Physics 464: Problem Set 11

1. Corrections to hydrogen energy levels. The gross (unperturbed) Hamiltonian has energy eigenvalues $E_n^{(0)}$ and energy eigenvectors $\psi^{(0)}_{nlm_lm_s}(r,\theta,\phi)$.

   (a) Evaluate the unperturbed energy levels $E_n^{(0)}$, $n = 1$, 2, and 3. Sketch and label them.

   (b) The so-called "Darwin" correction has no classical analog. It arises from a term in the Hamiltonian

   $$\mathcal{H}_D^{(1)} = \frac{Z\pi e^2\hbar^2}{2m_e^2c^2}\delta(\hat{\vec{R}}) \quad ,$$

   where $\delta(\hat{\vec{R}})$ is a the three-dimensional delta function and Z is the charge of the nucleus. Evaluate the matrix elements

   $$E_D^{(1)} = \langle\psi^{(0)}_{nlm_lm_s}|\mathcal{H}_D^{(1)}|\psi^{(0)}_{nlm_lm_s}\rangle$$

   and express the results in terms of $E_n^{(0)}$. For what values of n, l, and m_l is this matrix element nonzero?

   (c) As we discussed in class, there is also a relativistic correction to the gross Hamiltonian from the term

   $$\mathcal{H}_{mv}^{(1)} = -\frac{1}{8}\frac{p^4}{m^3c^2} \quad .$$

   The matrix elements are

   $$E_{mv}^{(1)} = -E_n^{(0)}\frac{Z^2\alpha^2}{n^2}\left(\frac{3}{4} - \frac{n}{l+1/2}\right) \quad .$$

   Evaluate these terms for the n=1, 2, and 3 levels.

   (d) The spin-orbit correction reduces to

   $$E_{s-o}^{(0)} = -E_n^{(0)}\frac{Z^2\alpha^2}{\hbar^2 n}\frac{\langle\vec{S}\cdot\vec{L}\rangle}{l(l+1/2)(l+1)} \quad .$$

   Evaluate the matrix elements

   $$\langle lsjm_j|\vec{S}\cdot\vec{L}|lsjm_j\rangle$$

   for $n = 1$, 2, and 3, where j is the quantum number of $\vec{J}^2$, and $\vec{J} = \vec{L} + \vec{S}$.

   (e) Show that the combined effects of the relativistic correction, the spin-orbit correction, and the Darwin term can be collected to yield

   $$E_{n,j}^{(1)} = \alpha^2\frac{m}{2\hbar^2}\frac{Z^4e^4}{n^4}\left(\frac{3}{4} - \frac{n}{j+1/2}\right) \quad .$$

   Why can you collect the terms together when you have apparently changed bases?

   (f) Finally, find the energy level shifts of n=1, 2, and 3 levels from the combined effects of the three $\mathcal{O}(\alpha^2)$ corrections and sketch them compared to the unperturbed energy levels.

2. The Zeeman Effect: Connecting Weak to Strong Field Regions.

   Consider a hydrogen atom in a uniform external magnetic field $\mathbf{B} = B_z\hat{\mathbf{z}}$, directed along the $\hat{z}$ axis. Our goal is to find what happens to the energies of the $2p$ ($n = 2$, $\ell = 1$) states of atomic hydrogen due to the combined effects of the spin-orbit interaction as well as the interaction of the magnetic moment of the atom and with the external applied field $\mathbf{B}$. The unperturbed (gross) Hamiltonian $\mathcal{H}^{(0)}$, has unperturbed eigenfunctions $\Psi^{(0)}_{n\ell jm_j}(\mathbf{r})$, for $n = 2$, $\ell = 1$. The s value is understood.

(a) The new terms may be written

$$\mathcal{H}^{(1)} = \mathcal{H}_{s-o} - \vec{\mu} \cdot \vec{B} = \mathcal{H}_{s-o} - \mu_z B_z,$$

where $\vec{\mu} = -\mu_B(\vec{L} + g_s\vec{S})$ and

$$\mathcal{H}_{s-o} = \frac{e^2}{2m_e^2c^2}(\frac{1}{r^3})\vec{S} \cdot \vec{L}\ .$$

Demonstrate which of the operators L^2, S^2, J^2, and J_z commute with $\mathcal{H}^{(1)}$ and hence with $\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)}$. What does this tell you about the nature of the eigenfunctions of $\mathcal{H}$, as compared with the unperturbed eigenfunctions $\Psi^{(0)}_{n\ell jm_j}(\mathbf{r})$?

(b) Write expressions for all the matrix elements of $\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)}$ in terms of the unperturbed energy eigenvalues, the "spin-orbit" energy, and the matrix elements

$$\langle\Psi^{(0)}_{n\ell jm_j}|S_z|\Psi^{(0)}_{n\ell j'm'_j}\rangle \text{ and } \langle\Psi^{(0)}_{n\ell jm_j}|L_z|\Psi^{(0)}_{n\ell j'm'_j}\rangle.$$

Calculate the required matrix elements of S_z and L_z by using the Clebsh-Gordon relationships you found on an earlier problem set. Order the states so that the matrix has the simplest block diagonal structure.

(c) Solve for all the roots, thereby obtaining the new energy eigenvalues of $\mathcal{H}$. Based on your results, draw an energy level diagram that shows how the energy levels change as the perturbations $\mathcal{H}_{s-o}$ and $-\mu_z B_z$ are included and the strength of the magnetic field changes. Use the fact that identical m_j values "repel" each other to connect weak field limit values to strong field limit values.

(d) Taking the weak field limit, show that your answer recaptures the results obtained in class. What results do you get in the strong field limit?