Midterm practice problems – STA 250/MTH 342 Fall 2017

1. Let $X_1, X_2, \ldots, X_n$ be i.i.d. data from a Beta$(\theta, 1)$ distribution. So the pdf of each observation is

$$\begin{aligned} f(x|\theta) &= \theta x^{\theta-1} \quad \text{for } 0 < x < 1 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

   (a) Find the MLE $\hat{\theta}(\boldsymbol{X})$ of θ.

   (b) What is the Fisher's information for θ from a single observation X_1?

   (c) Use the Fisher's information you found in (b), find an approximate sampling distribution of $\hat{\theta}(\boldsymbol{X})$ when n is large.

   (d) Find the sampling distribution of $Y = \max_{1 \le i \le n} X_i$ by giving its pdf.

   (e) Now suppose we do not observe the original random variables $X_1, X_2, \ldots, X_n$, but only observe Y, find the MLE $\hat{\theta}(Y)$ of θ.

2. Let $X_1, X_2, \ldots, X_n$ be i.i.d. data from a $N(\mu, \sigma^2)$ distribution where μ is unknown while $\sigma = 2$ is known.

   (a) What is the mean and variance of $T(\boldsymbol{X}) = \sum_{i=1}^{40}(X_i - \mu)^2$? What is the sampling distribution of $T(\boldsymbol{X})$?

   (b) Find $P\left(T(\boldsymbol{X}) > 200\right)$ in terms of the cdf of the sampling distribution of the $T(\boldsymbol{X})$ or in numeric values using a distribution table.

   (c) Find $P\left(T(\boldsymbol{X}) > 200\right)$ approximately by applying the central limit theorem.

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. data from a Poisson(θ) distribution. However, we do *not* observe the X_i's directly. Instead, for each X_i we observe

$$Y_i = \begin{cases} 0 & \text{if } X_i = 0 \\ 1 & \text{if } X_i \ge 1. \end{cases}$$

   (a) Find the likelihood function of θ for the sample $Y_1, Y_2, \ldots, Y_n$.

   (b) Find the MLE $\hat{\theta}(\boldsymbol{Y})$ for θ.

   (c) Denote by $p_0 = P(X_1 = 0)$. Find the MLE $\hat{p}_0(\boldsymbol{Y})$ for p_0, and find its mean squared error (MSE).

   (d) Denote by $p_1 = P(X_1 = 1)$. Find the MLE $\hat{p}_1(\boldsymbol{Y})$ for p_1.

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. data from a Geometric(p) distribution. So each has pmf

$$P(X_i = k) = p(1-p)^k, \quad k = 0, 1, 2, \ldots$$

where $p \in (0,1)$ is unknwon.

(a) Find the MLE for p. Is it unbiased?

(b) Apply Fisher's approximation to find the approximate normal sampling distribution of the MLE.

(c) If we take a Bayesian perspective and place a Beta(α, β) prior on p that is

$$\begin{aligned} \pi(p) &\propto p^{\alpha-1}(1-p)^{\beta-1} \quad \text{for } 0 < p < 1 \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

find the posterior distribution of p. Give its name and parameter(s), or its pdf (including the normalizing constant) for all $p \in \mathbb{R}$.

(d) Under this prior, what is the Bayes estimator for p under squared error loss?

(e) Now with a Uniform(0,1) prior on p, find an interval $[a, b]$ such that

$$\pi(a \leq p \leq b | X_1 = 0) = 0.90, \quad \text{with } \pi(p < a | X_1 = 0) = \pi(p > b | X_1 = 0).$$

Remark: Such an interval is called the 90% central credible interval of p.