STA 250/MTH 342 Intro to Mathematical Statistics
Assignment 6, Model Solutions

Solution 1: Exercise 8.7.6 (page 513)

We know that $n\hat{\sigma}_0^2/\sigma^2$ has a chi-square distribution with $n-1$ degrees of freedom. Therefore

$$\mathrm{Var}\left(\frac{n\hat{\sigma}_0^2}{\sigma^2}\middle|\,\mu,\sigma\right) = 2(n-1).$$

This implies

$$\mathrm{Var}\left(\hat{\sigma}_0^2\middle|\,\mu,\sigma\right) = \frac{2(n-1)\sigma^4}{n^2}, \quad \mathrm{Var}\left(\hat{\sigma}_1^2\middle|\,\mu,\sigma\right) = \mathrm{Var}\left(\frac{n}{n-1}\hat{\sigma}_0^2\middle|\,\mu,\sigma\right) = \frac{2\sigma^4}{n-1}.$$

We know that $\hat{\sigma}_1^2$ is an unbiased estimator for σ^2, so

$$\begin{aligned}
\mathsf{MSE}_{\hat{\sigma}_1^2}(\mu,\sigma^2) &= \mathrm{Var}\left(\hat{\sigma}_1^2\middle|\,\mu,\sigma\right) = \frac{2\sigma^4}{n-1}, \\
\mathsf{MSE}_{\hat{\sigma}_0^2}(\mu,\sigma^2) &= \mathrm{Var}\left(\hat{\sigma}_0^2\middle|\,\mu,\sigma\right) + \left(E(\hat{\sigma}_0^2) - \sigma^2\right)^2 = \frac{2(n-1)\sigma^4}{n^2} + \frac{\sigma^4}{n^2} = \frac{\sigma^4(2n-1)}{n^2}.
\end{aligned}$$

Since

$$\frac{2\sigma^4}{n-1} - \frac{(2n-1)\sigma^4}{n^2} = \frac{(3n-1)\sigma^4}{n^2(n-1)},$$

we see that for any $n \geq 2$, $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, $\mathsf{MSE}_{\hat{\sigma}_1^2}(\mu,\sigma^2) > \mathsf{MSE}_{\hat{\sigma}_0^2}(\mu,\sigma^2)$.

Solution 2: Exercise 8.2.6 (page 472)

When $t = 2$, X, Y, and Z have the same variance $2\sigma^2$. Since they all have mean zero, $X/\sqrt{2\sigma^2}$, $Y/\sqrt{2\sigma^2}$, and $Z/\sqrt{2\sigma^2}$ all have a standard normal distribution. Since they are independent, the sum of their squares, $V = (X^2+Y^2+Z^2)/(2\sigma^2)$ has a χ^2 distribution with three degrees of freedom. Therefore

$$\Pr(X^2+Y^2+Z^2 \leq 16\sigma^2) = \Pr(V \leq 8).$$

One finds on the table of the χ^2 distribution at the end of the textbook that, this probability is slightly greater than 0.95. Alternatively, one may use **R** to get an accurate value.

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> pchisq(8, df = 3)
[1] 0.9539883
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Solution 3: Exercise 8.3.6 (page 479) 10 points, 5 points each

(a)

Since $(X_i - \mu)/\sigma$ has a standard normal distribution for $i = 1, \cdots, n$, then $W = \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^2}$ has the χ^2 distribution with n degrees of freedom. The required probability can be rewritten as follows:

$$Pr(\frac{n}{2} \leq W \leq 2n)$$

Thus, when $n = 16$, we must evaluate $Pr(8 \leq W \leq 32) = Pr(W \leq 32) - Pr(W \leq 8)$, where W has the χ^2 distribution with 16 degrees of freedom. It is found from the table at the end of the book that $Pr(W \leq 32) = 0.99$ and $Pr(W \leq 8) = 0.05$.
Therefore, $Pr(8 \leq W \leq 32) = Pr(W \leq 32) - Pr(W \leq 8) = 0.99 - 0.05 = 0.94$.
(b)
By Theorem 8.3.1, $V = \frac{\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}{\sigma^2}$ has the χ^2 distribution with $n-1$ degrees of freedom. The required probability can be rewritten as follows:

$$Pr(\frac{n}{2} \leq V \leq sn)$$

Thus, when $n = 16$, we must evaluate $Pr(8 \leq W \leq 32) = Pr(W \leq 32) - Pr(W \leq 8)$, where V has the χ^2 distribution with 15 degree of freedom. It is found from the table that $Pr(V \leq 32) = 0.993$ and $Pr(V \leq 8) = 0.079$.
Therefore, $Pr(8 \leq W \leq 32) = Pr(W \leq 32) - Pr(W \leq 8) = 0.993 - 0.079 = 0.914$.

Solution 4: Problem 4.
(a). Recall the density function of the Gamma(a,b) distribution,

$$f(x) = \begin{cases} \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, & \\ 0, & \text{when } x \leq 0. \end{cases}$$

We have that when $z \leq 0$, $f_Z(z) = 0$, and when $z > 0$, We also have $a_1 = 1$, $b_1 = 2$,$a_2 = 3$, $b_2 = 2$

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,\mathrm{d}y \\ &= \int_0^z \frac{b_1^{a_1}}{\Gamma(a_1)} (z-y)^{a_1-1} e^{-b_1(z-y)} \frac{b_2^{a_2}}{\Gamma(a_2)} y^{a_2-1} e^{-b_2 y} \mathrm{d}y \\ &= 8\mathrm{e}^{-2z} \int_0^z y^2\,\mathrm{d}y \\ &= \frac{8}{3}\mathrm{e}^{-2z} y^3. \end{aligned} \tag{1}$$

This is a gamma distribution $Gamma(4, 2)$.
(b).
We have $E(Z) = \frac{4}{2} = 2$

$$E[Z] = E[X] + E[Y] = \frac{1}{2} + \frac{3}{2} = 2.$$

(c).

Denote f_W the pdf of W. Since neither gamma random variables nor uniform random variables have chance to take negative values, when $w < 0$, one has $f_W(w) = 0$.

Recall that U takes values only from $[0, 1]$ and X takes values only from $[0, \infty)$. When $0 \le w \le 1$, one solves the restrictions $0 \le w - x \le 1$ and $x \ge 0$ for x, to give $x \in [0, w]$.

$$f_W(w) = \int_{-\infty}^{\infty} f_U(w-x) f_X(x)\,\mathrm{d}x = \int_0^w 2(w-x)2\mathrm{e}^{-x}\,\mathrm{d}x = 2w + \mathrm{e}^{-2w} - 1.$$

When $w > 1$, one solves the restrictions $0 \le w - x \le 1$ and $x \ge 0$ for x, to give $x \in [w-1, w]$.

$$f_W(w) = \int_{-\infty}^{\infty} f_U(w-x) f_X(x)\,\mathrm{d}x = \int_{w-1}^w 2(w-x)2\mathrm{e}^{-x}\,\mathrm{d}x = \mathrm{e}^{-2w}(\mathrm{e}^2 + 1).$$

One may write f_W as a piecewise function

$$f_W(w) = \begin{cases} 0, & \text{if } w \in (-\infty, 0); \\ 2w + \mathrm{e}^{-2w} - 1, & \text{if } w \in [0, 1]; \\ \mathrm{e}^{-2w}(\mathrm{e}^2 + 1), & \text{if } w \in (1, \infty). \end{cases} \tag{2}$$

(d). Since U and X are independent, one has

$$\mathrm{Var}(W) = \mathrm{Var}(U) + \mathrm{Var}(X) = \frac{1}{18} + \frac{1}{4} = \frac{11}{36}.$$

Solution 5:

(a). For a random variable X which has a Poisson distribution $\mathrm{Poi}(\lambda)$, one takes it as the sum of n independent random variables $X_1, \cdots, X_\lambda$ which all have the distribution $\mathrm{Poi}(1)$. One has

$$E(X_i) = 1, \quad \text{and} \quad \mathrm{Var}(X_i) = 1.$$

According to the central limit theorem, if λ is large,

$$X = \sum_{i=1}^{n} X_i \qquad \text{has approximately a distribution } \mathcal{N}(\lambda, \lambda).$$

(b).

If $\lambda = 100$, X has approximately a distribution $\mathcal{N}(100, 100)$, so $\frac{X-100}{10}$ has approximately the distribution $\mathcal{N}(0, 1)$. Therefore

$$\Pr(X \le 90) = \Pr\left(\frac{X - 100}{10} \le -1\right) = \Phi(-1) \approx 0.159.$$

Solution 6:

(a). For a random variable X which has a Gamma distribution Gamma(m, β), one takes it as the sum of m independent random variables $X_1, \cdots, X_m$ which all have the distribution Gamma$(1, \beta)$. One has

$$E(X_i) = \frac{1}{\beta}, \quad \text{and} \quad \mathrm{Var}(X_i) = \frac{1}{\beta^2} < \infty.$$

According to the central limit theorem, if n is large,

$$X = \sum_{i=1}^{m} X_i \qquad \text{has approximately a distribution } \mathcal{N}\left(\frac{m}{\beta}, \frac{m}{\beta^2}\right).$$

(b). If $m = 50$ and $\beta = 4$, X has approximately a distribution $\mathcal{N}(12.5, 3.125)$, so $\frac{X-20}{2}$ has approximately the distribution $\mathcal{N}(0, 1)$. Therefore

$$\Pr(8 \le X \le 10) = \Pr\left(\frac{8 - 12.5}{\sqrt{3.125}} \le \frac{X - 12.5}{\sqrt{3.125}} \le \frac{10 - 12.5}{\sqrt{3.125}}\right) \approx \Phi(-1.4142) - \Phi(-2.545584) \approx 0.073196.$$

Solution 7:

(a). $\widehat{\beta} = \alpha/\bar{X}$ is the MLE for β.

(b). One has for a single observation X,

$$I(p) = -E_p\left(\frac{\mathrm{d}^2}{\mathrm{d}p^2} \log f(X|p)\right) = -E_p\left(\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\alpha}{\beta} - X\right)\right) = -E_p\left(-\frac{\alpha}{\beta^2}\right) = \frac{\alpha}{\beta^2}.$$

(c). As an MLE, $\widehat{\beta}$ has approximately the normal distribution

$$\mathcal{N}\left(\beta, \frac{\beta^2}{\alpha n}\right).$$

(d). We see that when $n = 40$, $\alpha = 5$, $\beta = 2$, approximately,

$$\frac{(\widehat{\beta} - 2)}{\sqrt{0.02}} \sim \mathcal{N}(0, 1).$$

Therefore

$$\Pr(|\widehat{\beta} - 2| < 0.1) = \Pr\left(\left|\frac{(\widehat{\beta} - 2)}{\sqrt{0.02}}\right| < \frac{0.1}{\sqrt{0.02}}\right) = 2\Phi(0.707) - 1.$$

The normal distribution table reads $\Phi(0.88) = 0.8106$, so

$$\Pr(|\widehat{\beta} - 2| < 0.1) \approx 2 \times 0.7602168 - 1 = 0.5204335.$$

Alternatively, one finds the value with **R**.

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> 2 * pnorm(0.707) - 1
[1] 0.5204335
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(e). According to the invariance of MLE, the MLE for θ is $\widehat{\theta} = (\frac{\alpha}{\overline{X}})^2$.

(f). According to the equation obtained in Question 4, one has

$$I(\theta) = \left(\frac{\mathrm{d}\beta}{\mathrm{d}\theta}\right)^2 I(\beta) = \frac{\alpha}{\beta^2}/(2\beta)^2 = \alpha/(4\theta^2).$$

(g). The approximate sampling distribution of $\widehat{\theta}$, as the sample size goes to infinity, is

$$\mathcal{N}\left(\theta, \frac{4\theta^2}{n\alpha}\right).$$

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