STA 250/MTH 342 Intro to Mathematical Statistics

Assignment 5, Model Solutions

Solution 1: Exercise 7.5.6 (page 425)

The likelihood function is

$$L(\sigma^2) = f_n(\mathbf{x}|\mu, \sigma^2) = \prod_{i=1}^{n} f(x_i|\mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\}.$$

One has

$$\log L(\sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2.$$

For simplicity, let $\phi = \sigma^2$.

$$\log L(\phi) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\phi - \frac{1}{2\phi}\sum_{i=1}^{n}(x_i - \mu)^2.$$

One takes the derivative to give

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\log L(\phi) = -\frac{n}{2\phi} + \frac{1}{2\phi^2}\sum_{i=1}^{n}(x_i - \mu)^2.$$

Let $\frac{\mathrm{d}}{\mathrm{d}\phi}\log L(\phi)\Big|_{\phi=\widehat{\phi}} = 0$ to give

$$\widehat{\phi} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2.$$

To verify that this is indeed the MLE, one approach is to check the second-order derivative,

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\log L(\phi)\Big|_{\phi=\widehat{\phi}} = \frac{n}{2\widehat{\phi}^2} - \frac{1}{\widehat{\phi}^3}\sum_{i=1}^{n}(x_i - \mu)^2 = \frac{n}{2\widehat{\phi}^2} - \frac{n\widehat{\phi}}{\widehat{\phi}^3} = -\frac{n}{2\widehat{\phi}^2} < 0.$$

This shows that $\hat{\phi}$ is a local maximizer of $\hat{\phi}$.

An even better approach can show that $\hat{\phi}$ is in fact the global maximer. Specifically, note that

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\log L(\phi) = -\frac{n}{2\phi} + \frac{1}{2\phi^2}\sum_{i=1}^{n}(x_i - \mu)^2 = -\frac{n}{2\phi^2}(\phi - \hat{\phi}).$$

Hence $\frac{\mathrm{d}}{\mathrm{d}\phi}\log L(\phi) < 0$ when $\phi > \hat{\phi}$ and $\frac{\mathrm{d}}{\mathrm{d}\phi}\log L(\phi) > 0$ when $\phi < \hat{\phi}$, implying that $\log L(\phi)$ is an increasing function on $\phi < \hat{\phi}$ and a decreasing function on $\phi > \hat{\phi}$. Therefore, $\hat{\phi}$ is the unique global maximizer of $\log L(\phi)$.

Solution 2: Exercise 7.5.8 (page 425)

(a). The likelihood function is

$$L(\theta) = f_n(\mathbf{x}|\theta) = \begin{cases} \exp\left(n\theta - \sum_{i=1}^{n} x_i\right), & \text{when } \theta < \min\{x_i\}; \\ 0, & \text{when } \theta \geq \min\{x_i\}. \end{cases}$$

We see that as θ increases from below towards $\min\{x_i\}$, $L(\theta)$ is increasing, while $L(\min\{x_i\}) = 0$, so the MLE of θ does not exist.

(b). The p.d.f. could be modified as

$$f(x|\theta) = \begin{cases} e^{\theta - x}, & \text{for } x \geq \theta; \\ 0, & \text{for } x < \theta. \end{cases}$$

Then the corresponding likelihood function becomes

$$L(\theta) = \begin{cases} \exp\left(n\theta - \sum_{i=1}^{n} x_i\right), & \text{when } \theta \leq \min\{x_i\}; \\ 0, & \text{when } \theta > \min\{x_i\}. \end{cases}$$

We see that L is now maximized at

$$\widehat{\theta} = \min\{x_i\}.$$

So now $\widehat{\theta} = \min\{X_i\}$ is the MLE.

Solution 3: Exercise 7.5.10 (page 425)

Given the p.d.f,

$$f(x|\theta) = \frac{1}{2} e^{-|x-\theta|} \text{ for } -\infty < x < \infty$$

The Likelihood function can be computed as:

$$L(x|\theta) = \prod_{i=1}^{n} \frac{1}{2} e^{-|x_i - \theta|} = \frac{1}{2^n} exp\left(-\sum_{i=1}^{n} |x_i - \theta|\right)$$

We cannot use the regular methods for getting the M.L.E. After the log transformation and the first order derivative, we cannot find the value of θ that maximize the likelihood function. However, according to the likelihood function itself, the M.L.E. of θ will be the value that minimizes $\sum_{i=1}^{n} |x_i - \theta|$. Median of $X_1, X_2, \cdots, X_n$ can minimize $\sum_{i=1}^{n} |x_i - \theta|$. The proof is below:

$$f(\theta) = \sum_{i=1}^{n} |x_i - \theta| \tag{1}$$

$$f`(\theta) = \sum_{i=1}^{n} \text{sign } |x_i - \theta| \text{ where sign } (x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

$$f`(\theta) = \text{number of } x_i \text{ smaller than } \theta - \text{number of } x_i \text{ bigger than } \theta \tag{2}$$

Therefore, median of $X_1, X_2, \cdots, X_n$ can minimize $\sum_{i=1}^{n} |x_i - \theta|$.

Solution 4: Exercise 7.5.12 (page 426)

The likelihood function is

$$L(\theta_1, \cdots, \theta_k) = \frac{n!}{n_1! \cdots n_k!} \theta_1^{n_1} \cdots \theta_k^{n_k}.$$

Since $\theta_1 + \cdots + \theta_k = 1$, we change variable as $\theta_i = u_i$ for $i = 1, \cdots, k-1$, and $\theta_k = 1 - u_1 - \cdots - u_{k-1}$. We have

$$\frac{\partial}{\partial u_i} \log L(\theta_1, \cdots, \theta_k) = \frac{\partial}{\partial \theta_i} \log L + \frac{\partial \log L}{\partial \theta_k} \frac{\partial \theta_k}{\partial u_i} = \frac{n_i}{\theta_i} - \frac{n_k}{\theta_k}.$$

Let $\frac{\partial}{\partial u_i} \log L = 0$ for all $i = 1, \cdots, k-1$ to give

$$\frac{n_1}{\theta_1} = \cdots = \frac{n_k}{\theta_k}.$$

Let $C = \frac{n_i}{\theta_i}$. We have

$$C = \sum_{i=1}^{k} \theta_i C = \sum_{i=1}^{k} n_i = n,$$

so

$$\widehat{\theta}_i = \frac{n_i}{n}.$$

(Not required.) To verify that this is the MLE, we simply need to verify that the $(k-1) \times (k-1)$ **Hessian matrix** H **is negative semi-definite**. Here, the (i, j)-entry of H is

$$H_{i,j} = \begin{cases} -\frac{n_i}{u_i^2} - \frac{n_k}{\theta_k^2}, & i = j; \\ -\frac{n_k}{\theta_k^2}, & i \neq j. \end{cases}$$

So for any $z = (z_i, \cdots, z_{k-1})^T \in \mathbb{R}^{k-1}$,

$$z^T H z = -\sum_{i=1}^{k-1} \frac{n_i}{u_i^2} z_i^2 - \frac{n_k}{\theta_k^2} \left(\sum_{i=1}^{k-1} z_i \right)^2 \leq 0.$$

Solution 5: Exercise 7.6.4 (page 441)

The probability that a given lamp will fail in a period of T hours is:

$$\begin{aligned} F(y < T) &= \int_0^T \beta e^{-\beta y} dy \\ &= -e^{-\beta y}|_0^T \\ &= 1 - e^{-\beta T} \end{aligned} \tag{3}$$

Therefore, $\beta = -\log(1-p)/T$.

The probability that exactly x lamps will fail is: $\binom{n}{x} p^x (1-p)^{n-x}$.

The MLE of this binomial distribution is:

$$\hat{p} = \frac{X}{n} = \frac{\sum_{i=0}^{n} x_i}{n}$$

Therefore, $\hat{\beta} = -\log(1 - \frac{\sum_{i=0}^{n} x_i}{n})/T$

MLE for Binomial Distribution

$$\begin{aligned} f_n(x|\theta) &= \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, x = \sum_{i=1}^{n} x_i \\ L(p; x) &= k + x \log p + (n-x) \log(1-p) \\ dL(p; x)/dp &= x/p - (n-x)/(1-p) = 0 \\ \hat{p} &= x/n, x = \sum_{i=1}^{n} x_i \end{aligned} \tag{4}$$

Check the second order derivative of the likelihood function to make sure that the $\hat{p}$ is the maximal solution instead of the minimal:

$$(dL(p; x)/dp)^2 = -xp^{-2} - (n-x)(1-p)^{-2} = -x\hat{p}^{-2} - (n-x)(1-\hat{p})^{-2} < 0$$

Therefore, $\hat{p}_{MLE} = \frac{X}{n} = \frac{\sum_{i=0}^{n} x_i}{n}$

Solution 6: Exercise 7.6.6 (page 441)

The distribution of $Z = (X - \mu)/\sigma$ will be a standard normal distribution. Therefore,

$$0.95 = Pr(X < \theta) = Pr(Z < \frac{\theta - \mu}{\sigma}) = \Phi(\frac{\theta - \mu}{\sigma})$$

Hence, from a table of the values of Φ it is found that $(\theta - \mu)/\sigma = 1.645$, then $\theta = \mu + 1.645\sigma$.

The MLE of normal distribution of unknown mean and variance are:

$$\hat{\mu} = \bar{X}_n, \hat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2)$$

Then, the MLE of the distribution of point θ is:

$$\hat{\theta} = \hat{\mu} + 1.645\hat{\sigma} = \bar{X}_n + 1.645\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

MLE for Normal Distribution with Unknown Means, or with Unknown Mean and Variance are explained very detailed in Example 7.5.5 and Example 7.5.6 of the Text.

Solution 7: Exercise 7.6.8 (page 441)

The likelihood function of α is

$$L(\alpha) = f_n(\mathbf{x}|\alpha) = \frac{1}{\Gamma(\alpha)^n}\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \exp\left(-\sum_{i=1}^{n} x_i\right).$$

So let $\frac{\mathrm{d}}{\mathrm{d}\alpha}\log L(\alpha)\Big|_{\alpha=\widehat{\alpha}} = 0$, and we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}\alpha}\log L(\alpha)\bigg|_{\alpha=\widehat{\alpha}} = -n\frac{\mathrm{d}}{\mathrm{d}\alpha}\log\Gamma(\alpha)\bigg|_{\alpha=\widehat{\alpha}} + \sum_{i=1}^{n}\log x_i = -n\frac{\Gamma'(\widehat{\alpha})}{\Gamma(\widehat{\alpha})} + \sum_{i=1}^{n}\log x_i.$$

To verify that $\hat{\alpha}$ is the MLE, note that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\log L(\alpha) = -n\left[\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(\hat{\alpha})}{\Gamma(\hat{\alpha})}\right]$$

which is > 0 on $\alpha < \hat{\alpha}$ and < 0 on $\alpha > \hat{\alpha}$. Thus $\hat{\alpha}$ is the unique global maximizer of $\log Ł(\alpha)$.

Now because MLE is invariant and because $\Gamma'(\alpha)/\Gamma(\alpha)$ is a one-to-one mapping on $\alpha \in (0, \infty)$, we have

$$\widehat{\left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right)} = \frac{\Gamma'(\widehat{\alpha})}{\Gamma(\widehat{\alpha})} = \frac{1}{n}\sum_{i=1}^{n}\log x_i.$$

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