Half-page derivation of the Thomas precession

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Introduction. — Composition of two non-collinear Lorentz boosts, results in a Lorentz transformation that is not a pure boost but a composition of a boost and a spatial rotation, known as the Wigner rotation [1]. As a consequence, a body moving on a curvilinear trajectory undergoes a rotational precession, that was first discovered by Thomas [2]. In the vast majority of textbooks this phenomenon is either omitted or described with very sophisticated mathematical tools, such as gyrogroups, associative-commutative groupoids, etc. [3]. Here we present a half-page derivation of the Thomas precession formula using only basic vector operations. Our approach is not only simple and clear, but also builds a better physical intuition of this relativistic effect.

Derivation. — Let us introduce three inertial observers: Alice, Bob and a cat and denote their reference frames by A, B, and C, respectively. We choose them such that A is non-rotated with respect to B, and B is non-rotated with respect to C (however in general C is inevitably going to be rotated with respect to A). Let Bob hold the cat and move with a constant velocity \boldsymbol{v} with respect to Alice. Unfortunatelly, at some point the cat decides to run away from Bob with an infinitesimally small velocity dv' with respect to him. It follows that Bob is moving with velocity -dv' with respect to the cat and Alice is moving with velocity -v with respect to Bob - see Fig. 1. Let us denote the cat's velocity in Alice's frame by v + dv. Since the cat is rotated relative to Alice, her velocity in the cat's frame \tilde{v} is yet to be determined: $\widetilde{\boldsymbol{v}} \neq -\boldsymbol{v} - \mathrm{d}\boldsymbol{v}$. The angle $\mathrm{d}\boldsymbol{\Omega}$ of that rotation equals:

$$d\mathbf{\Omega} = -\frac{\widetilde{\mathbf{v}}}{|\widetilde{\mathbf{v}}|} \times \frac{\mathbf{v} + d\mathbf{v}}{|\mathbf{v} + d\mathbf{v}|} \approx -\frac{1}{v^2} \widetilde{\mathbf{v}} \times (\mathbf{v} + d\mathbf{v}).$$
(1)

To derive the formula for the precession rate in Alice's frame we use the velocity composition law, which for the simplest case of motion along the x axis with the velocity V is given by:

$$u'^{x} = \frac{u^{x} - V}{1 - \frac{u^{x}V}{c^{2}}}, \quad u'^{y} = \frac{u^{y}\sqrt{1 - \frac{V^{2}}{c^{2}}}}{1 - \frac{u^{x}V}{c^{2}}}, \quad u'^{z} = \frac{u^{z}\sqrt{1 - \frac{V^{2}}{c^{2}}}}{1 - \frac{u^{x}V}{c^{2}}},$$

where u and u' are velocities of some object observed from the rest frame and a moving frame, respectively. For an arbitrary velocity V of the moving frame the transformation takes the form:

$$u' = \frac{\sqrt{1 - \frac{V^2}{c^2} \left(u - \frac{u \cdot V}{V^2} V \right) - \left(V - \frac{u \cdot V}{V^2} V \right)}}{1 - \frac{u \cdot V}{c^2}}.$$
 (2)



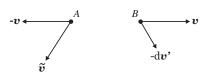


FIG. 1: Schematic diagram of mutual velocities between Alice, Bob and the cat.

where it is assumed that the primed and unprimed frames are mutually non-rotated. We will now follow two simple steps in order to express $\tilde{\boldsymbol{v}}$ appearing in Eq. (1) in terms of \boldsymbol{v} and $\mathrm{d}\boldsymbol{v}$. First we describe how the observers A and B observe C, and second how the observers B and C observe A. In the first step we use Eq. (2) to describe the transition from the frame A to B, which involves the following substitutions in the formula (2): $\boldsymbol{V} \to \boldsymbol{v}$, $\boldsymbol{u} \to \boldsymbol{v} + \mathrm{d}\boldsymbol{v}$, and $\boldsymbol{u'} \to \mathrm{d}\boldsymbol{v'}$. After leaving only the first order terms in $\mathrm{d}\boldsymbol{v}$ we get:

$$d\mathbf{v'} \approx \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(d\mathbf{v} - \frac{\mathbf{v} \cdot d\mathbf{v}}{v^2} \mathbf{v} \right) + \frac{1}{1 - \frac{v^2}{c^2}} \frac{\mathbf{v} \cdot d\mathbf{v}}{v^2} \mathbf{v}. \quad (3)$$

Secondly, we use again use the Eq. (2) applied to the transition from B to C that involves the following substitutions in (2): $\mathbf{V} \to \mathrm{d} \mathbf{v}', \ \mathbf{u} \to -\mathbf{v}$, and $\mathbf{u}' \to \widetilde{\mathbf{v}}$. Droping out higher order terms in $\mathrm{d} \mathbf{v}'$ leads to:

$$\widetilde{v} \approx -v + \frac{v \cdot dv'}{c^2} v - dv'.$$
 (4)

Substituting (3) to (4), then everything to (1) and dividing both sides of the resulting equation by dt we obtain:

$$\dot{\mathbf{\Omega}} = -\frac{1}{v^2} \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) \mathbf{v} \times \dot{\mathbf{v}}. \tag{5}$$

The above formula expresses the well-known Thomas precession of a body moving with variable velocity v(t) and observed by a fixed, inertial observer.

- [1] E. P. Wigner, Ann. Math. 40, 149 (1939).
- [2] L. H. Thomas, Nature **117**, 514 (1926).
- [3] Less technical, but still very complicated derivations can

be found in R. Ferraro, M. Thibeault, Eur. J. Phys. **20**, 143 (1999); J. Rhodes, M. Semon, Am. J. Phys. **72**, 943 (2004); K. O'Donnell, M. Visser, Eur. J. Phys. **32**, 1033 (2011).

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