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Global uniqueness for a two-dimensional inverse boundary value problem

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Dedicated to the memory of my mother, Rodica

Abstract

We show that the coefficient $\gamma(x)$ of the elliptic equation $\nabla \cdot (\gamma \nabla u) = 0$ in a two-dimensional domain is uniquely determined by the corresponding Dirichlet-to-Neumann map on the boundary, and give a reconstruction procedure. For the equation $\Sigma \partial_i (\gamma^{ij} \partial_j u) = 0$, two matrix-valued functions γ_1 and γ_2 yield the same Dirichlet-to-Neumann map if and only if there is a diffeomorphism of the domain which fixes the boundary and transforms γ_1 into γ_2.

Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let γ be a real-valued function in $L^\infty(\Omega)$ with a positive lower bound. The corresponding Dirichlet-to-Neumann map is the operator $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ defined by

$$\langle g, \Lambda_\gamma f \rangle = \int_\Omega \gamma \nabla v \cdot \nabla u, \tag{0.1}$$

where $f, g \in H^{1/2}(\partial\Omega)$, $\langle , \rangle$ denotes the bilinear pairing of $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega), u$ is the (weak) $H^1(\Omega)$ solution of the Dirichlet problem

$$\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = f, \tag{0.2}$$

and v is any function in $H^1(\Omega)$ with trace g on the boundary.

If Ω represents an inhomogeneous, isotropic body with conductivity γ then $\Lambda_\gamma f$ is the current flux at the boundary corresponding to a voltage potential f on $\partial\Omega$. Thus the operator Λ_γ represents the information which can be obtained from static voltage and current measurements at the boundary.

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In 1980, Calderón ([C]) posed the following problem: Decide whether γ is uniquely determined by Λ_γ and, if so, calculate γ in terms of Λ_γ. Precursors go at least as far back as a 1933 paper by Langer ([L]) who considered an analogous problem (formulated by Slichter) in a half-plane with analytic conductivity depending only on the depth; the case of cylindrical domains and γ constant along lines parallel to the axis was treated in [C-D-J]. Calderón proved injectivity of the derivative at $\gamma =$ constant of the map $\gamma \mapsto \Lambda_\gamma$ and gave an approximate reconstruction of γ.

The first general result on the nonlinear map $\gamma \mapsto \Lambda_\gamma$ was obtained by Kohn and Vogelius, who showed that if $\partial\Omega$ is C^∞ then Λ_γ determines $\frac{\partial^k \gamma}{\partial \nu^k}|_{\partial\Omega}$ for all $k \geq 0$ and used this to prove identifiability inside Ω of piecewise analytic conductivities ([K-V I], [K-V II]). More recently, an extension of these results to Lipschitz domains was proved by Alessandrini ([A II]).

Even outside the real-analytic category, the determination of γ and $\frac{\partial\gamma}{\partial\nu}$ on $\partial\Omega$ was a first step in much of the subsequent work on the problem. The substitution $u = \gamma^{-1/2}\tilde{u}$ in (0.2) yields $-\Delta\tilde{u} + q\tilde{u} = 0$ in Ω, with

$$q = \gamma^{-1/2}\Delta(\gamma^{1/2}); \tag{0.3}$$

if we define the corresponding Dirichlet-Neumann map Λ_q by:

$$\langle \tilde{g}, \Lambda_q \tilde{f} \rangle = \int_\Omega \nabla\tilde{v}\cdot\nabla\tilde{u} + q\tilde{v}\tilde{u}, \tag{0.4}$$

where $\tilde{u} \in H^1(\Omega)$ is the weak solution to the above Schrödinger equation with $\tilde{u}\mid_{\partial\Omega} = \tilde{f}$ and where $\tilde{v}$ is any function in $H^1(\Omega)$ with $\tilde{v}\mid_{\partial\Omega} = \tilde{g}$, then $\Lambda_q = \gamma^{-1/2}(\Lambda_\gamma + \frac{1}{2}\frac{\partial\gamma}{\partial\nu})\gamma^{-1/2}$. Thus, given Λ_γ one can obtain Λ_q by first calculating $\gamma|_{\partial\Omega}$ and $\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}$ and thereby reducing the inverse problem to that of recovering q from Λ_q. This reduction removes the unknown coefficient in (0.2) from the top order terms; we will return to it below.

In dimensions $n \geq 3$ (for conductivities smooth enough for the potential given by (0.3) to be reasonable), global uniqueness is by now understood: Sylvester and Uhlmann proved that if $\partial\Omega$ is C^∞ then Λ_γ uniquely determines γ in $C^\infty(\bar{\Omega})$ ([S-U II]). A simplification of their proof in [N-S-U] together with the results at the boundary in [S-U III] extended uniqueness to $\gamma \in W^{2,\infty}(\Omega)$; the condition on the boundary was relaxed to $\partial\Omega \in C^{1,1}$ in [N I] and $\partial\Omega$ Lipschitz in [A II]. Isakov ([I I]) showed that the weighted-$L^2(\mathbb{R}^n)$ estimate in [S-U II] could be replaced by known $L^2(\Omega)$ bounds on the regular fundamental solution of Ehrenpreis and Malgrange, thus extending the uniqueness result to a large class of partial differential equations. For Schrödinger operators, Jerison and Kenig (see [Ch]) showed that Λ_q uniquely determines $q \in L^p(\Omega)$ for $p > \frac{n}{2}$; this was extended to $q \in L^{n/2}(\Omega)$ in [L-N] (see [N II]). (If one abandons global uniqueness, more singular potentials can be treated: Chanillo ([Ch]) allows q in the Fefferman-Phong class Fp, $p > \frac{n-1}{2}$, provided $\|q\|_{Fp}$

is sufficiently small.) By combining the Jerison-Kenig result with Theorem 6 of the present paper, we prove global uniqueness for the inverse conductivity problem in dimensions $n \geq 3$ if $\partial\Omega$ is Lipschitz and $\gamma \in W^{2,p}(\Omega)$ for some $p > n/2$.

For a review of the uniqueness results known for some special cases of discontinuous γ see [I II]. A logarithmic stability estimate for γ known *a priori* to be in $W^{2,\infty}(\Omega)$ was obtained by Alessandrini ([A I], [A II]). Regarding the second half of Calderón's problem, a constructive method to determine γ in Ω from Λ_γ was given in [N I] and independently, for Schrödinger operators, by Novikov (announced in [H-N] and also, with sketches of proofs, in [No I]).

The global uniqueness question in two dimensions has, up to now, remained open. The difficulty stems from the fact that the inverse problem is no longer overdetermined in this case (a naive variable count shows that in n dimensions, the Schwartz kernel of the data Λ_γ depends on $2(n-1)$ variables while γ is a function of n variables). Intuitively, all the information in Λ_γ now needs to be used, while for $n \geq 3$ the large complex frequency information (in a sense to be made precise below) has been sufficient for global uniqueness: all of the proofs referred to above have relied on this feature of the overdeterminacy.

Local uniqueness in two dimensions was proved by Sylvester and Uhlmann ([S-U I]) for conductivities close to $\gamma_0 =$ constant in $W^{3,\infty}(\Omega)$, by Sun ([Su I]) for γ close to γ_0, where either $\gamma_0^{-1/2}\Delta(\gamma_0^{1/2}) =$ constant or $\gamma_0 = \exp(\operatorname{Re}\psi)$ with ψ a conformal map injective in $\bar{\Omega}$, and by Sun and Uhlmann [Su-U] for γ close to γ_0 in an open dense subset of $W^{3,\infty}_{\text{pos}}(\Omega)$. Global uniqueness has been shown to hold for pairs of conductivities in a dense open subset of $W^{3,\infty}_{\text{pos}}(\Omega) \times W^{3,\infty}_{\text{pos}}(\Omega)$ ([Su-U]), and for special conductivities γ with γ^α harmonic for some $\alpha \in \mathbb{R}$ ([S-U IV], [Su II]) or γ radial and Ω a disc ([Sy II]).

In this paper we prove the following:

THEOREM 1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. Let γ_1 and γ_2 be in $W^{2,p}(\Omega)$ for some $p > 1$, and have positive lower bounds.*

$$\text{If } \quad \Lambda_{\gamma_1} = \Lambda_{\gamma_2} \text{ then } \gamma_1 = \gamma_2.$$

The proof gives a constructive procedure for recovering γ from Λ_γ.

If the conductivity γ is anisotropic, then it is represented by a positive definite symmetric matrix $(\gamma^{ij}(x))$; the equation for the voltage u becomes:

$$\Sigma_{i,j} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) = 0 \text{ in } \Omega, \tag{0.5}$$

and the Dirichlet-to-Neumann map is defined by

$$\langle g, \Lambda_\gamma f\rangle = \sum_{i,j=1}^{n} \int_\Omega \gamma^{ij} \frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_j}, \tag{0.6}$$

where u, v are in $H^1(\Omega)$, u solves (0.5) and $u|_{\partial\Omega} = f, v|_{\partial\Omega} = g$. It was observed by L. Tartar (see [K-V III]) that if Φ is a diffeomorphism of Ω which fixes the boundary and $\Phi_*\gamma =: ((D\Phi)^t\gamma(D\Phi)/\det(D\Phi)) \circ \Phi^{-1}$, then $\Lambda_{\Phi_*\gamma} = \Lambda_\gamma$. Conversely, we can now prove:

THEOREM 2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with a C^3 boundary, and let γ_1, γ_2 be anisotropic C^3 conductivities in $\overline{\Omega}$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then there exists a C^3 diffeomorphism Φ of Ω such that*

$$\gamma_2 = \Phi_*\gamma_1 \quad \textit{and} \quad \Phi|_{\partial\Omega} = I. \tag{0.7}$$

For a review of the status of the corresponding conjecture in dimensions $n \geqq 3$, see [S-U V].

Theorem 2 was proved by Sylvester [Sy I] under the additional assumption

$$\|\log(\det\gamma_\ell)\|_{C^3} < \varepsilon(M, \Omega) \text{ for } \ell = 1, 2, \text{ with } M \geqq \|\gamma_\ell\|_{C^3} \text{ for } \ell = 1, 2, \tag{0.8}$$

and $\varepsilon(M, \Omega)$ sufficiently small. Combining his arguments (which reduce the problem to the isotropic case using isothermal coordinates) with our Theorem 1 easily yields Theorem 2.

In the remainder of this introduction we will describe the main results (Theorems 3, 2.1, 3.3, 4, 4.1, 5, 6) which go into our proof of Theorem 1, concluding with an outline of the reconstruction procedure.

The first step in our reconstruction from Λ_γ is, as indicated earlier, the determination of γ and $\frac{\partial\gamma}{\partial\nu}$ on the boundary $\partial\Omega$. The new result we need is given in Theorem 6 below and worked out in Section 5. Once γ and $\frac{\partial\gamma}{\partial\nu}$ are known on $\partial\Omega$, a simple construction (see §6) allows us to reduce the problem to one with γ identically equal to 1 near the boundary (in a slightly larger domain Ω_2, which we will continue to denote by Ω in the rest of the paper). This reduction is not really needed, but it helps simplify the proof. We then extend γ to be 1 outside Ω_2, and define q throughout $\mathbb{R}^2$ by (0.3).

We next need to discuss the forward problem and define the *non-physical* scattering transform $\mathbf{t}$ of q which, even though not directly measurable in experiments, will be a key intermediate object in our reconstruction.

In the construction of $\mathbf{t}$, which is of independent interest (for instance, in the exact solution of a family of nonlinear equations in 2 + 1 dimensions [B-L-M-P]) we do not require q to have compact support. In Section 1 we assume $q \in L^p(\mathbb{R}^2)$ for some p, $1 < p < 2$; in Sections 2, 3, and 4 we require $q \in L^p_\rho(\mathbb{R}^2)$ for some p, ρ with $1 < p < 2$ and $\rho > 1$, where $L^p_\rho(\mathbb{R}^2)$ is the weighted space with norm $\|f\|_{L^p_\rho} = \|\langle x\rangle^\rho f\|_{L^p}$, $\langle x\rangle = (1+|x|^2)^{1/2}$. To define $\mathbf{t}$, we need the family $\psi(x, \zeta)$ of exponentially growing solutions of the Schrödinger equation first introduced by Faddeev ([Fa]) and later rediscovered, via different routes, by Nachman and Ablovitz ([N-A]), Beals and Coifman ([B-C II]) and

Sylvester and Uhlmann ([S-U I, II]). For q in $L^p(\mathbb{R}^2)$, $1 < p < 2$, we define $\psi(x,\zeta)$ to be the solution (when it exists) of

$$(0.9)\qquad (-\Delta + q)\psi(x,\zeta) = 0 \text{ in } \mathbb{R}^2,$$

with $e^{-ix\cdot\zeta}\psi(x,\zeta) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$. Here $\zeta \in \mathcal{V} =: \{\zeta \in \mathbb{C}^2 \setminus 0 : \zeta^2 = 0\}$ and $\tilde{p}$ will denote the exponent given by $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}$. In [S-U I], Sylvester and Uhlmann proved existence of exponentially growing solutions assuming $\langle x\rangle^{1+\varepsilon}q$ bounded and ζ sufficiently large. We would like to remove the large ζ restriction. New (but elementary) estimates on convolution by Faddeev's Green function (Lemma 1.3) show that the problem (0.9) is equivalent to a Fredholm integral equation of the second kind. If ζ is such that (0.9) has a nontrivial solution h with $he^{-ix\cdot\zeta}$ in $W^{1,\tilde{p}}(\mathbb{R}^2)$ then it will be called an "exceptional point". For $\zeta \in \mathcal{V}$ nonexceptional and ξ in $\mathbb{R}^2$ we define the scattering transform (see also the introduction in [N I]) by

$$(0.10)\qquad \mathbf{t}(\xi,\zeta) = \int_{\mathbb{R}^2} e^{-ix\cdot(\xi+\zeta)}q(x)\psi(x,\zeta)\,dx.$$

In [T] it was pointed out that (for the two-dimensional zero energy problem of interest here) exceptional points will, in general, be present even for q with small $\|\langle x\rangle^{4+\varepsilon}q\|_{L^\infty}$ norm. We prove that zero-energy exceptional points are absent if and, roughly, only if the potential q comes from the conductivity problem! More precisely we have the following:

THEOREM 3. *Let q be a real-valued function in $L^p_\rho(\mathbb{R}^2)$, $1 < p < 2$, $\rho > 1$. The following are equivalent:*

(a) $q = (\Delta\psi_0)/\psi_0$ *for some* $\psi_0 \in L^\infty(\mathbb{R}^2)$ *with* $\psi_0 \geqq c_0 > 0$ a.e.

(b) *There are no exceptional points $\zeta \in \mathbb{C}^2$ with $\zeta^2 = 0$, and the scattering transform satisfies*

$$(0.11)\qquad |\,\mathbf{t}(-2\zeta_R,\zeta)| \leqq c|\zeta|^\varepsilon$$

for some $\varepsilon > 0$ and all sufficiently small $\zeta = \zeta_R + \zeta_I$ in $\mathcal{V}$.

Remarks: (i) We will use the notation $\psi_0(x) = \gamma^{1/2}(x)$ throughout the paper.

(ii) If condition (0.11) is left out of (b), then in the direct problem one needs to allow functions $\gamma^{1/2}$ with logarithmic growth at infinity, and that is outside the range of the present paper.

(iii) For $\zeta = 0$, we will show (see the proof of Lemma 3.5 (ii)) that, if (a) holds, then in fact any solution h of $h = -G_0 * (qh)$ (where G_0 denotes the standard Green's function for the Laplacian) with $\langle x\rangle^{-\beta}h$ bounded, for some $\beta < 1$, must vanish. We do not, however, assume this when proving (b) $\Rightarrow$ (a).

The proof of (a) $\Rightarrow$ (b) is based on the observation that, for potentials satisfying (a), we can factor $-\Delta + q$ as a product of first order operators in a manner compatible with our condition at infinity; absence of exceptional points is thereby reduced to a version of Liouville's theorem for pseudoanalytic functions (see Lemma 1.5). The vanishing of $\mathbf{t}$ for ζ near zero (0.11) is proved in Section 3. The proof of (b) $\Rightarrow$ (a) uses most of the machinery developed for inversion, and is at the end of Section 4.

Having shown the existence of $\mathbf{t}(\cdot, \zeta)$ at all $\zeta \in \mathcal{V}$ (for potentials satisfying (a)) we next describe how to recover γ from knowledge of $\mathbf{t}$. This step will require only the values of $\mathbf{t}(\xi, \zeta)$ with $\xi = -2\zeta_R$ and ζ in $\mathcal{V}$. In two dimensions, it is helpful to change notation as follows. The variety $\zeta^2 = 0$ in $\mathbb{C}^2$ can be parametrized as $\{(k, ik) : k \in \mathbb{C}\} \cup \{(k, -ik) : k \in \mathbb{C}\}$. We henceforth denote $\psi(x, (k, ik))$ by $\psi(x, k)$ and observe that (since q is real-valued) uniqueness for (0.9) yields $\psi(x, (-\bar{k}, i\bar{k})) = \overline{\psi(x, (k, ik))} = \overline{\psi(x, k)}$ so that, for reconstruction purposes, it suffices to work on the sheet $\zeta = (k, ik)$. We also introduce the functions $\mu(x, k) = \psi(x, k) \exp(-ikz)$; it will be convenient, throughout the paper, to switch back and forth between the notation $x = (x_1, x_2)$ for a point in $\mathbb{R}^2$ and the corresponding complex number $z = x_1 + ix_2$. For $\zeta = (k, ik)$ we will denote by $\mathbf{t}(k)$ the function $\mathbf{t}(-2\zeta_R, \zeta)$.

To recover γ from knowledge of $\mathbf{t}(k)$ (for all k in $\mathbb{C}\backslash 0$), we use the $\overline{\partial}$ method, initially introduced in one-dimensional inverse scattering (our paradigm for formally determined problems which are well understood) by Beals and Coifman ([B-C I]), and first extended to a two-dimensional problem by Ablowitz, Bar Yaacov and Fokas ([A-B-F]). The two-dimensional zero-energy Schrödinger problem considered here was initiated by Boiti, Leon, Manna and Pempinelli ([B-L-M-P]) and worked out by Tsai ([T]) (on a family of solutions which differ from $\psi(x, k)$) for potentials with $\|\langle x\rangle^{4+\varepsilon} q\|_{W^{1,\infty}}$ sufficiently small. For the positive energy problem see R. Novikov [No II] and the earlier references given there; for negative energy, and a result similar to the second half (b) $\Rightarrow$ (a) of our Theorem 3, see Grinevich and S. P. Novikov [G-N].

In Section 2 we prove (Theorem 2.1) that the $\overline{\partial}$ equation

$$\frac{\partial}{\partial \bar{k}}\, \mu(x, k) = \frac{1}{4\pi \bar{k}}\, \mathbf{t}(k) e_{-k}(x) \overline{\mu(x, k)} \tag{0.12}$$

holds in the weighted Sobolev space $W^{1,\tilde{p}}_{-\beta}(\mathbb{R}^2) = \{f : \langle x\rangle^{-\beta} f \in W^{1,\tilde{p}}(\mathbb{R}^2)\}$, if $k \neq 0$ and $\beta > \frac{2}{\tilde{p}}$; the function e_k is defined by

$$e_k(z) := \exp(i(kz + \bar{k}\bar{z})). \tag{0.13}$$

Equation (0.12) together with the high complex frequency estimate (1.1) is the basis for recovering μ from knowledge of $\mathbf{t}$. Existence and uniqueness for the problem (0.12) is, again, reduced to a form of Liouville's theorem for pseudoanalytic functions provided $\mu(\cdot, k)$ is bounded for k near zero. The behavior of $\mu(\cdot,k)$ and $\mathbf{t}(k)$ for k near zero is described in Theorem 3.3. The properties of the scattering transform obtained in the direct problem for our class of potentials are summarized in the following.

THEOREM 4. *Suppose q in $L^p_\rho(\mathbb{R}^2)$ $(1 < p < 2,\ \rho > 1)$ is of the form $q = (\Delta\psi_0)/\psi_0$ with $\psi_0 \in L^\infty(\mathbb{R}^2)$ and $\psi_0(x) \geqq c_0 > 0$. Then $\mathbf{t}(k)/\bar{k}$ is in $L^r(\mathbb{R}^2)$ for all $r \in (\tilde{p}', r_2)$, where*

$$r_2 = \tilde{p} \text{ if } \rho \geqq 5 - 4/p,\ r_2 = 4/(3-\rho) \text{ if } 1 < \rho < 5 - 4/p. \tag{0.14}$$

In Section 4, we prove the unique solvability of the inversion equation (0.12), given $\mathbf{t}$ as described by the result above (Theorem 4.1). Once μ is found, the usual method for recovering q is from the large k asymptotics of μ; knowing $\gamma\,|_{\partial\Omega}$ one could then solve a Dirichlet problem in Ω to obtain γ from q. This procedure requires more decay of $\mathbf{t}$ than is available, that is, more regularity of q than we have assumed. We take a different approach, making use of the behavior of $\mu(\cdot,k)$ for k near zero, analyzed in Section 3, where we show that, as $k \to 0$, $\lim \mu(x,k) = \gamma^{1/2}(x)$ (in the $W^{1,\tilde{p}}_{-\beta}$ topology).

The usefulness of the $\bar{\partial}$ method in the study of inverse boundary value problems came with the discovery that if q has support in Ω (or, more generally, is known outside Ω), then its scattering transform $\mathbf{t}$ can be calculated from the Dirichlet-to-Neumann map Λ_q on $\partial\Omega$ ([N I], [No I]). We have sharpened and simplified the proof in [N I] to allow Lipschitz boundaries and potentials in $L^p, p > 1$. The main result of Section 7 is the following. (See [N I] for the exterior problem which led to (0.17).) For $k \neq 0$, let S_k denote the operator

$$S_k f(x) = \int_{\partial\Omega} G_k(x-y) f(y)\, d\sigma(y)\,, \tag{0.15}$$

the single-layer operator corresponding to Faddeev's Green function

$$G_k(x) = \frac{e^{i(x_1+ix_2)k}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)}\, d\xi. \tag{0.16}$$

THEOREM 5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ and let $q \in L^p(\Omega)$ for some $p > 1$. Assume 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω. Then, for any $k \in \mathbb{C} \setminus 0$:*

(i) *$S_k(\Lambda_q - \Lambda_0)$ is a compact operator: $H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ (Λ_0 denotes the Dirichlet-to-Neumann map for $-\Delta$ in Ω);*

(ii) *The trace on $\partial\Omega$ of the function $\psi(\cdot,k)$ satisfies the integral equation*

$$\psi(\cdot,k)|_{\partial\Omega} = e^{izk} - S_k(\Lambda_q - \Lambda_0)\psi(\cdot,k); \tag{0.17}$$

(iii) *$I + S_k(\Lambda_q - \Lambda_0)$ is invertible on $H^{1/2}(\partial\Omega)$ if and only if k is not an exceptional point (of q, extended to be zero outside Ω).*

(iv) *If $\psi(\cdot,k)|_{\partial\Omega}$ has been determined, then $\mathbf{t}(k)$ can be recovered from the formula*

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{z}\bar{k}}(\Lambda_q - \Lambda_0)\psi(\cdot,k)\, d\sigma. \tag{0.18}$$

In view of Theorem 3, for potentials coming from the conductivity problem, the equation (0.17) will be uniquely solvable for any $k \neq 0$. On the other hand, $\Lambda_q \equiv \Lambda_\gamma$ by our reduction to the case $\gamma \equiv 1$ near the boundary. To accomplish the latter, we first need to recover $\gamma|_{\partial\Omega}$ and $\frac{\partial\gamma}{\partial\nu}\Big|_{\partial\Omega}$.

A number of results on reconstruction at the boundary have been proved, following the papers of Kohn and Vogelius mentioned earlier. When $\partial\Omega$ is C^∞, Sylvester and Uhlmann ([S-U III]) showed that Λ_γ determines $\gamma|_{\partial\Omega}$ for γ continuous in $\overline{\Omega}$ and also determines $\frac{\partial\gamma}{\partial\nu}\Big|_{\partial\Omega}$ if γ is Lipschitz continuous. The proof in [N I] allowed $\partial\Omega$ to be in $C^{1,1}$ and assumed γ Lipschitz for the recovery of $\gamma|_{\partial\Omega}$ and γ in $C^{1,1}$ for that of $\frac{\partial\gamma}{\partial\nu}\Big|_{\partial\Omega}$. Alessandrini ([A2]) only requires $\partial\Omega$ Lipschitz and $\gamma \in W^{1,p}(\Omega), p > n$, for determination of $\gamma|_{\partial\Omega}$ and $\gamma \in C^1(\overline{\Omega})$ for $\frac{\partial\gamma}{\partial\nu}\Big|_{\partial\Omega}$. We need to sharpen the latter slightly to allow γ in $W^{2,p}(\Omega), p > \frac{n}{2}$, and would like a constructive recovery method. The formulas in [N I][1] involved the single-layer operator as a convenient parametrix for Λ_1; this relied on the compactness of the double-layer operator, which we do not have for Lipschitz domains. Here we use the Neumann-to-Dirichlet map R for the Laplacian. To describe the theorem which we prove in Section 5, we need to introduce some additional notation. We give the result for any dimension $n \geq 2$. Let $\Omega^1, \ldots, \Omega^N$ be the connected components of the Lipschitz domain Ω. We define R on $\overset{\circ}{H}{}^{-1/2}(\partial\Omega) = \{h \in H^{-1/2}(\partial\Omega) : \langle h, 1\rangle_{\partial\Omega^j} = 0, j = 1, \ldots N\}$ by $Rh = w|_{\partial\Omega}$, with $w \in H^1(\Omega)$ the weak solution (unique modulo functions constant on each Ω^j) of $\Delta w = 0$, $\partial w/\partial\nu = h$. For an arbitrary $x_0 \in \partial\Omega^{j_0}$, let $U = B \times I \subset \mathbb{R}^n$ be a cylindrical neighborhood with coordinates chosen so as to have $\Omega \cap U = \Omega^{j_0} \cap U = \{(x', x_n) \in U : x_n < \varphi(x')\}$ with φ a Lipschitz function. Let $h \in L^2(\partial\Omega)$ have support in $\partial\Omega^{j_0} \cap U$. For any $\eta \in \mathbb{R}^{n-1} \times \{0\}$ let h_η be the function defined to be identically zero on $\partial\Omega^j$ for $j \neq j_0$, and

$$h_\eta(x) = h(x)e^{-ix\cdot\eta} - \frac{1}{|\partial\Omega^{j_0}|}\int_{\partial\Omega\cap U} h(y)e^{-iy\cdot\eta}\, d\sigma(y) \text{ for } x \in \partial\Omega^{j_0}; \tag{0.19}$$

note that, in the above coordinates, $\exp(-ix\cdot\eta) = \exp(-ix'\cdot\eta')$.

[1]Formulas (1.35), (1.36) in [N I] are stated incorrectly: they should be localized to a coordinate patch, with $\exp(ix\cdot\eta), \eta \in \mathbb{R}^{n-1}$, defined in terms of the corresponding coordinates $(x_1, \ldots, x_{n-1})$ on $\partial\Omega$. With this clarification, the proofs are correct as given.

THEOREM 6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Suppose γ is in $W^{1,r}(\Omega)$ for some $r > n$ and has a positive lower bound. Let $x_0 \in \partial\Omega$ and let U be a cylindrical neighborhood of x_0 as described above. Then:*

(i) *$\gamma|_{\partial\Omega\cap U}$ can be recovered from Λ_γ by the formula:*

$$\langle h,\gamma f\rangle = \lim_{\substack{|\eta|\to\infty\\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle h_\eta, R\Lambda_\gamma e^{-i<\cdot,\eta>} f\rangle, \tag{0.20}$$

where $f \in H^{1/2}(\partial\Omega)\cap C(\partial\Omega)$ and $h \in L^2(\partial\Omega)$ are assumed supported in $U \cap \partial\Omega$ and h_η is as defined in (0.19).

(ii) *If, moreover, $\gamma \in W^{2,p}(\Omega)$ for some $p > n/2$, then for any continuous functions f, g in $H^{1/2}(\partial\Omega)$ with support in $U \cap \partial\Omega$,*

$$\left\langle g, \frac{\partial\gamma}{\partial\nu} f\right\rangle = \lim_{\substack{|\eta|\to\infty\\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle g, e^{-i<\cdot,\eta>}(\gamma\Lambda_1 + \Lambda_1\gamma - 2\Lambda_\gamma)e^{i<\cdot,\eta>} f\rangle, \tag{0.21}$$

with Λ_1 the Dirichlet-to-Neumann map corresponding to $\gamma \equiv 1$.

We have set up the proof so as only to require the recent hard estimates on the Dirichlet and Neumann problems for the Laplacian, while for the variable coefficient equation standard weak solutions lore will suffice.

The reader interested in a more leisurely account of the proofs in this paper is referred to the earlier version [N III].

We conclude this introduction with a summary of the main steps in our reconstruction of γ in Ω from knowledge of Λ_γ:

(i) determine $\gamma|_{\partial\Omega}$ and $\left.\frac{\partial\gamma}{\partial\nu}\right|_{\partial\Omega}$ (formulas (0.20) and (0.21)), and reduce the problem (constructively) to one with $\gamma \equiv 1$ near the boundary (§6),

(ii) solve the integral equations (0.17) to determine $\psi(\cdot, k)$ on $\partial\Omega$ and obtain $\mathbf{t}(k)$ using formula (0.18) (Theorem 5),

(iii) solve the integral equations (Theorem 4.1):

$$\mu(x,k) = 1 + \frac{1}{8\pi^2 i}\int_{\mathbb{C}} \frac{\mathbf{t}(k')}{(k'-k)\bar{k}'} e_{-x}(k')\overline{\mu(x,k')}\, dk' \wedge d\bar{k}', \tag{0.22}$$

and obtain $\gamma(x)$ from the (absolutely convergent) integral formula

$$\gamma^{1/2}(x) = 1 + \frac{1}{8\pi^2 i}\int_{\mathbb{C}} \frac{\mathbf{t}(k)}{|k|^2} e_{-x}(k)\overline{\mu(x,k)}\, dk \wedge d\bar{k}. \tag{0.23}$$

This reconstruction method yields a logarithmic stability estimate, which will be published elsewhere.

1. The exponentially growing solutions

THEOREM 1.1. *Let $q \in L^p(\mathbb{R}^2)$, $1 < p < 2$, be such that there exists a real-valued $\psi_0 \in L^\infty(\mathbb{R}^2)$ with $q = (\Delta\psi_0)/\psi_0$, $\psi_0(x) \geqq c_0 > 0$ and $\nabla\psi_0 \in L^p(\mathbb{R}^2)$. Then for any $k \in \mathbb{C}\setminus 0$ there is a unique solution $\psi(x,k)$ of (0.9) with $e^{-ikz}\psi(\cdot,k) - 1$ in $L^{\tilde p} \cap L^\infty$. Furthermore, $e^{-ikz}\psi(\cdot,k) - 1 \in W^{1,\tilde p}$ and*

$$\|e^{-izk}\psi(\cdot,k) - 1\|_{W^{s,\tilde p}} \leqq c|k|^{s-1}\|q\|_{L^p} \tag{1.1}$$

for $0 \leqq s \leqq 1$ and k sufficiently large.

We need some notation and some lemmas. The exponent $\tilde p$ was defined as $1/\tilde p = 1/p - 1/2$. Denote by ∂ and $\overline{\partial}$ the differential operators $\partial = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$ and $\overline{\partial} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$; we will write $\overline{\partial}^{-1}$ for the solid Cauchy transform $\overline{\partial}^{-1} f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(w)}{w-z}\, dw \wedge d\bar w$ and similarly for ∂^{-1}.

LEMMA 1.2. (i) *For any f in L^p and any k in $\mathbb{C}$ there is a unique u in $L^{\tilde p}$ satisfying $(\partial + ik)u = f$. Now $u = (\partial + ik)^{-1} f =: e_{-k}\partial^{-1}(e_k f)$ and*

$$\|(\partial + ik)^{-1} f\|_{L^{\tilde p}} \leqq c\|f\|_{L^p}, \text{ with } c \text{ independent of } k. \tag{1.2}$$

(ii) *If v is a function in $L^{\tilde p}$ with ∂v in L^p, then for any $k \in \mathbb{C}\setminus 0$ there is a unique solution w in $L^{\tilde p}$ of $(\partial + ik)w = v$. Furthermore, $w \in W^{1,\tilde p}$ and*

$$\|w\|_{L^{\tilde p}} \leqq \frac{c}{|k|}(\|v\|_{L^{\tilde p}} + \|\partial v\|_{L^p}) \text{ and } \|\nabla w\|_{L^{\tilde p}} \leqq c\|\partial v\|_{L^p}. \tag{1.3}$$

Proof. (i) The estimate (1.2) follows from the Hardy-Littlewood inequality on fractional integration [St]. (The bounded function e_k was defined in (0.13).) Uniqueness is easily reduced to (an $L^{\tilde p}$ version of) Liouville's theorem.

(ii) The resolvent equation motivates the ansatz ([S-U I]) $w = (I - (\partial + ik)^{-1}\partial)v/ik$. Since ∂v was assumed in L^p, w is well-defined, is easily checked to satisfy the given equation, and the first of the estimates (1.3) follows from part (i). For the second, we first bound ∂w using $\partial w = (\partial + ik)^{-1}\partial v$; for $\bar\partial w$ we then rely on the $L^{\tilde p}$ boundedness of the Beurling transform $\bar\partial\partial^{-1}$. □

The above lemma allows us to prove some new estimates on the operator of convolution by $g_k(x) =: \exp(-izk)G_k(x)$; we want to use smoothing properties of this operator to compensate for the roughness of the potentials.

LEMMA 1.3. *For any f in $L^p(\mathbb{R}^2)$ and any ζ in $\mathbb{C}^2 \setminus 0$ with $\zeta^2 = 0$ there is a unique u in $L^{\tilde p}(\mathbb{R}^2)$ satisfying*

$$(-\Delta - 2i\zeta\cdot\nabla)u = f \text{ in } \mathbb{R}^2. \tag{1.4}$$

Furthermore, $u \in W^{1,\tilde p}$ and

$$\|u\|_{W^{s,\tilde p}} \leqq \frac{c}{|\zeta|^{1-s}}\|f\|_p \quad \text{for } \|\zeta\| \geqq \text{ const.} > 0,\ 0 \leqq s \leqq 1. \tag{1.5}$$

Proof. We may assume without loss of generality that $\zeta = (k, ik)$ for some k in $\mathbb{C} \setminus 0$. To verify uniqueness, note that any tempered distribution solution u_0 of (1.4) with $f \equiv 0$ must be of the form $u_0 = p(z) + e_{-k}q(\bar{z})$, with p, q polynomials in z, respectively $\bar{z}$. Thus, if $u_0 \in L^r$ for some $1 \leqq r < \infty, u_0 \equiv 0$. For existence, let $v = -\frac{1}{4}\bar{\partial}^{-1}f$, and note that $v \in L^{\tilde{p}}$ and $\partial v \in L^p$. Thus we can apply Lemma 1.2 (ii) to obtain the formula

$$(1.6) \qquad u = -\frac{1}{4ik}\left[\bar{\partial}^{-1} - (\partial + ik)^{-1}\partial\bar{\partial}^{-1}\right] f = g_k * f.$$

The estimates (1.5) then follow from (1.3) by interpolation. □

The following simple lemma will be useful at several points in the paper:

LEMMA 1.4. *If $f \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)$ with $1 < p_1 < 2 < p_2$, then the function $u = \bar{\partial}^{-1}f$ satisfies $\lim_{|z|\to\infty} u(z) = 0$,*

$$(1.7) \qquad \|u\|_{L^\infty} \leqq c_{p_1,p_2}(\|f\|_{L^{p_1}} + \|f\|_{L^{p_2}}), \quad \text{and}$$

$$(1.8) \qquad |u(z_1) - u(z_2)| \leqq c_{p_2}|z_1 - z_2|^{1-2/p_2}\|f\|_{L^{p_2}}.$$

Proof. The estimates (1.7) and (1.8) are proved in [V]. If $f \in L_1^{p_1} \cap L_1^{p_2}$, then $f \in L^1$ and the identity

$$(1.9) \qquad \bar{\partial}^{-1}f(z) = \frac{1}{z}\left[-\frac{1}{2\pi i}\int f(w)\,dw \wedge d\bar{w} + \bar{\partial}^{-1}(wf(w))\right] \quad \text{for } z \neq 0$$

together with (1.7) shows that $u = 0(1/|z|)$; the vanishing of u at infinity for general $f \in L^{p_1} \cap L^{p_2}$ then follows by a density argument. □

We are now ready to prove the absence of exceptional points for our class of potentials.

LEMMA 1.5. *Let $q \in L^p(\mathbb{R}^2)$ be such that there exists a real-valued function $\psi_0 \in L^\infty(\mathbb{R}^2)$ with $q = (\Delta\psi_0)/\psi_0$, $\psi_0(x) \geqq c_0 > 0$ and $\nabla\psi_0 \in L^p(\mathbb{R}^2)$.*

If h satisfies $(-\Delta + q)h = 0$ in $\mathbb{R}^2$ and $he^{-izk} \in W^{1,\tilde{p}}(\mathbb{R}^2)$ for some $k \in \mathbb{C}$, then $h \equiv 0$.

Proof. Without loss of generality, we may assume h to be real-valued. Define $v = (\psi_0\partial h - h\partial\psi_0)e^{-ikz}$. Then since $\nabla\psi_0 \in W^{1,p} \subset L^{\tilde{p}}$ we have $v \in L^{\tilde{p}}$ and, using the fact that h and ψ_0 are real-valued, we find

$$(1.10) \qquad \bar{\partial}v = (\bar{\partial}\psi_0/\psi_0)v - (e_{-k}\partial\psi_0/\psi_0)\bar{v}.$$

Thus v is a pseudoanalytic function. We use the standard trick of constructing the auxiliary function $w = \bar{\partial}^{-1}((\bar{\partial}\psi_0)/\psi_0 - (\partial\psi_0)e_{-k}\bar{v}/(\psi_0 v))$. Note that $\bar{\partial}^{-1}$ is being applied to a function which is in $L^p \cap L^{\tilde{p}}$; hence, by Lemma 1.4, w is bounded and continuous. The function ve^{-w} is then entire and in $L^{\tilde{p}}$, and so it must vanish; thus h/ψ_0 is antiholomorphic. The function $(h/\psi_0)e^{i\bar{z}\bar{k}}$ is then antiholomorphic and also in $L^{\tilde{p}}$; hence it too must vanish. □

Remarks: (i) For $k \neq 0$, it suffices to assume $he^{-izk} \in L^{\tilde{p}} \cap L^{\infty}$: the function $u = he^{-izk}$ solves $\overline{\partial}(\partial + ik)u = qu/4$, with the right side in L^p, since u is bounded; Lemma 1.3 then shows that $u = -g_k * (qu)$, so $he^{-izk} \in W^{1,\tilde{p}}$.

(ii) If $q \in L^p_\rho(\mathbb{R}^2)$ $(1 < p < 2, \rho > 1)$, then we will show in Lemma 3.1 that the condition $\nabla\psi_0 \in L^p$ follows from the boundedness of ψ_0.

Proof of Theorem 1.1. We have already shown uniqueness. To prove existence, write $\mu(z,k) = \psi(z,k)\exp(-izk)$. Then we need to solve

$$\mu - 1 = -g_k * (q\mu). \tag{1.11}$$

Multiplication by $q \in L^p$ is a compact operator: $W^{1,\tilde{p}} \to L^p$. Thus $I + g_k * (q\cdot)$ is a Fredholm operator of index zero on $W^{1,\tilde{p}}$; hence, by Lemma 1.5, it is invertible and we can define $\mu = 1 - [I + g_k * (q\,\cdot)]^{-1}(g_k * q)$. To prove (1.1), we first fix s_0 with $\frac{2}{\tilde{p}} < s_0 < 1$; then, by (1.5) and Sobolev imbedding $\|g_k * (qf)\|_{W^{s_0,\tilde{p}}} \leqq \tilde{c}\|q\|_{L^p}\|f\|_{W^{s_0,\tilde{p}}}|k|^{s_0-1}$. If $|k| > (2\tilde{c}\|q\|_{L^p})^{1/(1-s_0)}$ then

$$\|\mu - 1\|_{W^{s_0,\tilde{p}}} \leqq 2\|g_k * q\|_{W^{s_0,\tilde{p}}} \leqq 2c\|q\|_{L^p}|k|^{s_0-1} \leqq 1. \tag{1.12}$$

Now (1.1) follows by applying (1.5) to (1.11) and using (1.12) on the right. □

2. The $\overline{\partial}$ equation

THEOREM 2.1. *Let q be real-valued and in $L^p_\rho(\mathbb{R}^2)$, $1 < p < 2$, $\rho > 1$. Then for any $k \in \mathbb{C}\setminus 0$ which is not an exceptional point, equation* (0.12) *holds in the $W^{1,\tilde{p}}_{-\beta}$ topology, $\beta > 2/\tilde{p}$, with* $\mathbf{t}$ *the scattering transform defined by*

$$\mathbf{t}(k) = \int_{\mathbb{R}^2} e_k(x)q(x)\mu(x,k)\,dx. \tag{2.1}$$

It will be helpful to use the notation $\mathcal{F}f(k) = \frac{i}{2}\int_{\mathbb{C}} e_k(z)f(z)\,dz \wedge d\bar{z}$.

LEMMA 2.2. *Let $\alpha > 2/p'$, $\beta > 2/\tilde{p}$ (where p' denotes the exponent dual to p). The map $k \to (\partial + ik)^{-1}$ is differentiable on $\mathbb{C}$ in the strong operator topology: $L^p_\alpha \to L^{\tilde{p}}_{-\beta}$ and*

$$\frac{\partial}{\partial\bar{k}}(\partial + ik)^{-1}f(z) = -\frac{i}{\pi}\mathcal{F}f(k)e_{-k}(z). \tag{2.2}$$

Proof. We will write $k = k_1 + ik_2$. Let $D_j(k)$, $j = 1,2$, be the operators

$$D_j(k)f(z) = \frac{(-1)^{j-1}}{\pi}\int \frac{y_j - x_j}{\bar{w} - \bar{z}}e_k(w - z)f(w)\,dw \wedge d\bar{w}, w = y_1 + iy_2. \tag{2.3}$$

We would like to show $\frac{\partial}{\partial k_j}(\partial+ik)^{-1} = D_j(k)$. Clearly, $\|D_j(k)f\|_{L^\infty} \leqq \frac{2}{\pi}\|f\|_{L^1}$. Since $L^p_\alpha \subset L^1$ for $\alpha > \frac{2}{p'}$ and $\langle x\rangle^{-\beta} \in L^{\tilde{p}}$ for $\beta > \frac{2}{\tilde{p}}$,

$$(2.4)\quad \lim_{h\to 0}\left\|\left(\frac{(\partial+i(k+h))^{-1}-(\partial+ik)^{-1}}{h} - D_1(k)\right)f\right\|^{\tilde{p}}_{L^{\tilde{p}}_{-\beta}}$$

$$=\lim_{h\to 0}\int_{z\in\mathbb{R}^2}\left|\frac{\langle z\rangle^{-\beta}}{2\pi i}\int\frac{e_k(w-z)}{\bar{w}-\bar{z}}\left(\frac{e_h(w-z)-1}{h}-2i(y_1-x_1)\right)f(w)dw\wedge d\bar{w}\right|^{\tilde{p}}$$

$$=0$$

by two applications of dominated convergence. Thus

$$(2.5)\quad \frac{\partial}{\partial\bar{k}}(\partial+ik)^{-1}f = \frac{1}{2}(D_1(k)+iD_2(k))f = \frac{1}{2\pi}\int e_k(w-z)f(w)\,dw\wedge d\bar{w}.$$

□

The difficulty in differentiating $g_k * f$ is that, while the above appear to require $f \in L^1$, the function $\partial\bar{\partial}^{-1}f$ (see formula (1.6)) need not be in L^1 even for compactly supported f. Working in the smaller space L^p_α we show that any function in $\partial\bar{\partial}^{-1}L^p_\alpha$ can be written as the sum of an L^1 function and another which is less decaying but has a derivative in L^p_α; this saves the day, for we can apply $(\partial+ik)^{-1}$ to the latter, as done in the proof of Lemma 1.2 (ii).

LEMMA 2.3. *If $2/p' < \alpha < 1$ and $\alpha+1-2/p < \delta < 2/p'$, then*

$$(2.6)\quad \partial\bar{\partial}^{-1}L^p_\alpha \subset L^p_\alpha + \{u \in L^p_\delta : \partial u \in L^p_\alpha\}.$$

To prove Lemma 2.3 we need

LEMMA 2.4. *If $1-2/p < \delta < 2/p'$, then:*

(i) *$\bar{\partial}^{-1}$ is a bounded operator: $L^p_\delta \to L^p_{\delta-1}$, and*

(ii) *$\partial\bar{\partial}^{-1}$ is a bounded operator on L^p_δ.*

These are special cases of results in [N-W]; see also [S-U I].

Proof of Lemma 2.3. Let $f \in L^p_\alpha$. We first prove the identity

$$(2.7)\quad \partial\bar{\partial}^{-1}f = \langle z\rangle^{-\alpha}\partial\bar{\partial}^{-1}\langle z\rangle^{\alpha}f - \frac{\alpha}{2}\langle z\rangle^{-2}\bar{z}\bar{\partial}^{-1}f + \frac{\alpha}{2}\langle z\rangle^{-\alpha}\partial\bar{\partial}^{-1}\langle z\rangle^{\alpha-2}z\bar{\partial}^{-1}f.$$

Start with

$$(2.8)\quad \bar{\partial}\langle z\rangle^{\alpha}\bar{\partial}^{-1}f = \frac{\alpha}{2}\langle z\rangle^{\alpha-2}z\bar{\partial}^{-1}f + \langle z\rangle^{\alpha}f.$$

Since $\delta < \alpha$, $f \in L^p_\delta$, so $\bar{\partial}^{-1}f \in L^p_{\delta-1}$. The right side of (2.8) is in $L^p_{\delta-\alpha}$ and $1-\frac{2}{p} < \delta-\alpha < \frac{2}{p'}$, so we can apply $\partial\bar{\partial}^{-1}$, and (2.7) follows. The first term on the right of (2.7) is in L^p_α and our derivation shows that the remaining two terms are in L^p_δ; we need to verify that their ∂ derivatives are in L^p_α. Now

$$\partial[\langle z\rangle^{-2}\bar{z}\bar{\partial}^{-1}f] = -\langle z\rangle^{-4}\bar{z}^2\bar{\partial}^{-1}f + \langle z\rangle^{-2}\bar{z}\partial\bar{\partial}^{-1}f; \tag{2.9}$$

since $f \in L^p_\delta$, the right side of (2.9) is in $L^p_{\delta+1} \subset L^p_\alpha$. It remains to calculate

$$\begin{aligned}(2.10)\ &\partial[\langle z\rangle^{-\alpha}\partial\bar{\partial}^{-1}\langle z\rangle^{\alpha-2}z\bar{\partial}^{-1}f]\\ &= -\frac{\alpha}{2}\langle z\rangle^{-\alpha-2}\bar{z}\partial\bar{\partial}^{-1}\langle z\rangle^{\alpha-2}z\bar{\partial}^{-1}f + \frac{\alpha-2}{2}\langle z\rangle^{-\alpha}\partial\bar{\partial}^{-1}\langle z\rangle^{\alpha-4}|z|^2\bar{\partial}^{-1}f\\ &\quad + \langle z\rangle^{-\alpha}\partial\bar{\partial}^{-1}\langle z\rangle^{\alpha-2}\bar{\partial}^{-1}f + \langle z\rangle^{-\alpha}\partial\bar{\partial}^{-1}\langle z\rangle^{\alpha-2}z\partial\bar{\partial}^{-1}f =: a+b+c+d.\end{aligned}$$

Now $\langle z\rangle^{\alpha-2}z\bar{\partial}^{-1}f$ is in $L^p_{\delta-\alpha}$, $\partial\bar{\partial}^{-1}$ is bounded on $L^p_{\delta-\alpha}$ so $a \in L^p_{\delta+1}$. For b we have $\langle z\rangle^{\alpha-4}|z|^2\bar{\partial}^{-1}f$ in $L^p_{\delta-\alpha+1}$. Choose $\delta' \in (\alpha, \alpha+\frac{2}{p'})$; then, on the one hand $\delta' < \delta+1$ so $\langle z\rangle^{\alpha-4}|z|^2\bar{\partial}^{-1}f \in L^p_{\delta'-\alpha}$, and on the other hand $0 < \delta'-\alpha < \frac{2}{p'}$ so we can apply $\partial\bar{\partial}^{-1}$ to find that $b \in L^p_{\delta'} \subset L^p_\alpha$. For c, we have $\langle z\rangle^{\alpha-2}\bar{\partial}^{-1}f \in L^p_{\delta-\alpha+1} \subset L^p_{\delta'-\alpha}$, with δ' as above, so $\partial\bar{\partial}^{-1}(\langle z\rangle^{\alpha-2}\bar{\partial}^{-1}f) \in L^p_{\delta'-\alpha}$, which shows $c \in L^p_{\delta'} \subset L^p_\alpha$. Finally, from $\partial\bar{\partial}^{-1}f \in L^p_\delta$ we have $\langle z\rangle^{\alpha-2}z\partial\bar{\partial}^{-1}f \in L^p_{\delta-\alpha+1} \subset L^p_{\delta'-\alpha}$; hence d, too, is in L^p_α. □

LEMMA 2.5. *Let $2/p' < \alpha < 1$, $\beta > 2/\tilde{p}$. The map $k \mapsto g_k * \cdot$ is differentiable on $\mathbb{C}\setminus 0$ in the strong operator topology: $L^p_\alpha \to W^{1,\tilde{p}}_{-\beta}$ and*

$$\frac{\partial}{\partial\bar{k}}g_k * f = -\frac{1}{4\pi\bar{k}}\mathcal{F}f(k)e_{-k}. \tag{2.11}$$

Proof of Lemma 2.5. Given f in L^p_α, we use Lemma 2.3 to write $\partial\bar{\partial}^{-1}f = f_1 + f_2$ with $f_1 \in L^p_\alpha$, $f_2 \in L^p_\delta$ and $\partial f_2 \in L^p_\alpha$. Then we have

$$(\partial+ik)^{-1}(\partial\bar{\partial}^{-1}f) = (\partial+ik)^{-1}f_1 + \frac{1}{ik}f_2 - \frac{1}{ik}(\partial+ik)^{-1}\partial f_2. \tag{2.12}$$

(Note that $f_2 \in W^{1,p} \subset L^{\tilde{p}}$.) Lemma 2.2 now shows that the right side of (2.12) is differentiable in $L^{\tilde{p}}_{-\beta}$ for $k \neq 0$ and

$$\frac{\partial}{\partial\bar{k}}(\partial+ik)^{-1}\partial\bar{\partial}^{-1}f = -\frac{i}{\pi}e_{-k}\mathcal{F}(\partial\bar{\partial}^{-1}f)(k) = -\frac{i}{\pi}\frac{k}{\bar{k}}e_{-k}\mathcal{F}f(k). \tag{2.13}$$

Formula (1.6) now yields (2.11) and the differentiability of $g_k * f$ in $L^{\tilde{p}}_{-\beta}$. To prove differentiability in $W^{1,\tilde{p}}_{-\beta}$, we observe that $\bar{\partial}g_k * f = -(\partial+ik)^{-1}f/4$ so Lemma 2.2 can be used again; for the derivative with respect to z we rely on the identity $\partial(g_k * f) = -ik(g_k * f) - \bar{\partial}^{-1}f/4$. □

Proof of Theorem 2.1. We may assume $2/\tilde{p} < \beta < \rho - 2/p'$ (since the norm $\|\ \|_{W^{1,\tilde{p}}_{-\beta}}$ is decreasing in β) and choose α satisfying $2/p' < \alpha < \min(1, \rho-\beta)$. We rewrite (1.11) as $\langle x\rangle^{-\beta}\mu = \langle x\rangle^{-\beta} - K(k)(\langle\cdot\rangle^{-\beta}\mu)$, with $K(k)$ the operator $K(k)f = \langle x\rangle^{-\beta}g_k * (\langle\cdot\rangle^{\beta}qf)$. By our choice of α and β, we have $\langle x\rangle^{\beta}q(x) \in L^p_\alpha$, so multiplication by $\langle x\rangle^{\beta}q$ is a compact operator: $W^{1,\tilde{p}} \to L^p_\alpha$. It follows from this and Lemma 2.5 that $k \mapsto K(k)$ is differentiable on $\mathbb{C} \setminus 0$ in the uniform operator topology $W^{1,\tilde{p}} \to W^{1,\tilde{p}}$.

If $\tilde{h}$ is a solution in $W^{1,\tilde{p}}$ of $\tilde{h} = -K(k)\tilde{h}$, then $\langle x\rangle^{\beta}\tilde{h}$ satisfies $\langle x\rangle^{\beta}\tilde{h} = -g_k * (q\langle\cdot\rangle^{\beta}\tilde{h})$ so $\langle x\rangle^{\beta}\tilde{h}$ is in $W^{1,\tilde{p}}$ and hence vanishes, since k is assumed nonexceptional. Thus $I + K(k)$ is invertible on $W^{1,\tilde{p}}$ and

$$\frac{\partial}{\partial\bar{k}}(\langle x\rangle^{-\beta}\mu) = -(I+K(k))^{-1}\frac{\partial K}{\partial\bar{k}}\langle\cdot\rangle^{-\beta}\mu(\cdot,k). \tag{2.14}$$

Since $\partial K/\partial\bar{k}$ is an operator of rank 1, we only need to calculate the function $(I+K(k))^{-1}(e_{-k}(x)\langle x\rangle^{-\beta})$. Noting that $g_k * (e_{-k}f) = e_{-k}\overline{g_k * f}$ for f real-valued, we have

$$(I+K(k))^{-1}(e_{-k}\langle x\rangle^{-\beta}) = e_{-k}\overline{(I+K(k))^{-1}\langle\cdot\rangle^{-\beta}}. \tag{2.15}$$

Thus (0.12) holds in the $W^{1,\tilde{p}}_{-\beta}$ topology. □

3. Behavior near $k = 0$

We will need the following consequences of our assumptions on q.

LEMMA 3.1. *Let $q \in L^p_\rho(\mathbb{R}^2)$, $1 < p < 2$, $\rho > 1$. If $\langle x\rangle^{-a}\psi_0 \in L^\infty(\mathbb{R}^2)$ for some $a < \min(1,\rho-1)$ and $\Delta\psi_0 = q\psi_0$ (as distributions), then ψ_0 is continuous, there is a constant c_∞ such that ψ_0 satisfies*

$$\psi_0 = c_\infty - G_0 * (q\psi_0)\,, \tag{3.1}$$

with $G_0(x) =: -\frac{1}{2\pi}\log|x|$, and the following are equivalent:

(a) $\psi_0 \in L^\infty(\mathbb{R}^2)$,

(b) $\displaystyle\lim_{R\to\infty}\frac{1}{\pi R^2}\int_{|x|<R}\psi_0(x)\,dx$ *exists (and equals c_∞)*,

(c) $\displaystyle\int_{\mathbb{R}^2} q\psi_0 = 0$,

(d) $\psi_0 - c_\infty \in W^{1,\tilde{p}}(\mathbb{R}^2)$,

(e) $\nabla\psi_0 \in L^{\tilde{p}}_1(\mathbb{R}^2)$, *and*

(f) $\nabla\psi_0 \in L^p(\mathbb{R}^2)$.

We will use the following elementary estimate on G_0. (See [N III] for a proof.)

LEMMA 3.2. *If $f \in L^p_\rho(\mathbb{R}^2)$, $1 < p < 2$, $\rho > 1$, then:*

$$\text{(i)} \qquad \|G_0 * f + \frac{1}{2\pi}(\log|x|)\int_{\mathbb{R}^2} f\|_{L^{\tilde{p}}} \leqq c\|f\|_{L^p_\rho},$$

$$\text{(ii)} \qquad \|\nabla G_0 * f\|_{L^{\tilde{p}}} \leqq c\|f\|_{L^p}.$$

Proof of Lemma 3.1. The function $h = \psi_0 + G_0 * (q\psi_0)$ is harmonic and, by Lemma 3.2, we have (since $\rho - a > 1$)

$$h(x) = -\frac{1}{2\pi}\left(\int_{\mathbb{R}^2} q\psi_0\right)\log|x| \quad \mathrm{mod}(L^{\tilde{p}} + L^\infty_{-a}). \tag{3.2}$$

The standard argument using mean values over disks (see, e.g., [St-W, p. 42]) then shows that h is a constant, which we denote by c_∞. Part (ii) of Lemma 3.2 gives $\nabla\psi_0 \in L^{\tilde{p}}$, hence $\psi_0 \in W^{1,\tilde{p}}_{\mathrm{loc}}$ and continuity follows. Integrating h on the disk of radius R we find that, as $R \to \infty$,

$$c_\infty \pi R^2 = \int_{|x|<R} \psi_0(x)dx - \left(\frac{1}{2}R^2\log R - \frac{1}{4}R^2\right)\int_{\mathbb{R}^2} q\psi_0 + O(R^{2/\tilde{p}'}); \tag{3.3}$$

this shows that (b) and (c) are equivalent, and that (a) implies (c). By Lemma 3.2, (c) implies (d) which in turn implies (a) by Sobolev imbedding. Since $\bar{\partial}(\partial\psi_0) = q\psi_0/4$ and $\partial\psi_0 \in L^{\tilde{p}}$, if (c) holds then we can write (see formula (1.9)) $\partial\psi_0(z) = (\bar{\partial}^{-1}(wq(w)\psi_0(w))(z))/4z$, and similarly for $\bar{\partial}\psi_0$; thus (c) implies (e). To prove that (c) implies (f) we use Lemma 2.4: $wq\psi_0$ is in L^p, hence $\bar{\partial}^{-1}(wq\psi_0) \in L^p_{-1}$. If (e) holds, choose p_1 with $p < p_1 < 2$ and let $p_2 = (1/p_1 - 1/\tilde{p})^{-1}$; then $p_2 > 2$ and $\|\nabla\psi_0\|_{L^{p_1}} \leqq \|\langle x\rangle\nabla\psi_0\|_{L^{\tilde{p}}}\|\langle x\rangle^{-1}\|_{L^{p_2}} < \infty$. It follows that the Fourier transform $(\partial\psi_0)^\wedge(\xi)$ is in $L^{p_1'}$; from $(\partial\psi_0)^\wedge(\xi) = -i(q\psi_0)^\wedge(\xi)\big/(2(\xi_1 + i\xi_2))$, the continuity of $(q\psi_0)^\wedge(\xi)$ and $p_1' > 2$ we deduce $(q\psi_0)^\wedge(0) = 0$. The same argument shows that (f) implies (c). □

The main result of this section is the following:

THEOREM 3.3. *Suppose q in $L^p_\rho(\mathbb{R}^2)(1 < p < 2,\ \rho > 1)$ is of the form $q = (\Delta\psi_0)/\psi_0$ with $\psi_0 \in L^\infty(\mathbb{R}^2)$ and $\psi_0(x) \geqq c_0 > 0$. Then*

$$|\mathbf{t}(k)| \leqq c|k|^\varepsilon \tag{3.4}$$

for k close to zero, and (with c_∞ as in Lemma 3.1)

$$\|\mu(\cdot,k) - \psi_0/c_\infty\|_{W^{1,\tilde{p}}_{-\beta}} \leqq c|k|^\varepsilon, \tag{3.5}$$

for all ε, β satisfying

$$0 < \varepsilon < \min((\rho-1)/2, 2/p') \text{ and } \beta > 2/\tilde{p} + \varepsilon. \tag{3.6}$$

The proof will use the well-known device ([Si], [Che], [B-L-M-P], [T]) of separating from the operator of convolution by g_k a rank 1 piece responsible for its $\log k$ singularity at k equal zero. We define a modified Faddeev Green function $\tilde{g}_k(x) = g_k(x) + \ell(k)$, with $\ell(k)$ given by

$$\ell(k) = (\log|k| + \gamma_0)/2\pi \text{ for } |k| \leqq 1 \text{ and } \ell(k) = \gamma_0/2\pi \text{ for } |k| > 1, \tag{3.7}$$

where γ_0 denotes not a conductivity but the Euler constant!

LEMMA 3.4. *For any ε, $0 < \varepsilon < 1$, there is a constant C_ε such that for all $0 < |k| \leqq 1/2$ there is an inequality $|\tilde{g}_k(x) - G_0(x)| \leqq C_\varepsilon |k|^\varepsilon \langle x\rangle^\varepsilon$.*

Lemma 3.4 is proved in [N III] using the following identities:

$$\frac{1}{2\pi}\left\{\int_{|\xi|<2/|x|} \frac{e^{ix\cdot\xi}-1}{|\xi|^2}d\xi + \int_{|\xi|>2/|x|} \frac{e^{ix\cdot\xi}}{|\xi|^2}d\xi\right\} = -\gamma_0 \tag{3.8}$$

for any $x \in \mathbb{R}^2 \setminus 0$, while a contour integration yields, for any $k \in \mathbb{C} \setminus 0$,

$$\frac{1}{2\pi}\int_{|\xi|<2} \frac{d\xi}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} = -\log|k| \text{ if } |k| \leqq 1, \text{ and } = 0 \text{ if } |k| > 1. \tag{3.9}$$

Let $\tilde{K}(k)f = \langle x\rangle^{-\beta}\tilde{g}_k * (\langle\cdot\rangle^\beta qf)$ if $k \neq 0$, $\tilde{K}(0)f = \langle x\rangle^{-\beta}G_0 * (\langle\cdot\rangle^\beta qf)$.

LEMMA 3.5. *Let q, β, ε satisfy the hypotheses of Theorem* 3.3 *and in addition $\beta < \min(1, \rho - \varepsilon - 2/p')$. Then $\tilde{K}(k)$ is bounded on $W^{1,\tilde{p}}$ for all k,*

(i) $\|\tilde{K}(k) - \tilde{K}(0)\|_{W^{1,p}\to W^{1,\tilde{p}}} \leqq c|k|^\varepsilon$ *for* $|k| < 1/2$, *and*

(ii) $I + \tilde{K}(k)$ *is invertible on* $W^{1,\tilde{p}}$ *for* k *sufficiently small.*

Proof. Since $\langle x\rangle^{\beta+\varepsilon}q \in L^1 \cap L^p$, the boundedness of $\tilde{K}(0)$ on $W^{1,\tilde{p}}$ is elementary, and the boundedness of $K(k)$ on $W^{1,\tilde{p}}$ implies that of $\tilde{K}(k)$ for all $k \neq 0$. Majorizing $\langle x-y\rangle^\varepsilon$ by $\langle x\rangle^\varepsilon\langle y\rangle^\varepsilon$ we obtain (i) for the operator norm $W^{1,\tilde{p}} \to L^{\tilde{p}}$ using Lemma 3.4. From $\bar{\partial}((\tilde{g}_k - G_0) * (\cdot)) = -[(\partial + ik)^{-1} - \partial^{-1}]/4$, the strong differentiability of $(\partial + ik)^{-1}$ (Lemma 2.2) and the compactness of premultiplication by $\langle x\rangle^\beta q$ we obtain $\|\bar{\partial}(\tilde{K}(k) - \tilde{K}(0))\|_{W^{1,\tilde{p}}\to L^{\tilde{p}}} \leqq c|k|$.

(ii) Invertibility of $I + \tilde{K}(k)$ for small k will follow from (i) once we prove it for $I + \tilde{K}(0)$. So we assume $\tilde{h} \in W^{1,\tilde{p}}$ and $\tilde{h} = -\langle x\rangle^{-\beta}G_0 * (q(\langle\cdot\rangle^\beta\tilde{h}))$. Let $h = \langle x\rangle^\beta\tilde{h}$. Then $h = -G_0 * (qh)$. Differentiating and noting that $qh \in L^p$ we obtain $\nabla h \in L^{\tilde{p}}$. Proceeding as in the proof of Lemma 1.5, using Lemma 3.1(e) and (f), we show that h/ψ_0 is antiholomorphic; since $\beta < 1$, Liouville's theorem yields $h = c\psi_0$. Thus $c\psi_0 = -cG_0 * (q\psi_0) = c(\psi_0 - c_\infty)$, the latter by (3.1). Since ψ_0 is assumed bounded away from zero, c_∞ cannot be zero, so c, hence h, must vanish. □

Proof of Theorem 3.3. We may assume that, in addition to (3.6),

$$\beta < \min(1, \rho - \varepsilon - 2/p'). \tag{3.10}$$

Fix ε and β satisfying (3.6) and (3.10), and choose ε' such that

$$\varepsilon < \varepsilon' < \min((\rho - 1)/2, \beta - 2/\tilde{p}, \rho - \beta - 2/p'); \tag{3.11}$$

the hypotheses of Lemma 3.5 (as well as (3.10)) then hold for β and ε'.

For k close to zero we can now define a new family of solutions of the Schrödinger equation $\tilde{\mu}(x,k) = \langle x\rangle^{\beta}(I + \tilde{K}(k))^{-1}\langle\cdot\rangle^{-\beta}$. The uniqueness proved above shows that: $\langle x\rangle^{\beta}(I + \tilde{K}(0))^{-1}\langle\cdot\rangle^{-\beta} = \psi_0(x)/c_\infty$. Lemma 3.5 (i) then yields $\|\tilde{\mu}(\cdot,k) - \psi_0/c_\infty\|_{W^{1,\tilde{p}}_{-\beta}} \leqq c|k|^{\varepsilon'}$ for k small. The resolvent equation gives

$$(I+K(k))^{-1} = (I+\tilde{K}(k))^{-1} + (I+\tilde{K}(k))^{-1}[\tilde{K}(k) - K(k)](I+K(k))^{-1}. \tag{3.12}$$

The operator $\tilde{K}(k) - K(k)$ is of rank 1; applying (3.12) to $\langle x\rangle^{-\beta}$ we find

$$\mu(x,k) = (1 + \ell(k)\tau(k))\tilde{\mu}(x,k), \tag{3.13}$$

where

$$\tau(k) =: \int_{\mathbb{R}^2} q(x)\mu(x,k)\,dx. \tag{3.14}$$

We integrate (3.13) against q which yields $\tau(k) = (1 + \ell(k)\tau(k))\tilde{\tau}(k)$; thus we can solve for τ in terms of $\tilde{\tau}$ (the latter defined by (3.14) with $\tilde{\mu}$ in place of μ): $\tau(k) = \tilde{\tau}(k)/(1 - \ell(k)\tilde{\tau}(k))$. (A similar formula was obtained in [T].) Using the above small k estimate on $\tilde{\mu}$ and Lemma 3.1(c) we have $|\tilde{\tau}(k)| \leqq c|k|^{\varepsilon'}$. Substitution in the formula for τ yields, for k sufficiently small, $|\tau(k)| \leqq c|k|^{\varepsilon'}$. We use the latter (as well as the definition (3.7)) in (3.13) to bound the left side of (3.5) by const. $|k|^{\varepsilon'}|\log k| \leqq c|k|^{\varepsilon}$. We then return to formula (2.1) and use (3.5) and Lemma 3.1(c) to obtain (3.4). □

From the information we have obtained so far in the direct problem, the proof of Theorem 4 is now immediate:

Proof of Theorem 4. We write $\mathbf{t}(k) = \mathcal{F}q(k) + \mathcal{F}(q(\mu(\cdot,k) - 1))(k)$ and use (1.1) with $s > 2/\tilde{p}$, together with $q \in L^1 \cap L^p$. We thus find that $\mathbf{t}(k)/\bar{k}$ is in $L^r(|k| \geq k_0)$ for $r > \tilde{p}'$ and k_0 sufficiently large. Near $k = 0$, we have $|k|^{\varepsilon-1} \in L^r(|k| \leqq 1)$ if $r < 2/(1-\varepsilon)$ and from (3.6) we find (0.14). For k nonzero, $\mathbf{t}(k)$ is continuous: combine the differentiability of $\mu(\cdot,k)$ with the fact that $\langle x\rangle^{\beta}q \in L^1$. □

4. From t to γ

We now view $\mu(x,k)$ as pseudoanalytic functions of k parametrized by x. Given q as in Theorem 4, with $\psi_0 = \gamma^{1/2}$ normalized (see Lemma 3.1 (d)) so that $\psi_0 - 1 \in L^{\tilde{p}}(\mathbb{R}^2)$, we have constructed (Theorems 1.1, 2.1, 3.3) for every $x \in \mathbb{R}^2$ a function $\mu(x,\cdot)$ on $\mathbb{C}\setminus 0$ satisfying (0.12) pointwise, as well as

(i) $\mu(x,\cdot)-1\in L^r(\mathbb{C})$, $p'<r\leqq\infty$, and (ii) $\lim_{k\to 0}\mu(x,k)=\gamma^{1/2}(x)$. We will prove that, given $\mathbf{t}(k)$ as in Theorem 4, there is, for every x, a unique solution of (0.12) satisfying (i). In view of (ii), it will follow that if γ_1 and γ_2 have the same $\mathbf{t}$, then $\gamma_1\equiv\gamma_2$.

THEOREM 4.1. *Let $\mathbf{t}(k)$ be such that $\mathbf{t}(k)/\bar{k}\in L^r(\mathbb{C})$ for all $r\in(\tilde{p}',r_2)$, $r_2>2$. Fix $r_0>\max(p',2r_2/(r_2-2))$; for all $x\in\mathbb{R}^2$, there is a unique solution $\mu(x,\cdot)$ of (0.12) with $\mu(x,\cdot)-1\in L^{r_0}\cap L^\infty(\mathbb{C})$. Moreover, $\inf_{x,k}|\mu(x,k)|>0$,*

$$\sup_x\|\mu(x,\cdot)-1\|_{L^r}<\infty \quad \textit{for all } r\in(p',\infty], \textit{ and} \tag{4.1}$$

$$\sup_x|\mu(x,k_1)-\mu(x,k_2)|\leqq c|k_1-k_2|^\varepsilon \quad \textit{for } 0<\varepsilon<1-2/r_2. \tag{4.2}$$

If r_2 is as in (0.14), then $1-2/r_2=\min(2/p',(\rho-1)/2)$ and (4.2) gives the same range for ε as that obtained in (3.6)!

Proof of Theorem 4.1. Fix $x\in\mathbb{R}^2$ and $r_0>\max(p',2r_2/(r_2-2))$; let $t_x^\#(k)=\mathbf{t}(k)e_{-x}(k)\big/4\pi\bar{k}$. To show uniqueness, suppose ν is a solution of (0.12) with $\nu\in L^{r_0}\cap L^\infty(\mathbb{C})$. The function $w=\partial_{\bar{k}}^{-1}(t^\#\bar{\nu}/\nu)$ is bounded and continuous by Lemma 1.4; thus $\nu\exp(-w)$ is holomorphic in k except possibly at $k=0$, and in $L^{r_0}\cap L^\infty$. Therefore, $k=0$ is a removable singularity and $\nu\equiv 0$ by Liouville's theorem. For existence, we consider the integral equation (0.22). Since $t_x^\#$ is in $L^2(\mathbb{C})$, the operator $I-\partial_{\bar{k}}^{-1}(t_x^\#\,\bar{\cdot})$ is Fredholm of index zero on $L^{r_0}(\mathbb{C})$ (see Lemma 4.2 in [N III]); if $\nu\in L^{r_0}$ is in its null-space then $\nu=\partial_{\bar{k}}^{-1}(t^\#\bar{\nu})$. Since $t^\#\bar{\nu}\in L^{rr_0/(r+r_0)}$ for all $r\in(\tilde{p}',r_2)$, we can apply Lemma 1.4 to deduce that $\nu\in L^\infty(\mathbb{C})$ and conclude as above that ν must vanish. Therefore , we can define $\mu(x,\cdot)-1=[I-\partial_{\bar{k}}^{-1}(t_x^\#\,\bar{\cdot})]^{-1}(\partial_{\bar{k}}^{-1}t_x^\#)$, noting that $\partial_{\bar{k}}^{-1}(t_x^\#)\in L^{r_0}$. Now let

$$v(x,\cdot)=\partial_{\bar{k}}^{-1}[t_x^\#(\overline{\mu(x,\cdot)}-1)/(\mu(x,\cdot)-1)];$$

then $\|v(x,\cdot)\|_{L^\infty}$ is bounded independently of x, by (1.7), $\mu(x,\cdot)-1=e^{v(x,\cdot)}\partial_{\bar{k}}^{-1}(t_x^\#e^{-v(x,\cdot)})$, and (4.1) follows. Using the fact that $\mu(x,\cdot)$ is uniformly bounded in (0.22), and (1.8), we obtain the Hölder estimate (4.2). To show $|\mu|$ is bounded below, we let $\tilde{v}(x,\cdot)=\partial_{\bar{k}}^{-1}(t_x^\#\overline{\mu(x,\cdot)}/\mu(x,\cdot))$; then, by Lemma 1.4, $\tilde{v}(x,\cdot)$ is bounded independently of x and $\lim_{|k|\to\infty}\tilde{v}(x,k)=0$. The function $\mu\exp(-\tilde{v})$ is holomorphic for $k\neq 0$ and bounded on $\mathbb{C}$, hence entire. Using Lemma 1.4 once more, we find $\lim_{|k|\to\infty}\mu(x,k)\exp(-\tilde{v}(x,k))=1$; therefore

$$\mu(x,k)=e^{\tilde{v}(x,k)}, \text{ and} \tag{4.3}$$

$$|\mu(x,k)|\geqq\exp[-c(\|t^\#\|_{L^{r_1}}+\|t^\#\|_{L^{r_3}})] \quad \text{if } \tilde{p}'<r_1<2<r_3<r_2. \tag{4.4}$$

Note that (4.3) gives a formula for $\gamma^{1/2}(x)=\mu(x,0)$ in terms of $\mathbf{t}$ and the phase $\bar{\mu}/\mu$ of μ. □

Proof of Theorem 3. If $q \in L^p_\rho$ and ψ_0 are as in (a) then, by Lemma 3.1(f), we also have $\nabla\psi_0 \in L^p$ and there are no exceptional points, as shown in Lemma 1.5; moreover, (0.11) holds for $\varepsilon < \min((\rho-1)/2, 2/p')$, by Theorem 3.3. To prove $(b) \Rightarrow (a)$, assume $q \in L^p_\rho$ is real-valued and has no exceptional points k in $\mathbb{C}$. Then the proof of Theorem 1.1 shows that we can construct $\mu(\cdot, k)$, for all $k \in \mathbb{C} \setminus 0$, satisfying (1.1) and $(\Delta + 4ik\bar{\partial})\mu = q\mu$. Furthermore, by Theorem 2.1 we have the $\bar{\partial}$ equation (0.12) in $\mathbb{C}\setminus 0$, with $\mathbf{t}(k)$ continuous on $\mathbb{C}\setminus 0$. By combining the proof at large k of Theorem 4 with the assumption (0.11) we show that $\mathbf{t}(k)/\bar{k} \in L^r(\mathbb{C})$ for all $r \in (\tilde{p}', r_\varepsilon)$ where $r_\varepsilon = 2/(1-\varepsilon)$. Now we can use Theorem 4.1: define $\mu_0(x) = 1 + [\partial_{\bar{k}}^{-1}(t_x^{\#}\overline{\mu(x,\cdot)})]|_{k=0}$; then $\mu_0 \in L^\infty(\mathbb{R}^2)$, $\inf_x |\mu_0(x)| > 0$ and $\sup_x |\mu_0(x) - \mu(x,k)| \leqq c|k|^{\varepsilon'}$ for any $\varepsilon' < \varepsilon$. The latter implies that, as $k \to 0$, the distributions $(\Delta + 4ik\bar{\partial})\mu(\cdot, k)$ and $q\mu(\cdot, k)$ converge to $\Delta\mu_0$, respectively $q\mu_0$; thus $\Delta\mu_0 = q\mu_0$. Conditions (b)–(f) of Lemma 3.1 are therefore satisfied by μ_0, with $c_\infty \neq 0$. We define $\psi_0(x) = \mu_0(x)/c_\infty$.

We claim that ψ_0 is real valued. Indeed, let φ_0 be its imaginary part. Then φ_0 is a bounded solution of $(-\Delta + q)\varphi_0 = 0$ and thus satisfies (b)–(f) of Lemma 3.1; since the limit in (b) for ψ_0 equals 1, that for φ_0 is zero. Thus $\varphi_0 \in W^{1,\tilde{p}}$ (by (d)) and so must vanish, since $k = 0$ is, by assumption, not an exceptional point. It now follows that ψ_0 has a positive lower bound. □

5. Reconstruction at the boundary

LEMMA 5.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Assume γ is in $W^{1,r}(\Omega)$ for some $r > n$ and has a positive lower bound. For any $f \in H^{1/2}(\partial\Omega)$ and $h \in \overset{\circ}{H}{}^{-1/2}(\partial\Omega)$ there exists the identity:*

$$(5.1) \qquad \langle h, (\gamma - R\Lambda_\gamma) f\rangle = \int_\Omega u\nabla w \cdot \nabla\gamma,$$

where u is the $H^1(\Omega)$ solution of $\nabla \cdot (\gamma\nabla u) = 0$ in Ω with $u|_{\partial\Omega} = f$, $w \in H^1(\Omega)$ is a weak solution of $\Delta w = 0$ in Ω with $\frac{\partial w}{\partial \nu}|_{\partial\Omega} = h$, and R denotes the Neumann-to-Dirichlet map for the Laplacian.

Recall that the form $\langle , \rangle$ on $\partial\Omega$ is bilinear rather than sesquilinear. We will use the notation described in Theorem 6.

Proof of Lemma 5.1. Since w is a weak solution of the Neumann problem,

$$(5.2) \qquad \langle h, v|_{\partial\Omega}\rangle = \int_\Omega \nabla w \cdot \nabla v \text{ for any } v \in H^1(\Omega);$$

also $Rh = w|_{\partial\Omega}$ (modulo functions constant on each $\partial\Omega^j$) so, by (0.1),

$$(5.3) \qquad \langle Rh, \Lambda_\gamma f\rangle = \int_\Omega \gamma\nabla w \cdot \nabla u.$$

(The constants do not matter, since $\langle 1, \Lambda_\gamma f\rangle_{\partial\Omega^j} = 0$ for each j.) Multiplication by γ is a bounded operator on $H^1(\Omega)$ so we can apply (5.2) to $v = \gamma u$:

$$\langle h, \gamma f\rangle = \int_\Omega \gamma\nabla w \cdot \nabla u + u\nabla w \cdot \nabla\gamma. \tag{5.4}$$

Subtracting (5.3) from (5.4) and using the symmetry of R, we obtain (5.1) □

Note that, unlike (5.3) or (5.4), the right side of (5.1) involves no derivatives of u. For part (ii) of Theorem 6 we will need the following.

LEMMA 5.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Assume γ is in $W^{2,p}(\Omega)$ for some $p > \frac{n}{2}$ and has a positive lower bound. For any f,$g \in H^{1/2}(\partial\Omega)$ there exists the identity*

$$\left\langle g, \left(2\Lambda_\gamma - \Lambda_1\gamma - \gamma\Lambda_1 + \frac{\partial\gamma}{\partial\nu}\right) f\right\rangle = \int_\Omega 2v\nabla(u-u^0)\cdot\nabla\gamma + v(2u-u^0)\Delta\gamma, \tag{5.5}$$

where u, u^0, v are respectively the $H^1(\Omega)$ solutions of $\nabla\cdot(\gamma\nabla u) = 0$, $\Delta u^0 = 0$ and $\Delta v = 0$ in Ω, with $u|_{\partial\Omega} = u^0|_{\partial\Omega} = f$ and $v|_{\partial\Omega} = g$.

Proof. From (0.1) we have (since $\gamma v \in H^1(\Omega)$)

$$\langle g, \gamma\Lambda_1 f\rangle = \int_\Omega \nabla(\gamma v)\cdot\nabla u^0 \tag{5.6}$$

and, noting that $\gamma(2u - u_0)|_{\partial\Omega} = \gamma f$ and $\gamma(2u - u_0) \in H^1(\Omega)$:

$$\langle g, \Lambda_1\gamma f\rangle = \langle \Lambda_1 g, \gamma f\rangle = \int_\Omega \nabla v\cdot\nabla(\gamma(2u - u^0)). \tag{5.7}$$

If $\gamma \in W^{2,p}(\Omega), p > \frac{n}{2}$, then multiplication by $\frac{\partial\gamma}{\partial\nu}$ defines a bounded operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ satisfying

$$\left\langle g, \frac{\partial\gamma}{\partial\nu} f\right\rangle = \int_\Omega \nabla(uv)\cdot\nabla\gamma + uv\Delta\gamma, \tag{5.8}$$

for any u, v in $H^1(\Omega)$ with traces f, respectively g on $\partial\Omega$. Using (5.8) with $2u - u_0$ in place of u, together with (0.1), (5.6) and (5.7) yields (5.5). □

Proof of Theorem 6. (i) By Lemma 5.1 applied to h_η and $f_\eta = e^{i<\cdot,\eta>}f$,

$$|\langle h_\eta, (\gamma - R\Lambda_\gamma)f_\eta\rangle| \leqq \|u_\eta\|_{L^\infty(\Omega)}\|\nabla w_\eta\|_{L^2(\Omega)}\|\nabla\gamma\|_{L^2(\Omega)}, \tag{5.9}$$

where $\nabla\cdot(\gamma\nabla u_\eta) = 0$, $u_\eta|_{\partial\Omega} = f_\eta, \Delta w_\eta = 0$ and $\partial w_\eta/\partial\nu = h_\eta$. The weak solutions u_η satisfy $\|u_\eta\|_{L^\infty(\Omega)} \leqq c\|f\|_{L^\infty(\partial\Omega)}$ ([G-T, Thm. 8.16]). The operator taking h_η to the solution of the Neumann problem w_η is bounded from $L^2(\partial\Omega)$ to $H^{3/2}(\Omega)$ ([J-K]). The functions h_η converge weakly in $L^2(\partial\Omega)$ to zero by the Riemann-Lebesque lemma; the inclusion of $H^{3/2}(\Omega)$ in $H^1(\Omega)$ is compact, so the solutions w_η converge to zero in the $H^1(\Omega)$ norm. Thus

$$\lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle h_\eta, R\Lambda_\gamma f_\eta\rangle = \lim\langle h_\eta, \gamma f_\eta\rangle = \langle h, \gamma f\rangle. \tag{5.10}$$

(ii) Let $g_\eta = e^{-i<\cdot,\eta>}g$. As in part (i), we have $\|u_\eta\|_{L^\infty(\Omega)}$, $\|u^0_\eta\|_{L^\infty(\Omega)}$ and $\|v_\eta\|_{L^\infty(\Omega)}$ bounded uniformly in η. The function $u_\eta - u^0_\eta$ satisfies

$$\Delta(u_\eta - u^0_\eta) = -(\nabla\gamma/\gamma)\cdot\nabla u_\eta \text{ and } (u_\eta - u^0_\eta)|_{\partial\Omega} = 0; \tag{5.11}$$

as an operator from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$, Δ has a bounded inverse; hence

$$\|u_\eta - u^0_\eta\|_{H^1(\Omega)} \leqq c\left\|\frac{1}{\gamma}\nabla\gamma\cdot\nabla u_\eta\right\|_{H^{-1}(\Omega)} \leqq c\|u_\eta\|_{L^\infty(\Omega)} \leqq c\|f\|_{L^\infty(\partial\Omega)}. \tag{5.12}$$

Lemma 5.2 now yields

$$\begin{aligned}&\left|\left\langle g_\eta, \left(2\Lambda_\gamma - \Lambda_1\gamma - \gamma\Lambda_1 + \frac{\partial\gamma}{\partial\nu}\right) f_\eta\right\rangle\right| \\ &\qquad \leqq c\|f\|_{L^\infty(\partial\Omega)}\|\gamma\|_{W^{2,p}(\Omega)}(\|v_\eta\|_{L^{p_n}(\Omega)} + \|v_\eta\|_{L^{p'}(\Omega)})\end{aligned} \tag{5.13}$$

where p_n is the exponent defined by $1/p_n = 1/2 + 1/n - 1/p$. The solution operator for the Dirichlet problem is bounded from $L^2(\partial\Omega)$ to $H^{1/2}(\Omega)$ ([J-K], [F], [Ve]); the functions g_η converge weakly to zero in $L^2(\partial\Omega)$ so, by compactness, v_η converge to zero in the $L^2(\Omega)$ norm. Convergence in the $L^{p_n}(\Omega)$ and $L^{p'}(\Omega)$ norms (note that $p' \leqq p_n$) immediately follows if $p_n \leqq 2$, and also if $2 < p_n < \infty$ for in that case $\|v_\eta\|_{L^{p_n}(\Omega)} \leqq \|v_\eta\|^{(1-2/p_n)}_{L^\infty(\Omega)}\|v_\eta\|^{2/p_n}_{L^2(\Omega)}$. □

6. Reduction to the case $\gamma \equiv 1$ near the boundary

Having determined $\gamma|_{\partial\Omega}$ and $\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}$ we now choose a domain $\Omega_2 \supset \overline{\Omega}$ and extend γ to a conductivity in $W^{2,p}(\Omega_2)$ with $\gamma \equiv 1$ near $\partial\Omega_2$ and $\inf_{\overline{\Omega}_2}\gamma > 0$. Let Λ_2 be the corresponding Dirichlet-to-Neumann map on $\partial\Omega_2$. In this section we give a constructive way to determine Λ_2 from knowledge of Λ_γ on $\partial\Omega$.

It will be convenient here to denote by Ω_1, respectively Λ_1, the domain Ω and the Dirichlet-to-Neumann map Λ_γ given at the outset.

The Dirichlet-to-Neumann map corresponding to γ in the domain $\Omega_2\setminus\overline{\Omega}_1$ can be viewed as a 2×2 matrix of operators $\Lambda^{ij} : H^{1/2}(\partial\Omega_j) \to H^{-1/2}(\partial\Omega_i)$, $i, j = 1, 2$, defined as follows. Given $f_j \in H^{1/2}(\partial\Omega_j)$ for $j = 1, 2$, consider the two Dirichlet problems $\nabla\cdot(\gamma\nabla u_j) = 0$ in $\Omega_2\setminus\overline{\Omega}_1$, with boundary conditions $u_1|_{\partial\Omega_1} = f_1, u_1|_{\partial\Omega_2} = 0$, respectively $u_2|_{\partial\Omega_1} = 0, u_2|_{\partial\Omega_2} = f_2$. Then, for any $g_i \in H^{1/2}(\partial\Omega_i), i = 1, 2,$

$$\langle g_i, \Lambda^{ij}f_j\rangle_{\partial\Omega_i} =: (-1)^i\int_{\Omega_2\setminus\overline{\Omega}_1}\gamma\nabla v_i\cdot\nabla u_j, \tag{6.1}$$

where v_i are functions in $H^1(\Omega_2 \setminus \bar{\Omega}_1)$ with traces g_i on $\partial\Omega_i$ and zero (same as u_i) on the other boundary.

PROPOSITION 6.1. *Let $\overline{\Omega}_1 \subset \Omega_2$ be bounded Lipschitz domains in $\mathbb{R}^2$ and let $\gamma \in W^{2,p}(\Omega_2)$,$p > 1$, with $\gamma(x) \geqq c_0 > 0$. Then $\Lambda_1 - \Lambda^{11}$ is an invertible operator: $H^{1/2}(\partial\Omega_1) \to H^{-1/2}(\partial\Omega_1)$ and*

$$\Lambda_2 = \Lambda^{22} + \Lambda^{21}(\Lambda_1 - \Lambda^{11})^{-1}\Lambda^{12}. \tag{6.2}$$

The inverse of $\Lambda_1 - \Lambda^{11}$ turns out to be the single-layer operator on $\partial\Omega_1$ corresponding to the Green function for the Dirichlet problem in Ω_2; the identity (6.2) then follows from the definition (6.1). For complete details, we refer the reader to [N III].

7. From Λ to t

Proof of Theorem 5. Our starting point is Alessandrini's identity

$$\langle u_0|_{\partial\Omega}, (\Lambda_q - \Lambda_0)u|_{\partial\Omega}\rangle = \int_\Omega q u_0 u, \tag{7.1}$$

for any $H^1(\Omega)$ (weak) solutions u, u_0 of $(-\Delta + q)u = 0$ and $\Delta u_0 = 0$ in Ω; from definition (0.4) of the Dirichlet-to-Neumann map Λ_q, (7.1) is immediate.

To verify (0.18) we apply (7.1) to $u = \psi(\cdot, k)$ and $u_0 = \exp(i\bar{z}\bar{k})$, obtaining

$$\langle e^{i\bar{z}\bar{k}}, (\Lambda_q - \Lambda_0)\psi(\cdot, k)\rangle = \int_\Omega e^{i\bar{z}\bar{k}} q(x)\psi(x,k)\, dx = \mathbf{t}(k), \tag{7.2}$$

the latter since q vanishes outside Ω. We will need the following:

LEMMA 7.1. *Assume Ω is a bounded domain with a Lipschitz boundary. The single layer operator S_k corresponding to Faddeev's Green function is bounded: $H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)$ for $-1 \leqq s \leqq 0$.*

Proof. $G_k(x)$ differs from the classical $G_0(x)$ by a harmonic function, so it suffices to prove the lemma for the standard single layer operator S_0. For $s = 0$ this is a consequence (see, for instance, [Ve] or the appendix in [N I]) of the theorem of Coifman, McIntosh and Meyer. The statement for $s = -1$ is obtained by duality, and that for $-1 < s < 0$ by interpolation. □

We now return to the proof of Theorem 5. Let $x \notin \overline{\Omega}$. We apply (7.1) to $u_0(y) = G_k(x - y)$ and $u(y) = \psi(y, k)$ and find that, outside $\bar{\Omega}$,

$$(S_k(\Lambda_q - \Lambda_0)\psi(\cdot, k))(x) = \int_\Omega G_k(x-y)q(y)\psi(y,k)\, dy = e^{izk} - \psi(x,k). \tag{7.3}$$

The restriction of (7.3) to the boundary $\partial\Omega$ from outside yields (0.17). More generally, if $u = P_q f$ is the $H^1(\Omega)$ solution of $(-\Delta+q)u = 0$ in Ω with $u|_{\partial\Omega} = f$, we obtain, as above,

$$S_k(\Lambda_q - \Lambda_0)f(x) = \int_\Omega G_k(x-y)q(y)u(y)\,dy. \tag{7.4}$$

Choose p_1 such that $1 < p_1 < p$ and let r be defined by $1/r = 1/p_1 - 1/p$. Then $2 < r < \infty$ and by (7.4) we have the factorization $S_k(\Lambda_q - \Lambda_0) = \mathcal{R}\mathbf{G}_k\mathbf{q}\mathcal{I}P_q$, where $P_q : H^{1/2}(\partial\Omega) \to H^1(\Omega)$, $\mathcal{I}$ is the inclusion: $H^1(\Omega) \to L^r(\Omega)$, $\mathbf{q} : L^r(\Omega) \to L^{p_1}(\Omega)$ denotes the operator of multiplication by q, $\mathbf{G}_k : L^{p_1}(\Omega) \to W^{1,\tilde{p}_1}(\Omega)$ is convolution by G_k and $\mathcal{R} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator. The compactness of $\mathcal{I}$ therefore implies that of $S_k(\Lambda_q - \Lambda_0)$.

It remains to prove (iii). Assume k is an exceptional point; that is, there exists a nontrivial solution h of $(-\Delta + q)h = 0$ in $\mathbb{R}^2$ with he^{-ikz} in $W^{1,\tilde{p}}(\mathbb{R}^2)$. Then $h = -G_k * (qh)$ by Lemma 1.3, and the identity (7.1) applied to $h(y)$ and $G_k(x-y)$, with x first outside $\overline{\Omega}$ then approaching $\partial\Omega$ as above, shows

$$[I + S_k(\Lambda_q - \Lambda_0)]h|_{\partial\Omega} = 0. \tag{7.5}$$

Furthermore h will not vanish identically on $\partial\Omega$, for if it did, h would be an interior Dirichlet eigenfunction, hence zero in $\overline{\Omega}$ by hypothesis; that would leave h harmonic outside $\overline{\Omega}$, with $h|_{\partial\Omega} = 0$ and $\partial h/\partial\nu|_{\partial\Omega} = 0$; therefore $h \equiv 0$ everywhere, by unique continuation. Conversely, suppose a nonzero $h \in H^{1/2}(\partial\Omega)$ satisfies (7.5). Let $v(x) = -(S_k(\Lambda_q - \Lambda_0)h)(x)$ for x throughout $\mathbb{R}^2$. Then denoting by v_- (respectively v_+) the boundary values (as given by the trace operators) of v from inside (respectively outside) Ω we have, by (7.5), $v_- = v_+ = h$ on $\partial\Omega$ and, by the jump relations for the single layer potential (see [Ve]), $\frac{\partial v_+}{\partial\nu} - \frac{\partial v_-}{\partial\nu} = (\Lambda_q - \Lambda_0)h$. Furthermore, since v is harmonic inside Ω, we have $\frac{\partial v_-}{\partial\nu} = \Lambda_0 h$, which yields $\frac{\partial v_+}{\partial\nu} = \Lambda_q h$. Now define u in $\mathbb{R}^2$ by $u = P_q h$ in Ω, $u = v$ outside Ω. Then on $\partial\Omega$, $u_- = h = v_+ = u_+$ and $\frac{\partial u_-}{\partial\nu} = \Lambda_q h = \frac{\partial v_+}{\partial\nu} = \frac{\partial u_+}{\partial\nu}$, so u solves $(-\Delta + q)u = 0$ throughout $\mathbb{R}^2$. We claim that ue^{-izk} is in $W^{1,\tilde{p}}(\mathbb{R}^2)$. This will show that (7.5) has a solution only if k is exceptional. Let $\tilde{u} = -G_k * (qu)$. We know that $u|_\Omega$ is in $H^1(\Omega)$; it follows that $qu \in L^{p_1}$ for any p_1 with $1 < p_1 < p$ so that $e^{-izk}\tilde{u} \in W^{1,\tilde{p}_1}(\mathbb{R}^2)$. For $x \notin \overline{\Omega}$ we have, applying (7.1) once more, $-(S_k(\Lambda_q - \Lambda_0)h)(x) = \tilde{u}(x)$; the left side is, by definition, equal to $u(x)$ outside $\overline{\Omega}$. The function $u - \tilde{u}$ is thus zero outside $\overline{\Omega}$; it is also harmonic throughout $\mathbb{R}^2$. It follows that $u \equiv \tilde{u}$ everywhere, and $e^{-izk}u \in W^{1,\tilde{p}_1}(\mathbb{R}^2)$. In particular, u is bounded in Ω; returning to the definition of $\tilde{u}$ we obtain $e^{-izk}u \in W^{1,\tilde{p}}(\mathbb{R}^2)$, as claimed. □

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