# Final report

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Computer Simulations of Stochastic Processes

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**Problem 1** (Insurer-reinsurer risk process)**.** *Let us introduce a general two-dimensional risk process describing capital in time of two insurance companies: insurer $U_1(t)$ and reinsurer $U_2(t)$. In a general framework such a process can be considered as a system of two processes $\{U_1(t),\, U_2(t)\}_{t\geq 0}$ defined in the following way:*

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t - \begin{pmatrix} \sum_{k=1}^{N_1(t)} X_{1k} \\ \sum_{k=1}^{N_2(t)} X_{2k} \end{pmatrix}, \quad t \geq 0, \tag{1}$$

*where for the $i$-th ($i = 1,\, 2$) risk process the constant $u_i$ denotes its initial capital, $\{N_i(t)\}_{t\geq 0}$ is a homogenous Poisson process with intensity $\lambda_i$'s and $(X_{ik})_{k\geq 1}^{i=1,2}$ are i.i.d. positive random variables describing claim sizes with the mean value $\mu_i$. The parameters $c_i$'s stand for constant rates at which the premiums are collected per unit time for a respective risk process. The constant premium rate $c_i$ is given as follows: $c_i = (1+\theta_i)\mu_i\lambda_i$, where $\theta_i > 0$ is called the relative safety loading.*

*For our purposes, we assume that that insurer and reinsurer agree on a quota-share reinsurance contract for the whole business, which means that there is only one source of claim for both sides and the claims are splited proportionally between insurer and reinsurer with some constant parameter $\delta \in (0,1)$, e.g. $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}_{t\geq 0}$ are identical with intensity $\lambda > 0$ and $(X_{1k}, X_{2k}) = (\delta X_k, (1-\delta)X_k)$ with $E(X_k) = \mu$. Therefore the model (1) simplifies to the following one:*

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} (1+\theta_1)\delta \\ (1+\theta_2)(1-\delta) \end{pmatrix} \mu\lambda t - \begin{pmatrix} \delta \\ 1-\delta \end{pmatrix} \sum_{k=1}^{N(t)} X_k, \quad t \geq 0. \tag{2}$$

*The management board's main subject of interest is the question of the solvency of the company within some finite time interval $T$, e.g. $T = 1$ year. Therefore, from that point of view the most crucial factor is the probability of bankruptcy within some finite time interval. Mathematically, for the considered two-dimensional system (2) the moment of bankruptcy (or ruin) might be defined in few ways, for instance as:*

- *the first moment that at least one of the companies is at the ruin (one of capitals less or equal 0)*

$$\tau_{OR}(u_1, u_2) = \inf\{t \geq 0 : U_1(t) < 0 \text{ or } U_2(t) < 0\}, \tag{3}$$

- *the first moment that both companies are at the ruin (both capitals less or equal 0)*

$$\tau_{SIM}(u_1, u_2) = \inf\{t \geq 0 : U_1(t) < 0 \text{ and } U_2(t) < 0\}. \tag{4}$$

*For the above defined moments of ruin (3)-(4) we define the ruin probabilities in finite time $T$ as*

$$\psi_{OR}(u_1, u_2, T) = \mathbb{P}(\tau_{OR}(u_1, u_2) < T), \tag{5}$$

*and*

$$\psi_{SIM}(u_1, u_2, T) = \mathbb{P}(\tau_{SIM}(u_1, u_2) < T), \tag{6}$$

*respectively.*

*For the above presented model:*

1. *assume two inequalities: $u_1/\delta < u_2/(1-\delta)$ and $\theta_1 > (1+\theta_2)^2 - 1$,*

2. *implement a simulation method for the trajectories of the two-dimensional risk process (2) up to time $T$ with claims $(X_k)_{k\geq 1}$ distributed as:*

   - *an exponential random variable with cdf: $F(x) = 1 - \exp(-\beta x)$ for $x > 0$, with $\beta > 0$,*
   - *the Pareto random variable with cdf: $F(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha}$ for $x > 0$, where $\alpha \in (1, 2)$ and $\lambda > 0$,*
   - *a log-normal random variable with cdf: $F(x) = \Phi\left(\frac{\log y - \mu}{\sigma}\right)$ for $x > 0$, where $\Phi(\cdot)$ is normal standard cdf, $\mu \in R, \sigma > 0$,*
   - *the Burr random variable with cdf: : $F(x) = 1 - \left(\frac{\lambda}{\lambda + x^{\tau}}\right)^{\alpha}$ for $x > 0$, where $\alpha \in (1, 2)$ and $\lambda > 0$, $\tau > 0$,*

3. *implement the Monte Carlo method of estimating the ruin probabilities (5)-(6),*

4. *estimate the ruin probabilities for the models from point 2. (fix some reasonable parameters of random variables). Moreover, plot the graphs of ruin probabilities (5)-(6) as a functions of $u_1$ and $u_2$ and several fixed $T$'s (for instance, take three values: 'small', 'moderate' and 'large') by choosing such $u_1$, $u_2$ and $\theta_1$, $\theta_2$ that they fulfill assumption 1. and the estimated ruin probabilities decreases from $\sim 0.8$ to $\sim 0.05$.*

***References:***


*[1] F. Avram, Z. Palmowski, M. Pistorius A two-dimensional ruin problem on the positive quadrant, Insurance: Mathematics and Economics 42(1) (2008), 227–234. link*

*[2] K. Burnecki, M. Teuerle, A. Wilkowska De Vylder type approximation of the ruin probability for the insurer-reinsurer model, Mathematica Applicanda 47(1) (2019), 5–24.*

*[3] K. Burnecki, J. Janczura, R. Weron Building loss models in Statistical Tools For Finance and Insurance (2011), Springer link*

*[4] K. Burnecki, M. Teuerle Ruin probability in finite time in Statistical Tools For Finance and Insurance (2011), Springer link*

**Definition 1** (Mean square displacement)**.** *Mean square displacement (MSD)*

$$\begin{aligned}
\textit{EA-MSD}(\tau) &\stackrel{def}{=} \mathbb{E}\left\|X(\tau)-X(0)\right\|^2, \\
\textit{TA-MSD}(T,\tau) &\stackrel{def}{=} \frac{1}{T-\tau}\int_0^{T-\tau}\left\|X(t_0+\tau)-X(t_0)\right\|^2 dt_0
\end{aligned}$$

*The main difference between both types of MSD is that the ensemble average is a number (it's just the expectation; also, in many applications $X(0) = x_0 = const.$, and it's just a variance when $\mathbb{E}X(t) = const$), while time average is a random variable.*

*We will calculate their estimators. Specifically, for $N$ trajectories $[x_k(0), \ldots, x_k(n)], k = 1, 2, \ldots, N$ of length $n+1$, we calculate*

$$\textit{EA-MSD}(\tau) \stackrel{def}{=} \frac{1}{N}\sum_{k=1}^{N}\left\|x_k(\tau)-x_k(0)\right\|^2, \quad \tau = 0, 1, \ldots, n \tag{7}$$

$$\textit{TA-MSD}(k,\tau) \stackrel{def}{=} \frac{1}{n+1-\tau}\sum_{t_0=0}^{n-\tau}\left\|x_k(t_0+\tau)-x_k(t_0)\right\|^2, \quad \tau = 0, 1, \ldots, n, k = 1, \ldots, N, \tag{8}$$

$$\textit{EA-TA-MSD}(\tau) \stackrel{def}{=} \frac{1}{N}\sum_{k=1}^{N}\textit{TA-MSD}(k,\tau), \quad \tau = 0, 1, \ldots, n. \tag{9}$$

*$N$ is the number of trajectories, $n$ is their length, and $\tau$ is the parameter. Here, $\|\cdot\|$ denotes norm of given random variable/vector, e.g. Euclidean norm, that is for $\ell_2(\mathbb{R}^2) \ni \mathbf{X}(t) = \{X_1(t), X_2(t)\}$ we have $\|\mathbf{X}(t)\| = \sqrt{X_1^2(t) + X_2^2(t)}$ for every $t$.*

**Problem 2.** *Let us consider 2-dimensional Brownian motion $\{B_1(t), B_2(t)\}_{t\geq 0}$ with correlated coordinates: $\mathrm{Corr}(B_1(t), B_2(t)) = \rho \in [-1, 1]$.*
*Based on simulations of $N = 1000$ trajectories of length $n = 1001$ (discretisizing interval $[0, 100]$ of such a model you are required to:*

1. *plot empirical quantile lines function (look at eq. (1) and Figure 2 from [1]) for chosen coordinate.*
2. *compare for different $\tau$'s ensemble averaged MSD given by (7) and EA-TA-MSD given by (9). (via plotting)*
3. *on the same plot mark also sample confidence intervals for every $\tau$ of TA-MSD given by 8. (note that EA-TA-MSD is just average of TA-MSD)*
4. *Try to fit power law ($c \cdot \tau^\alpha$) behaviour to the EA-TA-MSD and present exponent $\alpha$.*
5. *For sample $[x_k(0), \ldots, x_k(n)], k = 1, 2, \ldots, N$ consider process on angles between time $t$ and $t + dt$:*

$$\varphi_k^{dt}(t) = \textit{angle between } x_k(t) \textit{ and } x_k(t+dt) - x_k(t), \quad t = 0, 1, \ldots, n.$$

*We can assume that every trajectory should behave the same and angles process should not depend on time. Plot histogram of such process $\phi^{dt}(\cdot)$ by joining all $\phi_k^{dt}(\cdot)$ together. If possible, try to find distribution of these angles. Repeat this procedure for different dt's (let's say $dt =$ time step and $10 \cdot$ time step) and compare if anything changes. [Authors consider adding a plot for clarification...]*

*For all your calculations consider three norms:*

- *Euclidean norm ($\ell_2$ norm)* $\|\mathbf{X}(t)\|_2 = \sqrt{X_1^2(t) + X_2^2(t)}$;
- *Taxicab norm ($\ell_1$ norm)* $\|\mathbf{X}(t)\|_1 = |X_1(t)| + |X_2(t)|$;
- *maximum norm ($\ell_\infty$ norm)* $\|\mathbf{X}(t)\|_\infty = \max\{|X_1(t)|, |X_2(t)|\}$;

*and three possible correlation coefficients $\rho = 0$ and $\rho = 0.5$ and $\rho = -0.7$.*

**References:**

[1] K.Burnecki, E.Kepten, J.Janczura, I.Bronshtein, Y.Garini, A.Weron (2012) "Universal Algorithm for Identification of Fractional Brownian Motion. A Case of Telomere Subdiffusion", Biophysical Journal 103, 1839–1847. Article