STA 250/MTH 342 Intro to Mathematical Statistics
Homework 10 Solutions

Solution 1: Exercise 9.5.8 (page 586)
We know when $\sigma^2 = \sigma_0^2, V = \dfrac{(n-1)S_n^2}{\sigma_0^2} \sim \chi^2_{n-1}$, regardless of the value of μ. Define $c = F^{-1}_{\chi^2_{n-1}}(1-\alpha_0)/(n-1)$, then δ is the test that rejects H_0 if $S_n^2/\sigma_0^2 \geq c \Longleftrightarrow V \geq F^{-1}_{\chi^2_{n-1}}(1-\alpha_0)$. Now we proof that the power function $\pi(\mu, \sigma^2|\delta)$ has the following properties:
(i) $\pi(\mu,\sigma^2|\delta) < \alpha_0$ if $\sigma^2 < \sigma_0^2$; (ii) $\pi(\mu,\sigma^2|\delta) = \alpha_0$ if $\sigma^2 = \sigma_0^2$; (iii) $\pi(\mu,\sigma^2|\delta) > \alpha_0$ if $\sigma^2 > \sigma_0^2$.
Proof: If $\sigma^2 = \sigma_0^2$, then V has the χ^2_{n-1} distribution. Hence,

$$\pi(\mu, \sigma_0^2|\delta) = \Pr(S_n^2/\sigma_0^2 \geq c|\mu, \sigma_0^2) = \Pr(V \geq F^{-1}_{\chi^2_{n-1}}(1-\alpha_0)|\mu, \sigma_0^2) = \alpha_0.$$

This proves (i) above. For (ii) and (iii), define

$$V^* = \frac{(n-1)S_n^2}{\sigma^2} \quad \text{and} \quad W = \frac{\sigma_0^2}{\sigma^2}.$$

Then $V = V^*/W$. First, assume that $\sigma^2 < \sigma_0^2$ so that $W > 1$. It follows that

$$\begin{aligned}
\pi(\mu,\sigma^2|\delta) &= \Pr(V \geq F^{-1}_{\chi^2_{n-1}}(1-\alpha_0)|\mu,\sigma^2) = \Pr(V^*/W \geq F^{-1}_{\chi^2_{n-1}}(1-\alpha_0)|\mu,\sigma^2) \\
&= \Pr(V^* \geq F^{-1}_{\chi^2_{n-1}}(1-\alpha_0)W|\mu,\sigma^2) < \Pr(V^* \geq F^{-1}_{\chi^2_{n-1}}(1-\alpha_0)|\mu,\sigma^2). \qquad (1)
\end{aligned}$$

Since V^* has the χ^2_{n-1} distribution, the last probability in (1) is α_0. This proves (ii). For (iii), let $\sigma^2 > \sigma_0^2$ so that $W < 1$. The less-than in (1) becomes a greater-than, and (iii) is proven.

Solution 2: Exercise 9.5.12 (page 586)

The testing statistic is

$$T = \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} \sim t_{16}.$$

Here $n = 17, \mu_0 = 3, \bar{X} = 3.2, s = \sqrt{0.09 \times \dfrac{17}{17-1}} = \dfrac{3\sqrt{17}}{40}$. Thus the observed value is $T = \dfrac{8}{3} \approx 2.667$. The rejection region has the form $\mathscr{R}(C) = \{T > C\}$. The p-value is $1 - F_{t_{16}}(2.667) = 0.008437$.

Solution 3: Exercise 9.6.4 (page 596)
The random variable $\bar{X}_m - \bar{Y}_n$ has a normal distribution with mean 0 and variance $(\sigma_1^2/m) + (k\sigma_1^2/n)$. Therefore, the following random variable has the standard normal distribution:

$$Z_1 = \frac{\bar{X}_m - \bar{Y}_n}{(\frac{1}{m} + \frac{k}{n})^{1/2}\sigma_1}$$

The random variable S_X^2/σ_1^2 has a χ^2 distribution with $m-1$ degrees of freedom. The random variable $S_Y^2/(k\sigma_1^2)$ has a χ^2 distribution with $n-1$ degrees of freedom. These two random variables are independent. Therefore, $Z_2 = (1/\sigma^2)(S_X^2 + S_Y^2/k)$ has a χ^2 distribution with $m+n-2$ degrees of freedom. Since Z_1 and Z_2 are independent, it follows that $U = (m+n-2)^{1/2}Z_1/Z_2^{1/2}$ has the t distribution with $m+n-2$ degrees of freedom.

Solution 4: Exercise 9.6.6 (page 596)
The testing statistic is

$$T = \frac{\sqrt{n+m-2}(\bar{X} - \bar{Y} - \lambda)}{\left(\frac{1}{m} + \frac{1}{n}\right)^{1/2}\sqrt{S_X^2 + S_Y^2}} \sim t_{n+m-2},$$

and the rejection region is $\mathscr{R}(C) = \{T < c_1 \text{ or } T > c_2\}$, where $c_2 \geq c_1$ are two constants.

Solution 5: Exercise 9.6.10 (page 596)
Let X_i stand for an observation in the calcium supplement group and let Y_j stand for and observation in the placebo group. The summary statistics are:

$$\begin{aligned} m &= 10, \\ n &= 11, \\ \bar{x}_m &= 109.9, \\ \bar{y}_n &= 113.9, \\ s_x^2 &= 546.9 \\ s_y^2 &= 1282.9. \end{aligned}$$

We would reject the null hypothesis if $U > T_{19}^{-1}(0.9) = 1.328$. The test statistic has the observed value $u = -0.9350$. Since $u < 1.328$, we do not reject the null hypothesis.

Solution 6: (a). We assume the p-value has a continuous distribution. Let A_i denote the event that "the i'th study report p-value < 0.01". So $\Pr(A_i) = 0.01$. Let $\tilde{A}$ denote the event that at least one of the 1000 studes reports $p < 0.01$. We have $\tilde{A} = A_1 \cup A_2 \cup \cdots \cup A_{1000}$. So

$$\begin{aligned} \Pr(\tilde{A}) = \Pr(A_1 \cup A_2 \cup \cdots \cup A_{1000}) = 1 - \Pr(A_1^c \cap A_2^c \cap \cdots \cap A_{1000}^c) = 1 - (1 - \Pr(A_1))^{1000} \\ = 1 - (1 - 0.01)^{1000} = 1 - 0.99^{1000} = 0.999957. \end{aligned}$$

(b). Similarly, $\Pr(\tilde{A}) = 1 - (1 - \Pr(A_1))^{1000} = 1 - (1 - 0.0001)^{1000} = 1 - 0.9999^{1000} = 0.095167$.

(c). *Proof*: For a fixed α, denote A_i the event that in the i'th study the p-value is less than $\alpha/1000$. So $\Pr(A_i) = \frac{\alpha}{1000}$. Denote $\tilde{A}$ the event that at least one of the 1000 studies reports $p < \alpha/1000$. Similar as before, we have

$$\Pr(\tilde{A}) = 1 - \left(1 - \frac{\alpha}{1000}\right)^{1000} = \int_0^\alpha \left(1 - \frac{t}{1000}\right)^{999} \mathrm{d}t \le \int_0^\alpha 1\mathrm{d}t = \alpha.$$

(d). Let B_i denote the event that the i'th study gives a "significant" result at level 1% and the following study also gives a "significant" result at level 1%. We have $\Pr(B_i) = 0.01^2$. Let $\tilde{B}$ denote the event that among the 1000 studies there is at least one whose p-value is less than 0.01 for both the original study and the follow-up study. One has

$$\Pr(\tilde{B}) = \Pr(B_1 \cup B_2 \cup \cdots \cup B_{1000}) = 1 - \Pr(B_1^c \cap B_2^c \cap \cdots \cap B_{1000}^c) = 1 - (1 - \Pr(B_1))^{1000}$$
$$= 1 - (1 - 0.0001)^{1000} = 1 - 0.9999^{1000} = 0.095167.$$

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