the Schwarzschild null oscillator

$$\mathcal{U}_{\mu}(r,t) = [1, -\mathrm{e}^{2\Lambda}] \cdot \mathrm{e}^{i\omega_s z^*} \tag{2.33}$$

The covariant derivative of this function is

$$\mathcal{U}_{\mu;\nu} = \mathcal{B}_{\mu\nu} + \xi_{\mu\nu} \tag{2.34}$$

where $\mathcal{B}_{\mu\nu}$ is the tensor

$$\mathcal{B}_{\mu\nu} = b(r)\begin{bmatrix}\mathrm{e}^{2\Phi} & -1\\ -1 & \mathrm{e}^{2\Lambda}\end{bmatrix}_{tr} \cdot \mathrm{e}^{i\omega_s z^*} \quad \text{and} \quad b(r) = ik\mathrm{e}^{2\Lambda} + \Phi' \tag{2.35}$$

The trace of $\mathcal{U}_{\mu;\nu}$ is $2/r$ while the determinant of $\mathcal{B}_{\mu\nu}$ is zero. As in section 2.3, $\mathcal{B}_{\mu\nu}$ operates as a spectator which can be removed from the problem. The result is similar to the derivation of the Lagrangian in equation (2.15) except this time derived from an appropriately chosen null four-vector.

A Dark Matter Profile

Estimation of the Far Field Constant: A value for α can be ascertained under the assumption that dark radiation is a good candidate for dark matter. From experimental data, a galaxy with a total mass M is composed of approximately 85% dark and 15% ordinary portions. If the galaxy is about the size of the Milky Way with a dark matter halo extending to a radius $r_g \approx 100,000$ light-years, then the estimate becomes

$$\frac{M}{M + \alpha M r_g} \approx 0.15 \qquad \rightarrow \qquad \alpha \approx 6.00 \times 10^{-21}\ \mathrm{m}^{-1} \tag{A.1}$$

This is quite small and shows that the force from conventional gravity and the dark energy becomes equal at $r_{eq} \approx 17,600$ ly [4].

Power Output of Sun and Earth: Radiated power attributed to the sun and earth is shown in the second column of Table 2. For the sun, this is close to one billion times the power radiated from documented sources alluded to in section 1.1. Even with large power outputs though, the last two columns of the table have been included to compare the average ordinary mass density of the two gravitating bodies relative to the mass density of the dark field at the radius. The striking feature here is the irrelevance of the dark field compared to the mass densities by 11-13 orders of magnitude[5]. This suggests that precision solar experiments such as measurements of orbital periods, or the advance of Mercury's perihelion, will not incur any significant changes resulting from the presence of dark flux. In summary, local detection of the dark field—either directly or indirectly—seems unlikely.

Gravitating Body	Power Radiated	Mass Density	Dark Mass Density (R)
Sun	$P_\odot = 3.23 \times 10^{35}$ W	$1,410$ kg/m^3	1.97×10^{-9} kg/m^3
Earth	$P_\oplus = 9.66 \times 10^{29}$ W	$5,513$ kg/m^3	7.02×10^{-11} kg/m^3

Table 2: *Dark power output and mass densities for the sun and earth. The last column estimates the effective dark mass density at the radius of each body based on the estimate of* α.

[4] The power formula predicts that any graviating body will radiate its mass in dark energy in a time $T = 1/\alpha c$. Using α in (A.1), this is 17,600 years.

[5] The energy density of the dark field at the sun's radius is $\sim 1.8 \times 10^8$ J/m^3. Compare this with the CMB having a value of 4.2×10^{-14} J/m^3.

Pseudo-Isothermal Profile: Of the many dark matter profiles, the simplest is the pseudo-isothermal model. Formulas for this and the integrated dark matter mass function are:

$$\rho_I(r) = \frac{\rho_c}{1 + \left(\dfrac{r}{r_c}\right)^2} \tag{A.2a}$$

$$M_I(r) = 4\pi\rho_c r_c^2 \left[r - r_c \tan^{-1}\left(\frac{r}{r_c}\right)\right] \tag{A.2b}$$

Under the new formalism the pseudo-isothermal profile is related to the ordinary

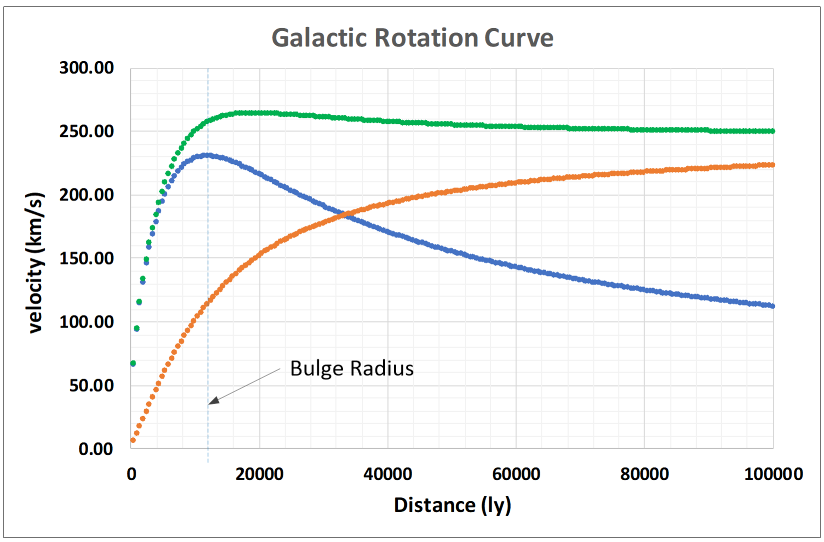

Figure 2: *Galactic rotation curve determined by the new formalism. Blue: Rotation curve from ordinary matter. Orange: Rotation curve from dark matter. Green: Combined rotation curve.*

mass function $M_m(r)$ of the galaxy by the relation $\rho_I(r) = \alpha M_m(r)/4\pi r^2$. In the limit of large r, the mass function tends to the total ordinary mass of the galaxy leading to

$$M_m(r) = M_m \left[\frac{r^2}{r_c^2 + r^2}\right] \quad \text{where} \quad \alpha M_m = 4\pi\rho_c r_c^2 \tag{A.3}$$

With full radial symmetry, the ordinary matter density may then be determined as

$$\rho_m(r) = \frac{1}{4\pi r^2}\frac{dM_m}{dr} = \frac{M_m}{2\pi r}\left[\frac{r_c^2}{(r_c^2 + r^2)^2}\right] \tag{A.4}$$

The construction of a galactic rotation curve for the combined mass follows from

$$v = \sqrt{\frac{G[M_m(r) + M_I(r)]}{r}} \tag{A.5}$$

For convenience, plots of the rotation curve for combined mass, and each mass independently, are illustrated in Figure 2. The plots are drawn for a galaxy similar to the Milky Way using an ordinary mass of $9 \times 10^{10} M_{\odot}$ and a bulge radius of 12 kly.

Dark Energy Density: The power formula in (1.10) is universal and may also be useful for a host of cosmological calculations including an estimate of the dark energy density in the observable universe. Using M_o as the total ordinary mass, and assuming T_o has the value of 13 billion years, then the observable universe with volume V_o will be characterized by an energy density

$$\varrho_{universe} = \frac{\alpha M_o c^3 T_o}{V_o} = 2.79 \times 10^{-5} \text{ J/m}^3 \tag{A.6}$$

Under modern estimates, roughly 35 percent of this is value appears as dark matter which has yet to empty into the background. This number will shrink further by including the possibility that radiated energy receives cosmological redshifts. On top of that is the question of how much of this energy—if any—is randomly absorbed by Schwarzschild radii of gravitating bodies. Overall, this appears to be a very good estimate considering its primitive nature.

B Weak Field Limit of the Schwarzschild Vacuum

An important exercise for the slightly modified Schwarzschild metric is to solve the problem in the weak field limit. Toward this end consider expansions of the metric tensor of the form

$$\mathrm{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + k_{\mu\nu} \tag{B.1}$$

In this expansion $h_{\mu\nu}$ is first order in the gravitational potentials while $k_{\mu\nu}$ includes whatever corrections exist to first order in ϵ. The forms of these two corrections can be deduced simply by inserting (B.1) into the field equations:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\left[\eta_{\mu\nu} + h_{\mu\nu} + k_{\mu\nu}\right]\mathcal{R} = \kappa\mathcal{D}^0_{\mu\nu} - \frac{\epsilon}{r^2}h_{\mu\nu} - \frac{\epsilon}{r^2}k_{\mu\nu} \tag{B.2}$$

Comparing terms on both sides of the equation implies that corrections to the metric tensor must be

$$h_{\mu\nu} = -\begin{bmatrix} r_s/r & 0 \\ 0 & r_s/r \end{bmatrix} \qquad\qquad k_{\mu\nu} = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \tag{B.3}$$

These corrections can also be deduced simply by expanding components of the metric in equation (2.2). Meanwhile, the weak form of the dark tensor becomes

$$\kappa\mathcal{D}^0_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{R} \tag{B.4}$$

This equation is identical to the weak field result and show that components of the Ricci tensor and the Ricci scalar have no preference as to whether they exist in Schwarzschild metric or a flat metric.

Now suppose we appeal to the theory of linearized gravity where $h_{\mu\nu}$ becomes a field in a flat background spacetime. Based on the inherent spherical symmetry of the Schwarzschild vacuum it will be convenient to work in a spherical-polar coordinate system where the Ricci tensor is approximated by

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}\left[h^{\sigma}_{\nu\,;\,\mu\sigma} + h^{\sigma}_{\mu\,;\,\nu\sigma} - h_{;\,\mu\nu} - \Box h_{\mu\nu}\right] \tag{B.5}$$

Inserting $h_{\mu\nu}$ and noting that its trace vanishes will verify that it makes no contribution to the Ricci tensor. This result is exemplified by the formula

$$\Box h_{\mu\nu} = h^{\sigma}_{\nu\,;\,\mu\sigma} + h^{\sigma}_{\mu\,;\,\nu\sigma} \tag{B.6}$$

As the Ricci scalar is a contraction on indices, then $\mathcal{R} = 0$ and therefore $h_{\mu\nu}$ plays no role in the weak field limit of the slightly modified Schwarzschild vacuum.

This result will not hold true for $k_{\mu\nu}$ which can also be inserted into equation (B.5) with all the labels changed. The calculation is facilitated from the divergence operation

$$k^{\sigma}_{\nu\,;\,\sigma} = [0, -2\epsilon/r, 0, 0] \equiv \zeta_{\nu} \tag{B.7}$$

Including the property that $k_{\mu\nu}$ is still traceless, the Ricci tensor becomes

$$\begin{aligned}\mathcal{R}_{\mu\nu} &= \frac{1}{2}\left[k^{\sigma}_{\nu\,;\,\mu\sigma} + k^{\sigma}_{\mu\,;\,\nu\sigma} - \Box k_{\mu\nu}\right] \\ &= -\frac{1}{2}\Box k_{\mu\nu} + \zeta_{\mu\,;\,\nu}\end{aligned} \tag{B.8}$$

The d'Alembertian acting on $k_{\mu\nu}$ is a somewhat tedius calculation. It possesses only spatial components and may be written

$$-\frac{1}{2}\Box k_{\mu\nu} = \frac{\epsilon}{r^2}\left[\mathbf{I} - 3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right] \tag{B.9}$$

Contraction on the two indices of the Ricci tensor indicates that the value of the Ricci scalar is wholly dependent on the gauge field

$$k^{\sigma\nu}_{\;;\,\sigma\nu} = \frac{2\epsilon}{r^2} = \zeta^{\nu}_{\;;\,\nu} = \mathcal{R} \tag{B.10}$$

The weak field Einstein equations determining the dark tensor can now be written as

$$G_{\mu\nu} = -\frac{1}{2}\Box k_{\mu\nu} + \zeta_{\mu\,;\,\nu} - \frac{1}{2}\eta_{\mu\nu}\zeta^{\sigma}_{\;;\,\sigma} = \kappa\mathcal{D}^{0}_{\mu\nu} \tag{B.11}$$