STA 250/MTH 342 Intro to Mathematical Statistics
Assignment 2, Model Solutions

Solution 1: The joint p.f. of the eight observations is given by Eq. (7.2.11). Since $n = 8$ and $y = 2$ in this exercise,

$$f_n(x|\theta) = \theta^2(1-\theta)^6 \quad \text{for } \theta > 0.$$

Therefore,

$$\begin{aligned} \xi(0.1|x) = Pr(\theta = 0.1|x) &= \frac{\xi(0.1)f_n(x|0.1)}{\xi(0.1)f_n(x|0.1) + \xi(0.2)f_n(x|0.2)} \\ &= \frac{0.7 * 0.1^2 * 0.9^6}{0.7 * 0.1^2 * 0.9^6 + 0.3 * 0.2^2 * 0.8^6} \\ &= 0.5418 \end{aligned} \tag{1}$$

It follows that $\xi(0.2|x) = 1 - \xi(0.1|x) = 0.4582$

Solution 2: Recall that the Gamma distribution with density

$$f(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}\mathrm{e}^{-\beta\theta}, \quad \theta > 0,$$

has mean α/β and variance α/β^2. So we have

$$\frac{\alpha}{\beta} = 10, \quad \text{and} \quad \frac{\alpha}{\beta^2} = 5,$$

which gives $\alpha = 20$ and $\beta = 2$. Therefore the prior density of θ is

$$f(\theta) = \frac{2^{20}}{\Gamma(20)}\theta^{19}\mathrm{e}^{-2\theta} \quad \text{for } \theta > 0.$$

Solution 3: The conditions of this exercise are precisely the conditions of Example 7.2.17 with $n = 8$ and $y = 3$. Therefore, the posterior distribution of θ is a beta distribution with parameter $\alpha = 4$ and $\beta = 6$.

Solution 4: We have for a fixed θ, the distribution of X is,

$$f(x|\theta) = \chi_{[\theta-\frac{1}{2},\theta+\frac{1}{2}]}(x),$$

where χ is the characteristic function. On the other hand, the prior distribution of θ is $\xi(\theta) = \frac{1}{10}\chi_{[10,20]}(\theta)$. One has the posterior distribution

$$\xi(\theta|x) \propto f(x|\theta)\xi(\theta) = \frac{1}{10}\chi_{[\theta-\frac{1}{2},\theta+\frac{1}{2}]}(x)\chi_{[10,20]}(\theta).$$

Note that $\chi_{[\theta-\frac{1}{2},\theta+\frac{1}{2}]}(x) = \chi_{[x-\frac{1}{2},x+\frac{1}{2}]}(\theta)$. So we have

$$\xi(\theta|x = 12) \propto \chi_{[x-\frac{1}{2},x+\frac{1}{2}]}(\theta)\chi_{[10,20]}(\theta) = \chi_{[11.5,12.5]}(\theta)\chi_{[10,20]}(\theta) = \chi_{[11.5,12.5]}(\theta).$$

Therefore

$$\xi(\theta|x = 12) = \chi_{[11.5,12.5]}(\theta).$$

Solution 5: The number of defects on a 1200-foot roll of tape has the same distribution as the total number of defects on twelve 100-foot rolls, and it is assumed that the number of defects on a 100-foot roll has the poisson distribution with mean θ. By theorem 7.3.2, the posterior distribution of θ is the gamma distribution for which the parameters are $2 + 4 = 6$ and $10 + 12 = 22$.

Solution 6: Let $x = (x_1, \cdots, x_n)$ be n observations. Denote θ_0 the mean of θ. We have

$$\begin{aligned} p(\theta|x) &\propto f(x|\theta)\xi(\theta) \propto \exp\left\{-\frac{1}{8}\sum_{i=1}^{n}(x_i-\theta)^2\right\}\exp\left\{-\frac{(\theta-\theta_0)^2}{2}\right\} \\ &\propto \exp\left\{\frac{1}{4}\left(\sum_{i=1}^{n}x_i\right)\theta - \frac{n}{8}\theta^2 - \frac{\theta^2}{2} + \theta_0\theta\right\} = \exp\left\{-\left(\frac{1}{2}+\frac{n}{8}\right)\theta^2 + \left(\theta_0 + \frac{1}{4}\sum_{i=1}^{n}x_i\right)\theta\right\} \\ &\propto \exp\left\{-\frac{(\theta-C_0)^2}{2\frac{4}{4+n}}\right\}. \end{aligned}$$

Here C_0 is a constant independent of θ. So $p(\theta|x)$ is a normal distribution with variance $\frac{4}{4+n}$. For reducing this variance to 0.01 which corresponds to the value 0.1 of standard deviation, one solves $\frac{4}{4+n} \leq 0.01$ to give $n \geq 396$.

Solution 7: Suppose that the prior distribution of θ is the Pareto distribution with parameters x_0 and α ($x_0 > 0$ and $\alpha > 0$). Then the prior p.d.f. $\xi(\theta)$ has the form

$$\xi(\theta) \propto 1/\theta^{\alpha+1} \text{ for } \theta \geq x_0$$

If $X_1, \cdots, X_n$ form a random sample from an uniform distribution on the interval $[0, \theta]$, then

$$f_n(x|\theta) \propto 1/\theta^n \text{ for } \theta > \max\{x_1, \cdots, x_n\}$$

Hence, the posterior p.d.f of θ has the form

$$\xi(\theta|x) \propto \xi(\theta)f_n(x|\theta) \propto 1/\theta^{\alpha+n+1}$$

for $\theta > \max\{x_1, \cdots, x_n\}$, and $\xi(\theta|x) = 0$ for $\theta \leq \max\{x_1, \cdots, x_n\}$. This posterior p.d.f can now be recognized as also being the Pareto distribution with parameters $\alpha + n$ and $\max\{x_1, \cdots, x_n\}$.

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