STA 250/MTH 342 Intro to Mathematical Statistics

Homework 8 Solutions

Solution 1: Exercise 9.2.12 (page 558)

For the case in which $\mathbf{X}$ is a random sample from a distribution with univariate p.d.f or p.f $f(x|\theta)$, we then have, for $i = 0$ or $i = 1$,

$$f_i(\mathbf{x}) = f(x_1|\theta_i)f(x_2|\theta_i)\cdots f(x_n|\theta_i)$$

For our problem, $X_1, \cdots, X_n$ are a random sample from the exponential distribution with unknown parameter θ, therefore,

$$f_i(\mathbf{x}) = \theta_i^n exp(-\theta_i \sum_{j=1}^{n} x_j) \text{ for } i = 1, 0$$

The likelihood ratio test has the following form: reject H_0 if $\frac{f_1(\mathbf{x})}{f_0(\mathbf{x}} > k$ where k is chosen so that the probability of rejecting H_0 is α_0 given $\theta = \theta_0$. The ratio of f_1 to f_0 is:

$$\frac{f_1(\mathbf{x})}{f_0(\mathbf{x})} = \frac{\theta_1^n}{\theta_0^n} exp([\theta_0 - \theta_1] \sum_{i=1}^{n} x_i)$$

Since $\theta_0 < \theta_1$ that is given in the question, the above ratio will be greater than k if and only if $\sum_{i=1}^{n} x_i$ is less than some other constant, c. That c is chosen so that $Pr(\sum_{i=1}^{n} x_i < c|\theta = \theta_0) = \alpha_0$. The distribution of $\sum_{i=1}^{n} x_i$ given $\theta = \theta_0$ is the gamma distribution with parameters n and θ_0. Hence, c must be the α_0 quantile of that distribution.

Solution 2:

(a). The likelihood ratio is

$$\text{LR} = \frac{f(x|H_1)}{f(x|H_0)} = \begin{cases} 0.5 & \text{if } x = x_1 \\ 1 & \text{if } x = x_2 \\ 1/3 & \text{if } x = x_3 \\ 2.5 & \text{if } x = x_4. \end{cases}$$

So we can order the x_i's in the corresponding LRs: x_4, x_2, x_1, x_3. Recall the gold miner analogy we talked about in class. The higher the LR, the higher the "gold-to-price" ratio, the stronger the support for the alternative, and thus should be included in the rejection region (or "purchased") first.

(b). Now the Neyman-Pearson lemma tells us that the most powerful test rejects when the LR$> K$ for some constant K. We just need to determine the value of K based on the Type I error threshold. Note that for $K \in [1, 2.5)$, say $K = 2$, the rejection region is $\{x_4\}$ and the type I error is $P(X = x_4|H_0) = 0.2$. So this is the most powerful level 0.2 test. For $K \in [0.5, 1)$, say $K = 0.6$, the rejection region is $\{x_2, x_4\}$, and the type I error is $P(X \in \{x_2, x_4\}|H_0) = 0.5$. Thus this is the most powerful level 0.5 test.

Solution 3:

(a). The null hypothesis $p = 0.5$ is simple whild the alternative hypothesis $p \neq 0.5$ is composite.

(b). Denote X to be the number of heads observed.

The level (size) of the test is

$$\alpha = P(\text{Reject } H_0 | H_0 \text{ true}) = P(X = 0 \text{ or } 10 | p = 0.5) = 0.5^{10} + (1 - 0.5)^{10} = 0.5^9.$$

(c). The power of the test is $P(\text{Reject } H_0 | H_1 \text{ true})$.

For the alternative $p = 0.4$,

$$\textit{power} = 1 - \beta = P(X = 0 \text{ or } 10 | p = 0.4) = 0.4^{10} + (1 - 0.4)^{10} = 0.4^{10} + 0.6^{10}.$$

For the alternative $p = 0.1$,

$$\textit{power} = 1 - \beta = P(X = 0 \text{ or } 10 | p = 0.1) = 0.1^{10} + (1 - 0.1)^{10} = 0.1^{10} + 0.9^{10}.$$

(d). The power of as a function of the probability of heads is

$$\pi(p) = p^{10} + (1 - p)^{10}.$$

Solution 4:

(a). False. The size of a test is the probability of rejecting the null hypothesis when the null hypothesis is true.

(b). False. The power of a test is the probability of rejecting the null hypothesis when the alternative hypothesis is true.

(c). False. The lower the level of a test, the lower the probability of a type I error. Usually we expect the type II error to increase.

(d). False. A type I error occurs only when the data falls in the rejection region while the null hypothesis is true.

(e). False. The power of a test is determined by the probability of a type II error, which depends on the distribution of the alternative hypothesis, not the null hypothesis.

Solution 5:

(a). The type I error is the probability that we reject H_0 when the proportion p of ledger sheets is indeed equal to 0.2.

(b). Denote A the number of the sheets in the first two that are error-free. Denote B the number of the sheets in the following two that are error-free. We have

$$\{\text{Reject } H_0\} = \{A = 2\} \cup \{A \neq 2, B = 2\},$$

so

$$\begin{aligned}\alpha &= P(\text{Reject } H_0|H_0 \text{ true}) = P(A=2|H_0) + P(A\neq 2|H_0)\times P(B=2|H_0)\\ &= (1-0.2)^2 + [1-(1-0.2)^2]\times(1-0.2)^2 = 0.8704.\end{aligned}$$

(c). The type II error is the probability that we fail to reject H_0 when the alternative hypothesis is true, i.e. when the proportion p of ledger sheets with errors is 0.3. We have

$$\beta = 1 - P(\text{Reject } H_0|H_a \text{ true}) = 1 - [(1-0.3)^2 + (1-(1-0.3)^2)\times(1-0.3)^2] = 0.2601.$$

(d). $\pi(p) = P(\text{Reject } H_0|H_a \text{ true}) = (1-p)^2 + [1-(1-p)^2](1-p)^2 = 1 - 4p^2 + 4p^3 - p^4.$

Solution 6:

$$H_0 : \sigma^2 = \sigma_0^2 = 1 \quad \text{vs} \quad H_1 : \sigma^2 = \sigma_1^2 = 2.$$

$$\Lambda = \frac{L(\sigma_1^2|\mathbf{x})}{L(\sigma_0^2|\mathbf{x})} = \frac{(2\pi)^{-n/2}\sigma_1^{-n}\exp\{-\frac{1}{2\sigma_1^2}\sum_{i=1}^n (X_i)^2\}}{(2\pi)^{-n/2}\sigma_0^{-n}\exp\{-\frac{1}{2\sigma_0^2}\sum_{i=1}^n (X_i)^2\}} = (\frac{\sigma_0}{\sigma_1})^n \exp\{(-\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_0^2})\sum_{i=1}^n X_i^2\} > k.$$

Since $\sigma_0^2 = 1 < 2 = \sigma_1^2$, $\Lambda > k$ if $\sum_{i=1}^n X_i^2 > c$. The test level $\alpha = 0.10 = P(\sum_{i=1}^n X_i^2 > c|\sigma^2 = 1)$.

$$\sum_{i=1}^n (\frac{X_i - 0}{\sigma})^2 \sim \chi_n^2 = Gamma(\frac{n}{2}, \frac{1}{2}),$$

$$\sum_{i=1}^n X_i^2 \sim \sigma^2\chi_n^2 = \chi_n^2 = Gamma(\frac{n}{2}, \frac{1}{2}).$$

Thus,

$$0.10 = P(\sum_{i=1}^n X_i^2 > c|\sigma^2 = 1) = 1 - F_{Gamma(\frac{n}{2},\frac{1}{2})}(c),$$

$$c = F^{-1}_{Gamma(\frac{n}{2},\frac{1}{2})}(0.9).$$

The rejection region forthe LR test at level $\alpha = 0.10$ is when $\sum_{i=1}^n X_i^2 > F^{-1}_{Gamma(\frac{n}{2},\frac{1}{2})}(0.9)$.

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