STA 250/MTH 342 Intro to Mathematical Statistics

Assignment 1, Model Solutions

Solution 1: Denote A the event that the student knew the answer, and B the event that he or she answered it correctly. Then $A \subset B$, so $P(AB) = P(A)$. We have

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(AB) + P(BA^c)} = \frac{P(A)}{p(A) + P(B|A^c)(1 - P(A))} = \frac{p}{p + \frac{1}{m}(1-p)}.$$

Solution 2: Let $t = 0$ represent 12 noon and $t = 1$ represent 1pm. Then the probability that the first to arrive has to wait longer than 10 minutes is the area of the white triangles in the following diagram. This is

$$2 \cdot \frac{1}{2}\left(\frac{5}{6}\right)^2 = \frac{25}{36}.$$

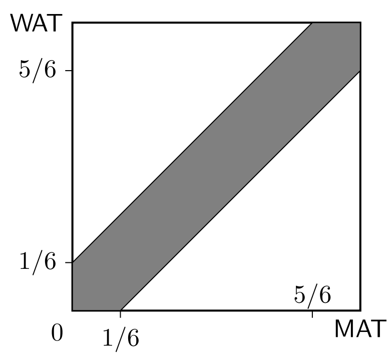

Figure 1: Problem 1. WAT=woman's arrival time; MAT=man's arrival time.

Solution 3: We have

$$E[X] = \int_{-\infty}^{x} 0 \cdot f_Z(z)\,\mathrm{d}z + \int_{x}^{\infty} z f_Z(z)\,\mathrm{d}z,$$

where $f_Z(z)$ is the probability density function of Z. Then,

$$E[X] = \int_{x}^{\infty} z \cdot \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{z^2}{2}}\,\mathrm{d}z.$$

By integration by parts (or noting that $\frac{\mathrm{d}}{\mathrm{d}z}(-\mathrm{e}^{-\frac{z^2}{2}}) = z\mathrm{e}^{-\frac{z^2}{2}}$), we have

$$E[X] = \frac{1}{\sqrt{2\pi}}\left(-\mathrm{e}^{-\frac{z^2}{2}}\right)\Big|_{x}^{\infty} = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}},$$

as desired.

Solution 4: By properties of conditional density, we have

$$\begin{aligned} f_X(x) &= \int_0^1 f_X(x|U=p) f_U(p)\,\mathrm{d}p = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \cdot 1\,\mathrm{d}p \\ &= \binom{n}{x} \frac{\Gamma(x+1)\Gamma(n-x+1)}{\Gamma(n+2)} \int_0^1 \frac{\Gamma(n+2)}{\Gamma(x+1)\Gamma(n-x+1)} p^{(x+1)-1}(1-p)^{(n-x+1)-1}\,\mathrm{d}p. \end{aligned}$$

Note that the integrand is the density function of the Beta distribution with parameter $(x+1, n-x+1)$, so the integral equals 1. Consequently,

$$f_X(x) = \frac{n!}{x!(n-x)!} \cdot \frac{x!(n-x)!}{(n+1)!} = \frac{1}{n+1},$$

as the probability mass function for $x \in \{0, 1, 2, \cdots, n\}$.

Solution 5: We claim that $X + Y$ is Poisson distributed with parameter $\lambda_1 + \lambda_2$. To see this, we compute

$$\begin{aligned} f_{X+Y}(z) &= \sum_{n=0}^{z} f_X(n) f_Y(z-n) = \sum_{n=0}^{z} \frac{\lambda_1^n}{n!} \mathrm{e}^{-\lambda_1} \cdot \frac{\lambda_2^{z-n}}{(z-n)!} \mathrm{e}^{-\lambda_2} \\ &= \frac{(\lambda_1+\lambda_2)^n (\lambda_1+\lambda_2)^{z-n}}{z!} \mathrm{e}^{-(\lambda_1+\lambda_2)} \sum_{n=0}^{z} \frac{z!}{n!(z-n)!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^n \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{z-n}. \end{aligned}$$

Note that the summand is the probability mass function of the binomial distribution with parameter $\left(z, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$, so the sum evaluates to 1. As a result,

$$f_{X+Y}(z) = \frac{(\lambda_1+\lambda_2)^z}{z!} \mathrm{e}^{-(\lambda_1+\lambda_2)},$$

which is the Poisson distribution with parameter $\lambda_1 + \lambda_2$.

Solution 6: By properties of a uniform distribution, we get $E[X] = \frac{1}{2}$, $E[X^2] = \frac{1}{3}$ (these are very straightforward to show, anyhow). Now, we calculate $E[Y|X]$:

$$E[Y|X] = \int_0^X y f_Y(y)\,\mathrm{d}y = \int_0^X y \cdot \frac{1}{X}\,\mathrm{d}y = \frac{1}{X} \cdot \frac{1}{2} y^2 \bigg|_0^X = \frac{X}{2}.$$

Next, we calculate $E[Y^2|X]$:

$$E[Y^2|X] = \int_0^X y^2 f_Y(y)\,\mathrm{d}y = \int_0^X y^2 \frac{1}{X}\,\mathrm{d}y = \frac{1}{X} \cdot \frac{1}{3} y^3 \bigg|_0^1 = \frac{X^2}{3}.$$

By the law of total expectation, we have

$$\begin{aligned} E[Y] &= E[E[Y|X]] = E[\frac{1}{2}X] = \frac{1}{2}E[X] = \frac{1}{4}, \\ E[Y^2] &= E[E[Y^2|X]] = E[\frac{1}{3}X^2] = \frac{1}{3}E[X^2] = \frac{1}{9}. \end{aligned}$$

Then, we apply the typical variance formula to get

$$\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2 = \frac{1}{9} - \left(\frac{1}{4}\right)^2 = \frac{7}{144}.$$

Note that one can also apply the following formula to find the variance:

$$\mathrm{Var}[Y] = E[\mathrm{Var}[Y|X]] + \mathrm{Var}[E[Y|X]].$$

∼∼END∼∼