STA 250/MTH 342 Intro to Mathematical Statistics
Assignment 3, Model Solutions

Solution 1: Exercise 7.3.12 (page 406)
Let $\mathbf{x} = (x_1, \cdots, x_{20})$ be the serving time of the 20 customers. Let $\bar{x} = \frac{1}{20}\sum_{i=1}^{20} x_i$ be the average serving time. For a fixed parameter θ, the probability density function of the average serving time is,

$$f(\mathbf{x}|\theta) = \prod_{i=1}^{20} \theta \mathrm{e}^{-\theta x_i} = \theta^{20}\mathrm{e}^{-20\bar{x}\theta}.$$

The prior distribution of θ is

$$\xi(\theta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}\mathrm{e}^{-\beta\theta} & \text{if } \theta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

We have $\alpha/\beta = 0.2$ and $\alpha/\beta^2 = 1$, which gives $\alpha = 1/25$ and $\beta = 1/5$. Therefore, given $\bar{x} = 3.8$, the posterior distribution is

$$p(\theta|\mathbf{x}) \propto f(\mathbf{x}|\theta)\xi(\theta) \propto \theta^{20}\mathrm{e}^{-76\theta}\theta^{-0.96}\mathrm{e}^{-0.2\theta} = \theta^{20.04-1}\mathrm{e}^{-76.2\theta} \text{ for } \theta > 0.$$

which is Gamma(20.04, 76.2).

Solution 2: Exercise 7.3.15 (page 406)
(a). (5 points)
One has

$$\int_{-\infty}^{\infty} \xi(\theta)\,\mathrm{d}\theta = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}\mathrm{e}^{-\beta/\theta}\,\mathrm{d}\theta \overset{t=\frac{\beta}{\theta}}{=\!=} \int_\infty^0 \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{\beta}{t}\right)^{-(\alpha+1)}\mathrm{e}^{-t}\left(-\frac{\beta}{t^2}\right)\mathrm{d}t$$
$$= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t = 1.$$

(b). (5 points)
One has

$$p(\theta|x) \propto f(x|\theta)\xi(\theta) \propto \frac{1}{\sqrt{\theta}}\exp\left\{-\frac{(x-\mu)^2}{2\theta}\right\}\theta^{-(\alpha+1)}\mathrm{e}^{-\beta/\theta} = \theta^{-(\alpha+\frac{1}{2}+1)}\exp\left\{-\frac{\frac{(x-\mu)^2}{2}+\beta}{\theta}\right\}$$

for $\theta > 0$, and $= 0$ otherwise. This is the variable part of an inverse gamma distribution with parameters $\alpha + 1/2$ and $\beta + (x-\mu)^2/2$. So $p(\theta|x)$ is still an inverse gamma distribution. When there are n i.i.d. observations from $f(x|\theta)$, we know that the posterior must also be an inverse gamma by induction.

Solution 3: Exercise 7.3.20 (page 407) Let $\mathbf{t} = (t_1, \cdots, t_{10})$ be the life time of the 10 components respectively. Denote $\bar{t} = (t_1 + \cdots + t_{10})/10$ the average life time. We have

$$p(\beta|\mathbf{t}) \propto f(\mathbf{t}|\beta)\xi(\beta) \propto \left(\prod_{i=1}^{10}\beta\mathrm{e}^{-\beta t_i}\right)\beta^{a-1}\mathrm{e}^{-b\beta} = \beta^{10+a-1}\mathrm{e}^{-(b+10\bar{t})\beta},$$

which is Gamma$(10+a, b+10\bar{t})$. Since the prior lifetime mean is 4, we have

$$4 = \mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T|\beta]] = \mathbb{E}\frac{1}{\beta} = \frac{b^a}{\Gamma(a)}\int_0^\infty \frac{1}{\beta}\beta^{a-1}\mathrm{e}^{-b\beta}\,\mathrm{d}\beta = \frac{b^a}{\Gamma(a)} \times \frac{\Gamma(a-1)}{b^{a-1}} = \frac{b}{a-1}. \tag{1}$$

The posterior lifetime mean is 5, so similar to (1) we have

$$5 = \mathbb{E}[T|\mathbf{t}] = \mathbb{E}[\mathbb{E}[T|\beta]|\mathbf{t}] = \mathbb{E}[\frac{1}{\beta}|\mathbf{t}] = \frac{b+10\bar{t}}{a+9}. \tag{2}$$

Since $\bar{t} = 6$, we combine (1) and (2) to obtain $a = 11$ and $b = 40$.

Solution 4: Exercise 7.4.2 (page 416)
We have

$$p(\beta|\mathbf{x}) \propto f(\mathbf{x}|\beta)\xi(\beta) = \binom{20}{1}\theta(1-\theta)^{19}\frac{\Gamma(15)}{\Gamma(5)\Gamma(10)}\theta^4(1-\theta)^9 \propto \theta^5(1-\theta)^{28},$$

which is Beta$(6, 29)$, so the posterior mean is $6/35$, which is just the Bayes estimate of θ with squared error loss function.

Solution 5: Exercise 7.4.10 (page 417)
The prior distribution is

$$\xi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}\mathrm{e}^{-\beta\theta} \quad \text{for } \theta > 0.$$

Since $\alpha/\beta = 0.2$ and $\alpha/\beta^2 = 1$, we have $\alpha = 1/25$ and $\beta = 1/5$. So the posterior distribution of θ is

$$p(\theta|\mathbf{t}) \propto f(\mathbf{t}|\theta)\xi(\theta) \propto \theta^{20}\mathrm{e}^{-20\bar{t}\theta}\theta^{-0.96}\mathrm{e}^{-0.2\theta} = \theta^{20.04-1}\mathrm{e}^{-76.2\theta} \quad \text{for } \theta > 0,$$

which is just the Gamma$(20.04, 76.2)$ distribution, and therefore the Bayes estimate of θ by squared error loss function is $\frac{20.04}{76.2} = 0.263$.

Solution 6: Exercise 7.4.12 (page 417)
(a). (5 points)
Applying Bayes' theorem, we have the two posteriors

$$p_A(\theta|\mathbf{x}) \propto \theta^{711}(1-\theta)^{290} \quad \text{for } 0 < \theta < 1,$$
$$p_B(\theta|\mathbf{x}) \propto \theta^{713}(1-\theta)^{290} \quad \text{for } 0 < \theta < 1,$$

which are Beta$(712, 291)$ and Beta$(714, 291)$ respectively.
(b). (5 points)
From the distribution we can compute the posterior means, which are the Bayes estimates: $\delta_A(x) = \frac{712}{1003} = 0.7099$, $\delta_B(x) = \frac{714}{1005} = 0.7104$.

(c). (5 points)
Suppose there are m voters who are in favor of the proposition. We have $0 \le m \le 1000$, and

$$p_A(\theta|\mathbf{x}) \propto \theta^{m+1}(1-\theta)^{1000-m} \quad \text{for } 0 < \theta < 1,$$
$$p_B(\theta|\mathbf{x}) \propto \theta^{m+3}(1-\theta)^{1000-m} \quad \text{for } 0 < \theta < 1.$$

The difference between the two estimates are

$$\left|\frac{m+2}{1003} - \frac{m+4}{1005}\right| = \frac{|2m-2002|}{1003 \times 1005} \le \frac{2002}{1003 \times 1005} = 0.001986.$$

Solution 7: Exercise 7.4.14 (page 416)
Since $\psi = \theta^2$, the posterior distribution of ψ can be derived from the posterior distribution of θ. The Bayes estimator $\hat{\psi}$ will then be the mean $E(\psi|\mathbf{x})$ of the posterior distribution of ψ. But $E(\psi|\mathbf{x}) = E(\theta^2|\mathbf{x})$, where the first expectation is calculated with respect to the posterior distribution of ψ and the second with respect to the posterior distribution of of θ. Since $\hat{\theta}$ is the mean of the posterior distribution of θ, it is also true that $\hat{\theta} = E(\theta|\mathbf{x})$. Finally, since

$$\text{Var}(\theta|\mathbf{x}) = E(\theta^2|\mathbf{x}) - E(\theta|\mathbf{x})^2 > 0$$

we have

$$\hat{\psi} = E(\theta^2|\mathbf{x}) > [E(\theta|\mathbf{x})]^2 = \hat{\theta}^2$$

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