Zadanie 1

b) $f_1(n) = 2^{100n}$

$$f_2(n) = \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{(n-1)n}{2}$$

$f_3(n) = n\sqrt{n}$

warunek wystarczający $\lim_{n\to\infty} \left| \frac{f(n)}{g(n)} \right| < \infty$

$$f_3(n) \underset{①}{\leq} f_2(n) \underset{②}{\leq} f_1(n)$$

$$① \lim_{n\to\infty} \frac{f_3(n)}{f_2(n)} = \lim_{n\to\infty} \frac{n\sqrt{n}\cdot 2}{(n-1)n} = \lim_{n\to\infty} \frac{2\sqrt{n}}{\sqrt{n}(\sqrt{n}-\underbrace{\tfrac{1}{\sqrt{n}}}_{\xrightarrow{n\to\infty} 0})} = 0 < \infty$$

$$② \lim_{n\to\infty} \frac{f_2(n)}{f_1(n)} = \lim_{n\to\infty} \frac{\frac{(n-1)n}{2}}{2^{100n}} = \lim_{n\to\infty} \frac{(n-1)n}{2\cdot 2^{100n}} = \lim_{n\to\infty} \frac{n^2-n}{2\cdot 2^{100n}} \overset{H}{=}$$

$$= \frac{1}{2}\lim_{n\to\infty} \frac{2n-1}{100\ln(2)\cdot 2^{100n}} \overset{H}{=} \frac{1}{2}\lim_{n\to\infty} \frac{2}{100\cdot\ln(2)\cdot\ln(2)\cdot 2^{100n}\cdot 100} =$$

$$= \lim_{n\to\infty} \frac{1}{10000\cdot(\ln(2))^2\cdot 2^{100n}} = 0$$

c) $f_1(n) = n^{\sqrt{n}}$

$f_2(n) = 2^n$

$f_3(n) = n^{10}\cdot 2^{\frac{n}{2}}$

$f_4(n) = \sum_{i=1}^{n}(i+1)$

$$f_4(n) \underset{①}{\leq} f_1(n) \underset{②}{\leq} f_3(n) \underset{③}{\leq} f_2(n)$$

wiemy, że $\sum_{i=1}^{n} 1 = n$ oraz $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

czyli $\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = \frac{n^2+n}{2} + \frac{2n}{2} = \frac{n^2+3n}{2}$

(1) $$\lim_{n\to\infty}\frac{f_4(n)}{f_1(n)} = \lim_{n\to\infty}\frac{\sum_{i=1}^{n}(i+1)}{n^{\sqrt{n}}} = \lim_{n\to\infty}\frac{\frac{n^2+3n}{2}}{n^{\sqrt{n}}} = \lim_{n\to\infty}\frac{n^2+3n}{2n^{\sqrt{n}}} =$$

$$= \lim_{n\to\infty}\left(\frac{n^2}{2n^{\sqrt{n}}}+\frac{3n}{2n^{\sqrt{n}}}\right) = \lim_{n\to\infty}\left(\underbrace{\frac{1}{2n^{\sqrt{n}-2}}}_{\downarrow n\to\infty \; 0}+\underbrace{\frac{3}{2n^{\sqrt{n}-1}}}_{\downarrow n\to\infty \; 0}\right) = 0$$

(2) $$\lim_{n\to\infty}\frac{f_1(n)}{f_3(n)} = \lim_{n\to\infty}\frac{n^{\sqrt{n}}}{n^{10}\cdot 2^{n/2}} = \lim_{n\to\infty}\frac{1}{n^{10-\sqrt{n}}\cdot\sqrt{2^n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} n^{10-\sqrt{n}}\cdot\sqrt{2^n}}$$

$$\lim_{n\to\infty} n^{10}\cdot\frac{\sqrt{2^n}}{n^{\sqrt{n}}} = \lim_{n\to\infty} n^{10}\cdot\lim_{n\to\infty}\sqrt{2^n\cdot n^{-2\sqrt{n}}} =$$

$$= \lim_{n\to\infty} n^{10}\cdot\lim_{n\to\infty}\sqrt{e^{\ln(2)n-2\sqrt{n}\ln(n)}} = \lim_{n\to\infty} n^{10}\cdot\lim_{n\to\infty}\sqrt{e^{n(\ln(2)-2\ln(n)\cdot n^{-1/2})}} =$$

$$= \lim_{n\to\infty} n^{10}\sqrt{e^{\lim_{n\to\infty} n(\ln 2-\underbrace{2\ln(n)n^{-1/2}}_{=\frac{\ln(n)}{\sqrt{n}}})}} = \quad , \text{ a } \lim_{n\to\infty}\frac{\ln(n)}{\sqrt{n}} = 0$$

$$= \lim_{n\to\infty} n^{10}\sqrt{e^{\lim_{n\to\infty} n\ln 2}} = \lim_{n\to\infty} n^{10}\cdot\sqrt{e^{\infty}} = \infty$$

czyli $$\frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} n^{10-\sqrt{n}}\cdot\sqrt{2^n}} \cdot \left[\frac{1}{\infty}\right] = 0$$

(3) $$\lim_{n\to\infty}\frac{f_3(n)}{f_2(n)} = \lim_{n\to\infty}\frac{n^{10}\cdot 2^{\frac{n}{2}}}{2^n} = \lim_{n\to\infty}\frac{n^{10}\cdot\cancel{\sqrt{2^n}}}{\sqrt{2^n}\cdot\cancel{\sqrt{2^n}}} \overset{H}{=} \lim_{n\to\infty}\frac{10n^9}{\frac{1}{2\sqrt{2^n}}\cdot\ln 2\cdot 2^n} =$$

$$= \lim_{n\to\infty}\frac{10n^9}{\ln 2\cdot 2^{\frac{1}{2}n-1}} \overset{H}{=} \lim_{n\to\infty}\frac{90n^8}{(\ln 2)^2\cdot 2^{\frac{1}{2}n-1}\cdot\frac{1}{2}} = \lim_{n\to\infty}\frac{90n^8}{(\ln 2)^2\cdot 2^{\frac{1}{2}n-2}} \overset{H}{=}$$

$$= \lim_{n\to\infty}\frac{720n^7}{(\ln 2)^3\cdot 2^{\frac{1}{2}n-3}} = \dots = 0$$

postępujemy analogicznie aż do otrzymania liczby w liczniku i wtedy zauważamy, że ta granica wynosi 0