The Radius of the Electron

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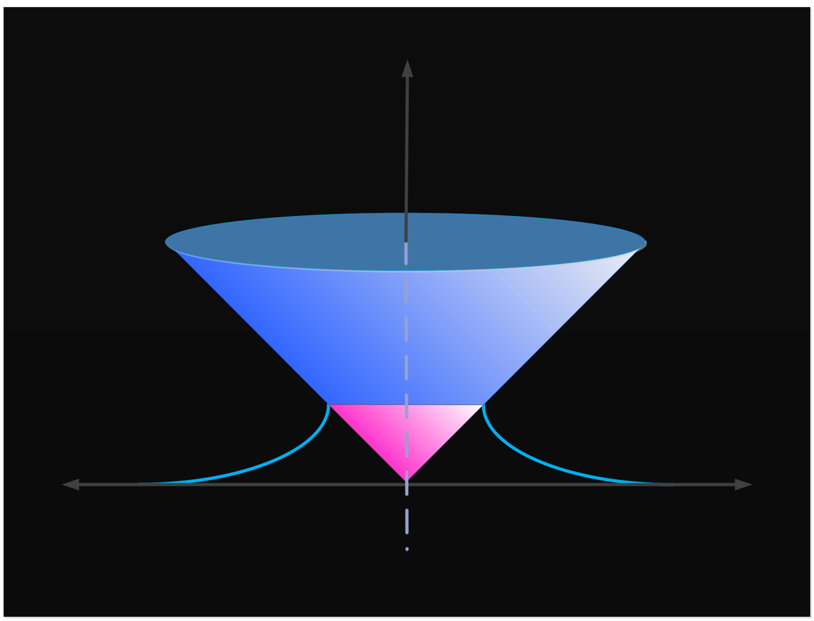

Abstract

Causality requires generalized Coulomb fields of the classical electron to be carriers of momentum and energy. The inclusion of this energy requires the use of vacuum gauge potentials having a physical interpretation as deformations of the vacuum in the neighborhood of the charge. Inherent similarities with classical theories of continuous media serve as a guide for the development of a *Vacuum Lagrangian*, reproducing all the results of conventional classical electron theory, with the additional benefit of eliminating problems of infinite self-energy and particle stability. The vacuum theory accomodates particle accelerations with the inclusion of an acceleration stress tensor, appearing as a perturbation to the velocity theory, and correctly generating additional acceleration terms of the symmetric stress tensor.

Vacuum
Gauge
Electrodynamics

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1 Theory of the Classical Vacuum

1.1 Momentum in the Electromagnetic Field

While macroscopic problems involving static electric fields are commonplace in electromagnetism, a model for a classical electron endowed with a static Coulomb field is highly suspect and doomed for failure. This follows because the electromagnetic force between elementary charge particles is well established as being mediated by massless spin-1 photons. These mediators are associated only with the acceleration fields of the particle and render the static velocity fields as superfluous and unneeded.

For a successful model of a classical electron, consider instead a theory where the generalized Coulomb fields are re-interpreted as momentum flux fields determined by multiplying the velocity fields by a fundamental surface charge density

$$\boldsymbol{\pi}_{\mathrm{E}} = \sigma_e \mathbf{E}_v \qquad\qquad \boldsymbol{\pi}_{\mathrm{B}} = \sigma_e \mathbf{B}_v \tag{1.1}$$

Such fields can be rationalized by re-structuring the problem of a classical electron as indicated by Figure 1. Here, an electron is created at the origin of coordinates at some time τ and moves to the right with four-velocity β^ν. The fields of this particle are

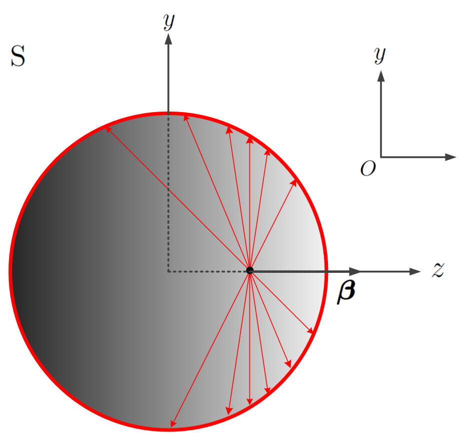

Figure 1: *Flux fields of a causal electron as it moves to the right. The observer at O currently has no knowledge of the existence of the particle.*

constrained by a causality light sphere expanding in all directions about the particle origin, and this requires the velocity fields to be carriers of momentum and energy as in equations (1.1). The light sphere is a scalar field which can be shown to be proportional to the retarded time

$$\vartheta = \vartheta(t_r/\gamma) \tag{1.2}$$

A covariant formula written in terms of the velocity portion of the field strength tensor—and inclusive of the light sphere—defines the **electromagnetic flux tensor**:

$$\pi^{\mu\nu} \equiv \sigma_e F_{M}^{\mu\nu} \cdot \vartheta \tag{1.3}$$

Technically, the light sphere is always required, but its usefulness is generally limited to problems involving covariant integrations. For this reason it is more practical to work almost exclusively with infinite space fields like (1.1), and insert the light sphere as needed.

The merits of modeling a classical electron with radiating velocity fields can be immediately realized by quantizing the particle with the assumed particle Hamiltonian given by

$$\mathcal{H}(\tau) = mc^2 \cdot \theta(\tau) - \frac{mc^2}{\tau_e}\tau \tag{1.4}$$

The step-function on the right side of this equation indicates that an electron of mass-energy mc^2 is created at time $\tau = 0$, and after time τ, has radiated an amount of field energy proportional to that time. If emitted radiation is composed of massless quanta having a single invariant energy $\boldsymbol{\varepsilon_{vac}}$, it becomes necessary to associate a frequency with the classical particle. Requiring the Dirac frequency ω_D determines $\boldsymbol{\varepsilon_{vac}} = \hbar/\tau_e$. Now write the complex exponential

$$\mathrm{e}^{i\mathcal{H}/\boldsymbol{\varepsilon_{vac}}} = \mathrm{e}^{i\alpha}\,\mathrm{e}^{-imc^2\tau/\hbar} \tag{1.5}$$

This is a Dirac oscillator with a global U(1) gauge transformation having a phase constant α equal to the fine structure constant. Ignoring the phase, the full Dirac wave function follows from a Lorentz transformation on the proper time and the inclusion of the spinor field ψ_o:

$$\psi(\boldsymbol{x}, t) = \psi_o \mathrm{e}^{-i(Et - \boldsymbol{p}\cdot\boldsymbol{x})/\hbar} \tag{1.6}$$

It is well known that the quantum wave function is not an observable, and we assume this is also true of the radiation field used to derive it.

1.2 Vacuum Gauge Potentials

A theory of the classical electron with fields given by (1.1) requires an application of the vacuum gauge condition

$$|\partial_\nu A^\nu| \equiv \sqrt{\mathrm{E}^2 - \mathrm{B}^2} \tag{1.7}$$

For transverse electromagnetic waves, the vacuum gauge and the Lorenz gauge are identical. This is not the case for the velocity fields of the electron where the vacuum gauge velocity potentials may be written

$$\boxed{A^\nu = \frac{eR^\nu}{\rho^2}} \tag{1.8}$$

In this equation R^ν is the four-vector from the retarded position of the electron to the present time event where the fields are evaluated. An important property of vacuum gauge potentials is their natural decomposition in terms of mutually orthogonal time-like and space-like components:

$$A^\nu = A^\nu_t + A^\nu_s \quad \text{where} \quad \begin{cases} A^\nu_t = \dfrac{e}{\rho}\beta^\nu \\[2ex] A^\nu_s = \dfrac{e}{\rho}U^\nu \end{cases}$$

The norms of both four-vectors are the same to within an overall sign and their similar structure suggests a reference to them as complimentary un-identical twins. Compatibility of vacuum gauge potentials with Einstein's energy-momentum relation can be exhibited by evaluating complimentary components at the electron radius $\rho = r_e$ and then forming the vector $S^\nu = p^\nu + mcU^\nu$. A contraction with lowered indices then implies $p^\nu p_\nu = m^2c^2$. The spacetime diagram of Figure 2 illustrates how the gauge field A^ν_s rotates the Liénard–Wiechert potentials onto the light cone.

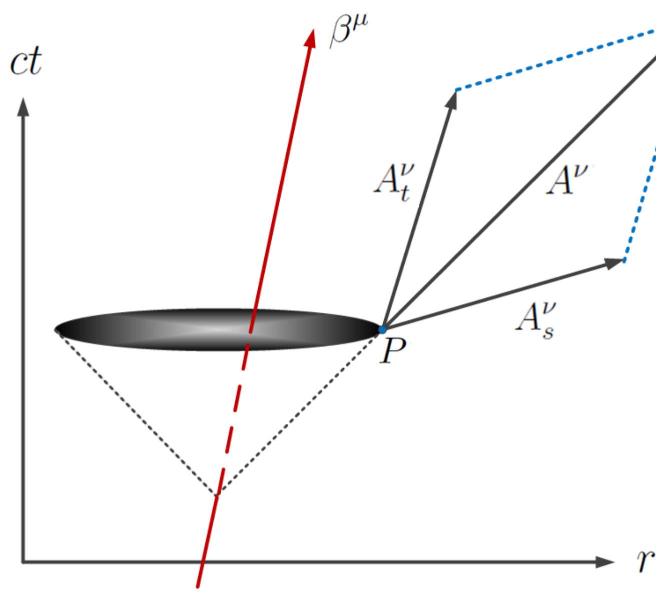

Figure 2: *Minkowski space diagram showing the addition of components of vacuum gauge potentials*

The importance of the gauge field reveals itself during particle accelerations where differential operations on the potentials determine

$$F^{\mu\nu}_v = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{1.9}$$

The suprising feature of this equation is the remarkeable property that the potentials have no ability to determine the acceleration field strength $F^{\mu\nu}_a$ regardless of the particle trajectory—and this property has far reaching consequences. Most importantly, it allows for the construction of a theory with fields in (1.1) without worrying about particle accelerations which satisfy their own set of equations. Potential formulations of each field are

$$\Box^2 A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J_e^\nu \tag{1.10a}$$

$$\Box^2 A_a^\nu - \partial^\nu \partial_\mu A_a^\mu = \frac{4\pi}{c} J_a^\nu \tag{1.10b}$$

The form of the acceleration four-current may be written

$$eJ_a^\nu \equiv -\frac{c}{2\pi}(a^\lambda A_\lambda) A^\nu \tag{1.11}$$

and is—to within a constant—the projection of the four-acceleration onto the velocity potentials. One may verify that the acceleration current satisfies $\partial_\nu J_a^\nu = 0$ as it must since $F_a^{\mu\nu}$ is anti-symmetric.

The theory of vacuum gauge potentials can be summarized by solving the acceleration equation in (1.10b) and writing the *total* vacuum gauge potentials for the most general motions of the classical electron

$$A^\nu_{total}(\boldsymbol{r}, t) = (1 - a^\lambda R_\lambda) A^\nu \tag{1.12}$$

The acceleration potentials are clearly visible in this equation and—like J_a^ν—are proportional to the velocity potentials.

1.3 Vacuum Stress and Strain Tensors

The nature of vacuum gauge potentials as radial null–potentials is an indication of their ability to describe the physics of a single electron. In fact, the vacuum gauge can be used to develop a theory of the vacuum by a simple re-definition of the velocity potentials as a ***vacuum displacement vector*** or ***vacuum dilatation vector***:

$$A^\nu \equiv 4\pi\sigma_e \mathcal{U}^\nu \quad \text{where} \quad \mathcal{U}^\nu = \frac{r_e^2}{\rho^2} R^\nu \tag{1.13}$$

The four-vector $\mathcal{U}^\nu$ is purely geometrical and describes a small orifice created by deforming the "medium" around the instantaneous retarded position of the particle out to a radius r_e—hereafter referred to as the classical radius of the electron.

Second rank tensors required for the new theory can be established beginning with the third rank tensor $\Psi^{\alpha\mu}{}_\nu$ composed of the four components of the dilatation along with two occurrences of the fourth rank, totally anti-symmetric, Levi-Civita symbol:

$$\Psi^{\alpha\mu}{}_\nu \equiv \tfrac{1}{2}\boldsymbol{\epsilon}^{\alpha\mu\sigma\tau}\boldsymbol{\epsilon}_{\sigma\tau\nu\kappa}\mathcal{U}^\kappa \tag{1.14}$$

Now consider a divergence operation on either of the last two indices. The inclusion of the metric is appropriate here to raise the last index with the result

$$\partial_\alpha \Psi^{\alpha\mu\nu} = \partial_\alpha \left[\mathrm{g}^{\nu\lambda} \Psi^{\alpha\mu}{}_\lambda \right] = \partial^\nu \mathcal{U}^\mu - \mathrm{g}^{\mu\nu} \partial_\alpha \mathcal{U}^\alpha \tag{1.15}$$

The first term on the right in this equation can be established as a unitless vacuum strain tensor:

$$\boxed{\eta^{\mu\nu} = \eta^{\dagger\,\nu\mu} \equiv \partial^{\mu}\mathcal{U}^{\nu}} \qquad \textbf{\textit{vacuum strain tensor}} \tag{1.16}$$

A concise representation of a vacuum stress tensor may then be determined from the entire right side of (1.15) with the inclusion of the modulus $\mu_e = 4\pi\sigma_e^2$:

$$\boxed{\mathit{\Delta}^{\dagger\,\mu\nu} \equiv \mu_e[\eta^{\nu\mu} - \mathrm{g}^{\mu\nu}\eta]} \qquad \textbf{\textit{vacuum stress tensor}} \tag{1.17}$$

The peculiar inversion of the indices on the previous two equations justifies the inclusion of the superscript (†) transpose operator in the definitions. For the stress tensor however, this and the un-transposed version $\mathit{\Delta}^{\mu\nu}$ will both be useful in the four-space theory of the electron.

Equation (1.17) defines a medium which can be characterized through comparisons with the 3-space theory of elastic media. For the 3-space theory, the medium is tacitly assumed to be embedded in a Galilean spacetime featuring independent temporal and spatial dimensions. For the vacuum theory under development here, equation (1.17) describes a deformable medium residing in four-dimensional Minkowski spacetime. While an elastic medium also comes complete with a unique undeformed ground state energy (or mass) density, this is an impossible requirement for the medium described by (1.17) which can be shown to possess both a restoring and admissibility property. Admissibility implies a negative force constant which, unlike Hooke's law, will allow the medium to create itself indefinitely. This medium can only move at the speed of light, and its specific density at any spacetime event will depend on the overall distribution of charged electrons, positrons, and other particles. Referring to this medium as the ***vacuum***, a precise definition can be given as:

> ***Vacuum– A self-generating deformable medium, embedded in Minkowski spacetime, and characterized by restoring and admissibility force constants having equal magnitudes.***

The vacuum will be shown to form stable electrons (and positrons) continously propagating electromagnetic momentum and energy via (1.1) during constant velocity motion, with additional (and more familiar) energy flux radiated during particle accelerations.

Explicit forms for the stress tensor may be determined by differentiating equation (1.13) or by differentiating timelike and spacelike components separately, and combining them together. The spacelike portion is symmetric and may be written in terms of differential operations on the scalar field φ. The timelike portion is easier to calculate and provides a single off-diagonal term to the final result:

$$\mathit{\Delta}^{\mu\nu} = \mu_e \frac{r_e^2}{\rho^2}\left[2\,U^{\mu}U^{\nu} - \beta^{\mu}\beta^{\nu} + U^{\mu}\beta^{\nu}\right] \tag{1.18}$$

Curiously, this tensor does not contain the metric as might be implied by its definition. Instead, the strain tensor carries the metric which is subtracted away.

Vacuum Dilatation: An important correspondence between the theory of elastic media and the theory presented here is determined by

$$\partial_\nu \mathcal{U}^\nu = \partial_\nu \mathcal{U}^\nu_s = \Box^2 \varphi \qquad \longleftrightarrow \qquad \boldsymbol{\nabla} \cdot \mathbf{u} = \boldsymbol{\nabla} \cdot \mathbf{u}_l = -\boldsymbol{\nabla}^2 \varphi \tag{1.19}$$

According to the 3-space theory, the divergence on the right measures the net distortion, or volumetric dilatation, of the medium. This suggests that the unitless four-divergence

$$\partial_\nu \mathcal{U}^\nu = \frac{r_e^2}{\rho^2} \tag{1.20}$$

is really a ***vacuum dilatation*** gauge condition—a measure of the amount by which the surrounding vacuum is deformed by the presence of the particle. This is an inherently positive quantity which has the effect of compressing the vacuum around the charge. As already indicated this is interpreted to mean that the dilatation is creating a microscopic spherical orifice (void) in the vacuum having a classical radius r_e. This is a very desirable feature of any classical theory of the electron to address—among other things—the well known problem of the divergent self-energy. Moreover, a particle radius in the context of an appropriately chosen gauge condition necessarily avoids the introduction of form factors and other senseless constructs based on macroscopic classical notions. The theory of vacuum dilatation can be compared with the Liénard-Wiechert potentials where the application of the Lorenz gauge condition $\partial_\nu A_t^\nu = 0$ generates a dilatation of zero and is unmistakably associated with a point-particle theory.

1.4 Differential Operations on the Vacuum Stress Tensor

The divergence operator applied to the second index of the vacuum stress tensor is an identity

$$\partial_\nu \mathit{\Delta}^{\mu\nu} = 0 \tag{1.21}$$

This can be proved from the definition in equation (1.17), or by applying ∂_ν directly to (1.18) where the divergence of each individual term can be shown to vanish. Applying the divergence operator to the first index is more complex and can be shown to be proportional to the acceleration four-current density J_a^ν when the position of the point-source is excluded. However, as with all terms involving particle accelerations, vacuum gauge electrodynamics will assign J_a^ν to an independent vacuum theory of acceleration strain. Withholding particle acceleration for now, a divergence on the first index follows from

$$\partial_\mu \mathit{\Delta}^{\mu\nu} = \partial_\mu \left[\tfrac{1}{2}(\mathit{\Delta}^{\mu\nu} + \mathit{\Delta}^{\nu\mu}) + \tfrac{1}{2}(\mathit{\Delta}^{\mu\nu} - \mathit{\Delta}^{\mu\nu})\right] = \tfrac{1}{2}\partial_\mu \mathit{\Delta}^{\mu\nu} + \tfrac{1}{2}\sigma_e \partial_\mu F_v^{\mu\nu} \tag{1.22}$$

From this it can only be concluded that

$$\partial_\mu \mathit{\Delta}^{\mu\nu} = 4\pi\sigma_e \frac{1}{c} J_e^\nu \tag{1.23}$$

where J_e^ν is the conventional point charge source current of the electromagnetic theory. This equation differs from the potential formulation of Maxwell's equations by a constant σ_e, yet derived from a completely different premise. The four equations are essentially the inhomogeneous Maxwell equations except that they identify two gauge invariant forms of momentum flux $\boldsymbol{\pi}_{\mathrm{E}}$ and $\boldsymbol{\pi}_{\mathrm{B}}$. As the homogeneous equations follow automatically from differential operations on vacuum gauge potentials, one finds

$$\boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{\mathrm{E}} = 4\pi\rho_e \qquad \boldsymbol{\nabla} \times \boldsymbol{\pi}_{\mathrm{B}} = 4\pi\mathbf{f}_e + \frac{1}{c}\frac{\partial \boldsymbol{\pi}_{\mathrm{E}}}{\partial t} \tag{1.24a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\pi}_{\mathrm{E}} = -\frac{1}{c}\frac{\partial \boldsymbol{\pi}_{\mathrm{B}}}{\partial t} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{\mathrm{B}} = 0 \tag{1.24b}$$

and it may be observed that the source $f_e^\nu = (\rho_e, \mathbf{f}_e)$ is now a point force density. An energy flux version of these equations follows by multiplying the above by an additional factor of $c/2$. In addition to momentum and energy flux in the fields, conventional classical electrodynamics also requires additional energy flux defined by the Poynting vector. This is still valid for the vacuum theory except the form of the equation must be slightly modified to read

$$\mathbf{S} = \frac{c}{\mu_e} \boldsymbol{\pi}_{\mathrm{E}} \times \boldsymbol{\pi}_{\mathrm{B}} \tag{1.25}$$

1.5 Covariant Integration of the Vacuum Stress Tensor

The radiation rate four-vector can be determined by integrations over both timelike and spacelike surfaces. First, consider the timelike 3-surface element illustrated in Figure 3 given by

$$d\sigma_\nu^t = U_\nu R^2 d\Omega\, dc\tau \tag{1.26}$$

This leads to the flux integral[1]

$$\mathcal{E}^\mu(\tau) = \oint_{\rho=r_e} \tfrac{1}{2}\mathit{\Delta}^{\dagger\mu\nu} d\sigma_\nu^t = -\tfrac{1}{2}\dot{\varrho} c\tau \beta^\mu \cdot \vartheta(\tau) \tag{1.28}$$

Of interest here is the conspicuous appearance of the minus sign. Moreover, there is

[1]This integral is easy to evaluate as the moving frame solid angle integral in an arbitrary reference frame is

$$\int_\Omega U^\mu d\Omega' = 0 \tag{1.27}$$

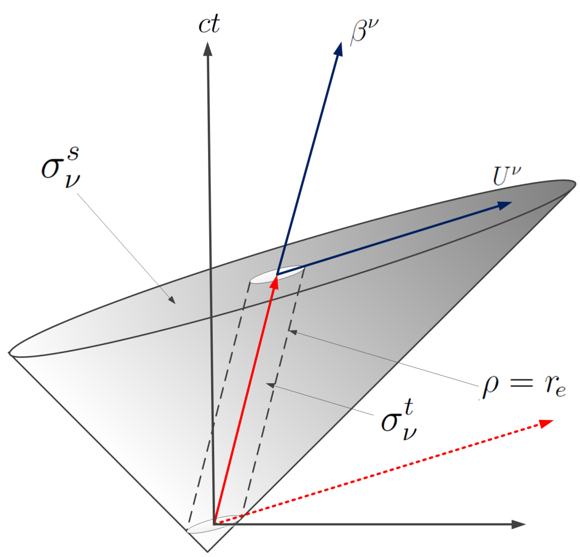

Figure 3: *Illustrating timelike and spacelike surfaces of integration for the vacuum stress tensor.*

a temptation to include an integration constant and write the total energy four-vector as

$$\mathcal{E}^\mu(\tau) = [mc^2 - \tfrac{1}{2}\dot{\varrho}c\tau]\,\beta^\nu \cdot \vartheta(\tau) = \mathcal{H}(\tau)\,\beta^\mu \cdot \vartheta(\tau) \tag{1.29}$$

where $\mathcal{H}$ is the invariant Hamiltonian for the system.

Vacuum power also results by choosing the spacelike hyper-surface pointing along the direction of the 4-velocity given by

$$d\sigma^s_\nu = \beta_\nu \rho^2 d\rho\, d\Omega' \tag{1.30}$$

Once again, $\mathcal{E}^\mu(\tau)$ is determined from the integral

$$\begin{aligned}\mathcal{E}^\mu(\tau) &= \tfrac{1}{2}\int \mathit{\Delta}^{\dagger\mu\nu} d\sigma^s_\nu \\ &= -\tfrac{1}{2}c\mu_e r_e^2 \beta^\mu \int \vartheta(\tau - \rho/c + \tau_e) d\rho\, d\Omega'\end{aligned} \tag{1.31}$$

To perform the last integral it is necessary to require the lower limit $\rho = r_e$, which is mandated by the vacuum dilatation. The upper integration limit can be anything since it is under control of the causality step. Adding an integration constant as before yields exactly the final result produced by the timelike integral.

2 Lagrangian Formulation of the Vacuum Gauge Electron

The correspondence between a radiation-based theory of the classical electron and the theory of elastic continua can be used as a guide for the development of a highly specialized Lagrangian formulation. A strictly inflexible requirement for a successful Lagrangian is the re-production of all the usual gauge invariant results of the conventional Maxwell-Lorentz electron theory. In particular, the vacuum Lagrangian $\mathcal{L}_{vac}$ must generate the symmetric stress tensor of the conventional theory. In addition, stresses associated with radiated vacuum energy must also be generated.

2.1 Derivation of the Vacuum Lagrangian

A specific form for $\mathcal{L}_{vac}$ could easily be postulated based on definitions (1.16) and (1.17), and knowledge of continuum mechanics, but a more suitable approach is to begin with the classical electromagnetic Lagrangian interacting with the source term

$$\mathcal{L}_{em} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - \frac{1}{c}J_e^{\nu}A_{\nu} \tag{2.1}$$

But stress and strain tensors can be inserted through the relations

$$\sigma_e F^{\mu\nu} = \varDelta^{\mu\nu} - \varDelta^{\nu\mu} = \mu_e(\eta^{\mu\nu} - \eta^{\nu\mu}) \tag{2.2}$$

Re-arranging constants in the interaction term leads to

$$\begin{aligned}\mathcal{L}_{em} &= -\frac{1}{4}(\varDelta^{\mu\nu} - \varDelta^{\nu\mu})(\eta_{\mu\nu} - \eta_{\nu\mu}) - \frac{1}{c}J_e^{\nu}A_{\nu} \\ &= -\frac{1}{2}\varDelta^{\mu\nu}\eta_{\mu\nu} + \frac{1}{2}\varDelta^{\nu\mu}\eta_{\mu\nu} - 4\pi f_e^{\nu}\mathcal{U}_{\nu} = \mathcal{L}_o + \mathcal{L}_x + \mathcal{L}_{int}\end{aligned} \tag{2.3}$$

It is easy to show that the term $\mathcal{L}_x$ produces no equations of motion and removing it from the theory is a tempting possibility. But if equivalence between the electromagnetic Lagrangian and the vacuum theory is a fundamental requirement, then it is more appropriate to include $\mathcal{L}_x$ along with $\mathcal{L}_o$ and define the equivalent two component ***Vacuum Lagrangian*** in terms of the field quantities $\mathcal{U}^{\nu}$ and $\eta^{\mu\nu}$:

$$\mathcal{L}_{vac} = -\frac{1}{2}\varDelta^{\mu\nu}\eta_{\mu\nu} - 4\pi f_e^{\nu}\mathcal{U}_{\nu} \tag{2.4a}$$

$$\mathcal{L}_x = \frac{1}{2}\varDelta^{\nu\mu}\eta_{\mu\nu} \tag{2.4b}$$

The legitimacy of the vacuum Lagrangian can be immediately verified by applying the Euler-Lagrange equations resulting in the momentum flux version of the Maxwell-Lorentz theory of (1.24)

$$\partial_{\mu}\varDelta^{\mu\nu} = 4\pi f_e^{\nu} \tag{2.5a}$$

$$\partial_{\mu}\varDelta^{\nu\mu} = 0 \tag{2.5b}$$

It seems reasonable to consider the vacuum Lagrangian and its equation of motion as a symmetry property of classical electron theory. If required, the conventional theory can be instantly recovered simply by subtracting equations (2.5).

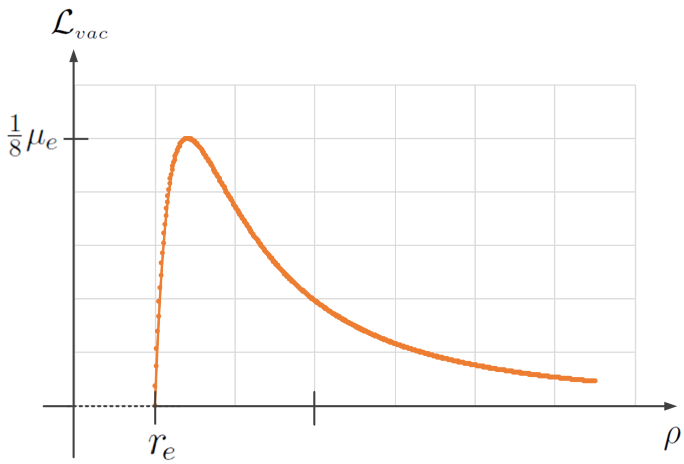

Figure 4: *Plot of $\mathcal{L}_{vac}$ versus the invariant scalar ρ. The value of the Lagrangian minimizes at the boundary $\rho = r_e$ where it vanishes. Inside the boundary assume $\mathcal{L}_{vac}$ is not defined.*

Lagrangian for a Moving Vacuum: The second term of the vacuum stress tensor is proportional to the metric and shows three positive definite space-space diagonal components. These terms represent the admissibility property of the stress tensor allowing the vacuum continuum to self-generate. The Lagrangian in equation (2.4a) can only accommodate this feature with an additional term. It must be linear in field quantities and, like the term $\mathcal{L}_x$, can have no effect on the equations of motion. A simple and meaningful possibility is to observe that the vacuum Lagrangian posesses an internal symmetry under the variation

$$\mathit{\Delta}^{\mu\nu} \longrightarrow \mathit{\Delta}^{\mu\nu} - \mu_e \mathrm{g}^{\mu\nu} \tag{2.6}$$

The resulting interacting velocity Lagrangian is then

$$\boxed{\mathcal{L}_{vac} \equiv -\frac{1}{2}\mathit{\Delta}^{\mu\nu}\eta_{\mu\nu} + \frac{1}{2}\mu_e\eta - 4\pi f_e^{\nu}\mathcal{U}_{\nu}} \tag{2.7}$$

The linear term should be properly referred to as a vacuum dilatation producing dilatation stresses which—as already indicated—function to propagate field energy away from the source. Its presence is reminiscent of a term added to a Lagrangian in point particle mechanics to represent forces of constraint. The analogy has some grey area but μ_e assumes the role of a Lagrange multiplier. For convenience, a plot of $\mathcal{L}_{vac}(\rho)$ is included in Figure 4.

2.2 Symmetric and Total Stress Tensors

The Noether current generated from invariance under the infinitesimal translation group $x^\mu \to x^\mu + \epsilon^\mu$ is the *canonical stress tensor*

$$T^{\mu\nu}_{vac} = \frac{\partial \mathcal{L}_{vac}}{\partial \eta_{\mu\lambda}} \eta^\nu{}_\lambda - \mathrm{g}^{\mu\nu}\mathcal{L}_{vac} \tag{2.8}$$

To facilitate the construction of an appropriate stress tensor based on (2.7) it is beneficial to begin by considering only terms quadratic in field quantites for which

$$T^{\mu\nu}_{vac} = \tfrac{1}{2}\mathrm{g}^{\mu\nu}\Delta^{\alpha\lambda}\eta_{\alpha\lambda} - \Delta^{\mu\lambda}\eta^\nu{}_\lambda \tag{2.9}$$

By design, the conservation law implied by Noether's theorem is $\partial_\mu T^{\mu\nu}_{vac} = 0$. It can be verified explicitly for the vacuum theory by considering only the second term on the right in (2.9) and differentiating:

$$\partial_\mu(\Delta^{\mu\lambda}\eta^\nu{}_\lambda) = \Delta^{\mu\lambda}\cdot\partial_\mu\eta^\nu{}_\lambda = \Delta^{\mu\lambda}\cdot\partial^\nu\eta_{\mu\lambda} = \tfrac{1}{2}\partial^\nu(\Delta^{\mu\lambda}\eta_{\mu\lambda}) \tag{2.10}$$

To determine the symmetric stress tensor from (2.8) it is only necessary to write out two individual pieces

$$\Theta^{\mu\nu}_{vac} = \mu_e\left[\tfrac{1}{2}\mathrm{g}^{\mu\nu}\eta^2 - \eta^{\mu\lambda}\eta^\nu{}_\lambda\right] \tag{2.11a}$$
$$T^{\mu\nu}_{D} = \mu_e\eta\eta^{\nu\mu} \tag{2.11b}$$

As with the conventional theory, $T^{\mu\nu}_D$ has a non-symmetric component and can be eliminated by showing that $\partial_\mu T^{\mu\nu}_D = 0$ which leaves only the symmmetric tensor. If symmetrizing this way is deemed unsatisfactory, it is reasonable to include contributions to $T^{\mu\nu}_D$ resulting from the excluded Lagrangian $\mathcal{L}_x$. In this case

$$\begin{aligned} T^{\mu\nu}_D &\longrightarrow \mu_e\eta\eta^{\nu\mu} + \Delta^{\lambda\mu}\eta^\nu{}_\lambda - \tfrac{1}{2}\mathrm{g}^{\mu\nu}\Delta^{\lambda\alpha}\eta_{\alpha\lambda} \\ &= \mu_e\left[\eta^{\lambda\mu}\eta^\nu{}_\lambda - \mathrm{g}^{\mu\nu}\eta^2\right] = \frac{1}{4\pi}F^{\lambda\mu}\partial_\lambda A^\nu \end{aligned} \tag{2.12}$$

The term on the far right lacks gauge invariance and is the same leftover term that must be removed in the conventional theory.

A divergence operation on the symmetric stress tensor is somewhat different than the conventional theory and follows by re-inserting the vacuum tensor so that

$$\partial_\mu\Theta^{\mu\nu}_{vac} = \mu_e\left[\tfrac{1}{2}\partial^\nu\eta^2 - \partial_\mu\eta\eta^{\nu\mu}\right] - \partial_\mu\Delta^{\mu\lambda}\eta^\nu{}_\lambda - \Delta^{\mu\lambda}\cdot\partial_\mu\eta^\nu{}_\lambda = -4\pi f^\lambda_e\eta^\nu{}_\lambda \tag{2.13}$$

The source term here is composed of the point force density f^λ_e contracted with the strain tensor. Nevertheless, in the language of conventional electrodynamics, it is exactly the Lorentz force density associated with a point electron interacting with its own fields. This implies an equivalence relation linking the electromagnetic and vacuum gauge theories

$$\Theta^{\mu\nu}_{vac} \equiv \Theta^{\mu\nu}_{em} \tag{2.14}$$

This equivalence will be extended to include particle accelerations in section 3.

Applying equation (2.8) to the propagation term of the Lagrangian is rather trivial here, providing an additional stress

$$\Lambda^{\mu\nu} \equiv \tfrac{1}{2}\Delta^{\dagger\,\mu\nu} \tag{2.15}$$

which stands on its own without need of any symmetrization. Placing this term next to the symmetric stress tensor then defines the ***Total Vacuum Stress Tensor***:

$$\boxed{\mathcal{T}^{\mu\nu} \equiv \mu_e \left[\tfrac{1}{2}\mathrm{g}^{\mu\nu}\eta^2 - \eta^{\mu\lambda}\eta^{\nu}{}_{\lambda}\right] + \Lambda^{\mu\nu}} \tag{2.16}$$

Since $\partial_\mu \Lambda^{\mu\nu} = 0$, the differential law for the total vacuum stress tensor is no different than the conventional theory

$$\partial_\mu \mathcal{T}^{\mu\nu} = -4\pi f_e^{\lambda}\eta^{\nu}{}_{\lambda} \tag{2.17}$$

A useful representation of $\Lambda^{\mu\nu}$ follows by grouping the components as[2]

$$\begin{aligned} \mathrm{U}_{\Lambda} &\equiv -\frac{1}{2}\mu_e \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{U}} & \qquad \frac{1}{c}\mathbf{S}_{V} &\equiv -\frac{1}{2}\mu_e \boldsymbol{\nabla}\mathcal{U} \\ \frac{1}{c}\mathbf{S}_{A} &\equiv \frac{1}{2}\mu_e \frac{\partial \boldsymbol{\mathcal{U}}}{\partial ct} & \qquad \widehat{\mathbf{T}}_{\Lambda} &\equiv \frac{\mu_e}{2}\left[-\boldsymbol{\nabla}\boldsymbol{\mathcal{U}} + \mathbf{1}\partial_\lambda \mathcal{U}^{\lambda}\right] \end{aligned} \tag{2.18}$$

Individual elements are the canonical work-energy density U_{Λ}, six "components" of the canonical energy flux vectors $\mathbf{S}_V$ and $\mathbf{S}_A$, and the canonical stress-strain tensor $\widehat{\mathbf{T}}_{\Lambda}$. Now add a well known representation of the symmetric stress tensor and write the total stress tensor as

$$\mathcal{T}^{\mu\nu} = \begin{bmatrix} \mathrm{U} & \frac{1}{c}\mathbf{S} \\ \frac{1}{c}\mathbf{S} & -\widehat{\mathbf{T}} \end{bmatrix} + \begin{bmatrix} \mathrm{U}_{\Lambda} & \frac{1}{c}\mathbf{S}_V \\ \frac{1}{c}\mathbf{S}_A & \widehat{\mathbf{T}}_{\Lambda} \end{bmatrix} \tag{2.19}$$

As a reminder all elements of the symmetric stress tensor are derived from vacuum gauge potentials. For example, energy flux in the vacuum theory should properly be determined from the formula

$$\mathbf{S} = \mu_e \boldsymbol{\nabla}\mathcal{U}^{\nu}\frac{\partial \mathcal{U}_{\nu}}{\partial t}$$

although this quantity has already been constructed previously from the flux fields.

[2]The subscript V appended to the energy flux $\mathbf{S}_V$ has been borrowed from the conventional theory and only serves as a means of distinction from the energy flux $\mathbf{S}_A$.

2.3 Particle Stability in the Rest Frame

The most straight forward application of the formalism in (2.19) is in the rest frame where the presence of the vacuum tensor provides a simple solution for a stable particle of radius r_e. Most of the energy flux components are zero and (2.19) reduces to

$$\mathcal{T}^{\mu\nu} = \begin{bmatrix} \mathrm{U} & 0 \\ & \\ 0 & -\widehat{\mathbf{T}} \end{bmatrix} + \begin{bmatrix} \mathrm{U}_\Lambda & \frac{1}{c}\mathbf{S}_V \\ & \\ 0 & \widehat{\mathbf{T}}_\Lambda \end{bmatrix} \tag{2.20}$$

Using $(c\tau, \boldsymbol{r})$ as rest frame coordinates, the total stress-energy at some time τ may be written as a combination of the light sphere and the infinite space tensor $\mathcal{T}^{\mu\nu}_M$. The integral to be evaluated takes the form

$$\mathcal{E}^{\mu\nu}(\tau) = \int \mathcal{T}^{\mu\nu}_M \cdot \vartheta(\tau - r/c + \tau_e) d^3r \tag{2.21}$$

Components of $\mathcal{T}^{\mu\nu}$ are determined entirely by the vacuum dilatation vector, which rquires that the radial integration have a lower limit r_e. As indicated by the illustration in Figure 5, this occurs at time $c\tau = 0$. Also shown is the upper integration limit chosen to be the finite time $c\tau + r_e$. A calculation of all 16 integrals given by (2.21) is

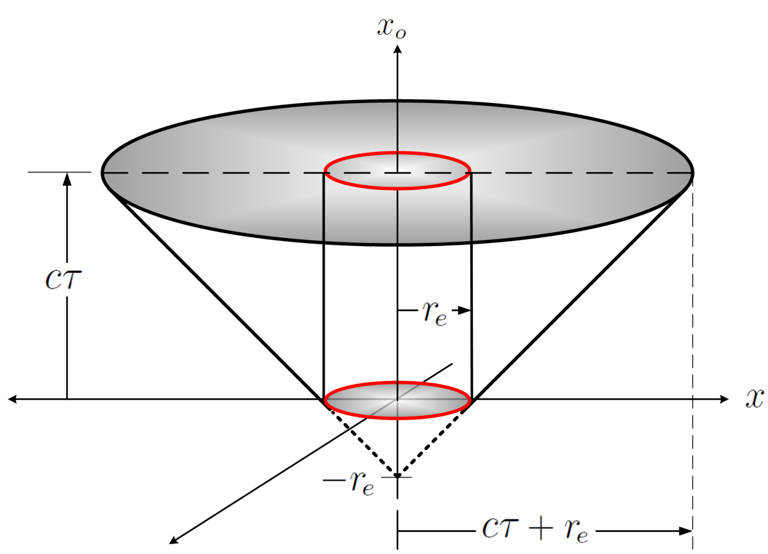

Figure 5: *Spacetime diagram indicating the expansion of the radial step $\vartheta(\tau - r/c + \tau_e)$ in the rest frame. The radius r_e divides the causal light cone into interior and exterior four-volumes.*

not necessary since several of the components share similar characteristics. Essential details of the integrations fit appropriately into 3 groups:

1. Total Energy:
****

$$\mathcal{T}^{\tau\tau} = \frac{1}{2}\mu_e(\partial_\nu \mathcal{U}^\nu)^2 - \frac{1}{2}\mu_e \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{U}} = \frac{1}{2}\mu_e\left[\frac{r_e^4}{r^4} - \frac{r_e^2}{r^2}\right]\cdot\vartheta$$

The energy density is zero at the electron radius. Integrals of each term over all space are complicated by the presence of the causality sphere and may be performed with different levels of mathematical rigor. Results generate the total invariant energy, or total particle Hamiltonian

$$\mathcal{H} = mc^2\cdot\vartheta(\tau) - \tfrac{1}{2}\dot{\varrho}c\tau \tag{2.22}$$

In this equation $\dot{\varrho} = 4\pi\mu_e r_e^2 = \mu_e a_e > 0$ is the total scalar momentum radiated per unit time.

2. Space-Space Terms:
****

Diagonal pressure terms are all similar. The component $\mathcal{T}^{xx}$ may be written

$$\mathcal{T}^{xx} = -\frac{1}{2}\mu_e(\partial_\nu \mathcal{U}^\nu)^2 - \mu_e\frac{\partial \mathcal{U}^\nu}{\partial x}\frac{\partial \mathcal{U}_\nu}{\partial x} + \frac{1}{2}\mu_e\left[\partial_\alpha \mathcal{U}^\alpha - \frac{\partial \mathcal{U}_x}{\partial x}\right]$$

$$= \frac{1}{2}\mu_e\left[\frac{r_e^4(-x^2+y^2+z^2)}{r^6} + \frac{r_e^2}{r^2} + \frac{r_e^2(x^2-y^2-z^2)}{r^4}\right]\cdot\vartheta$$

The second line shows clearly how the outward pressure from the symmetric stress tensor is removed by the vacuum strain at $r = r_e$. In the language of electrodynamics, this outward pressure is interpreted as resulting from the repulsive force of the elementary charge on itself. In the new language, a restoring force resulting from the deformed vacuum exerts an equal and opposite force providing for a stable particle. Angular integrations are not trivial here but final results for all three space-space diagonal terms are identical. We list the xx-components as:

$$\mathcal{E}^{xx}_{part} = \tfrac{1}{3}mc^2\cdot\vartheta(\tau) \qquad\qquad \mathcal{E}^{xx}_{vac} = \tfrac{1}{3}\dot{\varrho}c\tau \tag{2.23}$$

Like the diagonal terms, the off-diagonal terms are all similar to the $\mathcal{T}^{xy}$ component

$$\mathcal{T}^{xy} = -\mu_e\frac{\partial \mathcal{U}^\nu}{\partial x}\frac{\partial \mathcal{U}_\nu}{\partial y} - \frac{1}{2}\mu_e\frac{\partial \mathcal{U}_y}{\partial x} = \mu_e\left[-\frac{r_e^4 xy}{r^6} + \frac{r_e^2 xy}{r^4}\right]\cdot\vartheta$$

Here again the vacuum strain removes the symmetric (or electromagnetic) stress at $r = r_e$. Solid angle integrals then require all off-diagonal pressure terms to be zero.

3. Energy Flux:
****

In the rest frame, there is no contribution from the time-space and space-time components of the symmetric stress tensor. Contributions to the total energy come only from the vacuum stress tensor and are determined by

$$\frac{1}{c}\mathbf{S}_{V} = -\frac{1}{2}\mu_e \boldsymbol{\nabla}\mathcal{U} = -\frac{1}{2}\sigma_e \mathbf{E}$$

$$\frac{1}{c}\mathbf{S}_{A} = \frac{1}{2}\mu_e \frac{\partial \boldsymbol{\mathcal{U}}}{\partial c\tau} = \mathbf{0}$$

The vector $\mathbf{S}_V$ has been written in terms of the rest frame electric field vector. Since this vector is purely radial, the 3-space integral is zero. The vector $\mathbf{S}_A$ represents canonical energy flux based on explicit changes of the vacuum deformation with time and they vanish for the stationary electron.
****

To summarize, calculations in items 1–3 show that all terms of $\mathcal{T}^{\mu\nu}$ vanish at the radius r_e, except for the presence of residual energy flux propagating away from the source. Appropriate spatial integrations for all terms can be collected together in a matrix format as the total energy tensor:

$$\mathcal{E}^{\mu\nu}_{total} \equiv \mathcal{E}^{\mu\nu}_{part} + \mathcal{E}^{\mu\nu}_{vac}$$

$$= \begin{bmatrix} mc^2 & 0 & 0 & 0 \\ 0 & \frac{1}{3}mc^2 & 0 & 0 \\ 0 & 0 & \frac{1}{3}mc^2 & 0 \\ 0 & 0 & 0 & \frac{1}{3}mc^2 \end{bmatrix} \cdot \vartheta(\tau) + \begin{bmatrix} -\frac{1}{2}\dot{\varrho}c\tau & 0 & 0 & 0 \\ 0 & \frac{1}{3}\dot{\varrho}c\tau & 0 & 0 \\ 0 & 0 & \frac{1}{3}\dot{\varrho}c\tau & 0 \\ 0 & 0 & 0 & \frac{1}{3}\dot{\varrho}c\tau \end{bmatrix} \tag{2.25}$$

The term $\mathcal{E}^{\mu\nu}_{part}$ is a traceless mass-energy tensor just like the symmetric stress tensor from which it derives. Its components are representive of the energy associated with the locally compressed vacuum. In contrast, the term $\mathcal{E}^{\mu\nu}_{vac}$ represents an unstable continuum. In the language of deformable media, it refers to the total canonical stress imparted to the vacuum in a time $c\tau$ by the canonical momentum. The energy component is the total work done on the vacuum which is negative and assumed to result from the admissibility force constant.

2.4 Stability in a Moving Frame: Dirac Electron

Relative to a moving frame, the stability problem can be addressed by first defining the unitless quantities

$$\mathcal{G}^{\mu\nu}_1 \equiv 2\beta^\mu\beta^\nu - 2\,U^\mu U^\nu - \mathrm{g}^{\mu\nu} \tag{2.26a}$$

$$\mathcal{G}^{\mu\nu}_2 \equiv \beta^\mu\beta^\nu + \beta^\mu U^\nu - \mathrm{g}^{\mu\nu} = -\partial^\nu R^\mu \tag{2.26b}$$

The symmetric stress tensor and the vacuum tensor are then

$$\Theta_{1}^{\mu\nu} = \frac{1}{2}\mu_e \eta^2 \mathcal{G}_1^{\mu\nu} \qquad\qquad \Lambda^{\mu\nu} = -\frac{1}{2}\mu_e \eta \left[\mathcal{G}_1^{\mu\nu} - \mathcal{G}_2^{\mu\nu}\right] \tag{2.27}$$

and the total stress may be written

$$\mathcal{T}^{\mu\nu} = -(\mathcal{L}_o + \mathcal{L}_{\Lambda})\mathcal{G}_1^{\mu\nu} + \mathcal{L}_{\Lambda}\mathcal{G}_2^{\mu\nu} \tag{2.28}$$

Stability is determined by the first term which vanishes at $\rho = r_e$. What remains represents radiated stress associated with canonical energy flux. This tensor only has real meaning at the electron radius but its importance is paramount as the particle radius is synonymous with the vacuum boundary. Define this radiated stress by

$$\mathcal{E}_{rad}^{\mu\nu} \equiv \mathcal{L}_{\Lambda}\mathcal{G}_2^{\mu\nu}\Big|_{\rho=r_e} = -\frac{1}{2}\mu_e \partial^\nu R^\mu \tag{2.29}$$

It may be readily integrated over the boundary to determine the four-vector vacuum radiation rate. Using $\mathcal{V}_e$ as the volume inside the electron radius, the scalar contraction is an easily recognizable quantity

$$\mathcal{E}_{rad} = -\frac{mc^2}{\mathcal{V}_e} \tag{2.30}$$

More importantly however, $\mathcal{E}_{rad}^{\mu\nu}$ can be used as a vehicle to quantize the classical particle. This follows by writing it as a composition of the timelike and spacelike four-vectors

$$\mathcal{E}_{rad}^{\mu\nu} = \frac{mc^2}{4\pi r_e^3}\left[U^\nu\beta^\mu + \beta^\mu\beta^\nu - \beta^\lambda(\beta_\lambda + U_\lambda)\mathrm{g}^{\mu\nu}\right] \tag{2.31}$$

Terms multiplying the metric might be considered superfluous but they are necessary to enforce the overall form of the tensor as a quantity $X^{\mu\nu} - X\mathrm{g}^{\mu\nu}$. Now replace the four-velocity with the four-momentum using $p^\nu = mc\beta^\nu$ and make the quantum mechanical substitution

$$\boxed{U^\nu \longrightarrow \pm\gamma^\nu} \tag{2.32}$$

where γ^ν are the Dirac matrices. Choosing the minus sign, the result is a new tensor

$$\mathcal{E}_{Dirac}^{\mu\nu} = \frac{c}{4\pi r_e^3}\left[-\gamma^\nu p^\mu + \frac{1}{mc}p^\mu p^\nu - mc\,\mathrm{g}^{\mu\nu} + \gamma^\lambda p_\lambda \mathrm{g}^{\mu\nu}\right] \tag{2.33}$$

The term inside the brackets of (2.33) already contains the Dirac energy-momentum tensor which becomes apparent by operating on the left and right with Dirac spinor fields rendering

$$\frac{1}{mc}p^\mu p^\nu \bar{\psi}\psi - \mathsf{T}_{Dirac}^{\dagger\,\mu\nu} = 0 \tag{2.34}$$

where

$$\mathsf{T}^{\mu\nu}_{Dirac} = \bar{\psi}[\gamma^{\mu}p^{\nu} - \mathrm{g}^{\mu\nu}(\gamma^{\lambda}p_{\lambda} - mc)]\psi \tag{2.35}$$

One can also derive the Dirac Lagrangian directly from the contraction

$$\mathcal{L}_{Dirac} = \bar{\psi}\left[\frac{1}{c}\mathcal{E}_{Dirac} \cdot \mathcal{V}_{e}\right]\psi \tag{2.36}$$

The simplicity by which Dirac electron theory has emerged from the vacuum gauge electron is impressive. Of importance is the fact that $\mathcal{E}^{\mu\nu}_{Dirac}$ vanishes when operated on by a set of Dirac spinors. This can be interpreted to mean that stress associated with the flow of the vacuum field cannot be known to the quantum mechanical particle. One may consider a classical radiating particle or quantum-mechanical particle–but not both at the same time. This fact is also indicated more directly by the quantization procedure in (2.32) where the spacelike vector can be traced directly to the presence of the gauge field A^{ν}_{s}. In the classical theory, the gauge field propagates vacuum energy; but when the gauge field is removed in favor Dirac matrices, field energy yields to the quantum mechanical theory.

Total Energy Tensor in a Moving Frame: Lorentz transformation of $\mathcal{E}^{\mu\nu}$ in equation (2.25) proceeds by transforming individual portions of the tensor separately. In the rest frame, components of the traceless part are

$$\mathcal{K}^{\mu\nu}_{part} = mc^2\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\cdot\vartheta(\tau) \qquad \mathcal{P}^{\mu\nu}_{part} = \tfrac{1}{3}mc\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\cdot\vartheta(\tau) \tag{2.37}$$

Applying the general Lorentz transformation $\mathcal{X}^{\mu\nu} = L^{\mu}{}_{\alpha}\,\mathcal{X}^{\alpha\beta}\,L_{\beta}{}^{\nu}$ to both portions and combining derives the moving particle energy tensor

$$\mathcal{E}^{\mu\nu}_{part} = [mc^2\beta^{\mu}\beta^{\nu} + \tfrac{1}{3}mc^2(\beta^{\mu}\beta^{\nu} - \mathrm{g}^{\mu\nu})]\cdot\vartheta(\tau) \tag{2.38}$$

A scalar contraction with the metric may be implemented which—upon removal of a spurious factor—leads to the Einstein formula

$$\boxed{p^{\nu}p_{\nu} = m^2c^2} \tag{2.39}$$

A similar breakdown can be arranged for the vacuum terms

$$\mathcal{K}^{\mu\nu}_{vac} = -\tfrac{1}{2}\dot{\varrho}c\tau\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \qquad \mathcal{P}^{\mu\nu}_{vac} = \tfrac{1}{3}\dot{\varrho}\tau\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \tag{2.40}$$

Combining transformed quantities again shows that

$$\mathcal{E}^{\mu\nu}_{vac} = -\tfrac{1}{2}\dot{\varrho}c\tau\beta^\mu\beta^\nu + \tfrac{1}{3}\dot{\varrho}c\tau\left(\beta^\mu\beta^\nu - \mathrm{g}^{\mu\nu}\right) \tag{2.41}$$

Contracting both terms with the metric tensor as before then derives the general scalar relation

$$\boxed{\mathcal{K} = \tfrac{1}{2}\mathcal{P}c} \tag{2.42}$$

where both $\mathcal{K}$ and $\mathcal{P}$ are seen to be scalar invariants. Combining vacuum and particle terms leads to the symmetric total energy tensor

$$\begin{aligned}\mathcal{E}^{\mu\nu}_{total} = [mc^2\beta^\mu\beta^\nu + \tfrac{1}{3}mc^2(\beta^\mu\beta^\nu - \mathrm{g}^{\mu\nu})]\cdot\vartheta(\tau)&\\ - \tfrac{1}{2}\dot{\varrho}c\tau\beta^\mu\beta^\nu + \tfrac{1}{3}\dot{\varrho}c\tau\left(\beta^\mu\beta^\nu - \mathrm{g}^{\mu\nu}\right)&\end{aligned} \tag{2.43}$$

A single contraction with the four-velocity derives the total particle energy 4-vector

$$\boxed{\mathcal{E}^{\nu}_{total} = [mc^2\cdot\vartheta(\tau) - \tfrac{1}{2}\dot{\varrho}c\tau]\beta^\nu = \mathcal{H}\beta^\nu} \tag{2.44}$$

3 Accelerated Motions of the Electron

It is well known that an accelerating charge emits transverse electromagnetic waves over a range of frequencies. For the vacuum gauge particle these waves must be re-interpreted as transverse distortions of the vacuum travelling at light speed and independent of radiation associated with the velocity theory. A description of these variations will naturally require the inclusion of an acceleration strain tensor $\epsilon^{\mu\nu}$. One essential property of $\epsilon^{\mu\nu}$ is immediately available by requiring that it may not be a source of additional vacuum dilatation so that $\epsilon = 0$ only. This requirement may be used to classify classical radiation theory as an observable phenomena in contrast with the theory of dilatation radiation which is assumed to be unobservable.[3]

Naturally, $\epsilon^{\mu\nu}$ is expected to be functionally dependent on the four-acceleration of the particle, and as a preliminary calculation it will be useful to define the quantity $a^\nu_\perp$ orthogonal to the spacelike vector U^ν

$$a^\nu_\perp \equiv a^\nu + (a^\lambda U_\lambda)\,U^\nu \tag{3.1}$$

A link between $a^\nu_\perp$ and the velocity theory can be immediately established through sets of eigenvalue equations

$$\Delta^{\mu\nu}a^\perp_\nu = 0 \qquad \longleftrightarrow \qquad \eta^{\mu\nu}a^\perp_\nu = \eta a^\mu_\perp \tag{3.2a}$$

$$\Delta^{\mu\nu}a^\perp_\mu = 0 \qquad \longleftrightarrow \qquad \eta^{\mu\nu}a^\perp_\mu = \eta a^\nu_\perp \tag{3.2b}$$

[3]This section relies heavily on various contractions of the strain tensors $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu}$. These contractions have been evaluated explicitly in separate tables in Appendix B serving as a useful calculational tool.

In general, interactions of $a_\perp^\nu$ with other first and second rank tensors is quite limited. Evidence for this follows from orthogonality with the four-vectors R^ν and β^ν in addition to its defined orthogonality with U^ν.

3.1 Theory of Acceleration Strain

Beginning with the theory of potentials, suppose an acceleration strain tensor can be determined by defining $\varepsilon^{\mu\nu} \equiv \partial^\mu A_a^\nu$. Calculating ε is an extensive exercise in the use of implicit differentiation, but an application of the chain rule shows that

$$\varepsilon^{\mu\nu} = -a^\lambda R_\lambda(\partial^\mu A^\nu) - \partial^\mu(a^\lambda R_\lambda) \cdot A^\nu \tag{3.3}$$

Unfortunately, a serious problem with this equation lies in the fact that it leads to a non-zero vacuum dilatation $\varepsilon = -2(a^\lambda R_\lambda)\partial_\nu A^\nu$, with each of the separate terms on the right side of (3.3) making equal contributions. This violates the premise that acceleration strain may not dilate the vacuum.

A clue to resolving this problem is available from the well known anti-symmetry of the acceleration field strength tensor determined from $[\partial^\mu, A_a^\nu]$. This suggests a strategy to 'cleanse' $\varepsilon^{\mu\nu}$ of unwanted terms leaving an appropriate tensor $\epsilon^{\mu\nu}$ with the required property $\epsilon = 0$. As an initial (and final) guess, suppose all symmetric terms are separated from $\varepsilon^{\mu\nu}$ by

$$\varepsilon^{\mu\nu} = s^{\mu\nu} - \epsilon^{\nu\mu} \tag{3.4}$$

The anti-symmetric combination eliminates the symmetric term and leads to the equation

$$\partial_\mu[\epsilon^{\mu\nu} - \epsilon^{\nu\mu}] = \frac{4\pi}{c} J_a^\nu \tag{3.5}$$

The task of removing the symmetric terms depends critically on the expansion of the acceleration four-vector in terms of $a_\perp^\nu$ with the suprisingly simple result

$$\boxed{\epsilon^{\mu\nu} \equiv A^\mu a_\perp^\nu} \tag{3.6}$$

There are many interesting properties of this bi-linear quantity including $\epsilon = 0$ and vanishing determinants of all individual two-by-two minors which implies $\det \epsilon^{\mu\nu} = 0$. Divergences on each index of $\epsilon^{\mu\nu}$ are

$$\partial_\mu \epsilon^{\nu\mu} = \frac{4\pi}{c}(\tilde{J}_a^\nu - J_a^\nu) \qquad\qquad \partial_\mu \epsilon^{\mu\nu} = \frac{4\pi}{c}\tilde{J}_a^\nu \tag{3.7}$$

New Definitions for the Vacuum Theory: The previous calculations of strain from the theory of potentials can be reconciled with the vacuum theory by a rearrangement of constants and the introduction of the elastic modulus μ_e. Table 1 gives a summary of the new definitions. The most striking feature of the table is the emergence of momentum flux associated with transverse shear stresses on the

Names	Column 1	Column 2
Stress/Strain	$\mathit{\Delta}_a^{\mu\nu} = \mu_e \epsilon^{\mu\nu}$	$\epsilon^{\mu\nu} = \mathcal{U}^\mu a_\perp^\nu / c^2$
Force current densities	$\tilde{f}_a^\nu = \dfrac{\mu_e \eta}{4\pi c^2} a_\perp^\nu$	$f_a^\nu = -\dfrac{\mu_e \eta}{2\pi c^2 \rho^2} a^\lambda R_\lambda R^\nu$
Equations of motion.	$\partial_\mu \mathit{\Delta}_a^{\mu\nu} = 4\pi \tilde{f}_a^\nu$	$\partial_\mu \mathit{\Delta}_a^{\nu\mu} = 4\pi[\tilde{f}_a^\nu - f_a^\nu]$

Table 1: *Showing "normalized" definitions of acceleration stress and strain tensors along with associated force-densities and equations of motion.*

medium resulting from particle accelerations. This is consistent with the appearance of transverse momentum associated with shear waves in elastic media, and it means that Maxwell acceleration fields determined from

$$\partial_\mu [\mathit{\Delta}_a^{\mu\nu} - \mathit{\Delta}_a^{\nu\mu}] = 4\pi f_a^\nu \tag{3.8}$$

must now be re-intrepreted a momentum flux fields, similar to the fields in the velocity theory. According to vacuum gauge electrodynamics, all electromagnetic fields therefore transport momentum and Maxwell's equations for particle accelerations are

$$\boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{\mathrm{E}a} = 4\pi \rho_a \qquad \boldsymbol{\nabla} \times \boldsymbol{\pi}_{\mathrm{B}a} = 4\pi \mathbf{f}_a + \frac{1}{c}\frac{\partial \boldsymbol{\pi}_{\mathrm{E}a}}{\partial t} \tag{3.9a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\pi}_{\mathrm{E}a} = -\frac{1}{c}\frac{\partial \boldsymbol{\pi}_{\mathrm{B}a}}{\partial t} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{\mathrm{B}a} = 0 \tag{3.9b}$$

Formula's for acceleration flux vectors as functions of the dilatation vector may be written

$$\boldsymbol{\pi}_{\mathrm{E}a} = \mu_e [\mathcal{U} \boldsymbol{a}_\perp - \boldsymbol{\mathcal{U}} a_\perp^0] \tag{3.10a}$$

$$\boldsymbol{\pi}_{\mathrm{B}a} = \mu_e \boldsymbol{\mathcal{U}} \times \boldsymbol{a}_\perp \tag{3.10b}$$

Acceleration flux fields themselves are not to be associated with radiated flux as can be verified by dotting equations (3.10) with $\hat{\boldsymbol{n}}$. However, a generalization of the Poynting vector in (1.25) inclusive of acceleration flux fields reads

$$\mathbf{S} = \frac{c}{\mu_e}(\boldsymbol{\pi}_{\mathrm{E}} + \boldsymbol{\pi}_{\mathrm{E}a}) \times (\boldsymbol{\pi}_{\mathrm{B}} + \boldsymbol{\pi}_{\mathrm{B}a}) \tag{3.11}$$

This is exactly the Poynting vector of the conventional theory written in terms of a new set of fundamental quantities.

3.2 Generalized Vacuum Lagrangian and Stress Tensor

With appropriate definitions of acceleration stress and strain, a generalized vacuum Lagrangian may be determined beginning with the explicit form of the velocity Lagrangian:

$$\mathcal{L}_{vac} = -\tfrac{1}{2}\mu_e \left[\partial^\mu \mathcal{U}^\nu \partial_\mu \mathcal{U}_\nu - (\partial_\nu \mathcal{U}^\nu)^2\right] - 4\pi f_e^\nu \mathcal{U}_\nu \tag{3.12}$$

Under conditions where accelerated motions occur assume that $\partial^\mu \mathcal{U}^\nu$ receives a small anti-symmetric perturbation, first order in the quantity $a_\perp^\nu$, with the appearance of an acceleration four-current:

$$\partial^\mu \mathcal{U}^\nu \longrightarrow \partial^\mu \mathcal{U}^\nu - \mathcal{U}^\mu a_\perp^\nu + \mathcal{U}^\nu a_\perp^\mu \qquad\qquad f_e^\nu \longrightarrow f_e^\nu - f_a^\nu \tag{3.13}$$

Keeping only first order corrections in $a_\perp^\nu$ the modified velocity theory can be written

$$\begin{aligned}\mathcal{L}_{vac} = -\tfrac{1}{2}\mu_e\big[\partial^\mu \mathcal{U}^\nu \partial_\mu \mathcal{U}_\nu - (\partial_\nu \mathcal{U}^\nu)^2\big] - 4\pi f_e^\nu \mathcal{U}_\nu \\ + \mu_e\big[\partial^\mu \mathcal{U}^\nu\, \mathcal{U}_\mu a_\nu^\perp - \partial^\mu \mathcal{U}^\nu\, \mathcal{U}_\nu a_\mu^\perp\big] + 4\pi f_a^\nu \mathcal{U}_\nu\end{aligned} \tag{3.14}$$

All additional terms resulting from particle accelerations do not make contributions to the functional value of the Lagrangian, while derivatives with respect to the field quantity and its derivative are

$$\frac{\partial \mathcal{L}_{vac}}{\partial(\partial_\mu \mathcal{U}_\lambda)} = -\mu_e\big[\partial^\mu \mathcal{U}^\lambda - \mathrm{g}^{\mu\lambda}\partial_\nu \mathcal{U}^\nu - \mathcal{U}^\mu a_\perp^\lambda + \mathcal{U}^\lambda a_\perp^\mu\big] \tag{3.15}$$

$$\frac{\partial \mathcal{L}_{vac}}{\partial \mathcal{U}_\lambda} = \mu_e\big[\partial^\lambda \mathcal{U}^\mu - \partial^\mu \mathcal{U}^\lambda\big]\, a_\mu^\perp - 4\pi\left[f_e^\lambda - f_a^\lambda\right] \tag{3.16}$$

Appealing to equation (3.2), the second equation produces all the correct current densities necessary to write

$$\mu_e \partial_\mu \left[\partial^\mu \mathcal{U}^\lambda - \mathrm{g}^{\mu\lambda}\partial_\nu \mathcal{U}^\nu\right] = 4\pi f_e^\lambda = \partial_\mu \varDelta^{\mu\lambda} \tag{3.17a}$$

$$\mu_e \partial_\mu \left[\mathcal{U}^\mu a_\perp^\lambda\right] = 4\pi \tilde{f}_a^\lambda = \partial_\mu \varDelta_a^{\mu\lambda} \tag{3.17b}$$

$$\mu_e \partial_\mu \left[\mathcal{U}^\lambda a_\perp^\mu\right] = 4\pi[\tilde{f}_a^\lambda - f_a^\lambda] = \partial_\mu \varDelta_a^{\lambda\mu} \tag{3.17c}$$

Unfortunately, the generalized Lagrangian is not suitable for calculating an associated stress tensor from an equation like (2.8). The problem lies in the fact that acceleration stresses in the vacuum gauge are to be treated as small corrections to the velocity theory, and this same approach must be applied to the symmetric stress tensor. The velocity portion of the symmetric stress tensor has already been derived in section 2.2:

$$\Theta_{vac}^{\mu\nu} = \mu_e\left[\tfrac{1}{2}\mathrm{g}^{\mu\nu}\eta^2 - \eta^{\mu\lambda}\eta^{\nu}{}_{\lambda}\right] \tag{3.18}$$

The inclusion of particle accelerations follows immediately from the generalized strain

$$\zeta^{\mu\nu} \equiv \eta^{\mu\nu} - \epsilon^{\mu\nu} \tag{3.19}$$

The resulting stress tensor is

$$\Theta^{\mu\nu}_{vac} = \mu_e \left[\tfrac{1}{2}\mathrm{g}^{\mu\nu}\eta^2 - \eta^{\mu\lambda}\eta^{\nu}{}_{\lambda} + \eta^{\mu\lambda}\epsilon^{\nu}{}_{\lambda} - \epsilon^{\mu\lambda}\epsilon^{\nu}{}_{\lambda} + \epsilon^{\mu\lambda}\eta^{\nu}{}_{\lambda}\right] \tag{3.20}$$

which is exactly the symmetric stress tensor of the electromagnetic theory. One could also include the conjugate acceleration strain and write the variation

$$\eta^{\mu\nu} \longrightarrow \eta^{\mu\nu} - \epsilon^{\mu\nu} + \epsilon^{\nu\mu} \tag{3.21}$$

but all extra terms in $\Theta^{\mu\nu}_{vac}$ are zero and equation (3.20) remains unchanged. For a more concise repesentation of the symmetric stress tensor define

$$\mathcal{R}^{\mu\nu} \equiv \mu_e[\zeta^{\mu\lambda}\zeta^{\nu}{}_{\lambda}] \tag{3.22}$$

The total stress tensor including the radiation term is then

$$\boxed{\mathcal{T}^{\mu\nu} = \left[\tfrac{1}{4}\mathrm{g}^{\mu\nu}\mathcal{R} - \mathcal{R}^{\mu\nu}\right] + \Lambda^{\mu\nu}} \tag{3.23}$$

With the inclusion of particle accelerations, $\mathcal{T}^{\mu\nu}$ above can be used to determine the action integral $\mathcal{S}$ while simultaneously showing that accelerations do not contribute to the action. For this use the four-volume element $d^4V = \rho^2 d\rho\, d\Omega' c d\tau$ and write

$$\mathcal{S} = \frac{1}{c}\int \beta_\mu\, \mathcal{T}^{\mu\nu}\beta_\nu d^4V = \frac{1}{c}\int \mathcal{L}_{vac} d^4V \tag{3.24}$$

The full integration will show that the radiation term alone will depend on the square of the proper time. Instead, it is more appropriate to withold the proper time integral and determine the particle Hamiltonian from

$$\frac{d\mathcal{S}}{d\tau} = -\mathcal{H} \tag{3.25}$$

For a final calculation it will be necessary to adequately address the problem of the divergence of $\mathcal{T}^{\mu\nu}$ in vacuum gauge electrodynamics. If acceleration terms are withheld from a divergence operation on $\Theta^{\mu\nu}_{1}$, then an acceleration term must be included for the generalized tensor

$$\partial_\mu \Theta^{\mu\nu}_{vac} = -4\pi\left[f_e^{\lambda}\eta^{\nu}{}_{\lambda} + \eta f_a^{\nu}\right] \tag{3.26}$$

On the other hand, the propagation term can have no generalization and must remain as part of the velocity theory only.

A Derivatives of the Null Vector

The covariant derivative of R^ν is

$$\partial^\mu R^\nu = \mathrm{g}^{\mu\nu} - \frac{R^\mu \beta^\nu}{\rho} \tag{A.1}$$

The trace of the resulting matrix gives the 4-divergence $\partial_\nu R^\nu = 3$. In terms of individual components–and with the inclusion of a sign–a useful construction is:

$$-\partial^\mu R^\nu = \begin{bmatrix} -\dfrac{\partial R}{\partial ct} & -\dfrac{\partial \boldsymbol{R}}{\partial ct} \\ \\ \boldsymbol{\nabla} R & \boldsymbol{\nabla}\boldsymbol{R} \end{bmatrix} \tag{A.2}$$

where individual components are given by

$$\frac{\partial R}{\partial ct} = 1 - \frac{\gamma R}{\rho} \qquad\qquad \frac{\partial \boldsymbol{R}}{\partial ct} = \frac{-\gamma R \boldsymbol{\beta}}{\rho} \tag{A.3}$$

$$\boldsymbol{\nabla} R = \frac{\gamma \boldsymbol{R}}{\rho} \qquad\qquad \boldsymbol{\nabla}\boldsymbol{R} = \mathbf{1} + \frac{\gamma \boldsymbol{R}\boldsymbol{\beta}}{\rho} \tag{A.4}$$

The determinant of (A.2) can be written $\det[\partial^\mu R^\nu] = 0$. The divergence and curl of $\boldsymbol{R}$ may be written

$$\boldsymbol{\nabla} \cdot \boldsymbol{R} = 3 + \frac{\gamma \boldsymbol{R} \cdot \boldsymbol{\beta}}{\rho} = Tr[\boldsymbol{\nabla}\boldsymbol{R}] \tag{A.5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{R} = \frac{\gamma}{\rho} \boldsymbol{R} \times \boldsymbol{\beta} \tag{A.6}$$

Let $\boldsymbol{w}(ct_r)$ be the retarded position of a charged particle at time ct_r. The light cone condition is defined by

$$\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{w}(ct_r) \qquad\qquad R \equiv ct - ct_r \tag{A.7}$$

Derivatives of the retarded time with respect to present time coordinates are

$$\frac{\partial ct_r}{\partial ct} = \frac{\gamma R}{\rho} \tag{A.8}$$

$$\boldsymbol{\nabla} ct_r = \frac{-\gamma \boldsymbol{R}}{\rho} \tag{A.9}$$

B Contractions of Vacuum Strain Tensors

$$\eta^{\mu\nu} = \eta\left[\mathrm{g}^{\mu\nu} - \frac{R^\mu\beta^\nu}{\rho} - \frac{2R^\nu\beta^\mu}{\rho} + \frac{2R^\mu R^\nu}{\rho^2}\right] \tag{B.1}$$

$$\epsilon^{\mu\nu} = \eta\left[R^\mu a^\nu - \chi\frac{R^\mu\beta^\nu}{\rho} + \chi\frac{R^\mu R^\nu}{\rho^2}\right] \tag{B.2}$$

$\eta^{\mu\lambda}\eta_\lambda{}^\nu = \eta^2\left[\mathrm{g}^{\mu\nu} - \frac{R^\mu\beta^\nu}{\rho}\right]$	$\eta^{\mu\lambda}\eta^\nu{}_\lambda = \eta^2\left[\mathrm{g}^{\mu\nu} - \frac{1}{\rho}(R^\mu\beta^\nu + R^\nu\beta^\mu) + \frac{R^\mu R^\nu}{\rho^2}\right]$
$\eta^{\lambda\mu}\eta^\nu{}_\lambda = \eta^2\left[\mathrm{g}^{\mu\nu} - \frac{\beta^\mu R^\nu}{\rho}\right]$	$\eta^{\lambda\mu}\eta_\lambda{}^\nu = \eta^2\left[\mathrm{g}^{\mu\nu} - \frac{1}{\rho}(R^\mu\beta^\nu + \beta^\mu R^\nu)\right]$

Table 2: Contractions of $\eta^{\mu\nu}$.

$\eta^{\mu\lambda}\epsilon^\nu{}_\lambda = \eta\epsilon^{\nu\mu}$	$\epsilon^{\mu\lambda}\eta^\nu{}_\lambda = \eta\epsilon^{\mu\nu}$	$\epsilon^{\mu\lambda}\epsilon^\nu{}_\lambda = -\Theta_3^{\mu\nu}$
$\eta^{\mu\lambda}\epsilon_\lambda{}^\nu = 0$	$\epsilon^{\mu\lambda}\eta_\lambda{}^\nu = \eta\epsilon^{\mu\nu}$	$\epsilon^{\mu\lambda}\epsilon_\lambda{}^\nu = 0$
$\eta^{\lambda\mu}\epsilon^\nu{}_\lambda = \eta\epsilon^{\nu\mu}$	$\epsilon^{\lambda\mu}\eta^\nu{}_\lambda = 0$	$\epsilon^{\lambda\mu}\epsilon^\nu{}_\lambda = 0$
$\eta^{\lambda\mu}\epsilon_\lambda{}^\nu = -\eta\epsilon^{\mu\nu}$	$\epsilon^{\lambda\mu}\eta_\lambda{}^\nu = -\eta\epsilon^{\nu\mu}$	$\epsilon^{\lambda\mu}\epsilon_\lambda{}^\nu = 0$

Table 3: Contractions of $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu}$.