Practice Problems.

1. (a) $L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$

$\log L(\theta) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i$

$\frac{d\log L(\theta)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i = 0 \Rightarrow \hat{\theta} = -\frac{n}{\sum_{i=1}^{n}\log x_i}$, $\frac{d^2\log L(\theta)}{d\theta^2} = -\frac{n}{\theta^2} < 0$ for all θ. global maximizer

(b). $\frac{d^2\log f(x|\theta)}{d\theta^2} = -\frac{1}{\theta^2} \Rightarrow I(\theta) = -E\left(\frac{d^2\log f(x|\theta)}{d\theta^2}\right) = \frac{1}{\theta^2}$,

(c). $\hat{\theta} \underset{approx}{\sim} N\left(\theta, \frac{\tau^2(\theta)}{n}\right) = N\left(\theta, \frac{1}{nI(\theta)}\right) = N\left(\theta, \frac{\theta^2}{n}\right)$

(d) $P(Y \le y|\theta) = \prod_{i=1}^{n} P(X_i \le y|\theta) = \prod_{i=1}^{n}(y^\theta) = y^{n\theta}$ for $0<y<1$

$f_Y(y|\theta) = \frac{d}{dy}P(Y\le y|\theta) = \begin{cases} (n\theta)\cdot y^{n\theta-1} & \text{for } 0<y<1. \\ 0 & \text{otherwise.} \end{cases}$

(e). $L_Y(\theta) = (n\theta)\cdot y^{n\theta-1}$ $\log L_Y(\theta) = \log n + \log\theta + (n\theta-1)\log y$

$\frac{d\log L_Y(\theta)}{d\theta} = \frac{1}{\theta} + n\log y = 0 \Rightarrow \hat{\theta}(Y) = -\frac{1}{n\log Y}$

$\frac{d^2\log L_Y(\theta)}{d\theta^2} = -\frac{1}{\theta^2} < 0$ $\Rightarrow \hat{\theta}$ is global maximizer

2. (a) $$\frac{T(\underline{X})}{\sigma^2} = \frac{\sum_{i=1}^{40}(x_i-\mu)^2}{4} \sim \chi^2_{40}$$

$$\Rightarrow E\left(\frac{T(\underline{X})}{\sigma^2}\right) = 40 \qquad Var\left(\frac{T(\underline{X})}{\sigma^2}\right) = 80$$

$$\Rightarrow ET(\underline{X}) = 4 \times 40 = 160 \qquad Var\, T(\underline{X}) = 16 \times 80 = 1280$$

(b) $$P(T(\underline{X}) > 200) = 1 - P(T(x) \le 200)$$

$$= 1 - P\left(\frac{T(\underline{X})}{4} \le 50\right)$$

$$= 1 - F_{\chi^2_{40}}(50).$$

(c) $$P(T(\underline{X}) > 200) = P\left(\frac{T(\underline{X}) - 160}{\sqrt{1280}} > \frac{200-160}{\sqrt{1280}}\right)$$

$$\approx P\left(Z > \frac{40}{\sqrt{1280}}\right)$$

$$= 1 - \Phi\left(\frac{40}{16\sqrt{5}}\right)$$

$$\approx 1 - \Phi(1.12).$$

$$\approx 0.13.$$

3 4. (a) $P(Y_i=0)=P(X_i=0) = \dfrac{\theta^0 \cdot e^{-\theta}}{0!} = e^{-\theta}$.

$P(Y_i=1) = 1-e^{-\theta}$. So $P(Y_i=y_i) = e^{-\theta(1-y_i)} \cdot (1-e^{-\theta})^{y_i}$

$$L(\theta) = \prod_{i=1}^{n} P(Y_i=y_i) = \prod_{i=1}^{n} e^{-\theta(1-y_i)} (1-e^{-\theta})^{y_i}$$

$$= e^{-n\theta} \cdot (e^{\theta}-1)^{\sum_{i=1}^{n} y_i} \quad \text{for } \theta>0.$$

(b) $\log L(\theta) = -n\theta + \left(\sum_{i=1}^{n} y_i\right) \cdot \log(e^{\theta}-1)$.

$$\frac{d \log L(\theta)}{d\theta} = -n + \frac{\sum_{i=1}^{n} y_i}{e^{\theta}-1} \cdot e^{\theta} = 0.$$

$$\Rightarrow \left(\sum_{i=1}^{n} y_i\right) \cdot e^{\hat\theta} = n(e^{\hat\theta}-1).$$

$$\Rightarrow e^{\hat\theta} = \frac{n}{n-\sum_{i=1}^{n} y_i}.$$

$$\Rightarrow \hat\theta = \log\left(\frac{n}{n-\sum_{i=1}^{n} y_i}\right).$$

(c). $p_0 = P(X_j=0) = e^{-\theta}$

By the invariance property

$$\hat p_0 = e^{-\hat\theta} = \frac{n-\sum_{i=1}^{n} y_i}{n}.$$

$$E\hat p_0 = \cancel{\frac{n-np_0}{n}} \; \frac{n-n(1-p_0)}{n} = p_0,$$

$$Var\,\hat p_0 = \frac{n\,Var(Y_i)}{n^2} = \frac{p_0(1-p_0)}{n}$$

So $MSE_{\hat p_0}(p_0) = (E\hat p_0 - p_0)^2 + Var\,\hat p_0 = \dfrac{p_0(1-p_0)}{n}$.

(d) $p_1 = P(X_j = 1) = \theta e^{-\theta}$

By invariance property

$$\hat{p}_1 = \hat{\theta} e^{-\hat{\theta}} = \log\left(\frac{n}{n - \sum_{i=1}^{n} y_i}\right) \cdot \left(\frac{n - \sum_{i=1}^{n} y_i}{n}\right)$$

~~3~~ 4. (a) $L(p) = \prod_{i=1}^{n} p(1-p)^{x_i} = p^n (1-p)^{\sum_{i=1}^{n} x_i}$

$$\log L(p) = n \log p + \left(\sum_{i=1}^{n} x_i\right) \log(1-p)$$

$$\frac{\partial \log L(p)}{\partial p} = \frac{n}{p} - \frac{\sum_{i=1}^{n} x_i}{1-p} = 0 \Rightarrow n(1-\hat{p}) = \hat{p} \cdot \sum_{i=1}^{n} x_i$$

$$\Rightarrow \hat{p} = \frac{n}{n + \sum_{i=1}^{n} x_i} \qquad \frac{d^2 \log L(p)}{dp^2} = -\frac{n}{p^2} - \frac{\sum_{i=1}^{n} x_i}{(1-p)^2} < 0$$

$\hat{p}$ is Global maximizer

$$E\hat{p} \neq \frac{n}{n + \sum_{i=1}^{n} E x_i} = \frac{n}{n + n(\frac{1}{p} - 1)} = p.$$

So $\hat{p}$ is not unbiased

(b). ~~$I(p)=$~~

$$\log f(x_1|p) = \log p + x_1 \log(1-p).$$

$$\frac{d}{dp} \log f(x_1|p) = \frac{1}{p} - \frac{x_1}{1-p}.$$

$$\frac{d^2}{dp^2} \log f(x_1|p) = -\frac{1}{p^2} - \frac{x_1}{(1-p)^2}$$

$$I(p) = -E\left[\frac{d^2}{dp^2} \log f(x_1|p)\right] = \frac{1}{p^2} + \frac{\frac{1}{p} - 1}{(1-p)^2} = \frac{1}{p^2} + \frac{1}{p(1-p)} = \frac{1}{p^2(1-p)}$$

So by Fisher's approximation, for large n.

$$\hat{p} \sim_{approx} N\left(p, \frac{p^2(1-p)}{n}\right).$$

(c) $\pi(p|\underline{x}) \propto \pi(p) \cdot \prod_{i=1}^{n} f(x_i|p)$

$$\propto p^{\alpha+n-1}(1-p)^{\beta+\sum_{i=1}^{n}x_i-1}$$

$$\sim \text{Beta}(\alpha+n, \beta+\sum_{i=1}^{n}x_i)$$

(d) The Bayes estimator under squared error loss is the posterior mean

$$E(p|\underline{x}) = \frac{\alpha+n}{\alpha+n+\beta+\sum_{i=1}^{n}x_i}$$

(e). $\pi(p) \propto 1$. That is $\alpha=\beta=1$

$\Rightarrow$ ~~Post~~ Posterior $\pi(p|\underline{x})$ is $\text{Beta}(n+1, \sum_{i=1}^{n}x_i+1)$

Now $n=1$. $x_1=0$. So $\pi(p|x_1=0)$ is $\text{Beta}(2,1)$.

~~A 90% central credible interval~~

$$\pi(p|x_1=0) = \frac{\Gamma(2+1)}{\Gamma(2)\Gamma(1)} \cdot p^{2-1} \cdot (1-p)^{1-1} = 2p \quad \text{for } 0<p<1$$

So the c.d.f. of Beta(2,1) is

$$F(y) = \int_0^{y} 2p\,dp = p^2 \quad \text{for } 0<y<1.$$

~~Now choose~~ Thus $F^{-1}(0.05) = \sqrt{0.05}$

and $F^{-1}(0.95) = \sqrt{0.95}$.

So a central 90% credible interval is

$$[\sqrt{0.05}, \sqrt{0.95}]$$