Physics 464: Problem Set 9

1. Use Clebsch-Gordan coefficients to block diagonalize (decompose) the 4-dimensional $2 \otimes 2$ individual-basis operators $\vec{J}^2$, J_x, J_y, and J_z to their 4-dimensional block diagonal composite-basis form. Find the matrix which performs this block-diagonalization. Hints:
   - You may have to reorder your states in order to see the block-diagonal form.
   - $J_+ = S_{1+} + S_{2+}$, etc., where S are the spin-1/2 operators.
   - Find the operators by taking matrix elements, using for example, $\vec{J}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$.
   - $\langle m_1 m_2 | J_i | m_1' m_2' \rangle = \sum_{lm} \sum_{l'm'} \langle m_1 m_2 | lm \rangle \langle lm | J_i | l'm' \rangle \langle l'm' | m_1' m_2' \rangle$.
   - You are to find the matrix U such that $\mathcal{O}_{(\text{individual basis})} = U \mathcal{O}_{(\text{combined basis})} U^\dagger$, where $\mathcal{O}$ is any of the four operators above, and the same U works for all of them.

2. "Adding" $l = 1$ with $s = 1/2$: Use raising and lowering operators as well as orthogonality to express the combined basis states in terms of the individual basis states $|l = 1, m_l, s = 1/2, m_s\rangle$.

3. A spin-3/2 particle that exists in nature is the Δ^{++} particle, which is a bound state of three up-type quarks. The ++ designation indicates that it is charge +2, but that is not going to matter for us.
   (a) Find the operators $\hat{J}_z$, $\hat{J}_x$, $\hat{J}_y$ for the Δ^{++}.
   (b) If the Δ^{++} exists in the state $|j, m_z\rangle = |\frac{3}{2}, -\frac{1}{2}\rangle$, and a measurement of the particle's spin projection onto the $\hat{x}$ axis is made, what results are found and with what probatilities?
   (c) Each of the up quarks of which Δ^{++} is made is a spin-1/2 object. Express the state $|j, m_z\rangle = |\frac{3}{2}, -\frac{1}{2}\rangle$ in terms of the individual basis states for the three quarks:
   $$|s_1, m_{1z}\rangle = |\frac{1}{2}, \frac{1}{2}\rangle_1 \text{ or } |\frac{1}{2}, -\frac{1}{2}\rangle_1$$
   $$|s_2, m_{2z}\rangle = |\frac{1}{2}, \frac{1}{2}\rangle_2 \text{ or } |\frac{1}{2}, -\frac{1}{2}\rangle_2$$
   $$|s_3, m_{3z}\rangle = |\frac{1}{2}, \frac{1}{2}\rangle_3 \text{ or } |\frac{1}{2}, -\frac{1}{2}\rangle_3$$
   That is, find the Clebsh-Gordan coefficients that relate the $|s_1, m_{1z}, s_2, m_{2z}, s_3, m_{3z}\rangle$ basis to the $|j, m_j s_1, s_2, s_3\rangle$ basis that are needed for the state given.

4. Predict the rotational spectrum of HCl.

5. Evaluate
   (a) $exp[-i\hat{\vec{P}} \cdot \vec{a}/\hbar]|\vec{x}\rangle$, where $\hat{\vec{P}}$ is the three dimensional momentum operator, $\vec{a}$ is a real vector with units of length, and $|\vec{x}\rangle$ is a position state basis ket. Explain why this operator acts as a translation operator in position space.
   (b) $exp[-i\hat{\vec{X}} \cdot \vec{b}]|\vec{p}\rangle$, where $\hat{\vec{X}}$ is the three dimensional position operator, $\vec{b}$ is a real vector with units of inverse length, and $|p\rangle$ is a momentum state basis ket. Explain why this operator acts as a translation operator in momentum space.

6. Evaluate
   (a) $exp[-iS_x\theta/\hbar]\ |+\rangle$, where $S_x = \frac{\hbar}{2}\sigma_x$, where σ_x is the Pauli matrix, θ is a constant and $|+\rangle$ is the spin-up ket along the $\hat{z}$ axis. What happens when $\theta = 2\pi$? What do you think of that?
   (b) $exp[-iL_x\alpha/\hbar]\ |1\rangle$, where L_x is the 3x3 $\hat{x}$ direction operator for $l = 1$ and $|1\rangle$ is the $l = 1, m_l = 1\rangle$ ket with projection $m = 1$ along the $\hat{z}$ axis. What happens when $\alpha = 2\pi$? Compare this result to the one obtained in question (6a) above.