STA 250/MTH 342 Intro to Mathematical Statistics
Homework 9 Solutions

Solution 1:

(a). The likelihood function is

$$L(\theta) = \mathrm{e}^{-n\theta} \frac{\theta^{\sum_{i=1}^n X_i}}{\prod_{i=1}^n (X_i!)}.$$

The likelihood ratio is

$$\frac{L(\theta_1)}{L(\theta_0)} = \mathrm{e}^{-n(\theta_1-\theta_0)} \left(\frac{\theta_1}{\theta_0}\right)^{\sum_{i=1}^n X_i}.$$

(b). Since $\theta_1 > \theta_0$, for any $K \geq 0$, there exists some constant C such that

$$\frac{L(\theta_1)}{L(\theta_0)} > K \Leftrightarrow \sum_{i=1}^n X_i > C.$$

Therefore the rejection region has the form $\mathscr{R} = \mathscr{R}(C) = \{\sum_{i=1}^n X_i > C\}$. When H_0 holds,

$$\sum_{i=1}^n X_i \sim \text{Poisson}(n\theta_0),$$

So for determining a rejection region for a test at level α, one need to find the minimum C that satisfies

$$\alpha \geq \Pr\left[\mathscr{R}(C)|H_0\right] = \sum_{j>C} \mathrm{e}^{n\theta_0} \frac{(n\theta_0)^j}{j!}.$$

(c). Yes, since for any $\theta_1 > \theta_0$, the above derivation holds true and the test does not depend on θ_1.

Solution 2:

(a). The likelihood function is

$$L(\theta) = \left(\frac{m}{\theta}\right)^n \left(\prod_{i=1}^n X_i^{m-1}\right) \exp\left(-\frac{1}{\theta}\sum_{i=1}^n X_i^m\right).$$

The likelihood ratio, for any $\theta_1 > \theta_0$, is

$$\frac{L(\theta_1)}{L(\theta_0)} = \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left[\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\sum_{i=1}^n X_i^m\right].$$

The condition $\theta_1 > \theta_0$ implies that for any $K \geq 0$, there exists some C such that

$$\frac{L(\theta_1)}{L(\theta_0)} > K \Leftrightarrow \sum_{i=1}^n X_i^m > C.$$

Therefore the rejection region has the form $\mathscr{R} = \mathscr{R}(C) = \{\sum_{i=1}^{n} X_i^m > C\}$. We see that for any $\theta_1 > \theta_0$ and any level α, when H_0 happens, $\mathscr{R}(C)$ does not depend on θ_1. So the test based on $\mathscr{R}(C)$ is UMP.
(b) As $X_i^m \sim \text{Exponential}\left(\frac{1}{\theta}\right)$.
In fact, for $x > 0$,

$$f_{X_i^m}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Pr\left(X_i^m \le x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\Pr\left(X_i \le x^{1/m}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^{1/m}} \frac{1}{\theta} m t^{m-1} \mathrm{e}^{-t^m/\theta}\,\mathrm{d}t = \frac{1}{\theta}\mathrm{e}^{-x/\theta}.$$

(Note that $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)\mathrm{d}t = f(g(x))g'(x)$.)
We have already known that

$$X_i^m \sim \text{Exponential}\left(\frac{1}{\theta}\right) \Rightarrow \left(\sum_{i=1}^{n} X_i^m\right) \sim \text{Gamma}\left(n, \frac{1}{\theta}\right) \Rightarrow \frac{2}{\theta}\sum_{i=1}^{n} X_i^m \sim \chi^2_{2n}.$$

Therefore, for a fixed level α, we consider the inequality $\alpha \ge \Pr(\mathscr{R}(C)|H_0)$ to find the rejection region. We have

$$\alpha \ge \Pr\left(\mathscr{R}(C)|H_0\right) = \Pr\left(\sum_{i=1}^{n} X_i^m > C\right) = \Pr\left(\frac{2}{\theta_0}\sum_{i=1}^{n} X_i^m > \frac{2}{\theta_0}C\right) = 1 - F(\chi^2_{2n} = 2C/\theta_0),$$

of which the equality is achieved when $C = \theta_0[F^{-1}_{\chi^2_{2n}}(1-\alpha)]/2$ (F here is the CDF function for chi square distribution with 2n degree of freedom). So the rejection region is $\{\sum_{i=1}^{n} X_i^m > \theta_0[F^{-1}_{\chi^2_{2n}}(1-\alpha)]/2\}$. The rejection region is $\{\sum_{i=1}^{n} X_i^m > 50F^{-1}_{\chi^2_{40}}(0.95) = 2788\}$. The power function is

$$\pi(\theta_1) = 1 - \beta = \Pr(\mathscr{R}(\theta_0[F^{-1}_{\chi^2_{2n}}(1-\alpha)]/2)|\theta_1) = 1 - F_{\chi^2_{2n}}\left(\frac{\theta_0}{\theta_1}F^{-1}_{\chi^2_{2n}}(1-\alpha)\right) = 1 - F_{\chi^2_{40}}\left(\frac{5576}{\theta_1}\right).$$

Hint: calculate CDF in R:

```
>  qchisq(0.95,40)*50
[1] 2787.924
```

(c). For any positive integer n and a rejection region of level 0.05, the power at $\theta_1 = 105$ is

$$\beta(105) = F_{\chi^2_{2n}}\left(\frac{100}{105}F^{-1}_{\chi^2_{2n}}(0.95)\right).$$

For $\beta(105) \le 0.05$, one uses **R** to find that the minimum n is 4547.

```
sim <- 5000
n = sim
store <- rep(0,sim)
for (i in 1:n){
  a <- qchisq(0.95,2*i)*100/105
```

```
  store[i] = pchisq(a,2*i)
}
min(which(store <= 0.05 ))
[1] 4547
```

Solution 3:

The likelihood function is

$$L(p_1, \cdots, p_m) = \prod_{i=1}^{m} \binom{n_i}{X_i} p_i^{X_i} (1-p_i)^{n_i - X_i}.$$

We let Θ_0 and Θ_1 be the collections of all the possible values of $(p_1, \cdots, p_m)$ associated with H_0 and H_1 respectively. Let $\Theta = \Theta_0 \cup \Theta_1$.

$$\Theta_0 = \{(p_1, \cdots, p_m) | 0 \le p_1 = \cdots = p_m \le 1\},$$
$$\Theta = \{(p_1, \cdots, p_m) | 0 \le p_1, \cdots, p_m \le 1\}.$$

First, let's consider the H_1:

For each i, let $L_i(p_i) = \binom{n_i}{X_i} p_i^{X_i}(1-p_i)^{n_i - X_i}$ to give

$$\frac{\mathrm{d}}{\mathrm{d}p_i} \log L_i(p_i) = \frac{X_i}{p_i} - \frac{n_i - X_i}{1 - p_i}.$$

Let $\frac{\mathrm{d}}{\mathrm{d}p_i} \log L_i(p_i) = 0$ to give $p_i = X_i/n_i$. Since the derivative is strictly decreasing (we assume $n_i \ge 1$), we obtain the global maximizer $\hat{p}_i = X_i/n_i$ of L_i. $\hat{p}_i$ is the MLE for p under the alternative hypothesis.

Second, let's consider the H_0:

Since $L = \prod L_i$, the likelihood function is:

$$\max_{(p_1, \cdots, p_m) \in \Theta} = \prod_{i=1}^{m} \binom{n_i}{X_i} \hat{p}_i^{X_i} (1 - \hat{p}_i)^{n_i - X_i}.$$

Now consider

$$\frac{\mathrm{d}}{\mathrm{d}p} L(p, p, \cdots, p) = \frac{\mathrm{d}}{\mathrm{d}p} \left[\sum_{i=1}^{m} X_i \log p + \sum_{i=1}^{m} (n_i - X_i) \log(1-p) \right] = \frac{1}{p} \sum_{i=1}^{m} X_i - \frac{1}{1-p} \sum_{i=1}^{m} (n_i - X_i).$$

Obviously this derivative is a strictly decreasing function of p. We let the derivative be zero to find the global maximizer

$$\hat{p} = \frac{\sum_{i=1}^{m} X_i}{\sum_{i=1}^{m} n_i}.$$

$\hat{p}$ is the MLE of p under H_0. Given the likelihood function:

$$\max_{(p_1, \cdots, p_m) \in \Theta_0} = \prod_{i=1}^{m} \binom{n_i}{X_i} \hat{p}^{X_i} (1 - \hat{p})^{n_i - X_i}.$$

We need plug the two MLE for p into the likelihood function so that we get calculate the likelihood ratio. So the test should have a rejection region $\mathscr{R}(K) = \{\Lambda > K\}$, where

$$\Lambda = \frac{\max_{\Theta} L}{\max_{\Theta_0} L} = \prod_{i=1}^{m} \left(\frac{X_i \sum n_j}{n_i \sum X_j} \right)^{X_i} \left(\frac{(n_i - X_i) \sum n_j}{n_i \sum (n_j - X_j)} \right)^{n_i - X_i}.$$

Solution 4:

The likelihood function is

$$L(\mu_X, \mu_Y, \mu_W, \sigma_X, \sigma_Y, \sigma_W) =$$
$$\left(\frac{1}{\sqrt{2\pi}} \right)^{n_1+n_2+n_3} \frac{1}{\sigma_X^{n_1} \sigma_Y^{n_2} \sigma_W^{n_3}} \exp \left(-\sum_{i=1}^{n_1} \frac{(X_i - \mu_X)^2}{2\sigma_X^2} - \sum_{i=1}^{n_2} \frac{(Y_i - \mu_Y)^2}{2\sigma_Y^2} - \sum_{i=1}^{n_3} \frac{(W_i - \mu_W)^2}{2\sigma_W^2} \right).$$

We let Θ_0 and Θ_1 be the collections of all the possible values of $(p_1, \cdots, p_m)$ associated with H_0 and H_1 respectively. Let $\Theta = \Theta_0 \cup \Theta_1$.

$$\Theta_0 = \{\theta = (\mu_X, \mu_Y, \mu_W, \sigma_X, \sigma_Y, \sigma_W) | 0 < \sigma_X = \sigma_Y = \sigma_W < \infty\},$$
$$\Theta = \{\theta = (\mu_X, \mu_Y, \mu_W, \sigma_X, \sigma_Y, \sigma_W) | 0 < \sigma_X, \sigma_Y, \sigma_W < \infty\}.$$

Let

$$L_X(\mu_X, \sigma_X) = \frac{1}{\sqrt{2\pi}\sigma_X^{n_1}} \exp \left(-\sum_{i=1}^{n_1} \frac{(X_i - \mu_X)^2}{2\sigma_X^2} \right).$$

Similar as the derivation of MLE of the normal distribution, one obtains the global maximizer of L_X,

$$\hat{\mu}_X = \bar{X}, \quad \hat{\sigma}_X^2 = \frac{1}{n_1} \sum_{i=1}^{n_1} \left(X_i - \bar{X} \right)^2. \tag{1}$$

We similarly define L_Y and L_W, so $L = L_X L_Y L_W$, therefore the global maximizer of L over Θ is

$$\hat{\theta}_{\Theta} = \left(\bar{X}, \bar{Y}, \bar{W}, \frac{1}{n_1} \sum_{i=1}^{n_1} \left(X_i - \bar{X} \right), \frac{1}{n_2} \sum_{i=1}^{n_2} \left(Y_i - \bar{Y} \right), \frac{1}{n_3} \sum_{i=1}^{n_3} \left(W_i - \bar{W} \right) \right).$$

Now consider

$$\max_{\theta \in \Theta_0} L(\theta) = \max_{0<\sigma<\infty, \mu_X, \mu_Y, \mu_W} L(\mu_X, \mu_Y, \mu_W, \sigma, \sigma, \sigma).$$

Now that

$$\log L(\mu_X, \mu_Y, \mu_W, \sigma, \sigma, \sigma) = -\log(\sigma^{n_1+n_2+n_3}) - \sum_{i=1}^{n_1} \frac{(X_i - \mu_X)^2}{2\sigma^2} - \sum_{i=1}^{n_2} \frac{(Y_i - \mu_Y)^2}{2\sigma^2} - \sum_{i=1}^{n_3} \frac{(W_i - \mu_W)^2}{2\sigma^2}.$$

We have

$$\frac{\partial}{\partial \mu_X} \log L(\mu_X, \mu_Y, \mu_W, \sigma, \sigma, \sigma) = \sum_{i=1}^{n_1} \frac{X_i - \mu_X}{\sigma^2}.$$

The derivative is a strictly decreasing function of μ_X, and reaches zero when $\mu_X = \bar{X}$. So $\hat{\mu}_X = \bar{X}$. Similarly $\hat{\mu}_Y = \bar{Y}$ and $\hat{\mu}_W = \bar{W}$. We consider $\frac{\partial}{\partial \sigma} \log L$ to give

$$\frac{\partial}{\partial \sigma} \log L(\mu_X, \mu_Y, \mu_W, \sigma, \sigma, \sigma) = \frac{-n_1 - n_2 - n_3}{\sigma} + \frac{1}{\sigma^3} \left(\sum_{i=1}^{n_1} (X_i - \mu_X)^2 + \sum_{i=1}^{n_2} (Y_i - \mu_Y)^2 + \sum_{i=1}^{n_3} (W_i - \mu_W)^2 \right).$$

We plug $\hat{\mu}_X$, $\hat{\mu}_Y$, and $\hat{\mu}_W$ in, and let the derivative be zero to give the maximizer

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n_1} (X_i - \bar{X})^2 + \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2 + \sum_{i=1}^{n_3} (W_i - \bar{W})^2}{n_1 + n_2 + n_3}. \tag{2}$$

Since the derivative $\frac{\partial}{\partial \sigma} \log L$ is positive when $\sigma < \hat{\sigma}$ and negative when $\sigma > \hat{\sigma}$, $\hat{\sigma}$ is the global maximizer. Therefore

$$\max_{\theta \in \Theta_0} L(\theta) = L(\hat{\mu}_X, \hat{\mu}_Y, \hat{\mu}_W, \hat{\sigma}, \hat{\sigma}, \hat{\sigma}).$$

So we define the rejection region of the test to be $\mathscr{R}(K) = \{\Lambda > K\}$, where

$$\Lambda = \frac{\max_{\theta \in \Theta} L}{\max_{\theta \in \Theta_0} L} = \frac{L(\bar{X}, \bar{Y}, \bar{W}, \hat{\sigma}_X, \hat{\sigma}_Y, \hat{\sigma}_W)}{L(\bar{X}, \bar{Y}, \bar{W}, \hat{\sigma}, \hat{\sigma}, \hat{\sigma})}.$$

Here $\hat{\sigma}$ is defined in (2), $\hat{\sigma}_X$ is defined in (1), and $\hat{\sigma}_Y$ and $\hat{\sigma}_W$ are defined similarly as $\hat{\sigma}_X$.

Solution 5: Exercise 9.3.17 (page 567)

For every $\alpha_0 (0 < \alpha_0 < 1)$, if there exists a UMP test for the hypothesis $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$, then it must be a MP test for $H_0 : \theta = 0$ vs. $H_1 : \theta = \theta^* > 0$ at level of significance α_0.

The likelihood function for Cauchy distribution is

$$L(\theta) = \frac{1}{\pi[1 + (X - \theta)^2]}.$$

The likelihood ratio is

$$\frac{L(\theta^*)}{L(0)} = \frac{1 + X^2}{1 + (X - \theta^*)^2}.$$

Since $\frac{L(\theta^*)}{L(0)} > K \Leftrightarrow 1 + X^2 > K + K(X - \theta^*)^2 \Leftrightarrow (K-1)X^2 - 2\theta^* K X + K(\theta^*)^2 + K - 1 < 0$, we can then solve for K based on the level of significant α_0. Note that K is a function of θ^*, so there exists two different values $\theta_1 > 0$ and $\theta_2 > 0$ such that $K(\theta_1) \neq K(\theta_2)$, indicating that the two critical regions maximizing the power function for $H_1 : \theta = \theta_1 > 0$ and $H_1 : \theta = \theta_2 > 0$ are not the same. In other words, there is no single test procedure δ that maximizes the power function $\pi(\theta|\delta)$ simultaneously for every value of $\theta > 0$. Thus, for every $\alpha_0 (0 < \alpha_0 < 1)$, there does not exist a UMP test of these hypotheses at level of significance α_0.

Solution 6: Exercise 9.5.4 (page 585)

The testing statistic is

$$T = \frac{\sqrt{8}(\bar{X} - 0)}{s} \sim t_7.$$

The rejection region is

$$\mathscr{R} = \{|T| > F_{t_7}^{-1}(0.95) = 1.895\}.$$

The observed values are

$$\bar{X} = \frac{1}{8}\sum_{i=1}^{8} X_i = \frac{-11.2}{8} = -1.4,$$

$$s = \sqrt{\frac{1}{7}\sum_{i=1}^{8}\left(X_i - \bar{X}\right)^2} = \sqrt{\frac{1}{7}\sum_{i=1}^{8} X_i^2 - \frac{8}{7}(\bar{X})^2} = \sqrt{\frac{43.7 - 8 \times 1.4^2}{7}} = 2.001,$$

$$T = \frac{\sqrt{8}(-1.4)}{2.001} = -1.979 \in \mathscr{R},$$

so we reject H_0.

~~END~~