UNIT – I LOGICS

1. What is Discrete Mathematics?

Discrete Mathematics is a part of mathematics devoted to the study of discrete objects. Discrete means consisting of distinct or disconnected elements.

2. Define Declarative sentence.

A sentence which cannot be further broken down or split into simple sentences is called a declarative sentence.

3. What is meant by truth values or two valued logic?

The declarative sentences which have one and only one of two possible values (true & false) are called the truth values. The truth values true and false are denoted by T and F or 1 and 0 respectively.

4. Explain statements and atomic statements.

The declarative sentence to which it is possible to assign truth values true or false but not both are called statements.

(e.g.) (i) Banu is rich.(ii) Chennai is a city

Statements which do not contain any of the connectives are called the atomic statements or the primary statements.

(e.g.) P, Q, R *etc*.

5. Explain compound statements.

The statements which have connectives are called compound statements or molecular statements.

(e.g.) $P \wedge Q$, $P \to Q \wedge R$

6. Write the five logical operators.

i) Negation (NOT)

ii) Conjunction (AND)

iii) Disjunction (OR)

iv) Conditional Statement (IF THEN)

v) Biconditional Statement (IF AND ONLY IF)

7. Explain the connective Negation.

The negation is introduced by the word NOT. If P is a statement, then negation of P is denoted by P or : P and it should be read as NOT P. The truth table for negation is as follows

P	$\rceil P$
T	F
F	T

(e.g.) P: Today is Friday.

$\rceil P$: It is not that today is Friday

8. Explain the connective Conjunction.

Consider the two statements P and Q. Then the conjunction of P and Q is denoted by P $\wedge Q$ which is read as " P AND Q " or " P meets Q " .. The truth table for conjunction is as follows.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(e.g.) P: It is raining today Q: 2+2=4.

$P \wedge Q$: It is raining today and $2+2=4$.

9. Explain the connective Disjunction.

Consider the two statements P and Q. Then the disjunction of P and Q is denoted by $P \vee Q$ which is read as " P OR Q" or " P joins Q" . . The truth table for disjunction is as follows.

P	Q	$P \vee Q$

T	T	T
T	F	T
F	T	T
F	F	F

(e.g.) If P: It is raining today,

Q: 2+2=4. then $P \vee Q$: It is raining today or $2+2=4$.

10. Explain the Conditional statement.

Let P and Q be any two statements. Then the statement $P \rightarrow Q$ which is read as " IF P THEN Q" is called as the conditional statement. The truth table for conditional statement is as follows.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(e.g.) P: The sun is shining today Q: 2+7>4

$P \rightarrow Q$: If the sun is shining today then Q: $2+7>4$.

11. Explain Biconditional Statement.

Let P and Q be any two statements. Then the statement $P \leftrightarrow Q$ which is read as " P IF AND ONLY IF Q" is called as Biconditional statement. .

(e.g.) If P: The sun is shining today,

Q : 2+7>4 then

$P \leftrightarrow Q$: The sun is shining today if and only if *Q: 2+7>4*

The truth table for Biconditional is as follows

.P	Q.	$P \leftrightarrow Q$

T	T	.T
T	F	F .
F	T	F .
F	F	.T

12. Explain the term Tautology.

A statement formula which is always true irrespective of the truth values of the individual variables is called a tautology.

(e. g.) $\rceil P \vee P$ is a tautology .

13. Explain the term Contradiction.

A statement formula which is always false irrespective of the truth values of the individual variables is called contradiction.

(e. g.) $\rceil P \wedge P$ is a contradiction .

14. Explain Well Formed Formula (WFF).

The statement formula in which the order of finding the truth values are indicated by using parenthesis is called a Well Formed Formula.

(e. g.) $((P \to Q) \wedge R)$, $\left((P \to Q) \to (\rceil P \vee Q)\right)$

15. Write the rules for Well Formed Formula.

i) A statement variable standing alone is a well formed formula.

ii) If A is a well formed formula, then A is a well formed formula.

iii) If A and B are well formed formulae, then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $(A \leftrightarrow B)$ are also well formed formulae.

iv) A string of symbols containing the statement variables, connectives and parenthesis are a well formed formula iff it can be obtained by finitely many applications of the rules (i), (ii), (iii)

16. State which of the following are WFF.

i) $(\rceil P \wedge Q)$ ii) $((P \rightarrow Q) \wedge R)$ iii) $P \rightarrow Q \wedge R$ iv) $(P \rightarrow Q) \vee P \rightarrow Q)$

i) and ii) are WFF.

iii) is not WFF while $(P \rightarrow (Q \wedge R))$or $((P \rightarrow Q) \wedge R)$ is a WFF.

iv) is not a WFF while $((P \rightarrow Q) \vee (P \rightarrow Q))$ or $((P \rightarrow Q) \vee P) \rightarrow Q$ is a WFF.

17. Write the symbolic form of the following.

Let P: It is below freezing ; Q: It is snowing

i) It is below freezing and snowing: $P \wedge Q$

ii) It is below freezing but not snowing: $P \wedge \rceil Q$

iii) If it is below freezing, then it is also snowing: $P \rightarrow Q$

18. Construct the truth table for $P \wedge \rceil Q$

P	Q	$\rceil Q$	$P \wedge \rceil Q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

19. Explain Tautological implication.

A statement A is said to " tautologically imply a statement B if and only if $A \rightarrow B$is a tautology.

20. Explain Duality law.

Two statements A and A^* are said to be " dual " of each other if either one can be obtained from the other by replacing $\wedge$ by$\vee$, $\vee$ by$\wedge$, T by F and F by T.

21. Show that $(P \rightarrow Q) \rightarrow (\rceil P \vee Q)$ **is a tautology.**

i) To show $(P \rightarrow Q) \rightarrow (\rceil P \vee Q)$ is a tautology

$$(P \rightarrow Q) \rightarrow (\neg P \vee Q) \Leftrightarrow \neg P \vee Q \rightarrow \neg P \vee Q$$

$$\Leftrightarrow \neg(\neg P \vee Q) \vee (\neg P \vee Q)$$

$$\Leftrightarrow T$$

ii) To show $(\neg P \vee Q) \rightarrow (P \rightarrow Q)$ is a tautology.

$$(\neg P \vee Q) \rightarrow (P \rightarrow Q) \Leftrightarrow (\neg P \vee Q) \rightarrow (\neg P \vee Q)$$

$$\Leftrightarrow \neg(\neg P \vee Q) \vee (\neg P \rightarrow Q)$$

$$\Leftrightarrow T$$

22. Define NAND connective.

The word NAND is a combination of NOT and AND. The connective NAND is denoted by the symbol $\uparrow$ For any two statements P and Q, the NAND is defined as $P \uparrow Q \Leftrightarrow \neg(P \wedge Q)$. The truth table for the connective NAND is as follows.

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

23. Define NOR connective.

The word NOR is a combination of NOT and OR. The connective NOR is denoted by the symbol $\downarrow$. For any two statements P and Q, the NOR is defined as $P \downarrow Q \Leftrightarrow \neg(P \vee Q)$.

24. Show that i) $\neg(P \uparrow Q) \Leftrightarrow \neg P \downarrow \neg Q$, **ii)** $\neg(P \downarrow Q) \Leftrightarrow \neg P \uparrow \neg Q$

i) $\neg(P \uparrow Q) \Leftrightarrow \neg(\neg(P \wedge Q))$

$$\Leftrightarrow \neg(\neg P \vee \neg Q)$$

$$\Leftrightarrow \neg P \downarrow \neg Q$$

ii) $\rceil(P \downarrow Q) \Leftrightarrow \rceil(\rceil(P \vee Q))$

$\Leftrightarrow \rceil(\rceil P \wedge \rceil Q)$

$\Leftrightarrow \rceil P \uparrow \rceil Q$

25. Show $P \uparrow (\rceil P \to Q)$ in terms of NAND form.

$P \uparrow (\rceil P \to Q) \Leftrightarrow P \uparrow (\rceil(\rceil P) \vee Q)$

$\Leftrightarrow P \uparrow (P \vee Q)$

$\Leftrightarrow P \uparrow \rceil(\rceil(P \vee Q))$

$\Leftrightarrow P \uparrow \rceil(\rceil P \wedge \rceil Q)$

$\Leftrightarrow P \uparrow (\rceil P \uparrow \rceil Q)$

26. Define functionally complete set of connectives.

A set of connectives is said to be functionally complete set of connectives if every formula can be expressed in terms of equivalence formula containing the connectives only from that set.

(e. g.) $\{\wedge, \rceil\}, \{\vee, \rceil\}, \{\wedge, \vee, \rceil\}$ are functionally complete set of connectives

27. Define Disjunctive Normal Form.

Let A be a given formula, another formula B which is equivalent to A which consists of *sum of elementary product* is called a Disjunctive Normal Form.

28. Define Conjunctive Normal Form.

Let A be a given formula, another formula B which is equivalent to A which consists of *product of elementary sum* is called Conjunctive Normal Form.

29. Define Minterms.

Minterms consist of conjunction of each variable or its negation but not both.

(e. g.). $P \wedge Q$, $P \wedge \rceil Q$, $\rceil P \wedge \rceil Q$, $\rceil P \wedge Q$ are the minterms of two variables P, Q.

30. Define Maxterms.

Maxterms consist of disjunction of each variable or its negation but not both.

(e. g.) $P \vee Q$, $P \vee \rceil Q$, $\rceil P \vee \rceil Q$, $\rceil P \vee Q$ are the maxterms of two variables P, Q.

31. Define Principle Disjunctive Normal Form. (PDNF)

The sum of the minterms is called principle disjunctive normal form. (PDNF)

(e. g.) $(P \wedge Q) \vee (\rceil P \vee \rceil Q)$

32. Define Principle Conjunctive Normal Form. (PCNF)

The product of maxterms is called principle conjunctive normal form. (PCNF)

(e. g.) $(P \vee Q) \wedge (\rceil P \vee \rceil Q)$

UNIT II - THEORY OF INFERENCE

1. Define inference theory

The main aim of logic is to provide rules of inference to infer a conclusion from certain premises. The theory associated with rules of inference is known as inference theory.

2. What is called deduction?

If a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a deduction or a formal proof, and the argument or conclusion is called a valid argument or valid conclusion.

3. What is meant by " truth table technique" ?

The method to determine whether the conclusion logically follows from the given premises by constructing the relevant truth table is called " truth table technique" .

4. Explain rules for inferences theory

Rule 1: A given premises may be introduced at any point of derivation.

Rule 2: A formula S may be introduced in a derivation if S is tautologically implied by one or more of the preceding formulae in the derivation.

Rule 3: If we can derive S from R and a set of given premises then we can derive $R \rightarrow S$ from the set of premises alone.

5. Define indirect method of proof.

The method of using the rule of conditional proof and the notation of an inconsistent set of premises is called the indirect method of proof or proof by contradiction or reduction and absurdum.

6. What are the techniques of indirect method of proof?

i) Introduce the negation of the desired conclusion as a new premise.

ii) From the additional or new premise, together with the given premises, derive a contradiction.

iii) Assert the desired conclusion as a logical inference from the premises.

7. What is called " theory of inferences" ?

The analysis of the validity of the formula from the given set of premises by using derivation is called " theory of inferences" .

8. Define consistency of premises.

The given set of premises $P_1, P_2, ...P_m$ are said to be consistent if and only if $P_1 \wedge P_2 \wedge ...... \wedge P_m \Rightarrow T$.

9. Define inconsistency of premises.

The given set of premises $P_1, P_2, ...P_m$ are said to be inconsistent if and only if $P_1 \wedge P_2 \wedge ...... \wedge P_m \Rightarrow F$.

10. Define predicate logic and m-place predicate

The logic based upon the analysis of predicates in any statement is called predicate logic.

A predicate requiring m (m > 0) names is called an m-place predicate.

11. Define " Universally valid statement"

If the statement is valid for all the values, then the statement is called universally valid statement.

12. Define " existentially valid statements"

If the statement is valid for some of the values, then the statement is called " existentially valid statement" .

13. Define " simple statement function"

A simple statement function of one variable is defined to be an expression consisting of a predicate symbol and an individual variable.

14. Define quantifiers

Statements involving the words that indicate quantity such as ' all' ,' some' ,' none' , or ' one' are called quantifiers.

15. Define " Universal quantifiers"

The quantifier ' all' is the universal quantifier. We denote it by the symbol (∀ x).

The symbol (∀ x) represents the following phrases.

(i) For all x

(ii) For every x

(iii) For each x

(iv) Everything x

(v) Each thing x.

16. Define " existential quantifiers"

The quantifier ' some' is existential quantifier. We denote it by the symbol $(\exists x)$. The symbol $(\exists x)$ represents each of the following phrases.

(i)For some x

(ii) Some x such that

(iii) There exist an x such that.

17. What are the rules to obtain a well formed formula of predicate calculus?

(i)An atomic formula is a well formed formula.

(ii)If Q is a well formed formula, and then $\rceil Q$ is a well formed formula.

(iii) If A and B are well formed formulas, then $(A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B)$ are also well formed formulas.

(iv) If A is well formed formula, x is any variable then (x)A and (∃ x)A are well formed formulas.

(v) Only those formulas obtained by using rules (1) to (4) are well formed formulas.

18. Explain Bound and free variables

If the given formula containing a part of the form (x)P(x) or (∃ x)P(x), such a part is called an x-bound part of the formula. Any occurrence of the variable x in an x bound part of the formula is called bound variable (bound occurrence of x), otherwise it is called free variable.

19. Symbolize the expression:" x is the father of mother of y"

Let P(x): x is a person: F(x, y): x is the father of y; M(x, y): x is the mother of y.

We can note that in the expression " x is the father of mother of y" in between x and y there is a person.

Let Z be a person as the mother of Y.

P (z): Z is a person and

M(x,y): Z is the mother of y.

" x is the father of z and z is the mother of y" is true only for some Z.

We symbolize the above expression as $(\exists z)(P(z) \wedge F(x, z) \wedge M(z, y))$

20. Symbolize the following expressions: " For any given positive integer, there is a greater positive integer"

Let us consider the statements,

P(x):" x is a positive integer " : G(x, y):" x is greater than y"

$(\forall x)(P(x)) \rightarrow (\exists y)(P(y) \wedge G(y, x)))$

21. Express " 2 **is an irrational number" using quantifiers.**

Let p be the proposition," 2 is irrational" . Suppose that ┐ P is true. Then 2 is rational.

We will show that this leads to a contradiction. Under the assumption that 2 is rational,

There exists integers a and b with $\sqrt{2} = \frac{a}{b}$

Here a and b have no common factors.

when both sides of this equation are squared, we get $2= a^2/b^2$

$a^2 =2b^2$ implies a^2 is even.

Therefore, a=2c for some integer c.

2b2=4c2;b2 is even.

b must be even.

It has been shown that ┐ P implies that $\sqrt{2} = \frac{a}{b}$ where a and b have no common factors, and 2 divides a and b. This is contradiction since we have shown that ┐ P implies both r and┐ r, where r is the statement that a and b are integers with no common factor.

Hence┐ P is false.

P: 2 is irrational is true.

22. Explain the rules of quantifiers.

Let p(x) be any statement.

Rule US: Universal Specification (or) Universal Instantiation : $(\forall x)P(x) \Leftrightarrow P(y)$

Rule UG: Universal Generalization: $P(y) \Leftrightarrow (x)P(x)$

Rule ES: Existential Specification: $(\exists x)P(x) \Leftrightarrow P(y)$

Rule EG: Existential Generalization: $P(y) \Leftrightarrow (\exists x)P(x)$

23. Let P be " It is cold" and let Q be " It is raining" . Give a simple verbal sentence which describes each of the following statements.

a) $P \wedge Q$ **(b)** $P \vee Q$ **(c)** ┐ P **(d)Q**∨┐ P

(a) It is cold and raining.

(b) It is cold or it is raining.

(c) It is not cold.

(d) It is raining and it is not cold.

24. Let P be " Anand speaks telugu" and Q be " Anand speaks Hindi" .

Give a simple verbal sentence which describes each of the following.

(a) $P \vee Q$ **(b)** $P \wedge Q$ **(c) P**∧ ┐ Q **(d)**┐ P∨┐ Q

(a) Anand speaks Telugu or Hindi.

(b) Anand speaks Telugu and Hindi.

(c) Anand speaks Telugu but not Hindi.

(d) Anand does not speak Telugu or he does not speak Hindi.

25. Express the statement " Every student in this class has studied calculus" as a Universal quantification.

Let P(x): x has studied calculus.

Then the statement " Every student in this class has studied calculus" can be written as $(\forall x)$ P(x),

Where the universe of discourse consists of the students in the class. This statement can also be expressed as $(\forall x)(S(x) \rightarrow P(x))$ Where S(x) is the statement " x is in this class" P(x) is as before and the universe of discourse is the set of all students.

26. Symbolize the statements: " All men are giants"

G(x): x is a giant.

$M(x)$: x is a man

$(x)((M(x) \rightarrow G(x))$

UNIT III - LATTICES

1. Explain ordered pair.

An ordered pair consists of two elements in a given fixed order. Ordered pair is not a set consisting of two elements; it represents a point in a two – dimensional plane. Therefore $<x,y> \neq <y, x>$ where $x \neq y$.

2. Explain Cartesian product of sets.

The Cartesian product of sets A, B is written as $A \times B$. It is the set of all ordered pairs in which the first element belongs to A and the second element belongs to B.

(i.e.) $A \times B = \{(x,y) / x \in A, y \in B\}$

3. Prove that $A \times B \neq B \times A$.

Let $A = \{1,2,3\}$

$B = \{\alpha, \beta\}$

$A \times B = \{(1,\alpha), (1,\beta), (2,\alpha), (2,\beta), (3,\alpha), (3,\beta)\}$

$B \times A = \{(\alpha,1), (\alpha,2), (\alpha,3), (\beta,1), (\beta,2), (\beta,3)\}$

$\therefore A \times B \neq B \times A$.

4. Explain Binary Relation with an example.

A Binary Relation R from a set A to a set B is a subset of $A \times B$. (i.e.) Any subset of a set of ordered pairs defines a relation. When A = B, we say that R is a relation defined on A.

5. Explain properties of Binary Relation.

A relation R on a set A is

Reflexive if $xRx, \forall x \in A (or) (x, x) \in R$

Symmetric if $xRy \Rightarrow yRx$ for $x, y \in A$

Transitive if $xRy, yRz \Rightarrow xRz$ for $x, y, z \in A$

Irreflexive if $x \not R x, \forall x \in A (or) (x, x) \notin R$

Antisymmetric if $xRy, yRx \Rightarrow x = y$ for $x, y \in A$

6. Define Equivalence Relation.

A relation R on a set A is said to be an equivalence relation if it is

i. Reflexive
ii. Symmetric
iii. Transitive

7. Explain Partial Ordering Relation.

A relation R on a set X is said to be a partial ordering relation if it is

i. Reflexive
ii. Antisymmetric and Transitive

8. Give an example of a relation which is neither reflexive nor Irreflexive

Let A = {1, 2, 3}

Let R be a relation defined on A such that R = {(1,1), (1,2), (1,3), (2,1), (3,1)} is

i. Not reflexive (2,2), (3,3) does not belong to R.
ii. Not Irreflexive (1,1) belongs to R.

9. Give an example of a relation which is both symmetric and Antisymmetric.

Let A = {1, 2, 3} be a set and R be a relation defined on A such that R = {(1,1), (3,3)} which is

i. Symmetric, since xRy yRx
ii. Antisymmetric, since xRy, yRx x = y

10. Give an example of a relation which is tranisitive.

Let A = {1, 2, 3} be a set and R be a relation defined on A such that R = {(1,1), (1,2), (2,3), (1,3), (3,3)} is transitive since xRy, yRz xRz, x,y,z∈A.

11. Define lower bound of poset.

Let (L,≤) be a poset and let a,b ∈ L. If there exists an element c∈L, such that c ≤ a and c ≤ b then c is lower bound of a, b.

12. Define upper bound of poset.

Let (L,≤) be a poset a,b∈L. If there exist an element c∈L such that a ≤ c and b ≤ c then c is said to be upper bound of a, b.

13. Define Least Upper Bound (or) Supremum.

Let (L,≤) be a poset, and A⊆L. Any element x∈L is least upper bound for A if x is an upper bound of A and x ≤ y for any upper bound y of A.

14. Define Greatest Lower Bound (or) Infimum.

Let (L,≤) be a poset and A⊆L. Any element x∈L is the greatest lower bound for A, if x is a lower bound of A and y ≤ x for any lower bound y of A.

15. Define Lattice.

A lattice is a poset in which every pair of elements has greatest lower bound and least upper bound.

16. Explain Properties of Lattice.

Let (L,≤) be a lattice. The following properties hold good for a,b,c∈L.

Idempotent law

i. $a * a = a$
ii. $a \oplus a = a$

Associative law

i. $a * (b * c) = (a * b) * c$
ii. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

Commutative law

i. $a * b = b * a$
ii. $a \oplus b = b \oplus a$

Absorption law

i. $a \oplus (a * b) = a$
ii. $a * (a \oplus b) = a$

17. Name some Special Lattices.

Chain
Distributive Lattice
Complemented Lattice
Modular Lattice

18. Define Chain.

A lattice $(L,\leq)$ is called a chain if,

i. $a \leq b \leq c$
ii. $a \geq b \geq c, \forall a, b, c \in L$

19. Define Distributive Lattice.

A lattice $(L,*,\oplus)$ is called distributive lattice if the operations * and $\oplus$ are distributive over each other. That is,

i. $a * (b \oplus c) = (a * b) \oplus (a * c)$
ii. $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

20. Define Complemented Lattice.

A lattice (L,*,⊕ , 0,1) is called a complemented lattice if every element of L has at least one complement in L.

21. Define Modular Lattice.

A lattice (L,*,⊕) is called a modular lattice if $a \leq c$ $a \oplus (b * c) = (a \oplus b) * c$

22. State De Morgan' s Law over * and ⊕.

$(a * b)' = a' \oplus b'$

$(a \oplus b)' = a' * b'$

23. Write Distributive Inequality

Let (L,≤) be a lattice, for any a, b, c ∈ (L,≤), we have

1. $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$
2. $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

24. Write Modular Inequality.

Let (L,≤) be a lattice and any a, b, c ∈ L, we have, if $a \leq c$ then $a \oplus (b * c) \leq (a \oplus b) * c$.

25. Define sub lattice.

Let (L,*,⊕) be a lattice and let S ⊆ L be a subset of L. The algebra (S,*,⊕) is a sub lattice of (L,*,⊕) if S is closed under both operations * and ⊕.

26. Define Direct Product of Lattices.

Let (L,*, ⊕) and (S, ∧ , ∨) be two lattices. The algebraic system (LxS, g,+) in which the binary operations + and g on LxS are such that for any (a_1,b_1) and (a_2,b_2) in LxS.

$$(a_1,b_1)\ g\ (a_2,b_2) = (a_1 * a_2, b_1 \wedge b_2)$$

$$(a_1,b_1)\ g\ (a_2,b_2) = (a_1 \oplus a_2, b_1 \vee b_2)$$

is called the direct product of the lattices (L,*,⊕) and (S,∧,∨).

27. Define homomorphism of lattices.

Let (L,*,⊕) and (S,∧,∨) be two lattices. A mapping g : L   S is called a lattice homomorphism from the lattice (L,*,⊕) to (S,∧,∨) if for any a, b ∈ L. $g(a * b) = g(a) \wedge g(b)$ and $g(a \oplus b) = g(a) \vee g(b)$.

28. Define Complete Lattices.

A lattice is called complete if each of its non empty subsets has a least upper bound and greatest lower bound.

29. Define Boolean algebra.

A Boolean algebra is a lattice which is both distributive and complemented.

30. Define Hasse diagram or poset diagram.

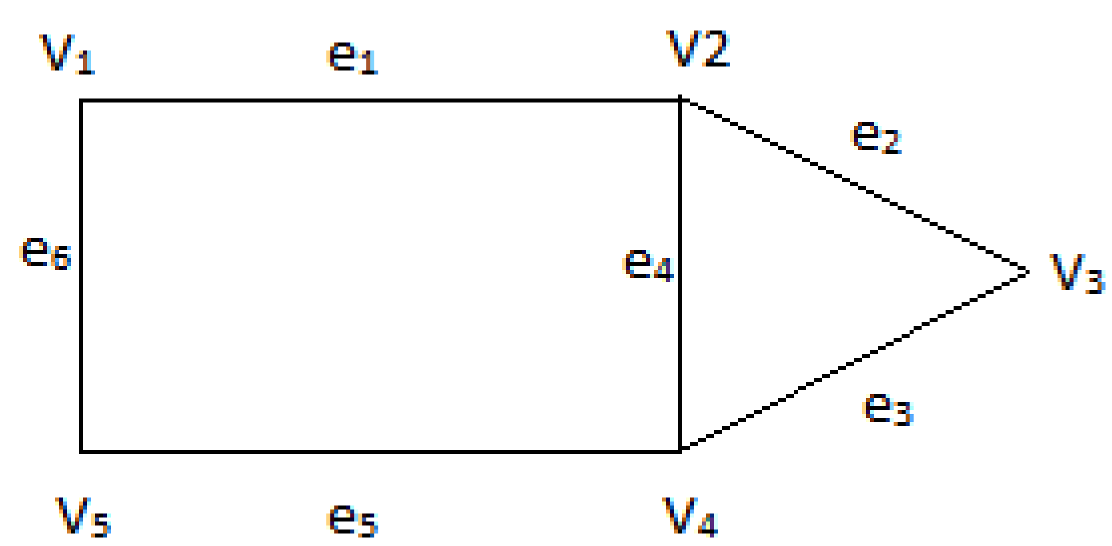
The graphical representation of a finite poset is called Hasse diagram or poset diagram.

UNIT – IV - GRAPH THEORY

1. Graph

A graph ' G' is an ordered pair G={V,E} where V is an non – empty set of vertices and E is the set of edges. The vertices are also known as nodes or points.

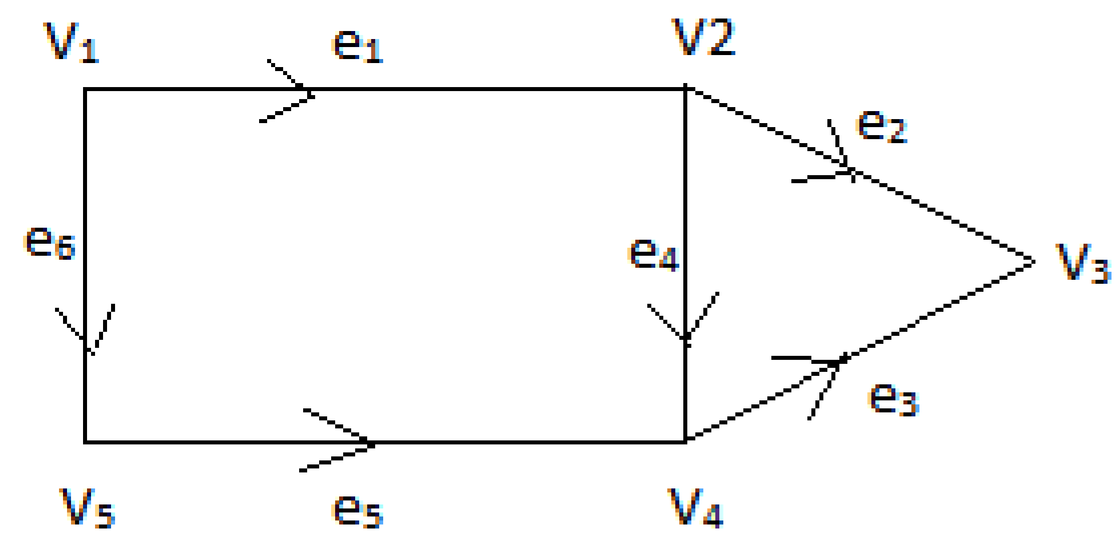
Example



Here G={V,E} where vertex set $V=\{V_1,V_2,V_3,V_4,V_5\}$ and edge set $E=\{e_1,e_2,e_3,e_4,e_5,e_6\}$.

2. Directed graph

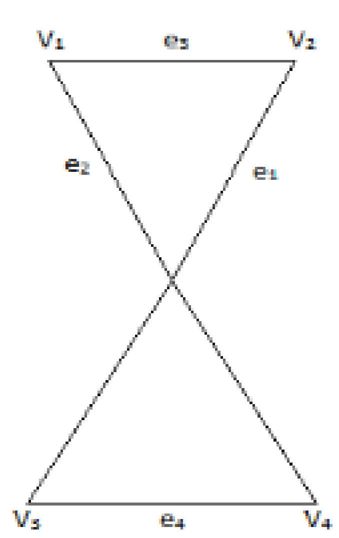
A graph in which every edge is directed is called a directed graph or digraph.



3. undirected graph

A graph in which every edge is not directed is called an undirected graph.

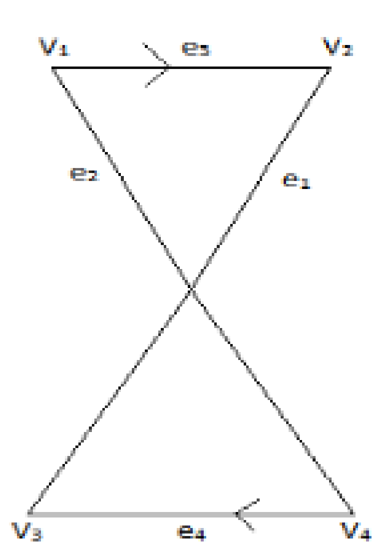
Example



4. MIXED GRAPH

If some edges are directed and some edges are not directed in a graph, then it is called a mixed graph.

Example



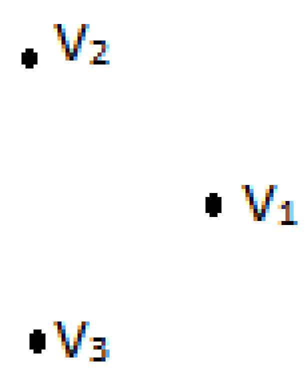
5. Finite graph

A graph is finite if both of its vertex set and edge set are finite; otherwise it is an infinite graph.

6. Null graph

A graph in which there is no edge is called a null graph.

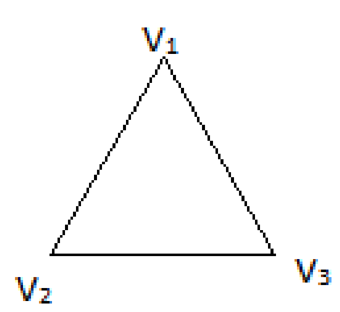
Example



7. Adjacent vertices

Any pair of vertices that are connected by an edge in a graph is called adjacent vertices.

Example



V_1, V_2 are adjacent vertices.

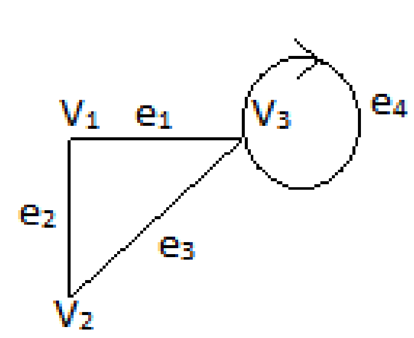
V_1, V_3 are adjacent vertices.

V_2, V_3 are adjacent vertices.

8. Self loop

An edge of a graph which joins a vertex to itself is called a self loop or loop.

Example

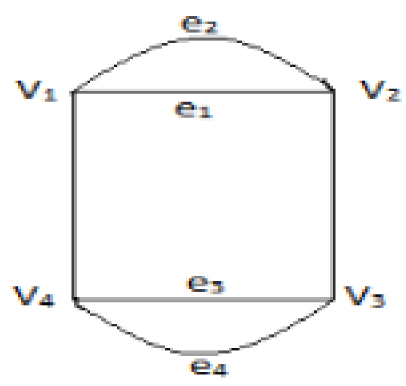


Here e_4 is a self loop

9. Parallel edges or Multiple edges

In a graph, if some pairs of vertices are joined by more than one edge, then the edges are called as parallel edges.

Example



e_1, e_2 are parallel edges.

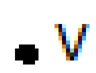
e_3, e_4 are parallel edges.

10. Trivial graph

A finite graph with only one vertex and no edges is called a trivial graph.

i.e $|E(G)| = 0$

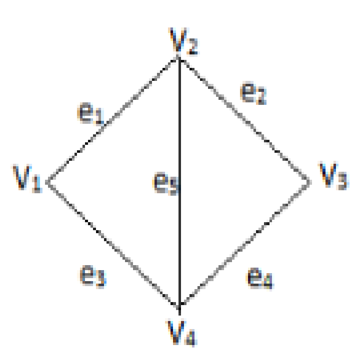
Example



11. Simple graph

A graph without any parallel edges and self loops is called a simple graph.

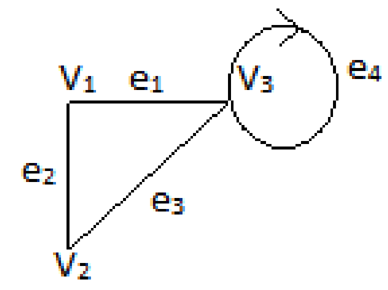
Example

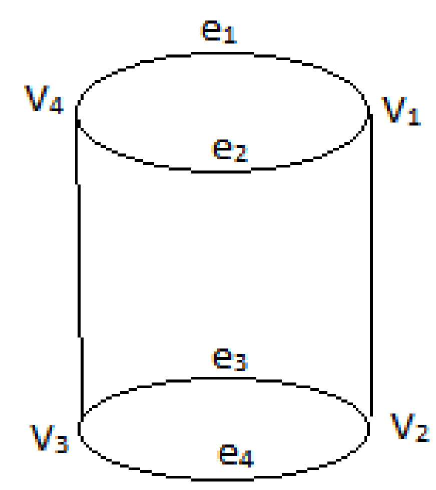


12. Multi – graph

A graph that contains some parallel edges (or) self loops (or) both is called as multi graph.

Example

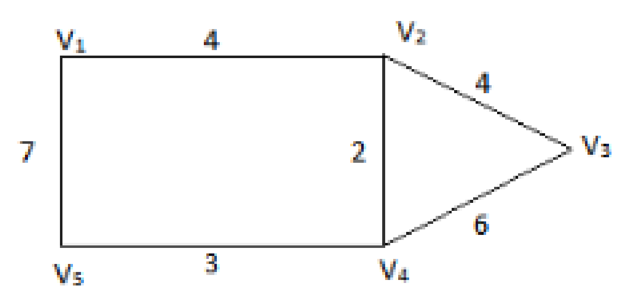




13. Weighted graph

A graph in which every edge is assigned a weight is called a weighted graph.

Example



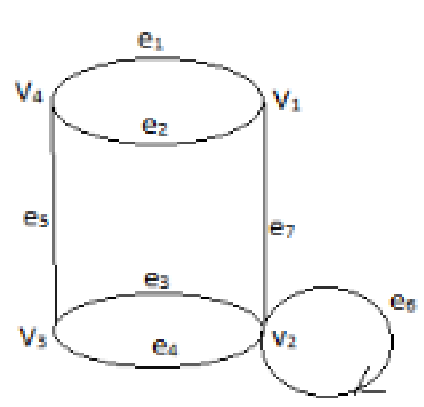
14. Degree of a vertex

Let V be a vertex in a graph ‘ G’ . Then the degree of V in G is the number of edges incident with V. (Each loop is counted twice.). Degree of V is denoted by deg(V) or d(V).

Example

In the figure below

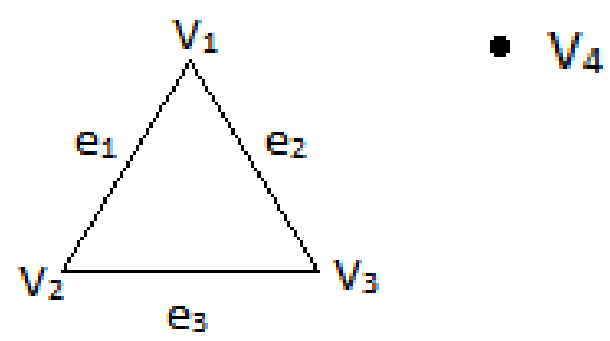
$d(V_1) = 3$, $d(V_2) = 5$, $d(V_3) = 3$, $d(V_4) = 3$



15. Isolated vertex

In any graph, if degree of a vertex is 0 (zero), then that vertex is called an isolated vertex.

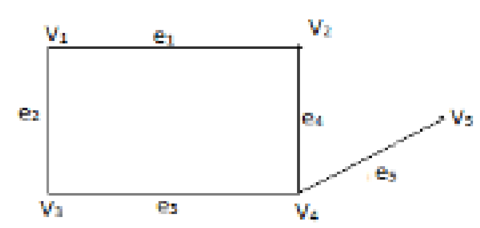
Example



Here V_4 is the isolated vertex.

16. Pendant vertex

In any graph, if the degree of a vertex is 1 (one), then that vertex is called pendant vertex.



Here V_5 is the pendant vertex in the example given above.

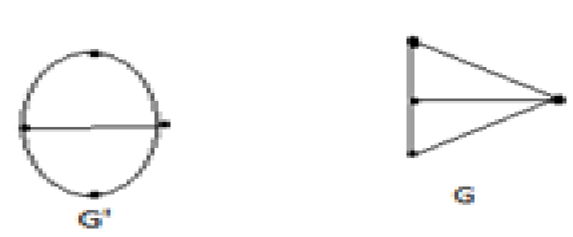
17. Degree sequence

If $V_1, V_2, \ldots \ldots .. V_n$ are n vertices of G, then the sequence $d_1, d_2 \ldots .. d_n$ is called the degree sequence of G, where $d_1 \le d_2 \le d_3 \ldots .. \le d_n$, $d_i = d(V_i)$.

18. Isomorphism in graph

Two graph G and G' are said to be isomorphic if there is a one – to – one correspondence between their vertices and edges, such that their incidence relationship is preserved.

Example



19. Sub graph

Let G be a graph, then H is said to be a sub graph of G, if all vertices and edges of H are in G and each edge of H has the same end vertices in H as in G.

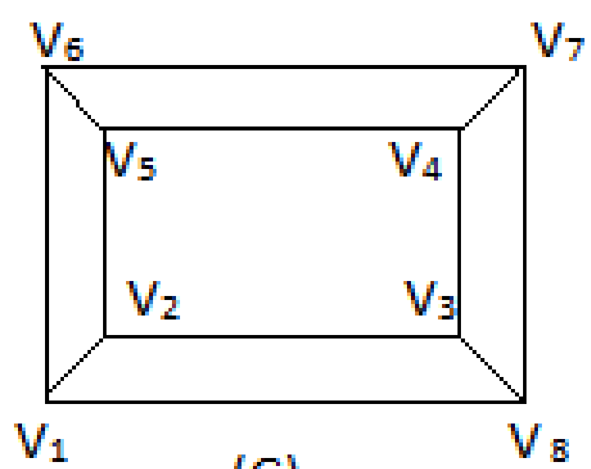
20. Edge disjoint sub graph

Two sub graphs H_1 and H_2 of G are said to be edge disjoint sub graph if H_1 and H_2 do not have any edges in common.

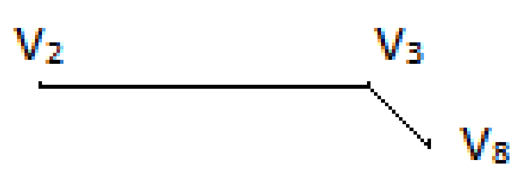
21. Vertex disjoint sub graph

Two sub graphs H_1 and H_2 of G are said to be vertex disjoint sub graphs if H_1 and H_2 do not have any vertices in common.

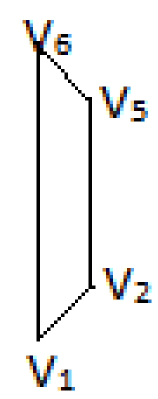
Example . Let G be a graph.



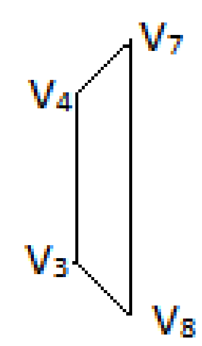
(G)



(H_1)



(H_2)



(H_3)

(i) Here H_1, H_2, H_3 are sub graphs of G.
(ii) H_1, H_2 are edge disjoint sub graph.
(iii) H_2, H_3 are vertex disjoint sub graph.
(iv) H_1, H_2 are edge disjoint but not vertex disjoint.

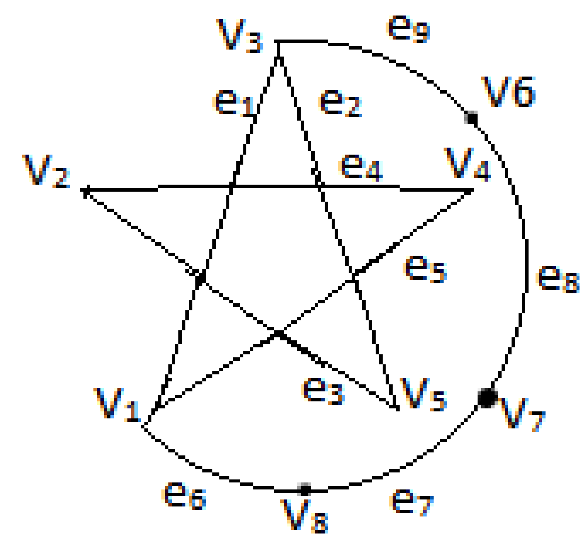
22. Walk

A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices such that each edge is incident with the preceding and the following vertices.

Note: In a walk, no edge appears more than once whereas the vertices can occur.

23. Open walk

If the terminal vertices of a walk (end vertices) do not coincide, then it is called an open walk.

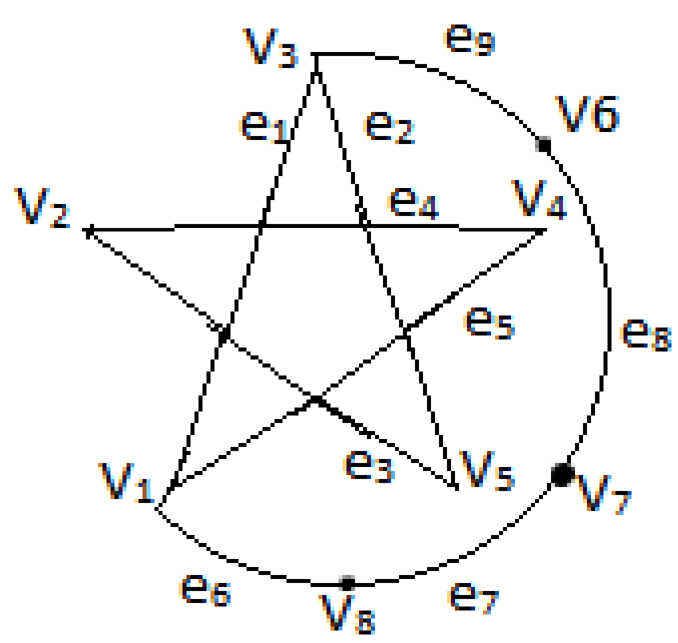


Open Walk : $V_1\, e_1\, V_3\, e_2\, V_5\, e_3\, V_2\, e_4\, V_4\, e_5\, V_1\, e_6\, V_8$

24. Closed walk

If the terminal vertices of a walk coincide, then it is called as a closed walk.

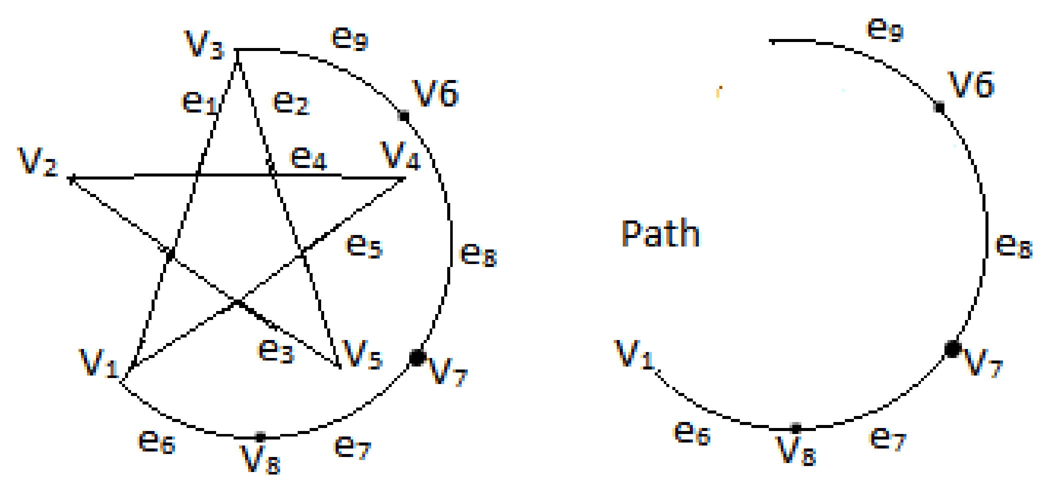
Example. Let G be a graph.



Closed Walk : $V_1\, e_1\, V_3\, e_2\, V_5\, e_3\, V_2\, e_4\, V_4\, e_5\, V_1$

25. Path

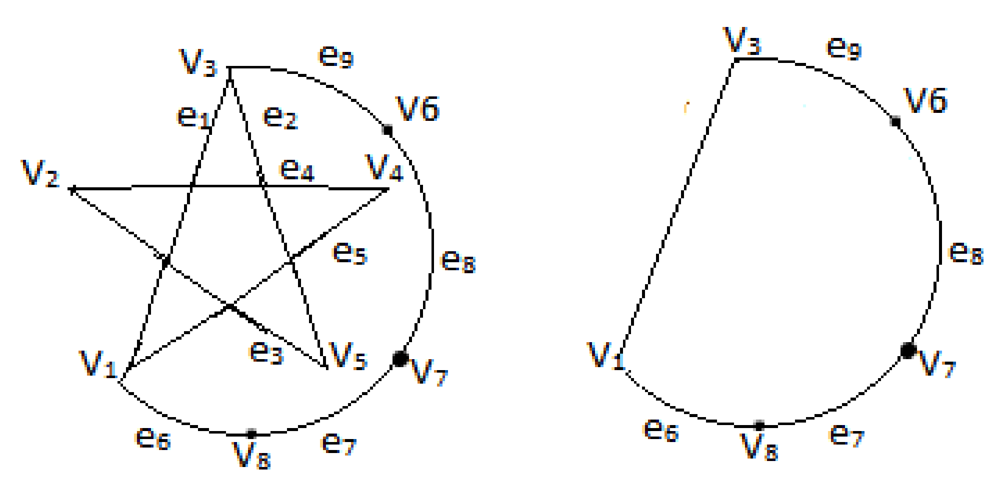
In an open walk, if all the vertices appear only once, then it is called a path. In a path the end vertices are of degree 1 and other vertices are of degree 2. A path cannot have a loop or self loop.



Path : V_3 e_9 V_6 e_8 V_7 e_7 V_8 e_6 V_1

26. Circuit

A closed walk in which no vertex appears more than once, except the end vertices is called as a circuit.

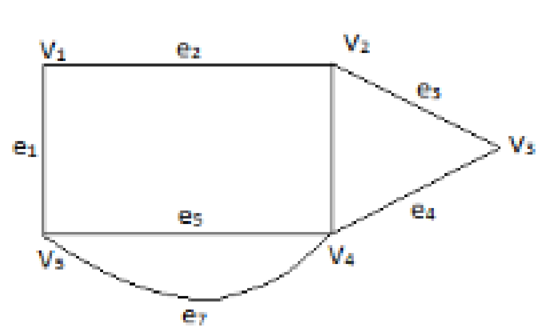


In a circuit all vertices are of degree 2.

27. Connected graph

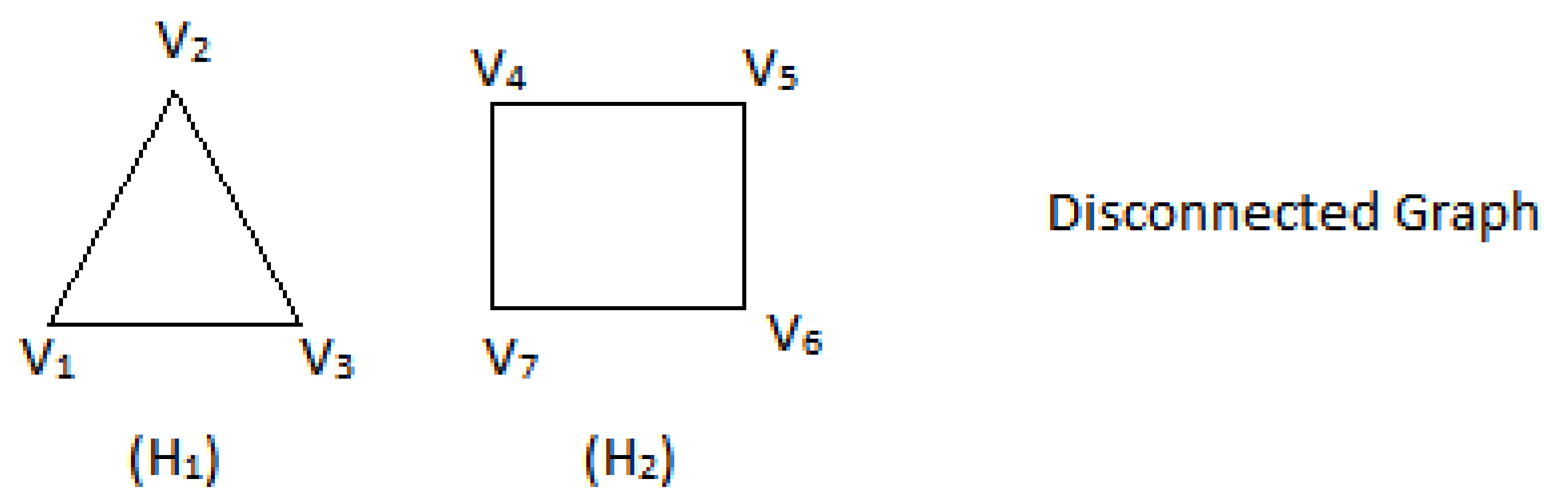
A graph G is said to be connected if there exist at least one path between every pair of vertices in G, otherwise G is disconnected.

Example



28. Disconnected graph

In a disconnected graph, there exist at least one pair of vertices which do not have any path between them. Consider the example given below,



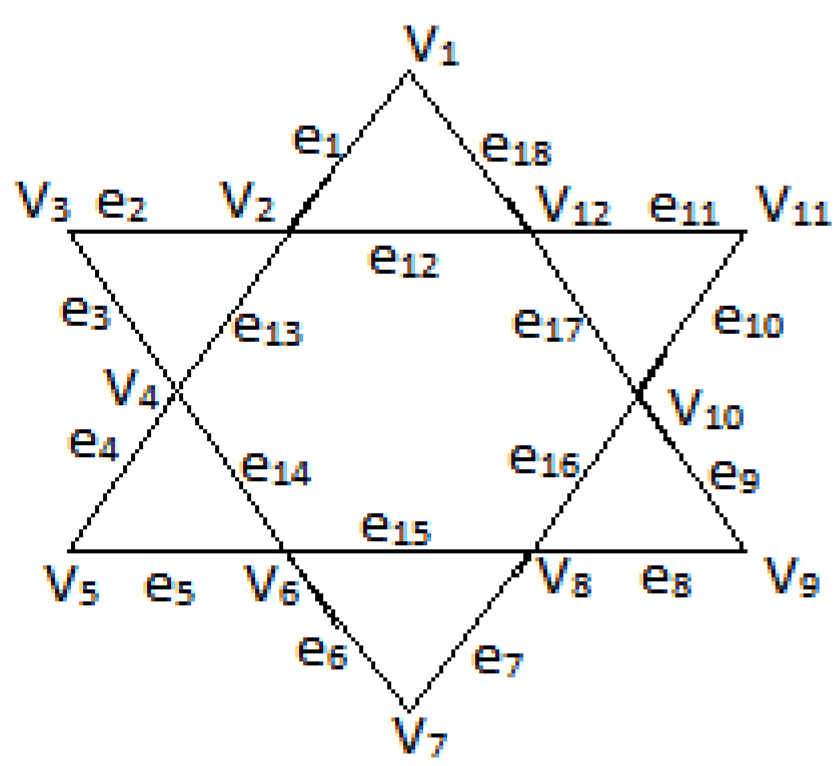
H_1 and H_2 are components. Any disconnected graph consist of 2 or more connected graph each of which is called a component.

29. Euler graph

A graph G is said to be Euler graph if it has a Euler circuit.

Euler circuit (Euler Tour) is a circuit in G which covers all the edges of G exactly once.

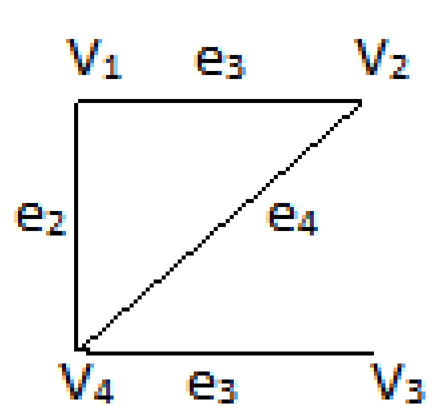
Example



30. Unicursal graph

Let G be an graph, an open walk covering all the edges in G exactly once is called a unicursal line. A graph which consists of unicursal line is called a unicursal graph.

Example

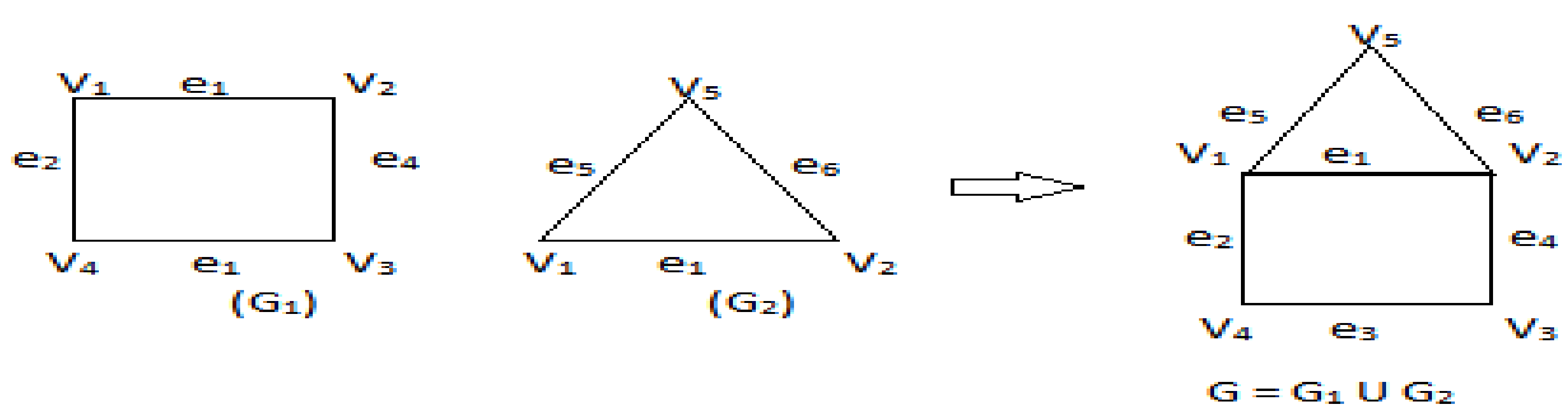


Unicursal line: (V_4 e_1 V_3 e_2 V_1 e_3 V_2 e_4 V_3)

31. Union of graphs

Let G_1 = (V_1, E_1) G_2 = (V_2, E_2). The union of G_1 and G_2 is a graph G = G_1 U G_2 whose vertex set

V = V_1 U V_2 and edge set E = E_1 U E_2



32. Intersection of graphs

The intersection of G_1 and G_2 is a graph G = G_1 I G_2 consisting only those vertices and edges which are in both G_1 and G_2.

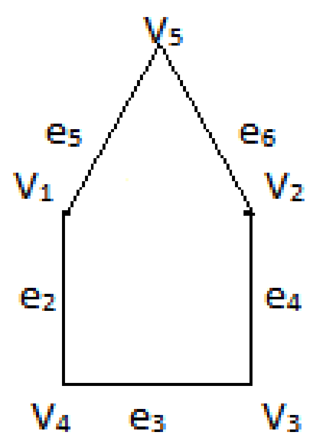
Example Consider the graph G_1, G_2 in the above example.

G_1 I G_2

33. Ring sum of graphs

The ring sum of two graphs G_1 and G_2 is a graph G = $G_1 \oplus G_2$ consisting of the vertex set V_1 U V_2 and edges that are either in G_1 or in G_2, but not in both.

Consider G_1 and G_2 in the above example, $G_1 \oplus G_2$ is as follows



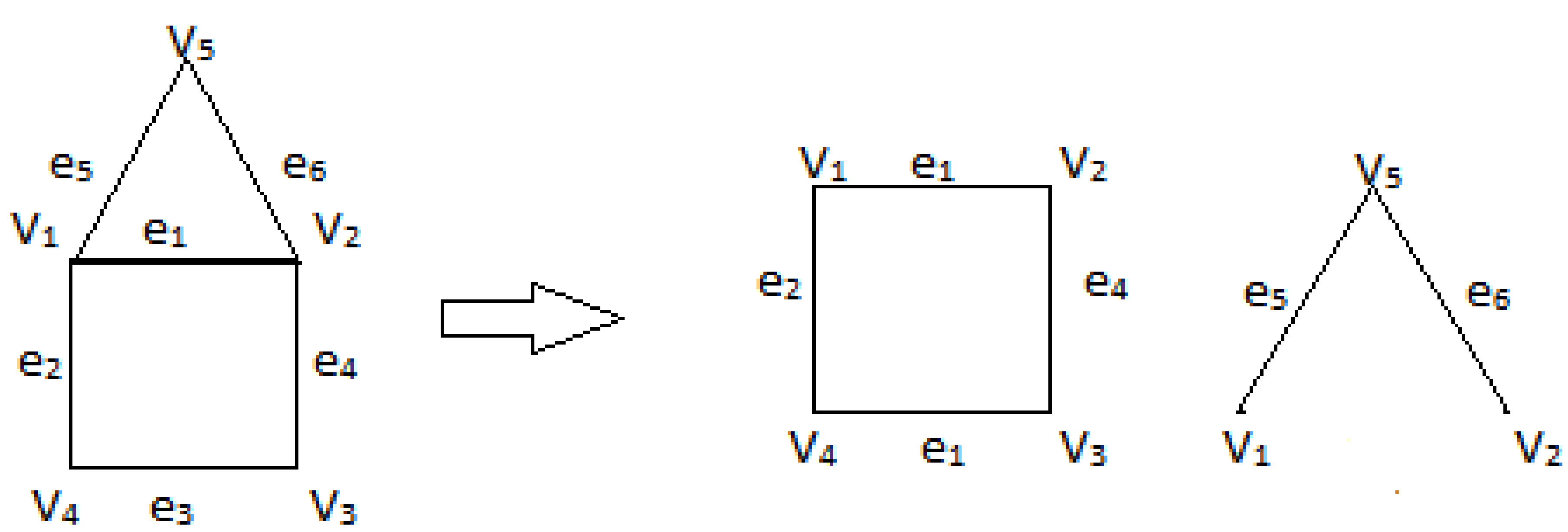
.

34. Decomposition

A graph G is said to have been decomposed into sub graphs g_1, g_2 if

(i) $g_1 \cup g_2 = G$
(ii) $g_1 \cap g_2$ = { Null graph }

Example. Let G be a graph.



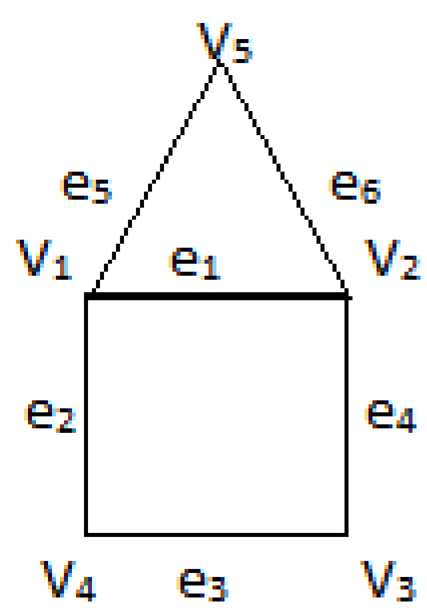
(G) (H1) (H2)

35. Complement

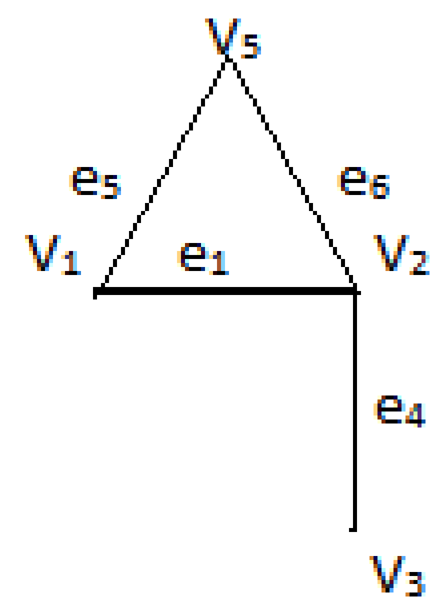
If H is a sub graph of G, then $G \oplus H$ is a sub graph, which consists of edges of G, that are not in H. $G \oplus H$ can also be written as $G - H$ and is called as complement of H in G.

36. Deletion

If V_i is the vertex of G, then $g = G - V_i$ is a sub graph of G obtained by deleting V_i from G. The deletion of a vertex V_i implies that deletion of those edges which are incident to V_i.



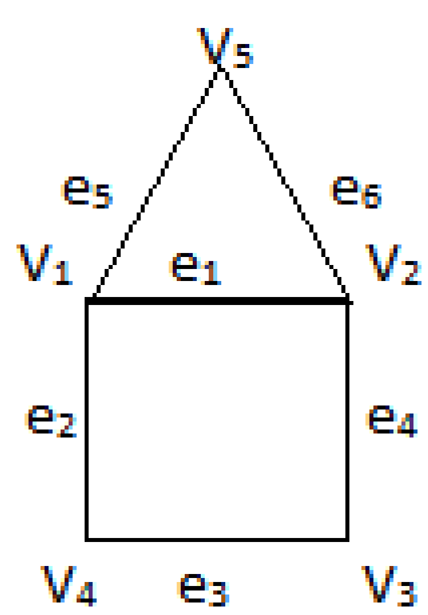
(G)



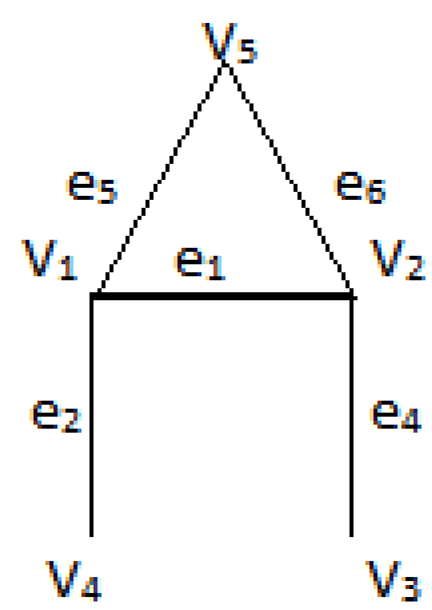
(g)

Example Here $g = G - V_4$

If e_i is an edge of G_1, then $g = G - e_i$ is a sub graph of G by deleting e_i from ' g' . Deletion of an edge does not imply deletion of the end vertices.



(G)



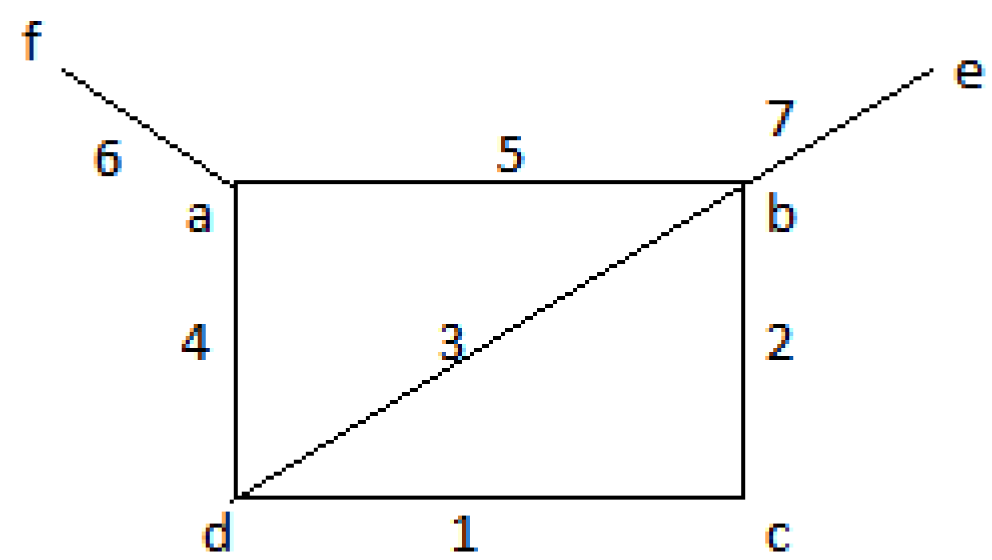
(g)

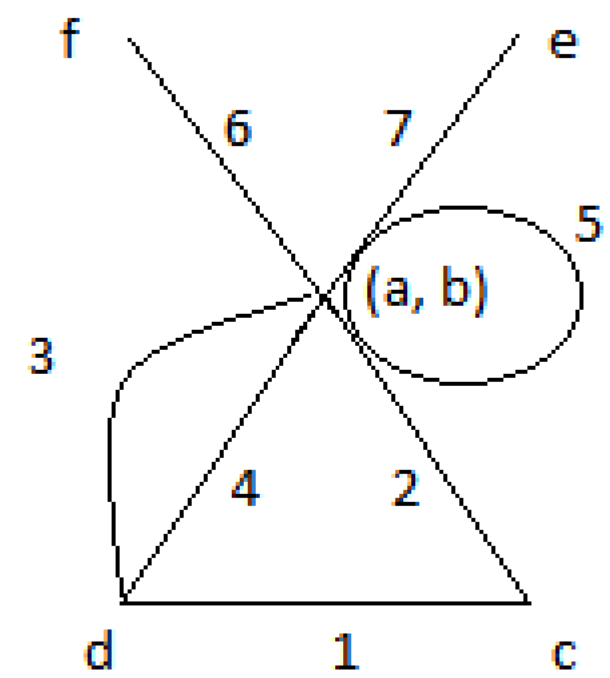
Example Here $g = G - e_3$

37. Fusion

A pair of vertices a, b in a graph are said to be fused (merged) if the two vertices are replaced by a single new vertex such that every edge that was incident on either a (or) b (or) that is incident on the new vertex.

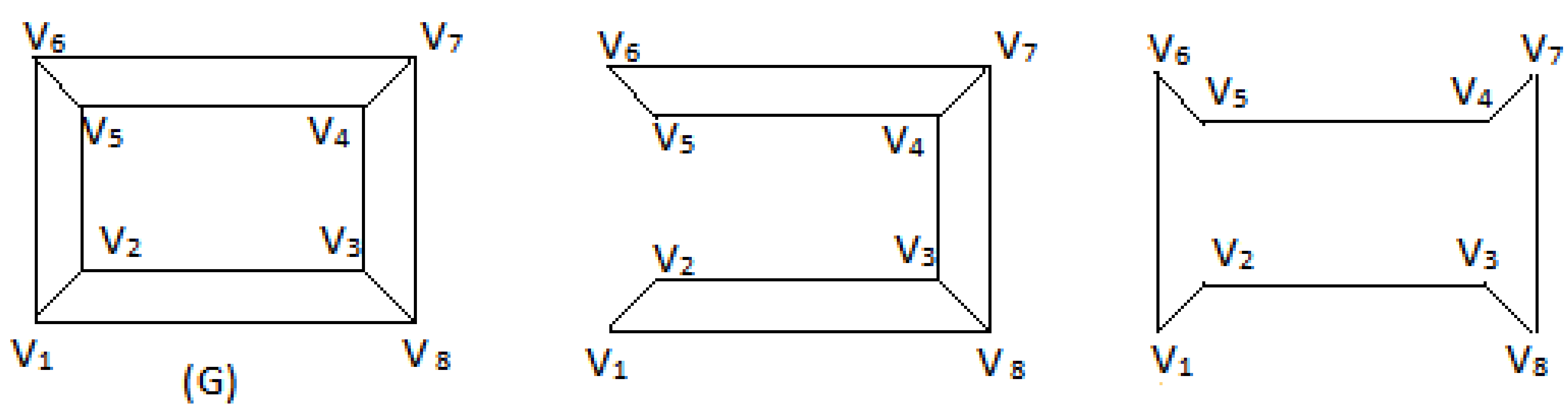
Note: The fusion of two vertices does not reduce the no. of edges.





38. Hamiltonian circuit

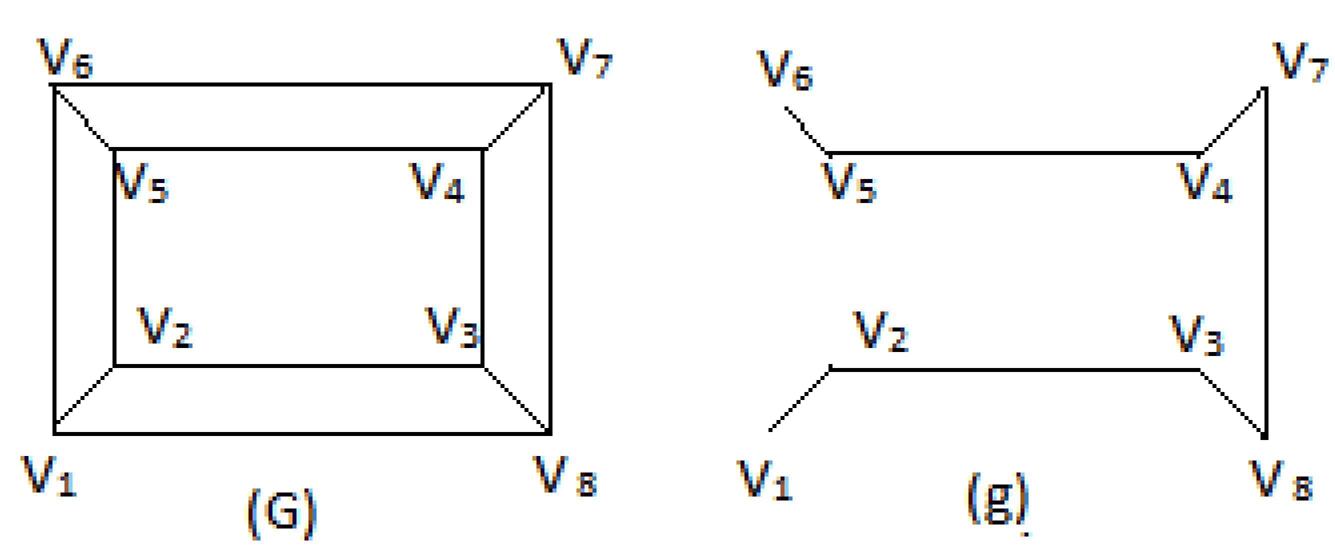
A Hamiltonian circuit is a circuit that covers all the vertices of G, exactly once.



Hamiltonian Circuits

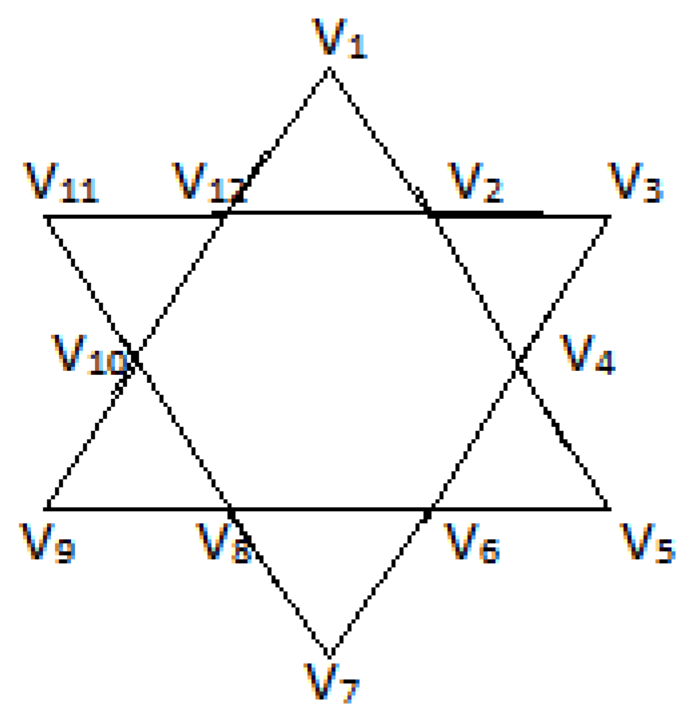
39. Hamiltonian path

The removal of an one edge in a Hamiltonian circuit is called a Hamiltonian path.

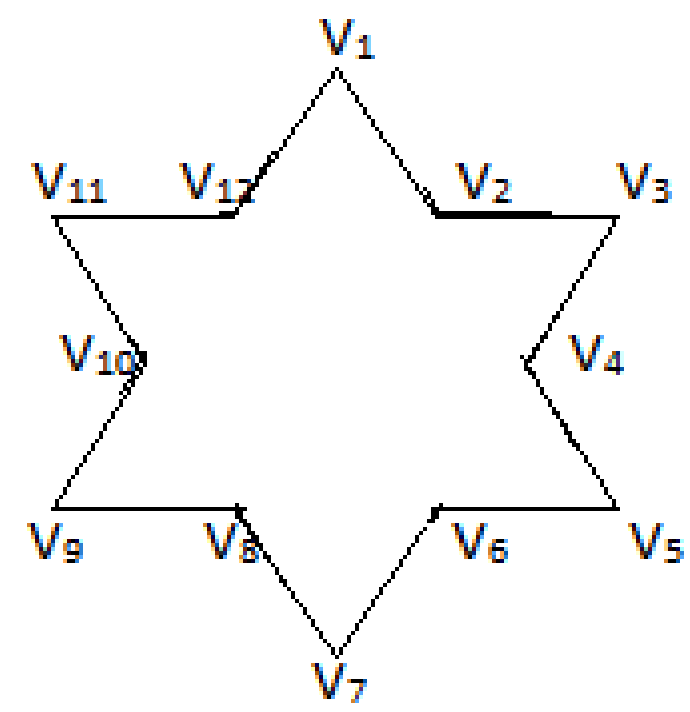


40. Hamiltonian graph

A graph which has Hamiltonian circuit is called Hamiltonian graph.



Hamiltonian Graph



Hamiltonian Circuit

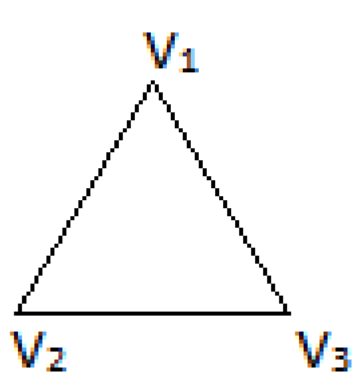
41. Complete graph

A graph G is said to be complete if there exists an edge between every pair of vertices of G.

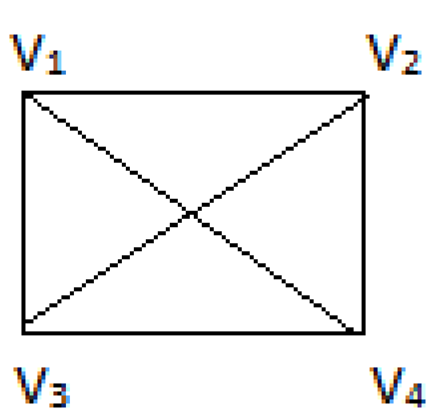
Example The complete graphs with (2, 3, 4, 5) vertices.

K_2

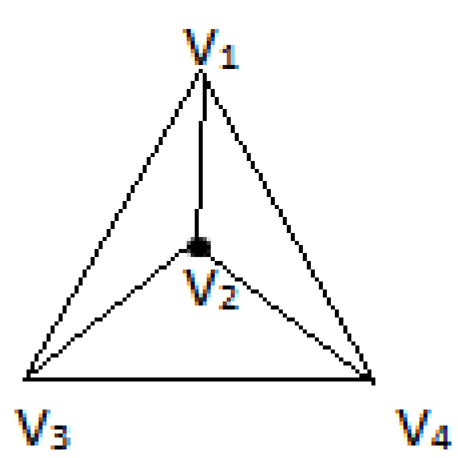
K_3



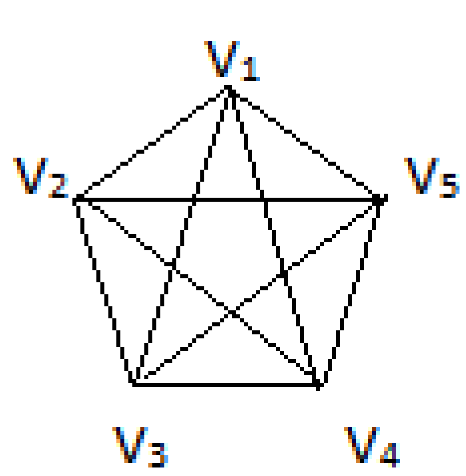
K_4



(or)



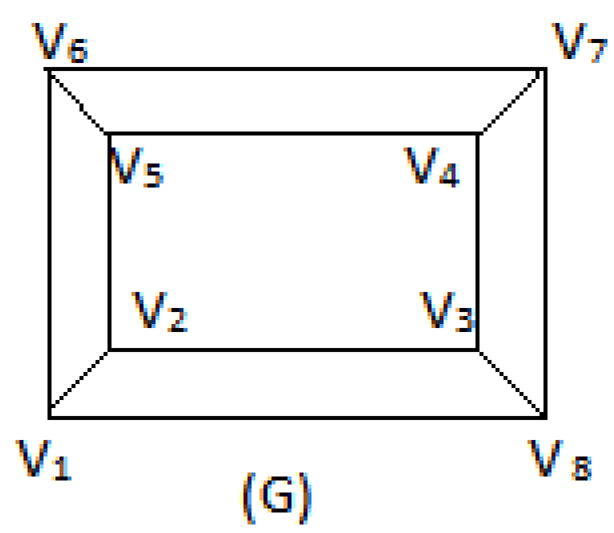
K_5



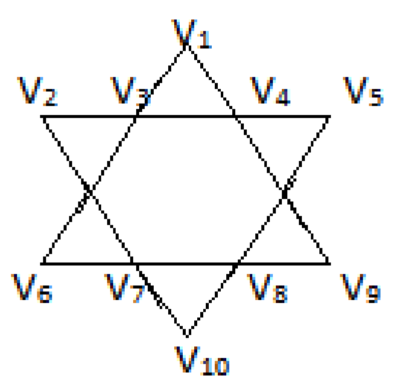
A complete graph G with n vertices is denoted by K_n.

42. Give an example of the following graphs

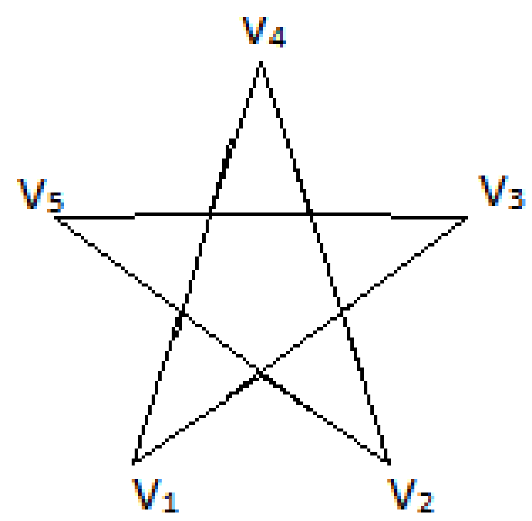
Hamiltonian



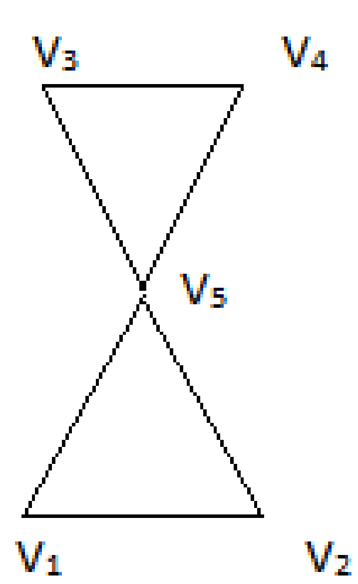
Euler



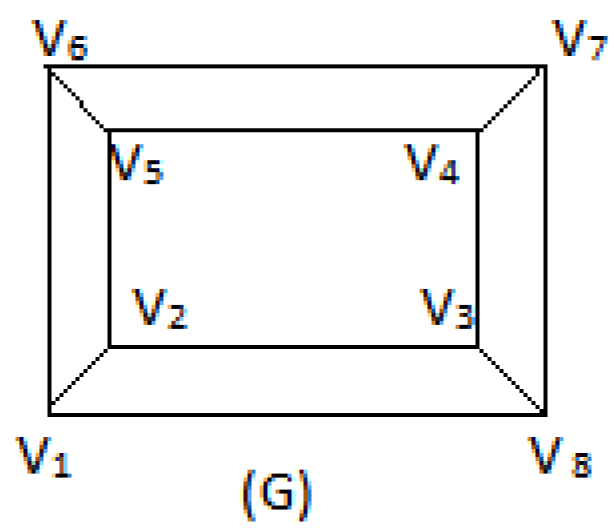
Both Euler and Hamiltonian



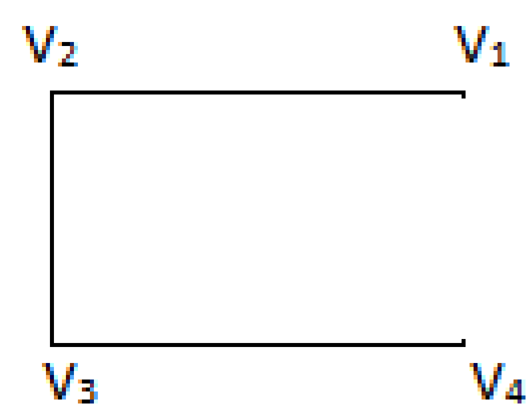
Euler and not Hamiltonian



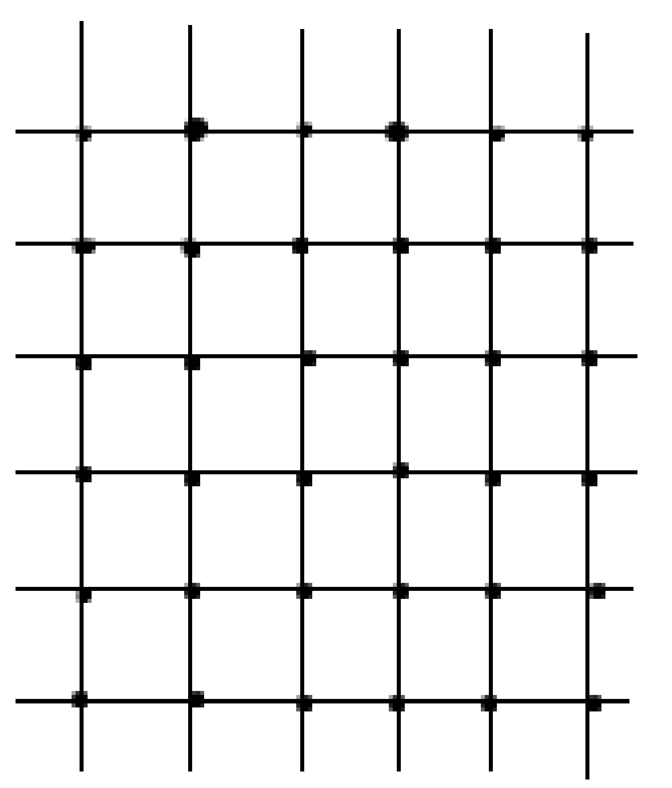
Hamiltonian but not Euler



Neither Euler nor Hamiltonian



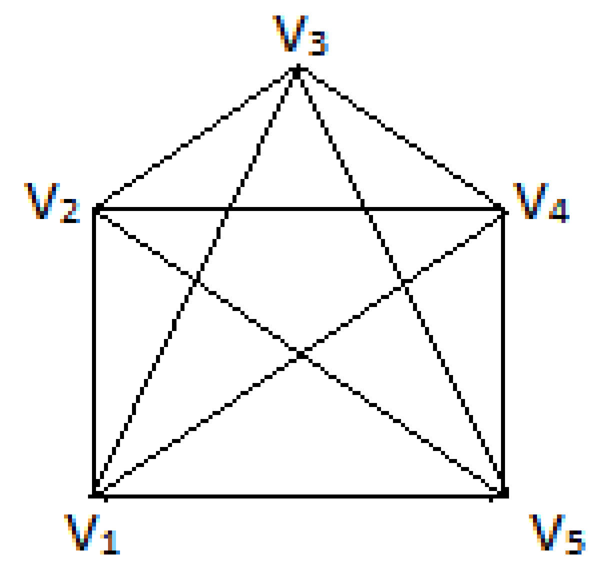
Infinite Graph

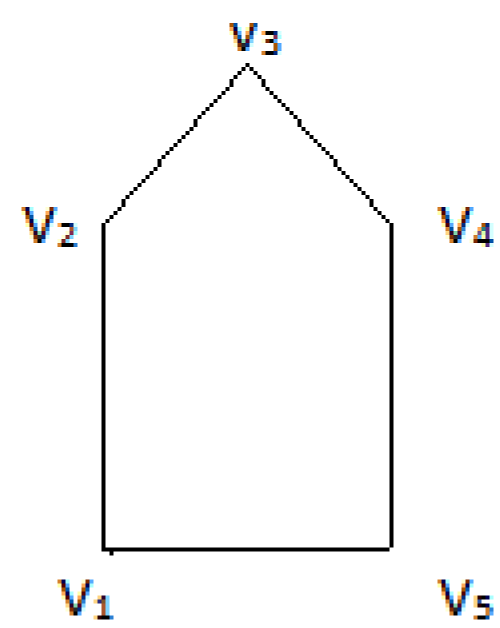


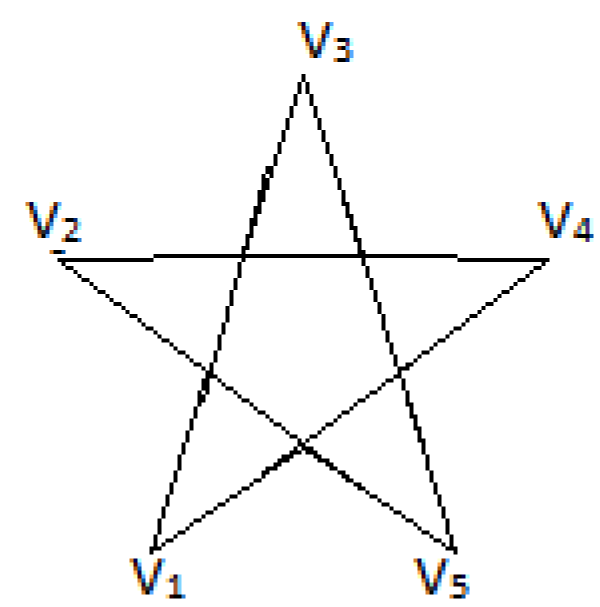
43. Find all the edge disjoint hamiltonian circuits in k_5.

We know that any complete graph with n vertices (K_n) will have $\frac{(n-1)}{2}$ edge disjoint Hamiltonian circuits. $\therefore$ For K_5, $\frac{(n-1)}{2} = \frac{(5-1)}{2} = 2$, there are two edge disjoint Hamiltonian circuits.

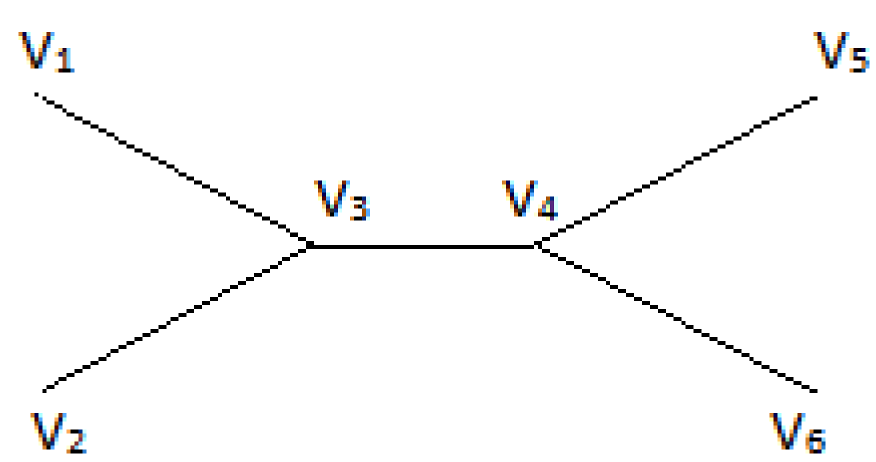
K5







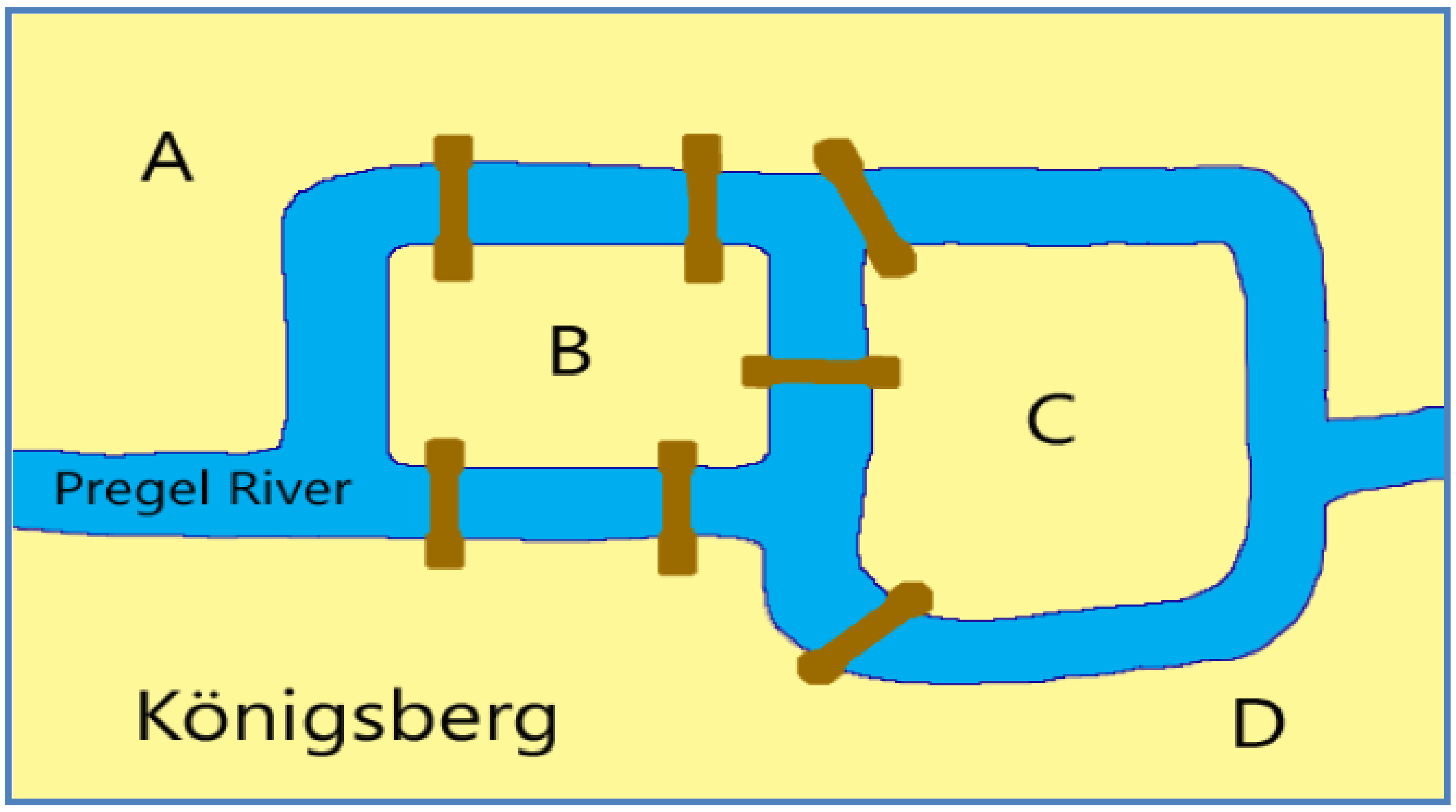
44. Draw a connected graph that becomes disconnected when an edge is removed from it.



45. Explain the applications of graph

(i) Konigsberg bridge problem

The problem was to start at any of the four land areas of the city A, B, C, or D; walk over each of the seven bridges exactly once and return to the same point.



Euler represented this problem by means of graph, in which the land areas correspond to

vertices and bridges correspond to edges.

Euler proved that a solution for this problem does not exist.

(ii) Utilities problem

There are three houses H_1, H_2, and H_3 each to be connected to each of the three utilities – water(W), gas(G) and electricity(E) by means of conduits.. The problem is to provide such conduits connect without any cross – over.

Conduits → edges

Houses and utility suppliers – vertices.

There is no solution for such a problem.

(iii) Electrical network problem

The properties of electrical network depends on

Nature and value of elements forming network such as capacitors, resistors etc.

Topology (or) the way these elements are connected.

The topology of a network can be represented by means of a graph, in which the junctions are represented by vertices and branches are represented by edges.

(iv) Seating arrangement problem

Nine members of a new club meet each day for lunch at a round table. They decided to sit such that every member has different neighbors at each lunch. How many days it would last (arrangement) ?

It is represented by means of a graph in which the members represent vertices and the relationship between them represents edges.

ANS

Two possible arrangements

1	2	3	4	5	6	7	8	9	1
1	3	5	2	7	4	9	6	8	1

By graph theory two more arrangements are possible.

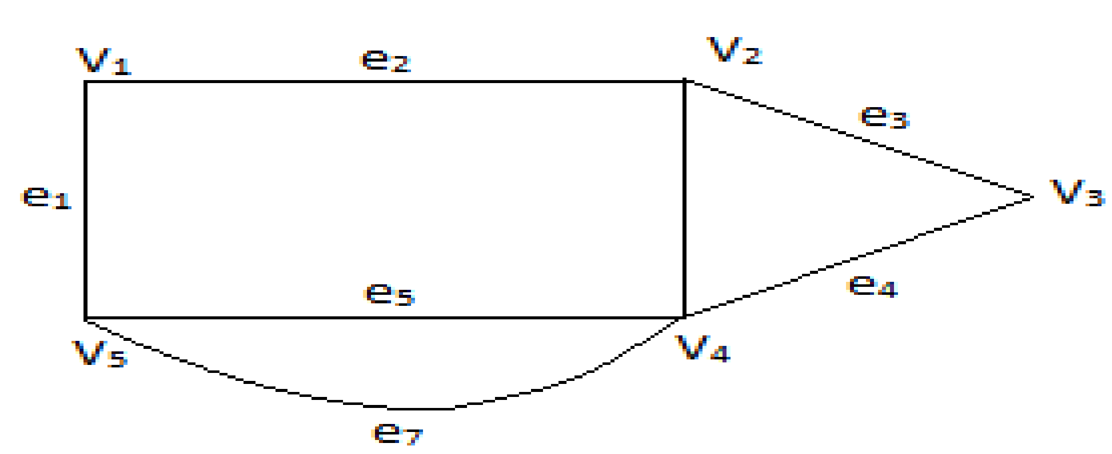
1	5	7	3	9	2	8	4	6	1
1	7	9	5	8	3	6	2	4	1

For any ‘ n’ people, the possible no. of arrangements is,

$$\frac{n-1}{2} \text{ odd}$$

$$\frac{n-2}{2} \text{ even}$$

46.Find the adjacency matrix and incidence matrix of the following graph



Adjacency matrix:

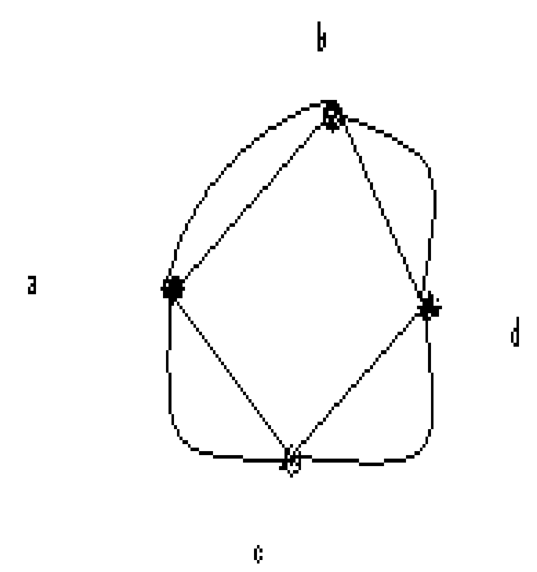
	V_1	V_2	V_3	V_4	V_5

V_1					
V_2	0	1	0	0	1
V_3	1	0	1	1	0
V_4	0	1	0	1	0
V_5	0	1	1	0	2
	1	0	0	2	0

Incidence matrix:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
V_1							
V_2	1	1	0	0	0	0	0
V_3	0	1	1	0	0	1	0
V_4	0	0	1	1	0	0	0
V_5	0	0	0	1	1	1	1
	1	0	0	0	1	0	1

47. What are the degrees of vertices in the graph?



deg(a)= deg(b) = deg(c)= deg(d) =4[4-regular graph]

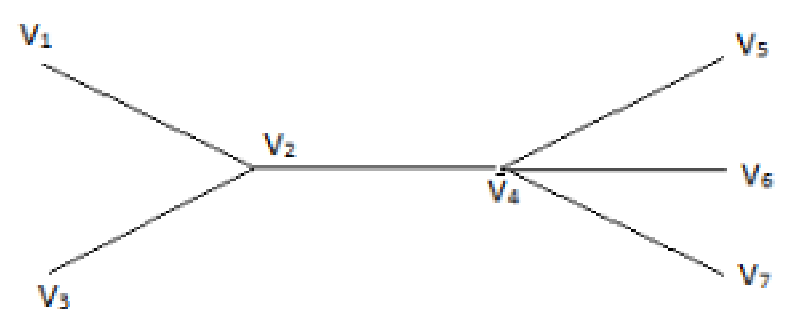
UNIT – V - TREES

1. Different definitions for a tree

A tree is a connected graph without any circuits.
A graph G which has only one path between every pair of vertices is called a tree.
A connected graph with n vertices and (n-1) edges is a tree.

A minimally connected graph is called a tree.

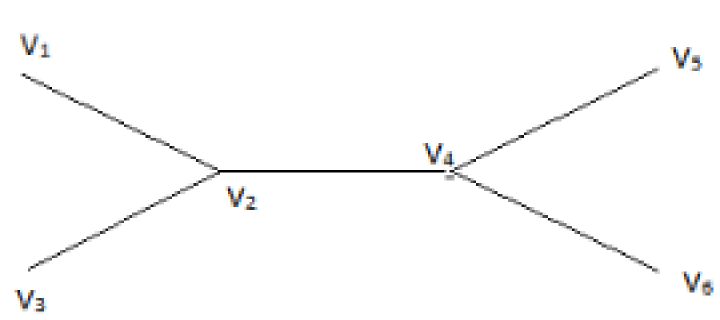
Example



2. Minimally connected

A connected graph is said to be minimally connected, if removal of any one edge from the graph, disconnects the graph.

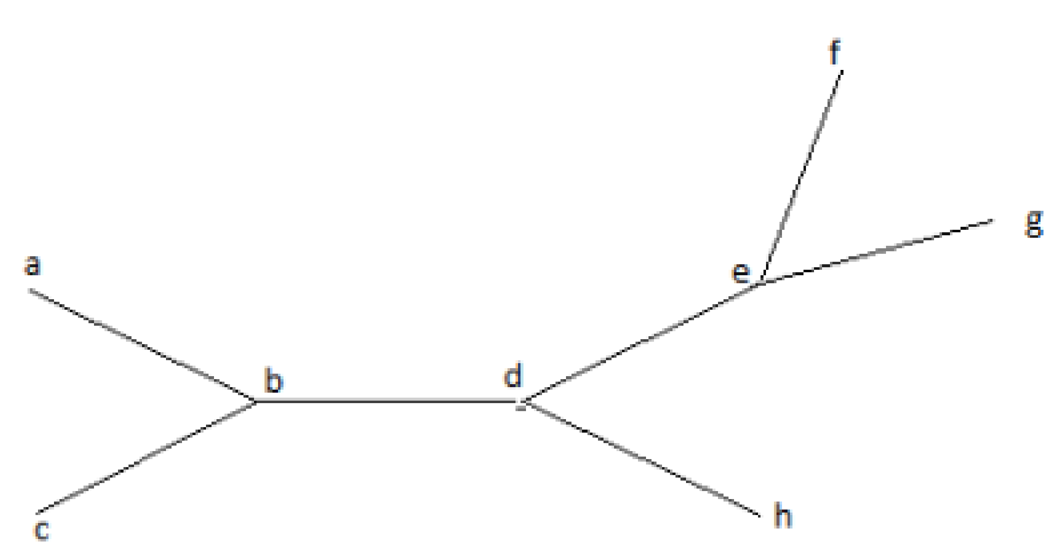
Example



3. Distance

In a connected graph G, the distance $d(V_i, V_j)$ between two of its vertices V_i and V_j is the length of the shortest path between them.

Example

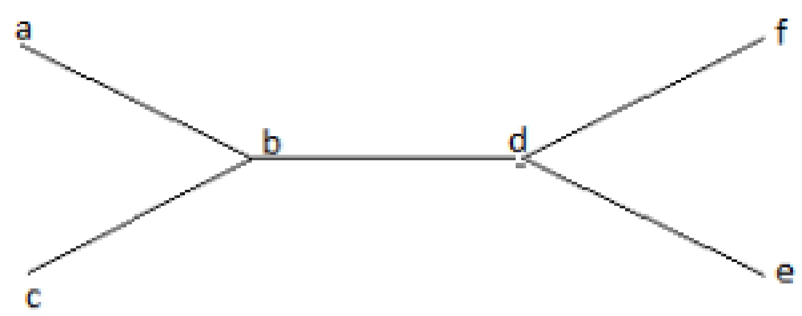


d(a, c) = 2; d(a, d) = 2; d(a, h) = 3; d(a, g) = 4; d(b, f) = 3; d(c, h) = 3

4. Eccentricity

The eccentricity E(V) of a vertex V in a graph ' G' is a distance from V to a vertex farthest from V in G. (i.e.) Eccentricity of a vertex is the maximum of the distance from V to the other vertices. (i.e.) $E(V) = \text{Max } d(V, V_i)$ where $V_i \in V$

Example

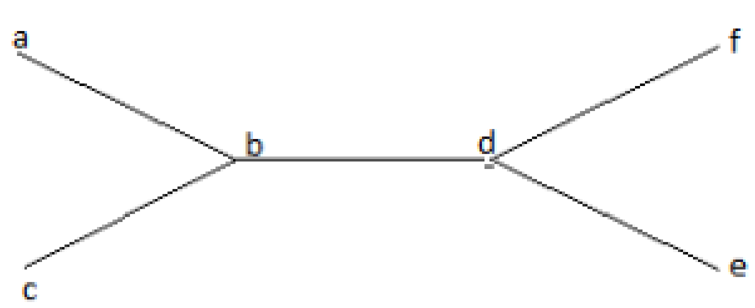


E(a) = 3; E(b) = 2; E(c) = 3; E(d) = 2; E(e) = 3; E(f) = 3

5. Centre of a tree

Let G be a connected graph, then a vertex in G with a minimum eccentricity is called centre of a tree.

Example



E(a) =3; E(b) = 2; E(c) = 3; E(d) = 2; E(e) = 3; E(f) = 3
Here b, d are the centers in a tree.

6. **Rooted tree**

A tree in which one vertex (called the root) is distinguished from all the other vertices is called a Rooted tree.

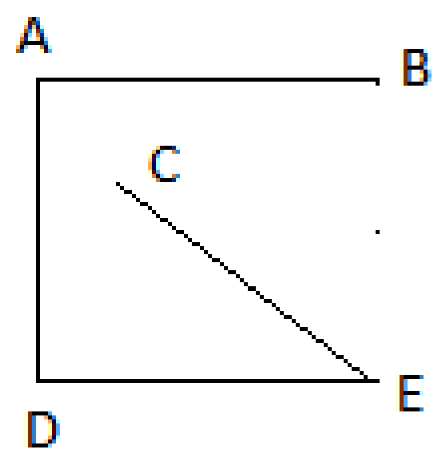
7. Binary tree

A Binary tree is a tree in which there is exactly one vertex of degree 2 and each of the remaining vertices is of degree 1 or 3.

8. Labeled graph

A graph in which each vertex is assigned a unique name or label is called labeled graph.

Example

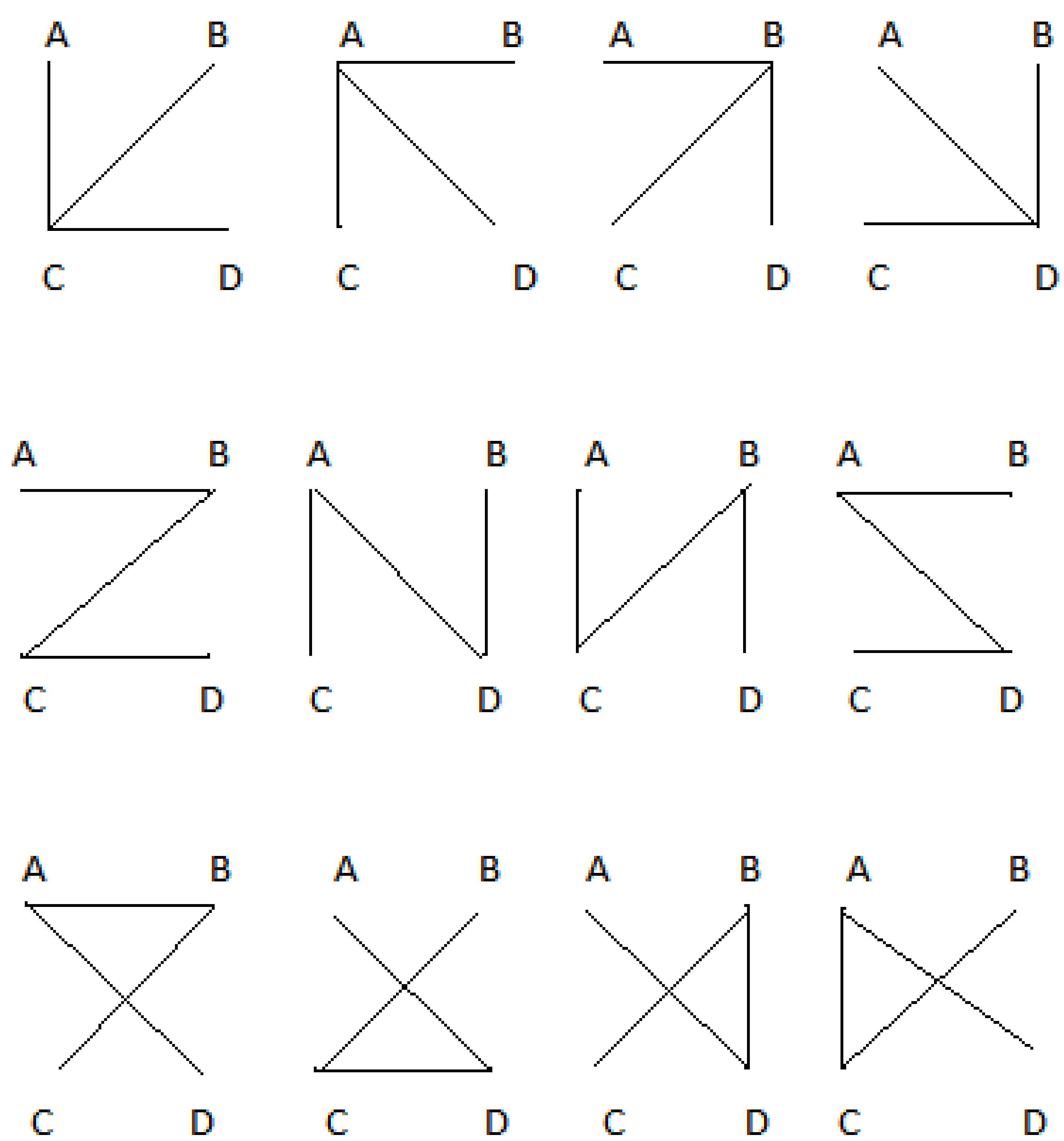


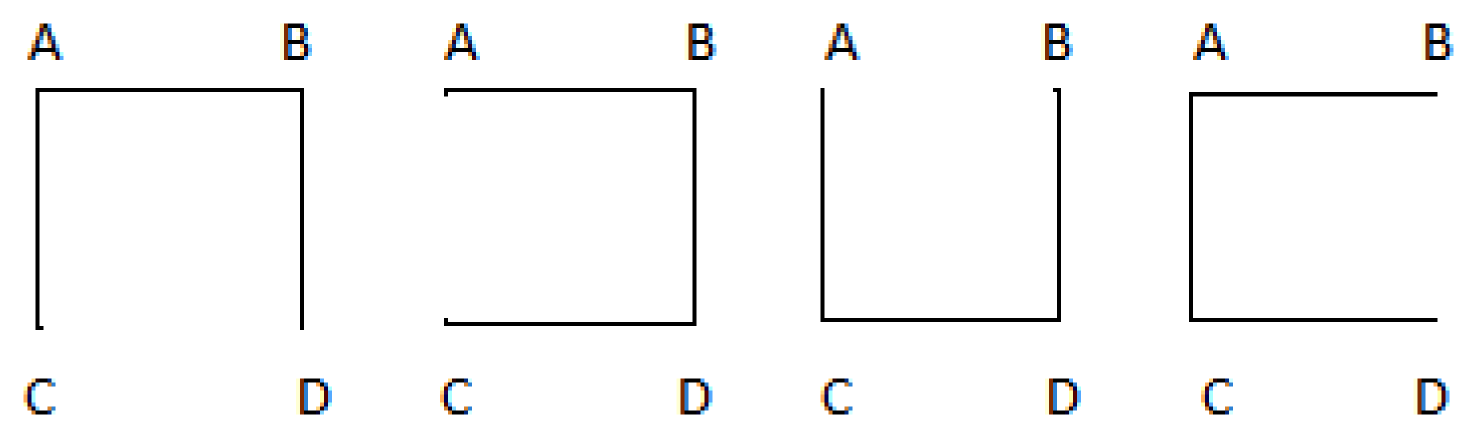
9. Cayley' s theorem

The number of labeled trees with n vertices is n^{n-2} , where $n \geq 2$.

Example

Draw all the labeled trees of 4 vertices when n = 4, n^{n-2} $4^{4-2} = 16$, labeled trees can be drawn.

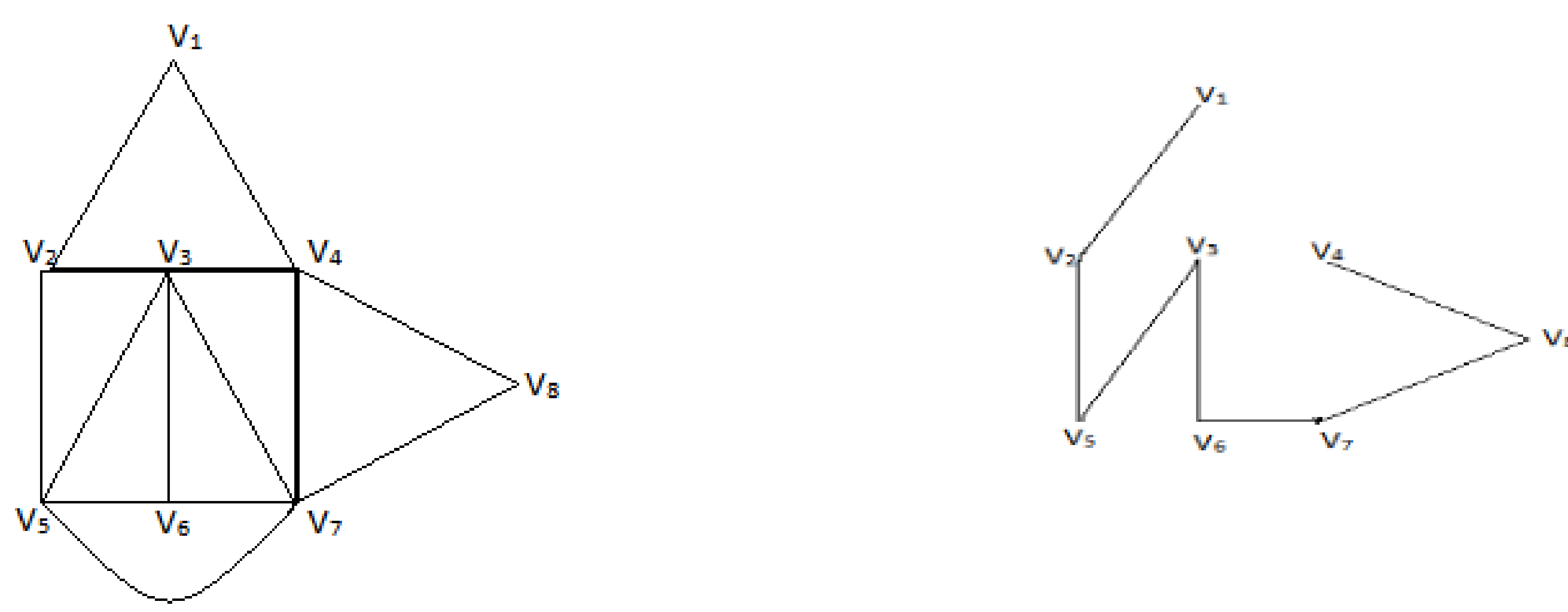




10. Spanning tree

A tree T is said to be a spanning tree of a connected graph ' G' if T is a sub graph of G without circuits and T contains all the vertices of G.

Example



11. Branch and chord

The edges in a spanning tree T are called the Branches of T. The edge of G which is not in the spanning tree is called Chord.

12. Rank and nullity

Let G be a graph with n vertices, e edges and k components. Then Rank and nullity is given by,

Rank (R) = n – k

Nullity (μ) = (e – n + k)

13. Weighted graph

Let G = {V, E} be a graph ' w' is a mapping from E R (w : E R) (R is a set of real numbers). ' w' is called as a weight function which assigns a number to each edge. The graph G together with a weight function is called as a weighted graph.

14. Shortest spanning tree

A spanning tree with the smallest weight in the weighted graph is called as shortest spanning tree or minimal spanning tree.

15. Kruskal' s algorithm

It is used to find a shortest spanning for a given weighted graph.

i) Arrange the edges of a graph with the increasing order of weights.

ii) The edge with least weight is included in the spanning tree T, which is to be constructed.

iii) Examine the next edge with least weight and include this edge in T, if it does not form a circuit with previously chosen edges.

iv) This procedure is continued until there are no edges to be included.

v) We get a spanning tree.

16. Prove that rank + nullity = no. Of edges in g

By definition of Rank and Nullity,

Rank + Nullity = n – k + e – n + k = e

Thus proved.

17. Distance between two spanning trees

The distance between two spanning trees T_i and T_j of a graph G is defined as the number of edges of G present in one tree but not in the other. It is denoted by $d(T_i, T_j)$

18. **Definition of Metric**

A function of two variables d(x,y) is said to be a metric if it satisfies the following conditions

(i) $d(x,y) \geq 0$ and $d(x,y) = 0$

iff x=y (Non-negativity)

(ii) $d(x,y) = d(y,x)$ (Symmetry)

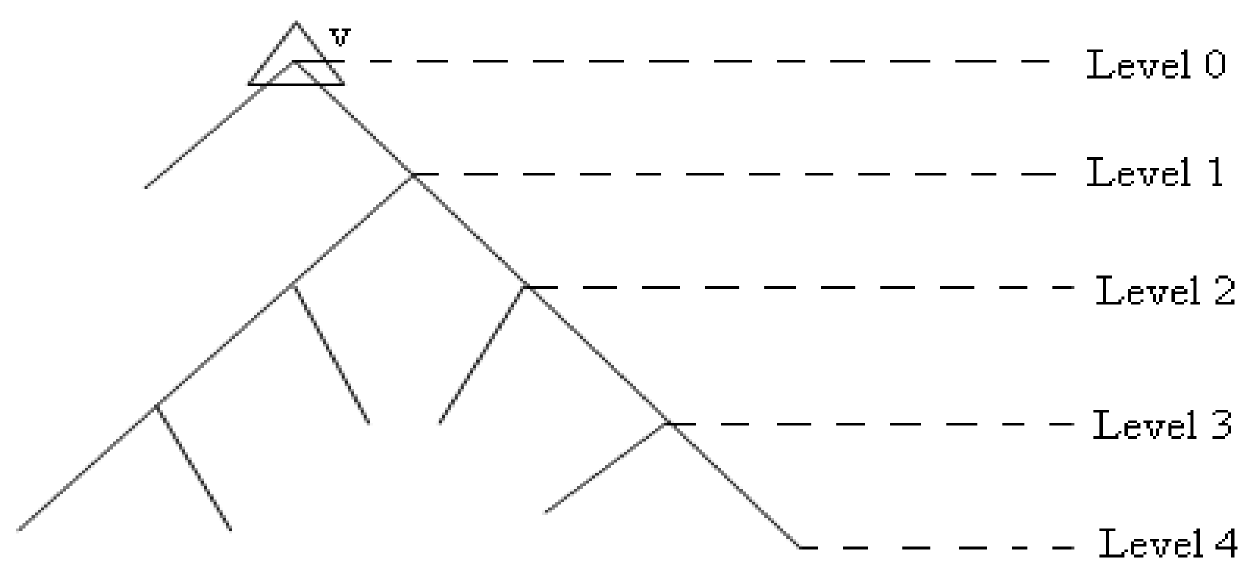
(iii) $d(x,y) \leq d(x,z) + d(z,y)\ \forall\ z$ (Triangle inequality).

19.Definition of level of a vertex

A binary tree T, a vertex v is said to level L. If v is at a distance L from the root vertex. Thus the root is at level v.

Example

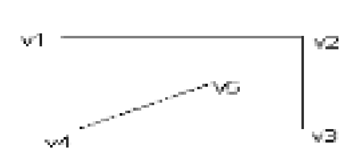
Consider a binary tree,



The figure shows vertices with 4 level binary tree.

20.Definition of labeled graph

A graph in which each vertex is assigned a unique name or label is called labelled graph.
Example



21. **Shortest spanning tree**

A spanning tree with the smallest weight in a weighted graph is called a shortest spanning tree or shortest distance spanning tree or minimal spanning tree.

22. **Central tree**

For a spanning tree T_0 of a graph G, let max $d(T_0,T_i)$ denote the maximal distance between T_0 and any other spanning tree of G. Then T_0 is called a central tree of G if max $d(T_0,T_i)$ $x \geq y$ max $d(T,T_j)$ for every tree T of G.

23.Radius and Diameter:

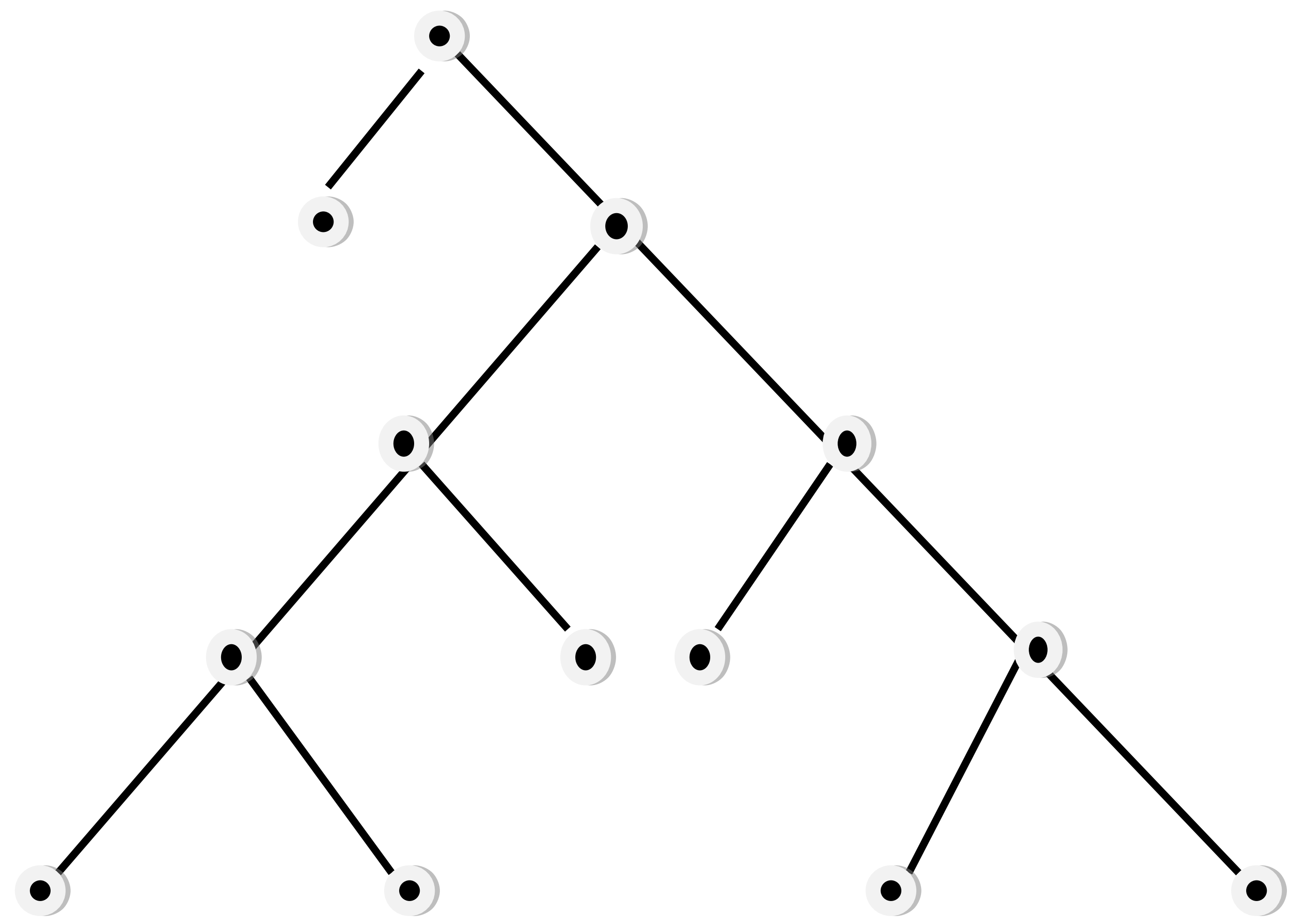
The eccentricity of a centre is a tree is defines as the radius of the tree.

The diameter of a tree T, in defined as the length of the longest path in T.

24. Explain briefly about level of binary trees?

In a binary tree a vertex V_i is said to be at level l_i if V_i is at a distance of l_i from the root. Thus the root is at level 0.

Eg: A, B – Vertex, from level binary tree.



Height of tree T = 4

For the above examples, it is clear that there can be only one vertex(the root) at level 0, at most two vertices at level 1, at most four vertices at level 2 and so on.

Therefore the maximum number of vertices possible in a k-level binary tree is

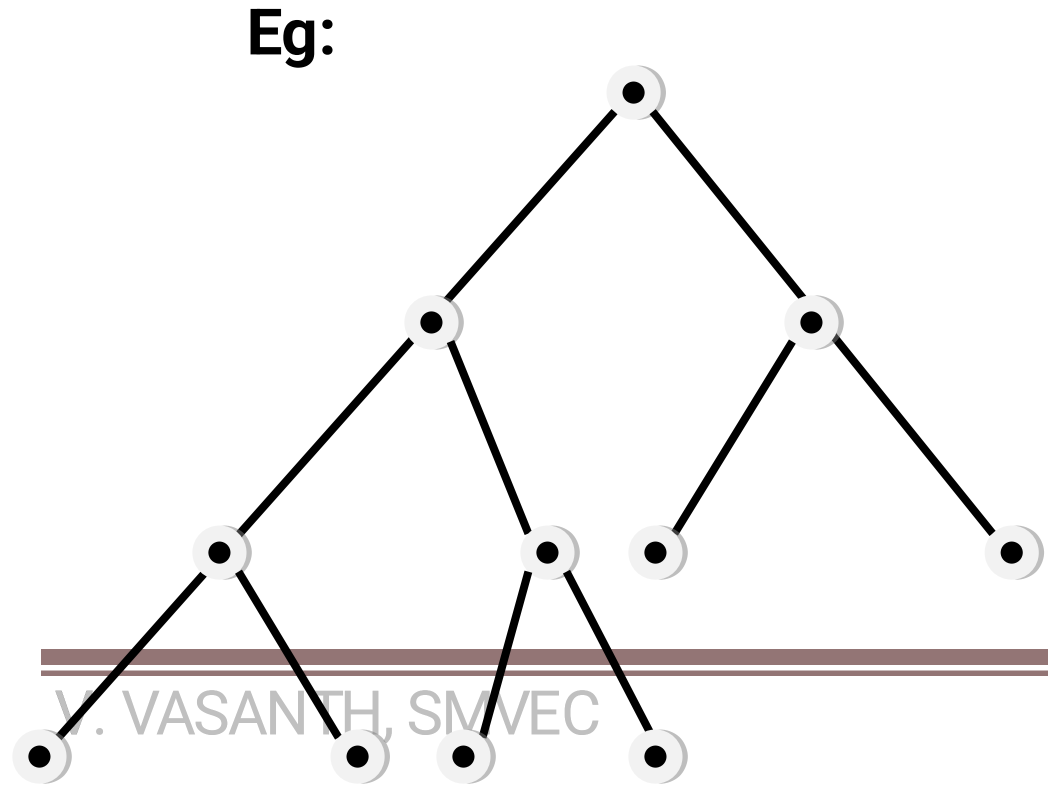
$$2^0+2^1+2^2+2^3+\ldots.+2^k >= n$$

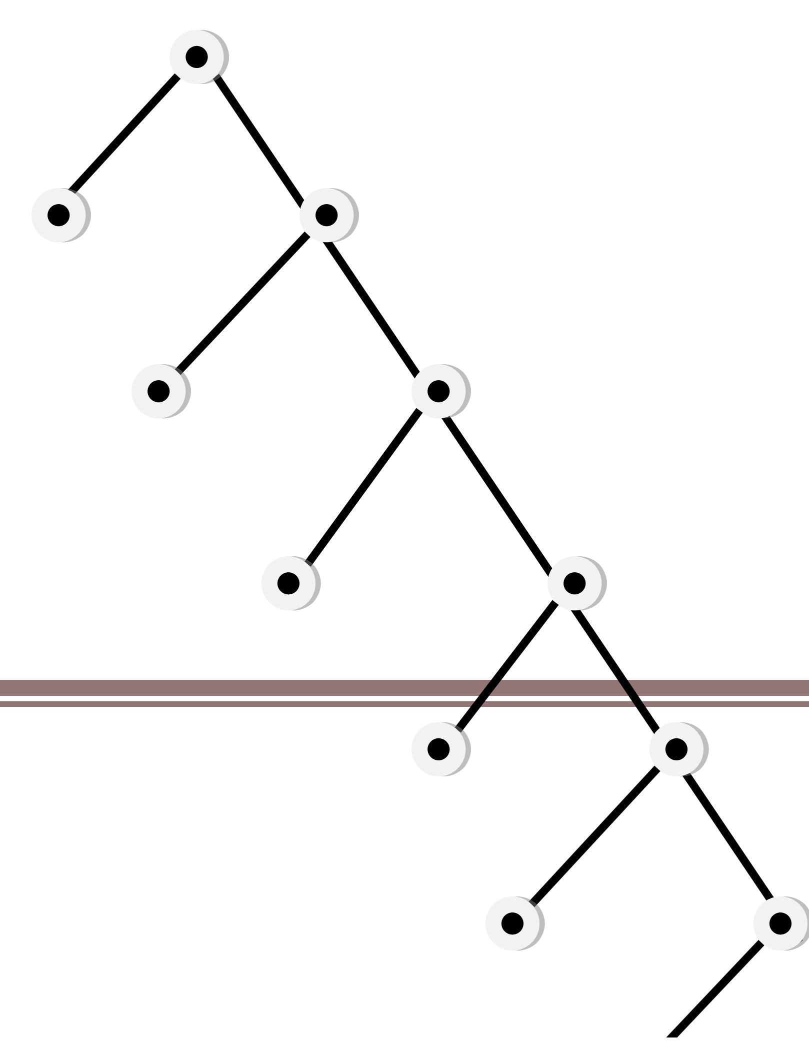
The maximum level, l_{max} of any vertex in a binary tree is called height of the tree.

The minimum possible height at an n-vertex binary tree is

$$\min l_{max} = [\log_2(n+1)-1] \text{ and } \max l_{max} = (n+1)/2$$

Eg:

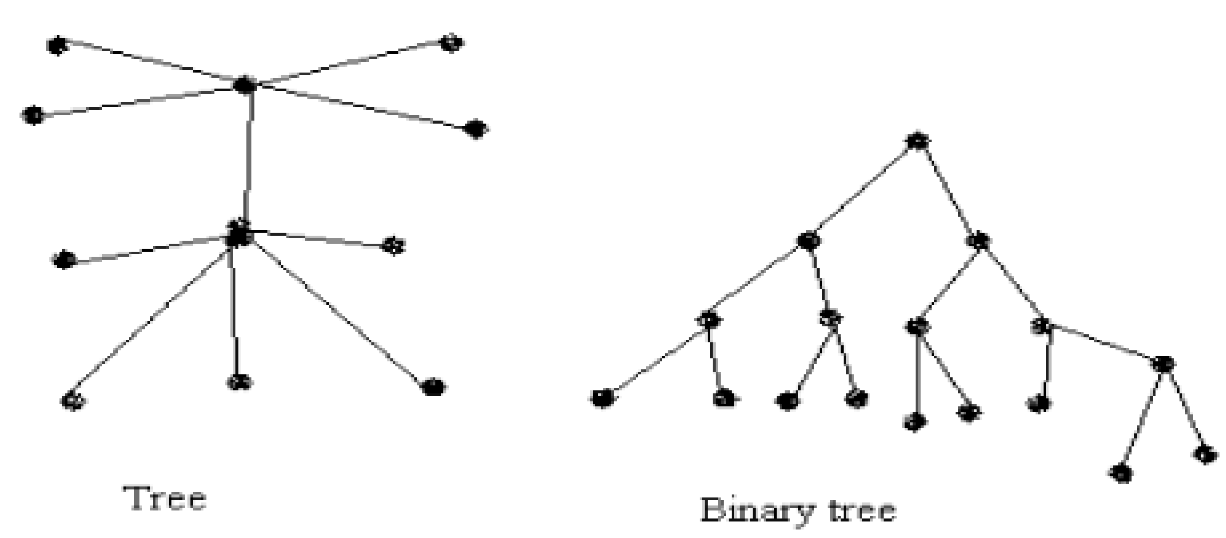




Min l_{max} =($\log^{12}_2$ -1)

Max l_{max}=(11-1)/2=5

25. Draw a tree with 9 pendant vertices and a binary tree with 9 pendant vertices



BIG QUESTIONS

UNIT-I LOGICS

Discrete Mathematics

Discrete Mathematics is a part of mathematics devoted to the study of discrete objects. Discrete means consisting of distinct or disconnected elements.

Declarative sentence

A sentence which cannot be further broken down or split into simple sentences is called a declarative sentence.

Truth values or two valued logic

The declarative sentences which have one and only one of two possible values (true & false) are called the truth values. The truth values true and false are denoted by T and F or 1 and 0 respectively.

Statements and atomic statements

The declarative sentence to which it is possible to assign truth values true or false but not both are called statements.

(e.g.) (i) Banu is rich.(ii) Chennai is a city

Statements which do not contain any of the connectives are called the atomic statements or the primary statements.

(e.g.) P, Q, R *etc*.

Connectives or logical operators.

There are five logical operators and they are:

i) Negation (NOT)

ii) Conjunction (AND)

iii) Disjunction (OR)

iv) Conditional Statement (IF THEN)

v) Biconditional Statement (IF AND ONLY IF)

Negation

The negation is introduced by the word NOT. If P is a statement, then negation of P is denoted by P or : P and it should be read as NOT P. The truth table for negation is as follows

P	$\rceil P$
T	F
F	T

(e.g.) P: Today is Friday.

$\rceil P$: It is not that today is Friday

Conjunction

Consider the two statements P and Q. Then the conjunction of P and Q is denoted by $P \wedge Q$ which is read as " P AND Q " or " P meets Q " .. The truth table for conjunction is as follows.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(e.g.) P: It is raining today

Q: 2+2=4.

$P \wedge Q$: It is raining today and $2+2=4$.

Disjunction

Consider the two statements P and Q. Then the disjunction of P and Q is denoted by $P \vee Q$ which is read as " P OR Q" or " P joins Q" . . The truth table for disjunction is as follows.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

(e.g.) If P: It is raining today,

Q: 2+2=4. then $P \vee Q$: It is raining today or $2+2=4$.

The above example is called exclusive OR.

Ex: I shall go to cinema or beach, this is called inclusive OR.

Conditional operators

Let P and Q be any two statements. Then the statement $P \rightarrow Q$ which is read as " IF P THEN Q" is called as the conditional statement. The truth table for conditional statement is as follows.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(e.g.) P: The sun is shining today

Q: 2+7>4

$P \rightarrow Q$: If the sun is shining today then Q : $2 + 7 > 4$.

Biconditional operator

Let P and Q be any two statements. Then the statement $P \leftrightarrow Q$ (or) $P \Leftrightarrow Q$ (or) which is read as " P IF AND ONLY IF Q" is called as Biconditional statement. .

(e.g.) If P: The sun is shining today,

Q : 2+7>4 then $P \leftrightarrow Q$ *: The sun is shining today if and only if 2+3>3*

The truth table for Biconditional is as follows

$.P$	$Q.$	$P \leftrightarrow Q$
T	T	$.T$
T	F	$F.$
F	T	$F.$
F	F	$.T$

Well formed formula(WFF)

The statement formula in which the order of finding the truth values are indicated by using parenthesis is called a Well Formed Formula.

(e. g.) $((P \to Q) \wedge R)$, $((P \to Q) \to (\rceil P \vee Q))$

Rules for Well Formed Formula

i) A statement variable standing alone is a well formed formula.

ii) If A is a well formed formula, then A is a well formed formula.

iii) If A and B are well formed formulae, then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $(A \leftrightarrow B)$ are also well formed formulae.

iv) A string of symbols containing the statement variables, connectives and parenthesis are a well formed formula if can be obtained by finitely many applications of the rules (i),(ii),(iii)

Statement Formula

A statement formula is an expression which is a string consisting of variables, (letters either all of them capital letters (or) all of them are lower case letters with (or) without subscripts),parenthesis and connective symbols(ù, $\wedge$, $\vee$, $\to$, $\leftrightarrow$) which produces a statement when the variables are replaced by statements.

Tautology

A statement formula which is always true irrespective of the truth values of the individual variables is called a tautology.

(e. g.) $\rceil P \vee P$爜a tautology .

Contradiction

A statement formula which is always false irrespective of the truth values of the individual variables is called contradiction.

(e. g.) $\rceil P \wedge P$爜a contradiction .

EQUIVALENCE FORMULA

1) Idempotent rule:

(i)$P \vee P \Leftrightarrow P$

(ii)$P \wedge P \Leftrightarrow P$

2) Commutative rule:

(i)$P \vee Q \Leftrightarrow Q \vee P$

(ii)$P \wedge Q \Leftrightarrow Q \wedge P$

3) Associative law:

(i)$P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$

(ii)$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$

4) Distributive law:

(i)$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

(ii)$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

5) Absorption law:

(i)$P \vee (P \wedge Q) \Leftrightarrow P$

(ii)$P \wedge (P \vee Q) \Leftrightarrow P$

6) Demorgan' s law:

(i) $\text{ù} (P \vee Q) \Leftrightarrow \text{ù} P \wedge \text{ù} Q$

(ii) $\text{ù} (P \wedge Q) \Leftrightarrow \text{ù} P \vee \text{ù} Q$

7) Dominance law:

(i)$P \vee T \Leftrightarrow T$

(ii)$P \wedge F \Leftrightarrow F$

8) Identity law:

(i)P ∧ T ⇔ P

(ii)P ∨ F ⇔ P

9) Conditional law as disjunction:

(i)P → Q ⇔ ù P ∨ Q

(ii)Q → P ⇔ ù Q ∨ P

10) Complement law:

(i)P ∧ ù P ⇔ F

(ii)P ∨ ù P ⇔ T

11) Double negation:

ù (ùP) ⇔ P

1. Construct the truth table for the following formula

(i) (ù P ∧ (ù Q ∧ R)) ∨ ((Q ∧ R) ∨ (P ∧ R)

P	Q	R	ù P	ù Q	ù Q ∧ R	ù P ∧ (ù Q ∧ R)	Q ∧ R	P ∧ R	(Q ∧ R) ∨ (P ∧ R)	ANS
T	T	T	F	F	F	F	T	T	T	T
T	T	F	F	F	F	F	F	F	F	F
T	F	T	F	T	T	F	F	T	T	T
T	F	F	F	T	F	F	F	F	F	F
F	T	T	T	F	F	F	T	F	T	T
F	T	F	T	F	F	F	F	F	F	F

F	F	T	T	T	T	T	F	F	F	T
F	F	F	T	T	F	F	F	F	F	F

(ii) ((P ∧Q) ∨ (ù P ∧Q)) ∨((P ∧ù Q) ∨ (ù Q ∧ù Q))

P	Q	ù P	ù Q	P∧ Q	ù P ∧Q	P∧ù Q	ù Q ∧ ù Q	(P∧Q)∨ (ù P∧Q)	(P∧ù Q) ∨ (ù Q∧ù Q)	ANS
T	T	F	F	T	F	F	F	T	F	T
T	F	F	T	F	F	T	T	F	T	T
F	T	T	F	F	T	F	F	T	F	T
F	F	T	T	F	F	F	T	F	T	T

The answer is a tautology.

2. Construct the truth table for

ù **(P∨ (Q∧R)) ((P∨Q) ∧(P∨R))**

P	Q	R	Q∧R	P∨ (Q∧R)	ù [P∨ (Q∧R)]	P∨Q	P∨R	(P∨Q) ∧ (P∨R)	ANSWER
T	T	T	T	T	F	T	T	T	F
T	T	F	F	T	F	T	T	T	F
T	F	T	F	T	F	T	T	T	F
T	F	F	F	T	F	T	T	T	F
F	T	T	T	T	F	T	T	T	F
F	T	F	F	F	T	T	F	F	F

F	F	T	F	F	T	F	T	F	F
F	F	F	F	F	T	F	F	F	F

It is a contradiction.

3. Show that ((P∨Q)∧ ù (ù P∧(ù Q∨ù R)))∨ (ù P∧ù Q)∨(ù P∧ù R))is a tautology.

Solution

((P∨Q)∧ ù (ù P∧ (ù Q∨ù R))) ∨((ù P∧ù Q) ∨(ù P∧ù R))

⇔ ((P∨Q)∧(P∨(Q∧R))) ∨(ù P∧(ù Q∨ù R)) {By Demorgan' s +Distributive law}

⇔ ((P∨Q)∧ (P∨Q)∧(P∨R)) ∨(ù P∧ (ù Q∨ù R)){By Distributive law}

⇔ ((P∨Q)∧ (P∨R)) ∨(ù P∧(ù Q∨ù R)) {By Idempotent rule}

⇔ (P∨ (Q∧R))∨(ù P∧(ù Q∨ù R)) {By Distributive law}

⇔ (P∨(Q∧R))∨ ù (P∨(Q∧R)) {By Demorgan' s law}

⇔ T {By Complement law}

.:This is a tautology.

4. Show that P → (Q → R) ⇔ P → (ù Q∨R) ⇔ (P∧Q) → R

Solution

P → (Q → R) ⇔ P → (ù Q∨R) {Condition as Disjunction}

⇔ ù P ∨(ù Q∨R) {Condition as Disjunction}

⇔ (ù P∨ù Q)∨ R {Associative law}

⇔ ù (P∧Q)∨ R {Demorgan' s law}

⇔ (P∧Q) → R {Condition as Disjunction}

5.Show that the following implication is a tautology.

(P → (Q → R)) ⇒ ((P → Q) → (P → R))

Solution

It is enough to prove that

$(P \rightarrow (Q \rightarrow R) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$ is a tautology

$(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$

$\Leftrightarrow [P \rightarrow (\neg Q \vee R)] \rightarrow [(\neg P \vee Q) \rightarrow (\neg P \vee R)]$ {Condition as disjunction}

$\Leftrightarrow [\neg P \vee (\neg Q \vee R)] \rightarrow [\neg (\neg P \vee Q) \vee (\neg P \vee R)]$ {Condition as disjunction}

$\Leftrightarrow \neg [(\neg P \vee \neg Q) \vee R] \vee [(P \wedge \neg Q) \vee (\neg P \vee R)]$ {Condition as disjunction & DeMmorgan's law}

$\Leftrightarrow [(P \wedge Q) \vee R] \vee [(\neg P \vee R) \vee (P \wedge \neg Q)]$ {Demorgan's &Commutative law}

$\Leftrightarrow [(P \wedge Q) \vee R] \vee [((\neg P \vee R) \vee P) \wedge [(\neg P \vee R) \vee \neg Q)]$ {Distributive law}

$\Leftrightarrow [(P \wedge Q) \vee R] \vee [((\neg (P \vee P) \vee R) \wedge ((\neg P \vee \neg Q) \vee R)]$ {Associative law}

$\Leftrightarrow [(P \wedge Q) \vee R] \vee [(T \vee R) \wedge (\neg (P \wedge Q) \vee R)]$ {Complement &Demorgan's law}

$\Leftrightarrow [(P \wedge Q) \vee R] \vee [T \wedge (\neg (P \wedge Q) \vee R)]$ {Associative &Dominance law}

$\Leftrightarrow [(P \wedge Q \vee R] \vee \neg [(P \wedge Q) \vee R]$ {Identity law}

$\Leftrightarrow T$ {Complement}

6. Write the equivalence formula

$P \wedge (Q \leftrightarrow R)$

$P \wedge (Q \leftrightarrow R) \Leftrightarrow P \wedge [(Q \rightarrow R) \wedge (R \rightarrow Q)]$

$\Leftrightarrow P \wedge [(\neg Q \vee R) \wedge (\neg R \vee Q)]$ {Condition as disjunction}

$\Leftrightarrow P \wedge [((\neg Q \vee R) \wedge \neg R) \vee ((\neg Q \vee R) \wedge Q)]$ {Distributive law}

$\Leftrightarrow P \wedge [((\neg R \wedge \neg Q) \vee (\neg R \wedge R)) \vee ((Q \wedge \neg Q) \vee (Q \wedge R))]$

{Distributive law}

$\Leftrightarrow P \wedge [((\neg R \wedge \neg Q) \vee F) \vee ((F \vee (Q \wedge R))]$ {Complement law}

$\Leftrightarrow P \wedge [(\neg R \wedge \neg Q) \vee (Q \wedge R)]$ {Identity law}

$\Leftrightarrow P \wedge [\neg (R \vee Q) \vee (Q \wedge R)]$ {Demorgan' s law}

$\Leftrightarrow (P \wedge \neg (R \vee Q) \vee (P \wedge (Q \wedge R))$ {Distributive law}

7.Express the following in terms of {functionally complete set of connectives}

(i) $P \uparrow P$

$P \uparrow P \Leftrightarrow \neg (P \wedge P)$

$P \uparrow P \Leftrightarrow \neg P$

(ii) $(P \uparrow P) \uparrow (P \uparrow Q)$

$(P \uparrow P) \uparrow (P \uparrow Q) \Leftrightarrow \neg (P \wedge P) \uparrow \neg (P \wedge Q)$

$\Leftrightarrow \neg P \uparrow \neg (P \wedge Q)]$

$\Leftrightarrow \neg (\neg P \wedge \neg (P \wedge Q))$

$\Leftrightarrow P \vee (P \wedge Q)$ {Demorgan' s law}

$(P \uparrow P) \uparrow (P \uparrow Q) \Leftrightarrow P$

(iii) $(P \uparrow Q) \uparrow (P \uparrow Q)$

$(P \uparrow Q) \uparrow (P \uparrow Q) \Leftrightarrow \neg [(P \uparrow Q) \wedge (P \uparrow Q)]$

$\Leftrightarrow \neg [\neg (P \wedge Q) \wedge \neg (P \wedge Q)]$

$\Leftrightarrow \neg [\neg (P \wedge Q)]$

$(P \uparrow Q) \uparrow (P \uparrow Q) \Leftrightarrow P \wedge Q$

(iv) $(P \uparrow P) \uparrow (Q \uparrow Q)$

$(P \uparrow P) \uparrow (Q \uparrow Q) \Leftrightarrow \neg (P \wedge P) \uparrow \neg (Q \wedge Q)$

$\Leftrightarrow \neg [\neg P \wedge \neg Q]$

$(P \uparrow P) \uparrow (Q \uparrow Q) \Leftrightarrow P \vee Q$

(v) $(P \downarrow Q) \downarrow (P \downarrow Q)$

$(P \downarrow Q) \downarrow (P \downarrow Q) \Leftrightarrow \neg (P \vee Q) \downarrow \neg (P \vee Q)$

⇔ ù [ù (P∨Q) ∨ ù (P∨Q)]

⇔ ù [ù (P∨Q)]

(P↓ Q)↓ (P↓ Q) ⇔ P∨Q

(vi)(P↓ P)↓ (Q↓ Q)

(P↓ P)↓ (Q↓ Q) ⇔ ù (P∨P)↓ ù (Q∨Q)

⇔ ù [ù P∨ù Q]

(P↓ P)↓ (Q↓ Q) ⇔ P∧Q

8. Express P → (ù P → Q) in terms of NAND form.

Solution

P → (ù P → Q) ⇔ P → [ù (ù P) ∨ Q]

⇔ P → [P∨Q]

⇔ ùP∨ (P∨Q)

⇔ ùP∨ù [ù (P∨Q)]

⇔ ùP∨ù (ùP∧ùQ)

⇔ ùP∨(ùP↑ ùQ)

⇔ ù [ù (ùP∨ (ùP↑ ùQ))]

⇔ ù [P∧ù (ùP↑ ùQ)]

⇔ P↑ ù (ùP↑ ùQ)

⇔ P↑ ù [ù (P∧P)↑ ù (Q∧Q)]

⇔ P↑ ù [(P↑ P)↑ (Q↑ Q)]

⇔ P↑ [((P↑ P)↑ (Q↑ Q)) ∧ ((P↑ P)↑ (Q↑ Q))]

P → (ù P → Q) ⇔ P↑ ù [((P↑ P)↑ (Q↑ Q))↑ ((P↑ P)↑ (Q↑ Q))]

9. Prove that {↑ }is a functionally complete set of connectives.

Solution

The { ù, ∧} and { ù, ∨} are functionally complete set of connectives.

It is enough to prove that " ù, ∧ , ∨" can be expressed in terms of ↑ alone**.**

(i) ùP

ùP ⇔ ù (P ∧ P)

ù P ⇔ P↑ P

(ii) P ∧ Q

P ∧ Q ⇔ ù [(P ∧ Q)]

⇔ ù (P↑ Q)

⇔ ù [(P ↑ Q) ∧ (P ↑ Q)]

P ∧ Q ⇔ (P↑ Q) ↑ (P ↑ Q)

(iii) P ∨ Q

P ∨ Q ⇔ [ù (P ∨ Q)]

⇔ ù ['I P ∧ ù Q]

⇔ ù P ↑ ù Q

P ∨ Q ⇔ (P↑ P) ↑ (Q ↑ Q)

{↑ }is a functionally complete set of connectives.

10. Prove that {↓ } is a functionally complete set of connectives.

Solution:-

We know that, { ù , ∧} and { ù , ∨} are functionally complete set of connectives.

It is enough to prove that " ù , ∧ , ∨" can be expressed interms of ↓ alone **.**

(i) ù P

ù P ⇔ (P ∨ P)

ù P ⇔ P↓ P

(ii) (P ∧ Q)

$(P \wedge Q) \Leftrightarrow \neg(\neg(P \wedge Q))$

$\Leftrightarrow \neg[\neg P \vee \neg Q]$

$\Leftrightarrow \neg P \downarrow \neg Q$

$(P \wedge Q) \Leftrightarrow (P \downarrow P) \downarrow (Q \downarrow Q)$

(iii)$P \vee Q$

$P \vee Q \Leftrightarrow \neg[\neg(P \vee Q)]$

$\Leftrightarrow \neg[P \downarrow Q]$

$\Leftrightarrow (P \downarrow Q) \downarrow (P \downarrow Q)$

$\{\downarrow\}$ is a functionally complete set of connectives.

11. Obtain the PDNF & PCNF for $(Q \rightarrow P) \wedge (\neg P \wedge Q)$

Truth table

P	**Q**	**$Q \rightarrow P$**	**$\neg P$**	**$\neg P \wedge Q$**	**$(Q \rightarrow P) \wedge (\neg P \wedge Q)$**	**Min Terms**
T	T	T	F	F	F	$P \wedge Q$
T	F	T	F	F	F	$P \wedge \neg Q$
F	T	F	T	T	F	$\neg P \wedge Q$
F	F	T	T	F	F	$\neg P \wedge \neg Q$

PDNF:

S: No PDNF

$\neg S: (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

PCNF:

$S: \neg(\neg S)$

$\Leftrightarrow \neg[(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)]$

$S \Leftrightarrow (\neg P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (P \vee \neg Q) \wedge (P \vee Q)$

12. Obtain the PDNF & PCNF for: $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

$(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

$\Leftrightarrow (P \vee R) \wedge [(Q \rightarrow P) \wedge (P \rightarrow Q)]$

$\Leftrightarrow (P \vee R) \wedge [(\neg QP) \wedge (\neg P \vee Q)]$

$\Leftrightarrow [(P \vee R) \vee (\neg Q \wedge Q)] \wedge [(\neg Q \vee P) \vee (\neg R \wedge R)] \wedge [(\neg P \vee Q) \vee (R \wedge \neg R)]$

$\Leftrightarrow [\{(P \vee R) \vee \neg Q\} \wedge \{(P \vee R) \vee Q\}] \wedge [\{(\neg Q \vee P) \vee \neg R\} \wedge \{(\neg Q \vee P) \vee R\}]$

$\wedge [\{(\neg P \vee Q) \vee R\} \wedge \{(\neg P \vee Q) \vee \neg R\}]$

$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$

$\wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee \neg R)$

PCNF:

$S \Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

$\neg S: (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

PDNF:

$S: \neg(\neg S)$

$S: (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

UNIT-II

The Theory of Inference

The main aim of logic is to provide rules of inference, or principles of reasoning. Here, we are concerned with the inferring of a conclusion from given premises.

We are going to check the logical validity of the conclusion, from the given set of premises by making use of equivalence rule and implication rule, the theory associated with such things is called inference theory.

Direct Method of Proof

When a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a direct proof.

Indirect Method of Proof

1] Method of contradiction

In order to show that a conclusion C follows logically from the premises H1,H2,.....,Hm, we assume that C is false and consider ┐C as an additional premises, If the new set of premises gives contradict values, then the assumption ┐C is true does not hold simultaneously with H1∧ H2∧ … … … ∧ Hm being true.

Therefore, C is true whenever H1∧ H2∧ H3∧ … … … ∧ Hm is true. Thus C follows logically from the premisesH1,H2,… … … ,Hm.

2] Method of Contra Positive

In order to prove H1∧ H2∧ … … ∧ Hm **=>** C, if we prove ┐C ┐ **(**H1∧ H2∧ … … … ∧ Hm **)** then the original problem follows. This method is called contra positive method.

Rules of inference

1] Rule P: A premise may be introduced at any point in the derivation.

2] Rule T: A formula S may be introduced at any point in the derivation if S is tautologically implied by any one or more of the preceding formulas.

3] Rule CP: If S can be derived from R and set of premises, then R—> S can be derived from the set of premises

Remark

1] Rule CP means Rule of Conditional Proof.

2] Rule CP is also called the deduction theorem.

3] In general, whenever conclusion is of the form R—>S (in terms of conditional). We should apply Rule CP .In such case, R is taken as conditional premises and S can be derived from the given premises and R.

Implication Rules

1	P, P—>Q=>Q	(Modus Phones)
2	┐Q, P—>Q=>┐P	(Modus Tollens)

3	$\neg P, P \vee Q \Rightarrow Q$	(Disjunctive Syllogism)
4	$P \to Q, Q \to R \Rightarrow P \to R$	Hypothetical Syllogism or Chain Rule)
5	$P, Q \Rightarrow P \wedge Q$	
6	$P \wedge Q \Rightarrow P, Q$	
7	$P, Q \Rightarrow P \vee Q$	
8	$P \wedge \neg Q \Rightarrow \neg (P \to Q)$	

Note

In the derivation, we should use all the rules but exactly once. Also, the order is immaterial.

Consistency and inconsistency of premises

A set of formulaeH1,H2, ...,Hm is said to be inconsistent if their conjunction implies contradiction.

(ie) $H1 \wedge H2 \wedge H3 \wedge \ldots \ldots \ldots \wedge Hm \Rightarrow R \wedge \neg R$ for some formula R. Note that ($R \wedge \neg R \Leftrightarrow F$)

A set of formulae H1,H2,... ... ... Hm is said to be consistent if their conjunction implies tautology.

A set of formulae H1,H2,... ... ... Hm is said to be consistent if it is not consistent.

Indirect Method of Proof

Validity by Indirect Method

In order to show that a conclusion C follows logically from the premisesH1,H2,... ... ... Hm, we assume C is false and consider $\neg C$ as an additional premises. If $H1 \wedge H2 \wedge \ldots \ldots \ldots \wedge Hm \wedge \neg C$ is a contradiction, then C follows logically fromH1,H2,... ... ... Hm.

Note

The technique of indirect method of proof runs as follows,

1] Introduce the negation of the desired conclusion as a new premise.

2] From the new premise, together with the given premises, derive a contradiction.

3] Assert the desired conclusion as a logical conclusion as a logical inference from the premises.

The Predicate Calculus

The predicate calculus deals with the study of predicates.

Consider the following statement:

" Ram is a boy"

In the above statement," is a boy" is the predicate and subject " Ram" by r, then the statement " Ram is a boy" can be represented as B(r).

Remark

Always we denote predicates by capital letters and the subjects by small letters.
In general, any statement of the type " S is P" where P is a predicate and S is the subject can be denoted by P(S).

Some More Examples

1]" X is a man"

Here, Predicate is " is a man" and it is denoted by M. Subject is " X" and it is denoted by X.

Therefore the given statement " X is a man" can be denoted by M(x).

If there is only one name of object associated with a predicate then it is known as1- place predicate.

In general, an n-place predicate is a predicate requiring n names of objects when n>0. It is denoted by P(a1 ,a2 ,..... ... ... an) Where a1,a2,... ... ... an are the names of the objects associated with predicate and P is a predicate.

The Statement Functions, Variables, Quantifiers

Statement Function

A simple statement function of one variable is defined to be an expression consisting of a predicate symbol and an individual variable.

Quantifiers

Quantifier is one which is used to quantify the nature of variables.

There are 2 important quantifiers which are for " all" and for " some" where " some" means " at least one" .

Universal Quantifier

The quantifier " for all x" is called universal quantifier. It is denoted by the symbol " (∀ x or(x))" . The universal quantifier is equivalent to each of the following phrases.

For all x.
For every x.
For each x.
Everything x is such that
Each thing x is such that

Example 1

" Every apple is red"

The above statement can be restated as

For all x, if x is an apple then x is red --------- (*)

Now, we will translate it into symbolic form using universal quantifier.

Define A(x): x is an apple.

R(x): x is red.

Therefore we write(*) into symbolic form as

$\forall xAx \rightarrow Rx$

Existential Quantifier

The quantifier for " some x" is called the existential quantifier. It is denoted by the symbol " (∃ x)". The existential quantifier is also equivalent to each of the following phrases.

For some x.
Some x such that.
There exists an x such that.
There is an x such that.
There is at least one x such that.

Example 1

" Some men are clever"

The above statement can be restated as

" There is an x such that x is a man and x is clever" . --------- (*)

We will translate it into symbolic form using existential quantifier.

Let M(x): x is a man

C(x): x is clever

Therefore we write (B) into symbolic form as

$\exists$ xMx$\wedge$ Cx

Predicate Formula

The n- place predicate along with n individual variables is called an n- place predicate formula.

For example(x1,x2,… … … ,xn) denotes an n- place predicate formula in which the letter P is an n- place predicate and x1,x2,… … … ,xn are individual variables.

Example

P (x, y), R (x), Q (x, y, z) are all predicate formulae.

Remark

Predicate formulae are also known as atomic formulae.
A 0-place predicate formula is a special case of predicate formula.

Free and Bound Variables

1] The variable is said to be bound if it is connected with either universal ($\forall$ x) or existential($\exists$ x) quantifier.

2] The scope of the quantifier is the formulae immediately following the quantifier.

3] The variable which is not concerned with any quantifier is called free variable.

The Universe of Discourse

Variables which are quantified stands for only those objects which are members of a particular set or class. Such a set is called the universe of discourse or domain or simply universe.

The universe may be, the class of human beings, or members, or some other objects. The truth value of a statement depends upon the universe.

The Theory of Inference for Predicate Calculus

The rules of equivalent formulae, tautological implications, Rule P, Rule T, Rule CP, indirect method of proof for statement calculus can also be used here. Apart from this, we have some more rules to be followed, involving quantifiers. The rules of specification called US and ES are used to eliminate quantifiers, whereas the rules of generalization called UG and EG are to be used to prefix the correct quantifier.

We shall now see the rules of generalization and specification.

1] Universal Generalization (UG)

$A(y) \Leftrightarrow (x)\, A(x)$

Provided x is not free in any premises and if x is free in a prior step which resulted from use of ES, then no variables introduced by that use of ES appear free in A(x).

2] Existential Generalization (EG)

$A(y) \Leftrightarrow (\exists\, x)A(x)$

3] Universal Specification (US) (or) Universal instantiation

$(x)A(x) \Leftrightarrow A(y)$

4] Existential Specification

$\exists\, xAx \Leftrightarrow A(y)$

Provided y is not free in any premises and also not free in any prior step of the derivation.

1. S.T $R \vee S$ follows logically from the premises $C \vee D$, $C \vee D$ � $\neg H$, $\neg H$ � $(A \wedge \neg B)$,

$A \wedge B$ � $R \vee S$

Solution

The given premises are:

{1} C∨D

{2} C∨D � ¬H

{3} H � A∧ ¬B

{4} A∧ B � R∨ S

C: R∨S

1. {1}	C∨D	rule P
2. {2}	(C∨D) � ¬H	rule P
3. {1,2}	¬H	rule T (P, P � Q ⇒ Q)
4. {3}	¬H � (A∧ ¬B)	rule P
5. {1,2,3}	A∧ ¬B	rule T (P, P � Q ⇒ Q)
6. {4}	A∧ ¬B	rule P
7. {1,2,3,4}	R ∨S	rule T (P, P � Q ⇒ Q)

Therefore R∨ S is a valid conclusion for the given premises.

2. S.T R ∧ (P∨ Q) is a valid conclusion from the premises P ∨Q, Q � R, P � M, ¬M

Solution

The given premises are:

{1} P∨Q

{2} Q � R

{3} P � M

{4} ¬ M

C: R∧ (P∨Q)

1. {1}	P∨Q	rule P
2. {1}	¬P � Q	rule T (P- � Q ⇔ ¬P∨Q)
3. {2}	Q � R	rule P
4. {1,2}	¬P � R	rule T (P- � Q, Q � R ⇒ P � R)
5. {1,2}	¬R � P	rule T (P- � Q ⇔ ¬Q � ¬P)
6. {3}	P � M	rule P
7. {1,2,3}	¬R � M	rule T (P- � Q, Q � R ⇒ P � R)
8. {1,2,3}	¬M � R	rule T (P- �Q ⇔ ¬Q � ¬P)
9. {4}	¬M	rule P
10. {4,1,2,3}	R	rule T (P, P- � Q ⇒ Q)
11. {4,1,2,3,1}	R ∧ (P ∨Q)	rule T(P, Q ⇒ P∧ Q)

Therefore R ∧ (P ∨ Q) is a valid statement for the given premises.

3. If there was a ball game, then travelling was difficult. If they arrived on time, then traveling was not difficult. They arrived on time. Therefore there was no ball game. Show that the statement consist of a valid argument.

Solution

P: There was a ball game.
Q: The travelling was difficult.
R: They arrived on time.
Given premises:
{1} P � Q
{2} R � ¬Q
{3} R
C: ¬P

1. {1}	P � Q	(Rule P)
2. {2}	R � ¬Q	(Rule P)
3. {2}	Q � ¬R	(Rule T)
4. {1,2}	R � ¬P	(Rule T)
5. {1,2}	R � ¬P	(Rule T)
6. {3}	R	(Rule P)
7. {3,1,2}	¬P	(Rule T)

4. Derive the following premises P � (Q � S) using rule CP. P � (Q � R), Q � (R � S)

Solution

Given Premises:
{1} P � (Q � R)
{2} Q � (R � S)
Additional premises {3} P
C: Q � S

1. {3}	P	(Rule P)
2. {1}	P � (Q � R)	(Rule P)
3. {3,1}	Q � R	(Rule T)
4. {2}	Q � (R∨S)	(Rule P)
5. {2}	Q (¬R∨S)	(Rule T)
6. {2}	¬Q∨ (¬R∨ S)	(Rule T)

7. {2} ¬Q ∨ (S ∨ ¬R) (Rule T)

8. {2} (¬Q ∨ S) ∨ ¬R (Rule T)

9. {2} (¬R) ∨ (¬Q ∨ S) (Rule T)

10. {2} R � (¬Q ∨ S) (Rule T)

11. {2} R � (QS) (Rule T)

12. {3,1,2} Q � (QS) (Rule T)

13. {3,1,2} Q � (¬Q ∨ S) (Rule T)

14. {3,1,2} ¬Q ∨ (¬Q ∨ S) (Rule T)

15. {3,1,2} (¬Q ∨ ¬Q) ∨ S (Rule T)

16. {3,1,2} ¬Q ∨ S (Rule T)

17. {3,1,2} Q � S (Rule T)

5. State the premises are inconsistent

1.) If jack misses many classes through illness then he fails high school

2.) If jack fails high school then he is uneducated

3.) If jack reads lot of books then he is not uneducated

4.) If jack misses many classes and reads of books

Solution:

P: Jack misses many classes

Q: Jack fails high school

R: He is uneducated

S: Jack reads lot of books

6. Using indirect method, ¬(P � Q) � ¬(R ∧ S), Q- � (P ∨ ¬R), R. C:P � Q

Solution

Given premises:

{1} ¬(P � Q) � (¬R ∧ S)

{2} Q � P ∨ ¬R

{3} R

{4} ¬(P ↔ Q)

{4} (P � Q) � ¬(Q � P)

1. {1} ¬(P � Q) � ¬(R ∨ S) (Rule P)
2. {1} (R ∨ S) � (P � Q) (Rule T)
3. {4} (P � Q) � ¬(Q � P) (Rule P)

4. {1,4} (R∨S) � ¬(Q�P) (Rule T)

5. {2} (Q� P) ∨ ¬R (Rule P)

6. {2} ¬R∨(Q�P) (Rule T)

7. {2} R� (Q�P) (Rule T)

8. {2} ¬(Q� P) � ¬R (Rule T)

9. {1,4,2} (R∨S) � ¬R (Rule T)

10. {1,4,2} R ¬(R∨S) (Rule T)

11. {3} R (Rule P)

12. {3,1,4,2} ¬(R∧S) (Rule T)

13. {3,1,4,2} ¬R∧ ¬S (Rule T)

14. {3,1,4,2} ¬R (Rule T)

15. {3,1,4,2} ¬R∧R (Rule T)

16. {3,1,4,2} F (Rule T)

7. Show that (X) M(X) logically form the premises (X) (H(X) → M(X)) and (X) H(X).

Solution

Conclusion: (X) M(X)

{1} (X) H(X) → M(X)

{2} (X) M(X)

1. {1} (X) (H(X) → M(X))
2. {1} H(Y) → M(Y)
3. {2} (X) H(X)
4. {2} H(Y)
5. {2,1} M(Y)
6. {2,1} (X) M(X)

UNIT III - LATTICES

Ordered Pair

Let A and B be any two sets, consider a pair (a, b) in which the first element a is from A and the second element b is from B. Then (a, b) is called an ordered pair.

Relation

Consider the particular ordered pair of (x, y) $\in$ R where R is a relation which can be defined as X RY which may be read as x related to y.

Equivalence Relation

A relation R defined on a set X is said to be equivalence relation if it satisfies the following condition

i. **Reflexive**
ii. **Symmetric**
iii. **Transitive**

i) Reflexive

A binary relation R in a set X is said to be reflexive $\forall$ x $\in$x, then

(x, x)$\in$ R that is, x R x

ii) Symmetric

A binary relation R in a set X is said to be symmetric $\forall$ x, y $\in$ x, whenever

(x, y)$\in$ R then <y, x>$\in$ R that is, x R y then y R x.

iii) Transitive

A binary relation R in a set X is said to be transitive if $\forall$ x, y ,z$\in$ x, whenever

(x, y)$\in$ R and (y, z)$\in$ R then (x, z)$\in$ R that is, x R y and y R z then x R z.

iv) Anti Symmetric

A relation R in a set X is said to be anti symmetric if $\forall$ x, y $\in$ x whenever x R y and

y R x then x = y.

Partial Ordered Relation (or) Partial Ordering

A relation R in a set X is said to be partial ordered relation or partial ordering if it satisfies the following conditions

(i)Reflexive (ii)Anti Symmetric (iii)Transitive

Irreflexive

A relation R in a set X is said to be irreflexive if $\forall$ x $\in$x, <x, x>$\notin$ R.

Partial ordered set (or) Po set:

A set L on which a partial ordering is defined as less than or equal to is called partial ordered set (or) Poset. It is denoted by $(L, \leq)$.

Lower Bound

Let $(L, \leq)$ be a Po set and a, b$\in$ L if there exist an element c $\in$L such that $c \leq a and c \leq b$ then c is said to be lower bound for a and b.

Greatest lower Bound (GLB)

Let $(L, \leq)$ be a Poset and $A \subseteq L$, any element $x \in L$ is said to be greatest lower bound for A, if x is the lower bound for A and $x \geq y$ where y is any lower bound for A.

Example

X = {-1, 0, 1, 2, 3, 4}

A= {0, 1}

LB= {0, -1}

GLB= {0}

Upper Bound

Let $(L, \leq)$ be a Po set and a, b$\in$ L if there exist an element C $\in$L such that $c \geq a and c \geq b$ then c is said to be upper bound for a and b.

Least Upper Bound (LUB)

Let $(L, \leq)$ be a Po set and A is containing $A \in L$, any element $x \in L$ is said to be greatest lower bound for A, if x is the lower bound for A and $x \leq y$ where y is any lower bound for A.

Example

X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

A= {6, 7, 8}

UB= {8, 9, 10}

LUB= {8}

NOTE

Let A and B be any two elements in a po set if least upper bound and greatest upper bound of A and B exist and it is denoted by

glb {a, b}= $a * b$ ($*$ - meet or product)

lub {a, b}= $a \oplus b$ ($\oplus$ - join or sum)

Lattice

A lattice is a poset in which every pair of element have least upper bound and greatest lower bound.

Properties of Lattices

1) Idempotent law

i) $a * a = a$

ii) $a \oplus a = a$

2) Commutative law

i) $a * b = b * a$

ii) $a \oplus b = b \oplus a$

3) Associative law

i) $a*(b*c)=(a*b)*c$

ii) $a\oplus(b\oplus c)=(a\oplus b)\oplus c$

4) Absorption law

i) $a*(a\oplus b)=a$

ii) $a\oplus(a*b)=a$

Theorem 1

Let $(L,\leq)$ be a lattice in which $*,\oplus$ denote the operation of the meet and join respectively for any element $a,b\in L$.

Prove that $a\leq b \Leftrightarrow a*b=a \Leftrightarrow a\oplus b=b$ (or)

Let $(L,\leq)$ be a lattice for any element $a,b\in L$ then the following are equivalent:

i) $a\leq b$

ii) $a*b=a$

iii) $a\oplus b=b$

Proof

Given : $(L,\leq)$ be a lattice and $a,b\in L$.

To Prove

i) $a\leq b \Rightarrow a*b=a$

ii) $a*b=a \Rightarrow a\oplus b=b$

iii) $a\oplus b=b \Rightarrow a\leq b$

Proof of (i)

$a\leq b \Rightarrow a*b=a$

Assume that $a\leq b$

To Prove

$a * b = a$

As $a \leq a$ and $a \leq b$

$a = \text{lb}\{a, b\}$

$a \leq a * b \rightarrow 1$

By definition,

$a * b \leq a \rightarrow 2$

from 1 and 2

$\boxed{a * b = a}$

$\therefore \; a \leq b \Rightarrow a * b = a$

Proof of (ii)

$a * b = a \Rightarrow a \oplus b = b$

Assume $a * b = a$

Consider,

$a \oplus b = (a * b) \oplus b$

$a \oplus b = b$ (absorption law)

$\therefore \; a * b = a \Rightarrow a \oplus b = b$

Proof of (iii)

$a \oplus b = b \Rightarrow a \leq b$

Assume $a \oplus b = b$

lub{a, b}=b

ub{a, b}=b

$\Rightarrow a \leq b$

$a \oplus b = b \Rightarrow a \leq b$

Theorem 2 [Isotonicity property]

In any lattice $(L, \leq)$ the operation $*$ and $\oplus$ if $b \leq c$ then prove that

i. $a * b \leq a * c$
ii. $a \oplus b \leq a \oplus c$

Proof.

we know that

$a * a \leq a$ and

If $a \leq b \Leftrightarrow a * b = a$ and $a \oplus b = b$ and

as $b \leq c \Leftrightarrow b * c = b$ and $b \oplus c = c \rightarrow$ **1**

Proof of (i):

To prove $a * b \leq a * c$

Consider

$$a * b = (a * a) * (b * c)$$

$$= a * (a * b) * c$$

$$= a * (b * a) * c$$

$$= (a * b) * (a * c)$$

$$= \text{glb}\{(a * b), (a * c)\}$$

$\boxed{a * b \leq a * c}$

Proof of (ii):

To prove $a \oplus b \leq a \oplus c$

consider

$$a \oplus c = (a \oplus a) \oplus (b \oplus c)$$

$= a\oplus(a\oplus b)\oplus c$

$= a\oplus(b\oplus a)\oplus c$

$= (a\oplus b)\oplus(a\oplus c)$

$= \text{lub}\{(a\oplus b),(a\oplus c)\}$

$\therefore\ a\oplus c\geq a\oplus b$

$\boxed{a\oplus b\leq a\oplus c}$

Theorem 3 (Inequality distributive law):

Let (L, ≤) be a lattice and for any a,b,c belongs to L(a,b,c∈L)

Prove that the following inequalities

i)a⊕(b* c) $\leq (a\oplus b)*(a\oplus c)$

$ii)\ a*(b\oplus c)\geq(a*b)\oplus(a*c)$

Given

$(L,\leq)$ is a lattice and a,b,c∈L

To prove

i)a⊕(b* c) $\leq (a\oplus b)*(a\oplus c)$

$ii)\ a*(b\oplus c)\geq(a*b)\oplus(a*c)$

Proof of (i)

To prove a⊕(b* c) $\leq (a\oplus b)*(a\oplus c)$

As $a\leq a\oplus b$ and $a\leq a\oplus c$

$a = \text{lb}\{a\oplus b, a\oplus c\}$

$a\leq \text{glb}\{a\oplus b, a\oplus c\}$

$a\leq(a\oplus b)*(a\oplus c)$ ________1

As $b * c \leq b$ and $b \leq a \oplus b$

$b * c \leq b \leq a \oplus b$ ________ 2

As $b * c \leq$ and $c \leq a \oplus c$

$b * c \leq c \leq a \oplus c$ ____________ 3

From 2 and 3 $b * c \leq (a \oplus b) * (a \oplus c)$ _________ 4

From 1 and 4

$(a \oplus b) * (a \oplus c)$ is the upper bound of a and $b * c$

$(a \oplus b) * (a \oplus c)$ is the upper bound of $a \oplus (b * c)$

$\boxed{a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)}$

Proof of (ii)

As $a \geq a * b$ and $a \geq a * c$

$a \geq (a * b) \oplus (a * c)$ _______ 1

As $b \geq a * b, c \geq (a * c)$

$b \oplus c \geq (a * b) \oplus (a * c)$ _____ 2

$(a * b) \oplus (a * c)$ is the upper bound of a and $b \oplus c$

$\boxed{a * (b \oplus c) \geq (a * b) \oplus (a * c)}$

Theorem 4

In a lattice (L,≤) show that

i) $(a * b) \oplus (c * d) \leq (a \oplus c) * (b \oplus d)$

ii) $(a * b) \oplus (b * c) \oplus (c * a) \leq (a \oplus b) * (b \oplus c) * (c \oplus a)$

Given

Let (L,≤) be a lattice

To prove

i) $(a*b)\oplus(c*d)\leq(a\oplus c)*(b\oplus d)$

ii) $(a*b)\oplus(b*c)\oplus(c*a)\leq(a\oplus b)*(b\oplus c)*(c\oplus a)$

Proof of i)

$(a*b)\oplus(c*d)\leq(a\oplus c)*(b\oplus d)$

As $a\leq(a\oplus c)$ and $b\leq b\oplus d$

$a*b\leq(a\oplus c)*(b\oplus d)$ ——— 1

As $c\leq a\oplus c$ and $d\leq(b\oplus d)$

$c*d\leq(a\oplus c)*(b\oplus d)$ ——— 2

From1 and 2

$(a\oplus c)*(b\oplus d)$ is the upper bound of $a*b$ and $c*d$

$\boxed{(a*b)\oplus(c*d)\leq(a\oplus c)*(b\oplus d)}$

Proof of (ii)

$(a*b)\oplus(b*c)\oplus(c*a)\leq(a\oplus b)*(b\oplus c)*(c\oplus a)$

As $a*b\leq a$ and $a\leq a\oplus b$

$a*b\leq a\leq a\oplus b$ ——— 3

As $a*b\leq b$ and $b\leq b\oplus c$

$a*b\leq b\leq b\oplus c$ ——— 4

As $a*b\leq a$ and $a\leq a\oplus c$

$a*b\leq a\leq c\oplus a$ ——— 5

From 3,4&5

$a*b\leq(a\oplus b)*(b\oplus c)*(c\oplus a)$

Similarly

$b*c\leq(a\oplus b)*(b\oplus c)*(c\oplus a)$

$c * a \leq (a \oplus b) * (b \oplus c) * (c \oplus a)$

$$\boxed{(a * b) \oplus (b * c) \oplus (c * a) \leq (a \oplus b) * (b \oplus c) * (c \oplus a)}$$

Theorem 5

Let (L,≤) be a lattice ∀ a,b,c∈ L then show that the following inequalities

i) $a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$

ii) $a \geq c \Leftrightarrow a * (b \oplus c) \geq (a * b) \oplus c$

Given

(L,≤) be a lattice ∀ a,b,c∈ L

To Prove

i) $a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$

ii) $a \geq c \Leftrightarrow a * (b \oplus c) \geq (a * b) \oplus c$

Proof of(i):

To prove

$a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$

WKT, $a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$

Given that $a \leq c \Leftrightarrow a * c = a \Leftrightarrow a \oplus c = c \rightarrow 1$

Consider

$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$ (By inequality distributive law)

$= (a \oplus b) * c$(by 1)

$$\boxed{a \oplus (b * c) \leq (a \oplus b) * c}$$

Proof of (ii)

$a \geq c \Leftrightarrow a * (b \oplus c) \geq (a * b) \oplus c$

Given that $c \leq a \Leftrightarrow c * a = c$ and $c \oplus a = a$

Consider, $a*(b\oplus c)\geq(a*b)\oplus(a*c)$ (By Theorem 3)

$= (a*b)\oplus c$

$\boxed{a*(b\oplus c)\geq(a*b)\oplus c}$

Theorem6.

Show that in a lattice $a\leq b$ and $c\leq d$ then $a*c\leq b*d$

Given

Let (L,≤) be a lattice .

Given that $a\leq b$ and $c\leq d$

To prove

$a*c\leq b*d$

Proof

WKT $a\leq b \Leftrightarrow a*b=a$ and $a\oplus b=b$

As $c\leq d \Leftrightarrow c*d=c$ and $c\oplus d=d$

Consider, $a*c=(a*b)*(c*d)$

$=a*(b*c)*d$ (assosiative)

$=a*(c*b)*d$ (commutative)

$a*c=(a*c)*(b*d)$ (assosiative)

$a*c=\text{glb}\{a*c, b*d\}$

$\boxed{a*c\leq b*d}$

Theorem 7

In a lattice if $a\leq b\leq c$ then show that

i) $a\oplus b=b*c$

ii) $(a*b)\oplus(b*c)=(a\oplus b)*(a\oplus c)$

Given

Let $(L,\leq)$ be a lattice and $a\leq b\leq c$.

To prove

i) $a\oplus b=b*c$ ii) $(a*b)\oplus(b*c)=(a\oplus b)*(a\oplus c)$

Proof

Wkt ,$a\leq b \Leftrightarrow a*b=a$ and $a\oplus b=b$

and $b\leq c \Leftrightarrow b*c=b$ and $b\oplus c=c$

and $a\leq c \Leftrightarrow a*c=a$ and $a\oplus c=c$

Proof (i)

To prove $a\oplus b=b*c$

Consider, $a\oplus b=b$;

$$\boxed{a\oplus b=b*c}$$

Proof (ii)

$(a*b)\oplus(b*c)=(a\oplus b)*(a\oplus c)$

LHS $=(a*b)\oplus(b*c)$

$= a\oplus b$ (by 1)

$=b$

RHS$=(a\oplus b)*(a\oplus c)$

$=b*c$

$=b$

LHS=RHS

$$\boxed{(a*b)\oplus(b*c) = (a\oplus b)*(a\oplus c)}$$

Some special lattices

Chain

A lattice $(L,\leq)$ is called a chain if $\forall$ $a,b\in L$ either $a \leq b$ or $b \leq a$.

Distributed lattice

A lattice $(L,*,\oplus)$ is called a distributive lattice the operation $*$ and $\oplus$ distribute over each other

$i) a*(b\oplus c) = (a*b)\oplus(a*c)$
$ii) a\oplus(b*c) = (a\oplus b)*(a\oplus c)$

Theorem 8

Every chain is a lattice

To prove: Every chain is a lattice

Let $(L,\leq)$ be a chain. $\forall$ $a,b\in L$ either $a \leq b$ or $b \leq a$.

It is enough to prove every ordered pair have glb and lub .

Case (i):

Assume that $a \leq b$

As $a \leq a$ and $a \leq b$

$a=\text{lb}\{a,b\}$

$a\leq \text{glb}\{a,b\}$

$a\leq a*b$

By definition, $a*b\leq a$

$\therefore \boxed{a=a*b}$

As $b\geq a$ and $b \geq b$

$b = \text{ub}\{a,b\}$

$b\geq \text{lub}\{a,b\}$

$b \geq a \oplus b$

By definition, $a \oplus b \geq b$

$\therefore$ $\boxed{a \oplus b = b}$

Case (ii) assume b ≤ a

As b ≤ a and b ≤ b

b=lb$\{a, b\}$

$b \leq$ glb$\{a, b\}$

$b \leq a * b$

By definition $a * b \leq b$

$\therefore$ $\boxed{a * b = b}$

As $a \geq a$and a ≥ b

a = ub$\{a, b\}$

$a \geq$ lub$\{a, b\}$

$a \geq a \oplus b$

By definition, $a \oplus b \geq b$

$\therefore$ $\boxed{a \oplus b = a}$

Therefore in both cases (a,b) have lub and glb.

Theorem 9

Every chain lattice is a distributive lattice ∀ a, b, c ϵ L

Given

Let (L, ≤) be a chain lattice.

Let a, b, c ϵ L

To prove

Let (L,≤) be a distributive lattice

$i) a * (b \oplus c) = (a * b) \oplus (a * c)$
$ii) a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

Proof

Since (L,≤) be a chain lattice a ≤ b ≤ c or a ≥ b ≥ c

Case (i): let a ≤ b ≤ c

Wkt a ≤ b ⇔ a * b=a and a ⊕ b=b

and b ≤ c ⇔ b* c=b and b ⊕ c=c

and a ≤ c ⇔ a* c=a and a ⊕ c=c

Proof of (i) $a * (b \oplus c) = (a * b) \oplus (a * c)$

$LHS = a * (b \oplus c)$

$= a * c$

$= a$

$\text{RHS} = (a * b) \oplus (a * c)$

$= a \oplus a = a$

LHS=RHS

Proof of (ii)

$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

$LHS = a \oplus (b * c) = a \oplus b = b$

$RHS = (a \oplus b) * (a \oplus c) = b * c = b$

LHS=RHS

Case(ii)

Let c ≥ b ≥ a

$c \leq b \Leftrightarrow c * b = c$ and $c \oplus b = b$

$b \leq a \Leftrightarrow b * a = b$ and $b \oplus a = a$

$c \leq a \Leftrightarrow c * a = c$ and $c \oplus a = a$

Proof of (i)

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$
$$LHS = a * (b \oplus c) = a * b = b$$
$$RHS = (a * b) \oplus (a * c) = b \oplus c = b$$
$$LHS = RHS$$

Proof of (ii)

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$
$$LHS = a \oplus (b * c) = a \oplus c = a$$
$$RHS = (a \oplus b) * (a \oplus c) = a * a = a$$
$$LHS = RHS$$

Theorem 10

Let $(L, *, \oplus)$ be a distributive lattice and $\forall$ $a,b,c \in L$ prove that
$((a * b) = (a * c)) \wedge ((a \oplus b) = (a \oplus c)) \Rightarrow b = c$

Given

Let $(L, *, \oplus)$ be a distributive lattice and $\forall$ $a,b,c \in L$

$(a * b)=(a * c)$ and $(a \oplus b)=(a \oplus c)$

To prove : b=c

Proof

Consider,

$b = b \oplus (a * b)$ (associative)

$= b \oplus (a * c)$ (given)

$= (b \oplus a) * (b \oplus c)$ (distributive)

$= (a \oplus b) * (b \oplus c)$ (commutative)

$= (a \oplus c) * (b \oplus c)$ (given)

$= c \oplus (a * b)$ (distributive)

$= c \oplus (a * c)$ (given)

$= c$

$\therefore \boxed{b = c}$

Complemented lattice

A lattice $(L, *, \oplus, 0, 1)$ is called complemented lattice, if every element of L has at least one complement in L.

(i.e) $a * a' = 0$

$a \oplus a' = 1$

Note:

i) Demorgans' s law

i) $(a * b)' = a' \oplus b'$

ii) $(a \oplus b)' = a' * b'$

ii) Identity law

i) $a * 1 = 1 * a = a$

ii) $a \oplus 0 = 0 \oplus a = a$

iii) Null law

i) $a * 0 = 0$

ii) $a \oplus 1 = 1$

iv) Involution law

$(a')' = a$

Theorem 11

Show that the Demorgan's law

i) $(a*b)' = a' \oplus b'$

ii) $(a \oplus b)' = a' * b'$

holds in a complemented, distributive lattice (Boolean algebra).

Given

Let $(L, \leq)$ be a complemented, distributive lattice

To prove

i) $(a*b)' = a' \oplus b'$

ii) $(a \oplus b)' = a' * b'$

Proof of (i)

$(a*b)' = a' \oplus b'$

It is enough to prove

a) $(a*b)*(a' \oplus b') = 0$

b) $(a*b) \oplus (a' \oplus b') = 1$

Proof of (a)

$(a*b)*(a' \oplus b') = 0$

Consider LHS = $(a*b)*(a' \oplus b')$

$= ((a*b)*a') \oplus ((a*b)*b')(dis)$

$= (a*(b*a')) \oplus (a*(b*b'))(asso)$

$= (a*(a'*b)) \oplus (a*(b*b'))(commu)$

$= ((a*a')*b) \oplus (a*(b*b'))(asso)$

$= (0*b) \oplus (a*0)(a*a' = 0, b*b' = 0)$

$= 0 \oplus 0 (a * 0 = 0)$

$= 0$

$= RHS$

$\boxed{(a * b) * (a' \oplus b') = 0}$

Proof of (b)

$(a * b) \oplus (a' \oplus b') = 1$

L.H.S= $(a * b) * (a' \oplus b')$

$= ((a' \oplus b) \oplus a) * ((a' \oplus b') \oplus b) (dis)$

$= (a' \oplus (b' \oplus a)) * (a' \oplus (b' \oplus b)) (asso)$

$= (a' \oplus (a \oplus b')) * (a' \oplus (b' \oplus b)) (commu)$

$= ((a' \oplus a) \oplus b') * (a' \oplus (b' \oplus b)) (asso)$

$= (1 \oplus b') * (a' \oplus 1) (a' \oplus a = 1, b' \oplus b = 1)$

$= 1 * 1 (a \oplus 1 = 1)$

$= 1$

$= R.H.S$

$\boxed{(a * b) \oplus (a' \oplus b') = 1}$

Proof of (ii)

$(a \oplus b)' = a' * b'$

It is enough to prove

a) $(a \oplus b) * (a' * b') = 0$

b) $(a \oplus b) \oplus (a' * b') = 1$

Proof of (a)

$\left(a \oplus b\right) * \left(a' * b'\right) = 0$

L.H.S = $\left(a \oplus b\right) * \left(a' * b'\right)$

$= ((a' * b') * a) \oplus ((a' * b') * b)(dis)$

$= (a' * (b' * a)) \oplus (a' * (b' * b))(asso)$

$= (a' * (a * b')) \oplus (a' * (b * b'))(commu)$

$= ((a' * a) * b') \oplus (a' * (b * b'))(asso)$

$= (0 * b') \oplus (a' * 0)(a' * a = 0, b' * b = 0)$

$= 0 \oplus 0$

$= 0$

$= R.H.S$

$\boxed{\left(a \oplus b\right) * \left(a' * b'\right) = 0}$

Proof of (b)

$(a \oplus b) \oplus (a' * b') = 1$

$LHS = (a \oplus b) \oplus (a' * b')$

$= ((a \oplus b) \oplus a') * ((a \oplus b) \oplus a')(dis)$

$= (a \oplus (b \oplus a')) * (a \oplus (b \oplus b'))(asso)$

$= (a \oplus (b \oplus a')) * (a \oplus (b \oplus b'))(commu)$

$= ((a \oplus a') \oplus b) * (a \oplus (b \oplus b'))(asso)$

$= (1 \oplus b) * (a \oplus 1)(a \oplus a' = 1, b \oplus b' = 1)$

$= 1 * 1(null)$

$= 1$

$= RHS$

$\boxed{(a\oplus b)\oplus(a'*b')=1}$

Theorem 12

Show that in a complemented , distributive lattice the following are equivalent.

$a\le b \Leftrightarrow a*b'=0 \Leftrightarrow a'\oplus b=1 \Leftrightarrow b'\le a'$

Given

Let L be a complemented, distributed lattice $\forall a,b,a',b'\in L$

To prove

$a\le b \Leftrightarrow a*b'=0 \Leftrightarrow a'\oplus b=1 \Leftrightarrow b'\le a'$

It is enough to prove

(i) $a\le b \Rightarrow a*b'=0$

(ii) $a*b'=0 \Rightarrow a'\oplus b=1$

(iii) $a'\oplus b=1 \Rightarrow b'\le a'$

(iv) $b'\le a' \Rightarrow a\le b$

Proof of (i)

$a\le b \Rightarrow a*b'=0$

wkt $a\le b \Leftrightarrow a*b=a$ and $a\oplus b=b$

Consider $a\oplus b=b$

Post multiply by b' on both sides

$(a\oplus b)*b'=b*b'$

$(a'*a)\oplus(b'*b)=b*b'(dis)$

$(a*b')\oplus(b*b')=b\oplus b'(comm)$

$(a*b')\oplus 0=0(b*b'=0)$

$(a*b')\oplus 0=0(a\oplus 0=0)$

$\boxed{a \le b \Rightarrow a * b' = 0}$

Proof of(ii):

$a * b' = 0 \Rightarrow a' \oplus b = 1$

Assume $a * b' = 0$

Taking complement on both sides

$(a * b')' = 0'$

$a' \oplus (b')' = 1 (\text{Demorgons law})$

$a' \oplus b = 1 (\text{involution law})$

$a' \oplus b = 1$

$\boxed{a * b' = a \Rightarrow a' \oplus b = 1}$

Proof of (iii):

$a' \oplus b = 1 \Rightarrow b' \le a'$

consider,

$a' \oplus b = 1$

post multiply by b' on both sides

$(a' \oplus b) * b' = 1 * b'$

$(b' * a') \oplus (b' * b) = 1 * b'$

$(a' * b') \oplus (b * b') = 1 * b' (comm)$

$(a' * b') \oplus 0 = b'$

$a' * b' = b'$

$\Rightarrow b' \le a'$

$\boxed{a' \oplus b = 1 \Rightarrow b' \le a'}$

Proof of (iv):

$b' \le a' \Rightarrow a \le b$

Assume $b' \leq a'$

(i.e) $a * b' = b'$

Take complement on both sides

$(a * b')' = (b')'$

$(a')' \oplus (b')' = b$(Demorgons law)

$a \oplus b = b$(involution law)

$\Rightarrow a \leq b$

$\boxed{b' \leq a' \Rightarrow a \leq b}$

Theorem 13:

Simplify the Boolean expression

$(a * b)' \oplus (a \oplus b)'$

Given

Let L be a complemented and distributive lattice $\forall a, b, a', b' \in L$

To prove

$(a * b)' \oplus (a \oplus b)' = (a' \oplus b') \oplus (a' * b')$(Demorgons law)

$= ((a' \oplus b') \oplus a') * ((a' \oplus b') \oplus b')$(Distributive)

$= ((b' \oplus a') \oplus a') * ((a' \oplus b') \oplus b')$(commu)

$= (b' \oplus a') * (a' \oplus b')$(idem)

$= (a' \oplus b') * (a' \oplus b')$(commu)

$= (a' \oplus b')$(idem)

$= (a * b)'$

$\boxed{(a * b)' \oplus (a \oplus b)' = (a * b)'}$

Theorem 14

In a distributive lattice, $(a*b)\oplus(b*c)\oplus(c*a)=(a\oplus b)*(b\oplus c)*(c\oplus a)$

Given

Let L be a distributive lattice where a,b,c∈L.

To prove

$(a*b)\oplus(b*c)\oplus(c*a)=(a\oplus b)*(b\oplus c)*(c\oplus a)$

Proof

$LHS=(a*b)\oplus(b*c)\oplus(c*a)$

$=(a*b)\oplus(c*b)\oplus(c*a)$ (commutative)

$=(a*b)\oplus(c*(b\oplus a))$ (distributive)

$=(a*b)\oplus c)*((a*b)\oplus(b\oplus a))$ (distributive)

$=((c\oplus a)*(c\oplus b))*(((a*b)\oplus b)\oplus a)$ (distributive,associative)

$=((c\oplus a)*(b\oplus c))*(b\oplus a)$ (associative)

$=((c\oplus a)*(b\oplus c))*(a\oplus b)$ (commutative)

$=(c\oplus a)*((b\oplus c)*(a\oplus b))$ (associative)

$=(c\oplus a)*((a\oplus b)*(b\oplus c))$ (commutative)

$=((c\oplus a)*(a\oplus b))*(b\oplus c)$ (associative)

$=(a\oplus b)*((c\oplus a)*(b\oplus c))$ (commutative)

$=(a\oplus b)*(b\oplus c)*(c\oplus a)$ (associative,commutative)

$=RHS$

$\therefore\ \boxed{(a*b)\oplus(b*c)\oplus(c*a)=(a\oplus b)*(b\oplus c)*(c\oplus a)}$

Theorem 15:

In a complemented distributive lattice, if a ≤ b then show that

1. $a*b'=0$

2. $a' \oplus b = 1$

Given

Let L be a distributive lattice where a,b,c∈L

To prove

1. $a * b' = 0$

2. $a' \oplus b = 1$

Proof

Wkt if $a \leq b \Leftrightarrow a * b = a$ and $a \oplus b = b$

Proof of 1

$a * b' = 0$

consider, $a \oplus b = b$

post multiplying by b' on both sides

$(a \oplus b) * b' = b * b'$

$(b * a) \oplus (b' * b) = b' * b$ (distributive)

$(a * b') \oplus 0 = 0$ (commutative)

$\boxed{a * b' = 0}$ (b'*b=0)

Proof of 2

$a' \oplus b = 1$

consider, $a * b = a$

Post adding by a' on both sides,

$(a * b) \oplus a') = a \oplus a'$

$(a' \oplus a) * (a' \oplus b) = a \oplus a'$ (distributive)

$1 * (a' \oplus b) = 1$ ($a \oplus a' = 1$)

$a' \oplus b = 1$ (1*a=a)

$\therefore \boxed{a' \oplus b = 1}$

Modular lattice

A lattice (L, *, ⊕) is said to be modular, if $a \leq c \Leftrightarrow a \oplus (b * c) = (a \oplus b) * c$

Theorem 16

In a distributive lattice, show that if $a \leq c \Leftrightarrow a \oplus (b * c) = (a \oplus b) * c$

Or show that a distributive lattice is modular.

Given

Let L be a distributive lattice where a, b, c∈L.

To prove

L is modular.

(i.e.), $a \leq c \Leftrightarrow a \oplus (b * c) = (a \oplus b) * c$

Proof

$Wkt\ if\ a \leq c \Leftrightarrow a * c = a$ and $a \oplus c = c$

Consider,

$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

$= (a \oplus b) * c$

$\therefore \boxed{a \oplus (b * c) = (a \oplus b) * c}$

Lattice has algebraic system:

Algebraic system

A lattice is an algebraic system (L, *, ⊕) with two binary operations on L then the following laws are satisfied

(i)Associative (ii) Commutative (iii) Absorption.

Sub lattice:

Let (L, *, ⊕) be a lattice and let S is contained in L(S<=L) be a subset of L, the algebra (S, *, ⊕) is a sub lattice of (L, *, ⊕) if and only if S is closed under both the operations (*, ⊕).

Direct product of lattices:

Let (L,*, ⊕) and (S, *, ⊕) be a two lattices, the algebraic system (L· S,.,+) which the binary operations . and + on L· S such that for any ordered pair (a_1,b_1) and $(a_2,b_2) \in$ L· S.

1. $(a_1, b_1).(a_2.b_2) = (a_1 * a_2, b_1 \wedge b_2)$

2. $(a_1, b_1) + (a_2, b_2) = (a_1 \oplus a_2, b_1 \vee b_2)$

Is called the direct product of lattice (L, *, ⊕), (S, $\wedge, \vee$)

Theorem 17

Show that the direct product of any two distributive lattices is a distributive lattice.

Proof.

Let (L, *, ⊕) and (S, $\wedge, \vee$) be two distributive lattice

Let (L· S, .,+) be the direct product of L and S.

Let x, y, z$\in$ L· S.

Let

$$x = (a_1, b_1)$$
$$y = (a_2, b_2)$$
$$z = (a_3, b_3)$$

To prove L· S is a distributive lattice

1. $x.(y+z) = (x.y) + (x.z)$

2. $x + (y.z) = (x+y).(x+z)$

Proofof1.

$$x.(y+z) = (x.y)+(x.z)$$

$$LHS = x.(y+z)$$

$$= (a_1, b_1).((a_2, b_2)+(a_3, b_3))$$

$$= (a_1, b_1).(a_2 \oplus a_3, b_2 \vee b_3)$$

$$= (a_1 * (a_2 \oplus a_3), b_1 \wedge (b_2 \vee b_3))$$

$$= ((a_1 * a_2) \oplus (a_1 * a_3), (b_1 \wedge b_2) \vee (b_1 \wedge b_3))$$

$$= ((a_1 * a_2), (b_1 \wedge b_2)) + ((a_1 * a_3), (b_1 \wedge b_3))$$

$$= ((a_1, b_1).(a_2, b_2)) + ((a_1, b_1).(a_3, b_3))$$

$$= (x.y)+(x.z)$$

$$= RHS$$

$$\therefore \boxed{x.(y+z) = (x.y)+(x.z)}$$

Proof of 2.

$$x+(y.z) = (x+y).(x+z)$$

$$LHS = x+(y.z)$$

$$= (a_1, b_1)+((a_2, b_2).(a_3, b_3))$$

$$= (a_1, b_1).(a_2 * a_3, b_2 \wedge b_3)$$

$$= (a_1 \oplus (a_2 * a_3), b_1 \vee (b_2 \wedge b_3))$$

$$= ((a_1 \oplus a_2) * (a_1 \oplus a_3), (b_1 \vee b_2) \wedge (b_1 \vee b_3))$$

$$= ((a_1 \oplus a_2), (b_1 \vee b_2)).((a_1 \oplus a_3), (b_1 \vee b_3))$$

$$= ((a_1, b_1)+(a_2, b_2)).((a_1, b_1)+(a_3, b_3))$$

$$= (x+y).(x+z)$$

$$= RHS$$

$\therefore \boxed{x + (y.z) = (x + y).(x + z)}$

$\therefore$ L $\cdot$ S is distributive lattice.

Hasse Diagram or Lattice Diagram

Every finite lattice can be pictured in a poset diagram is called Hasse diagram or lattice diagram or partially ordered set diagram.

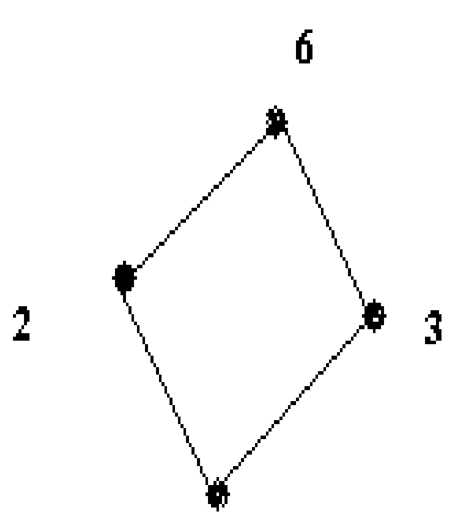
Example:

Let n be a positive integer and S_n be the set of all devices of n. Let $\leq$ denote the divisibility relation , then the ordered pair $(S_n, \leq)$ is a lattice.

1. Let n=6

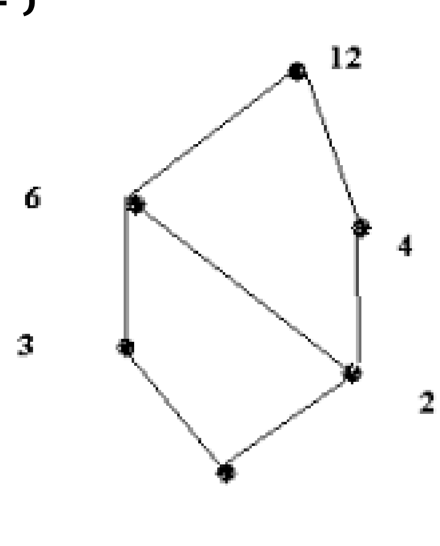
S_6={ 1,2,3,6}

$(S_6 , \leq)$



2.Let n=12, $(S_{12} , \leq)$

S_{12} ={1,2,3,4,6,12 }



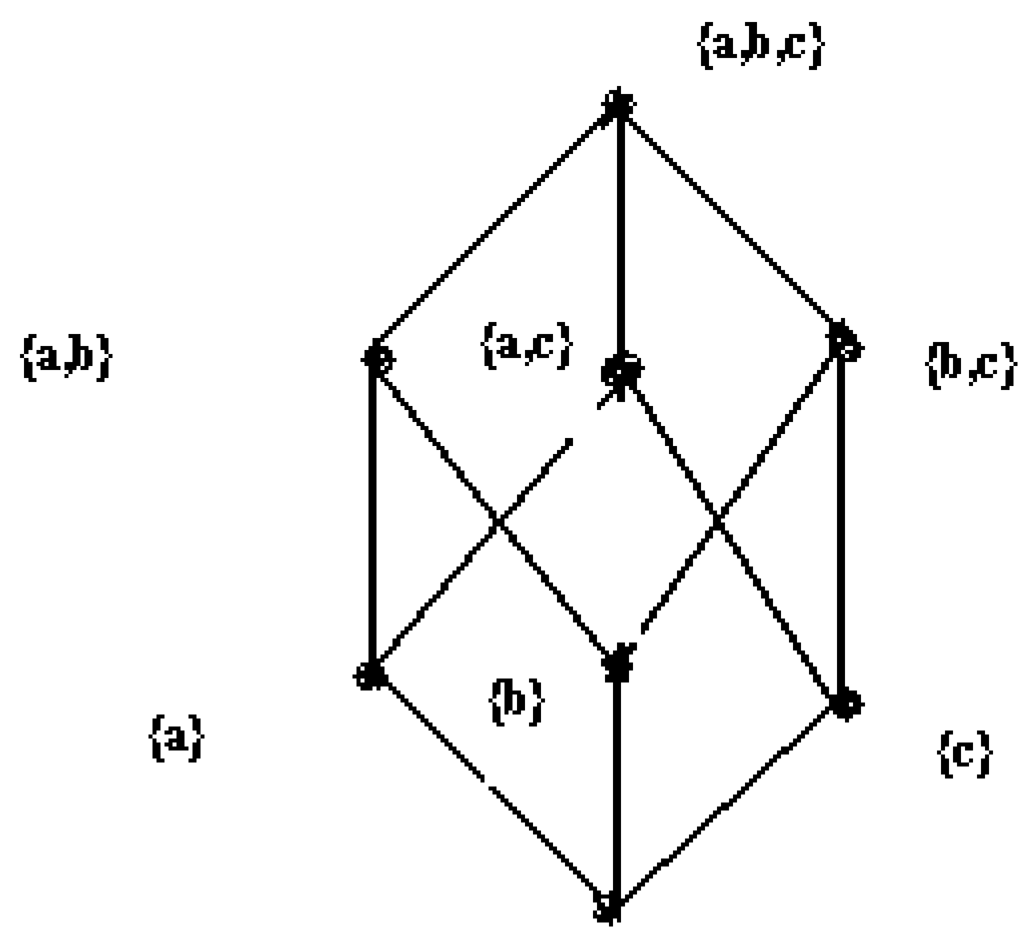
1.Draw the lattice diagram P (S,$\leq$) where S = {a,b,c}

Solution

Let S = {a,b,c} be set of three elements

The partition of S be

P (S) = { {a} , {b} , {c} , {a,b} , {a,c} , {b,c} , {a,b,c} , { } }



Properties of lattices

Absorption law

$(i) a \oplus (a * b) = a$

$(ii) a * (a \oplus b) = a$

Proof of (i)

As $a \le a, a * b \le a$

$a \oplus (a * b) \le a$

By definition,

$a \oplus (a * b) \ge a$

$\therefore\ a \oplus (a * b) = a$

Proof of (ii)

$a * (a \oplus b) = a$

As $a \le a, a \le a \oplus b$

$a \le a * (a \oplus b)$

By definition,

$a \ge a * (a \oplus b)$

$\therefore\ a * (a \oplus b) = a$

Commutative law

$(i) a * b = b * a$

$(ii) a \oplus b = b \oplus a$

Proof of (i)

$a * b$ = greatest lower bound (glb)of a and b
=glb of b and a
$\therefore$ a*b=b*a

Proof of (ii)

$a \oplus b$ = lowest upper bound (lub) of a and b
=lub of b and a
$\therefore \ a \oplus b = b \oplus a$

Idempotent law

$(i) a * a = a$

$(ii) a \oplus a = a$

Proof of (i) & (ii)

As $a \leq a, a * a \leq a$
As $a \geq a, a \oplus a \geq a$
$a * a$ = glb of a*a
$a \oplus a$ = lub of a$\oplus a$
$\therefore \ a * a = a = a \oplus a$

Identity law

$(i) a * 1 = 1 * a = a$

$(ii) a \oplus 0 = 0 \oplus a = a$

Proof of (i)

As $a * 1 \leq a$

As $a \leq a, a \leq 1$

$a \leq a * 1$

$\therefore \ a * 1 = a$

Proof of (ii)

By definition,

$a \oplus 0 \geq a$

$As\ a \leq a, 0 \leq a$

$a \oplus 0 \leq a$

$\therefore \ a \oplus 0 = a$

Null law

$(i)\ a * 0 = 0$

$(ii)\ a \oplus 1 = 1$

Proof of (i)

By definition,

$a \oplus 1 \geq a$ and $a \oplus 1 \geq 1$

$1 \geq a, 1 \geq 1$

$1 \geq a \oplus 1$

$\therefore\ a \oplus 1 = 1$

Proof of (ii)

By definition

$a * 0 \leq a$ and $a * 0 \leq 0$

$0 \leq a, 0 \leq 0$

$0 \leq a * 0$

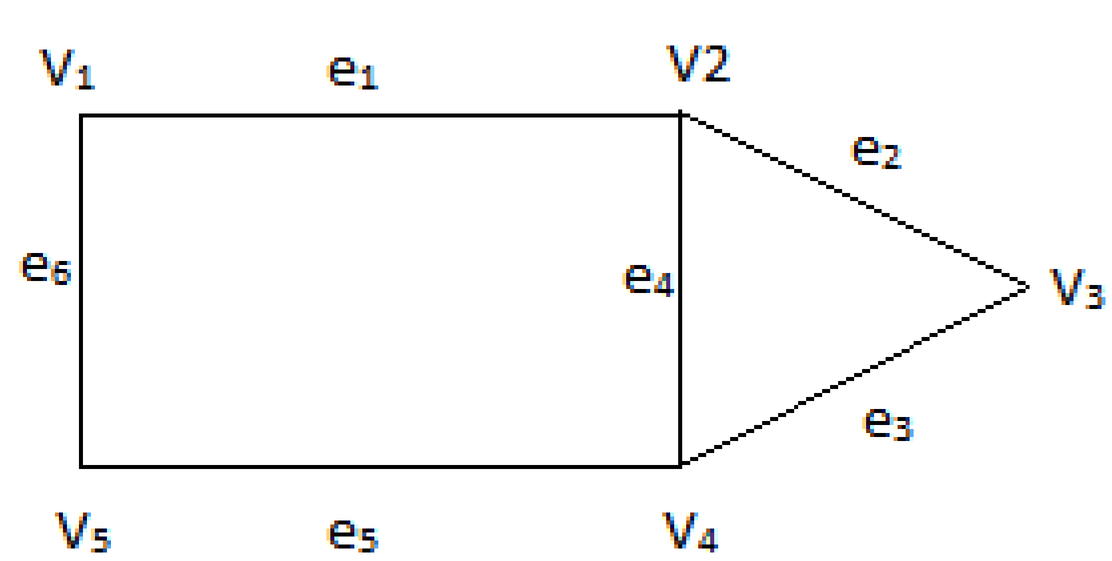
$\therefore\ a * 0 = 0$

UNIT-IV

GRAPH THEORY

Graph

A graph ' G' is an ordered pair G=(V,E) where V is an non – empty set of vertices and E is the set of edges. The vertices are also known as nodes or points.
eg:-

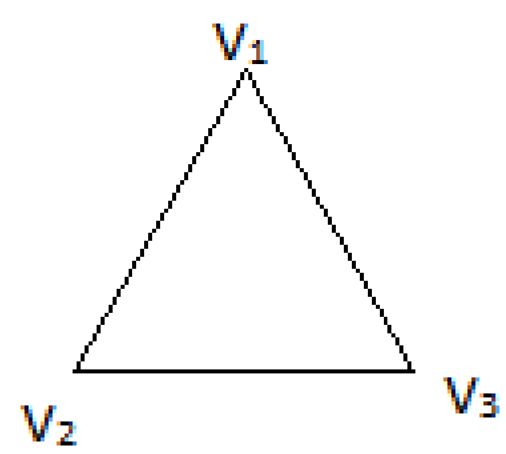


Here G=(V,E) where vertex set $V=\{V_1,V_2,V_3,V_4,V_5\}$ and edge set $E=\{e_1,e_2,e_3,e_4,e_5,e_6\}$.

Adjacent vertices

Any pair of vertices that are connected by an edge in a graph is called adjacent vertices.

eg:-

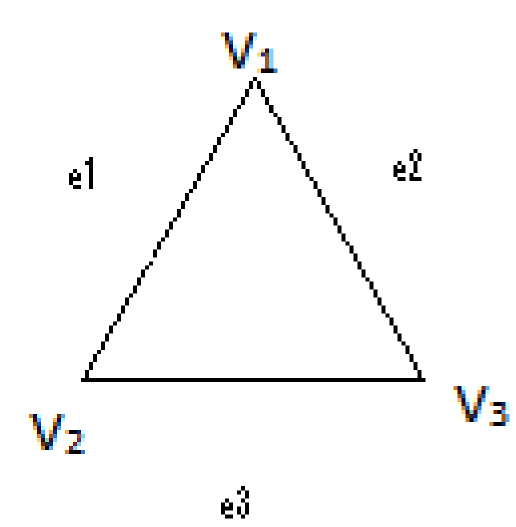


(i) V_1, V_2 are adjacent vertices.
(ii) V_1, V_3 are adjacent vertices.
(iii) V_2, V_3 are adjacent vertices.

Adjacent edges

Any pair of edges that are connected by a common vertex in a graph is called adjacent edges.

eg:-

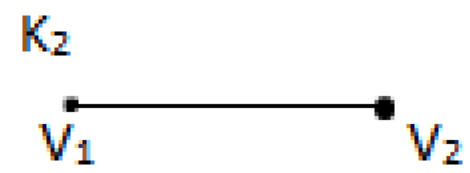


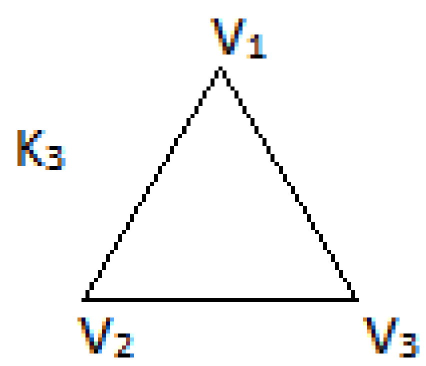
(i) e_1, e_2 are adjacent edges
(ii) e_1, e_3 are adjacent edges
(iii) e_2, e_3 are adjacent edges

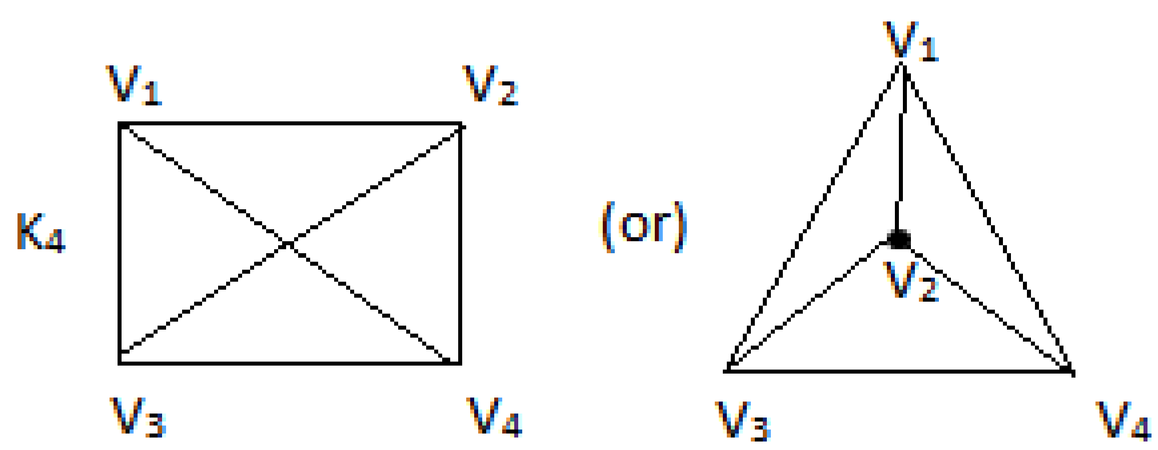
Complete graph

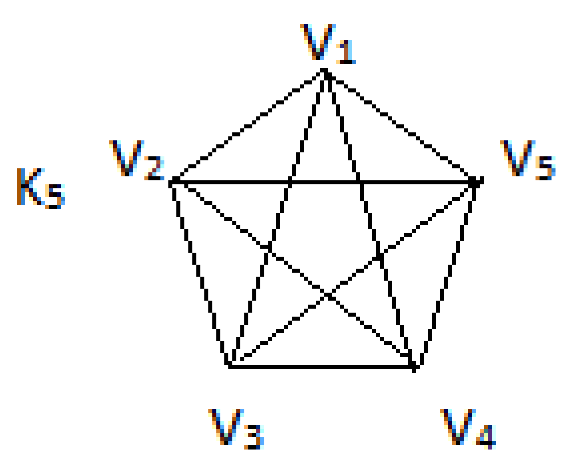
A graph G is said to be complete if there exists an edge between every pair of vertices of G.

eg:- The complete graphs with (2, 3, 4, 5) vertices.





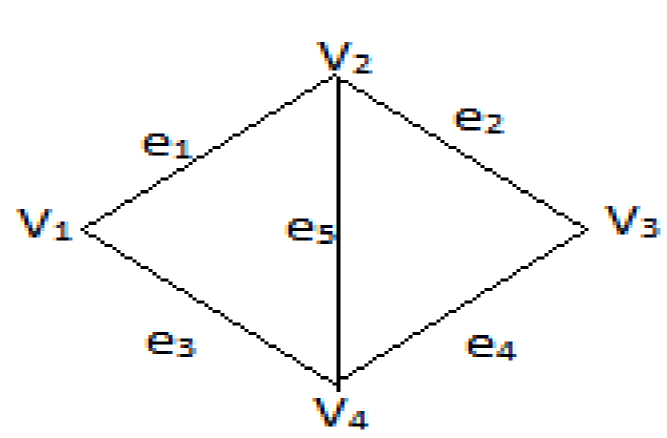




Simple graph

A graph without any parallel edges and self loops is called a simple graph.

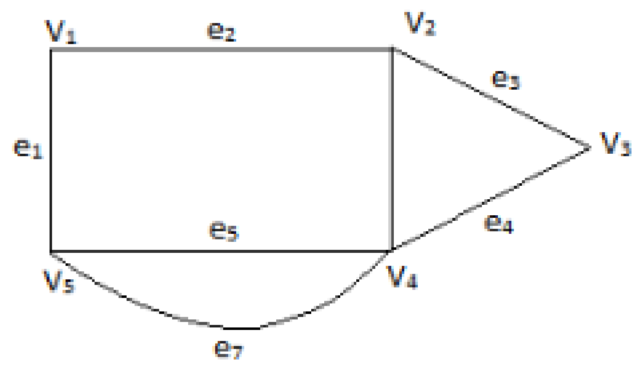
eg:-



Connected graph

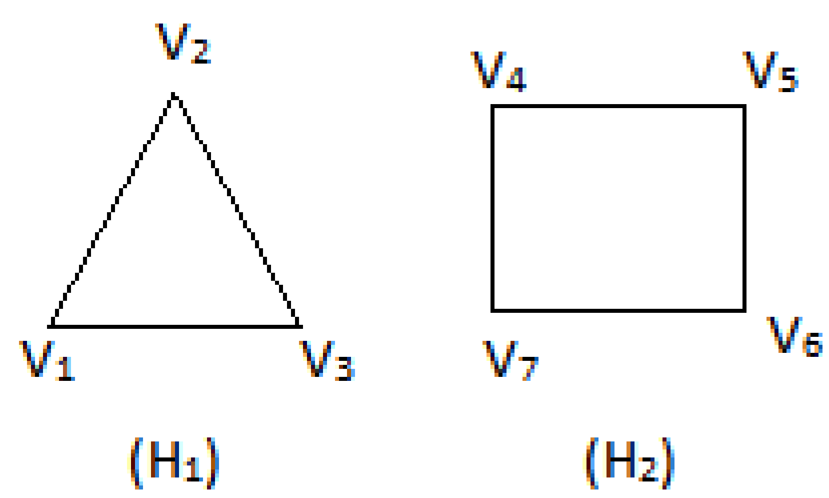
A graph G is said to be connected if there exist at least one path between every pair of vertices in G, otherwise G is disconnected.

Example.



Disconnected graph

In a disconnected graph, there exist at least one pair of vertices which do not have any path between them. Consider the example given below,



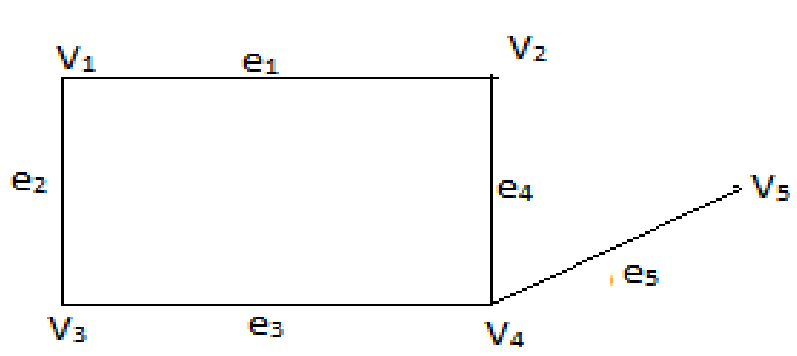
H_1 and H_2 are components. Any disconnected graph consists of 2 or more connected graph each of which is called a component.

Sub graph

Let G be a graph, then H is said to be a sub graph of G, if all vertices and edges of H are in G and each edge of H has the same end vertices in H as in G.

Degree of a vertex

The degree of the vertex *v* is the number of edges incident to that vertex *v*.



$d(v_1)=2$, $d(v_2)=2$, $d(v_3)=2$, $d(v_4)=3$, $d(v_5)=1$,

Isolated vertex

In any graph, if degree of a vertex is zero, then that vertex is called an isolated vertex.

Pendant vertex

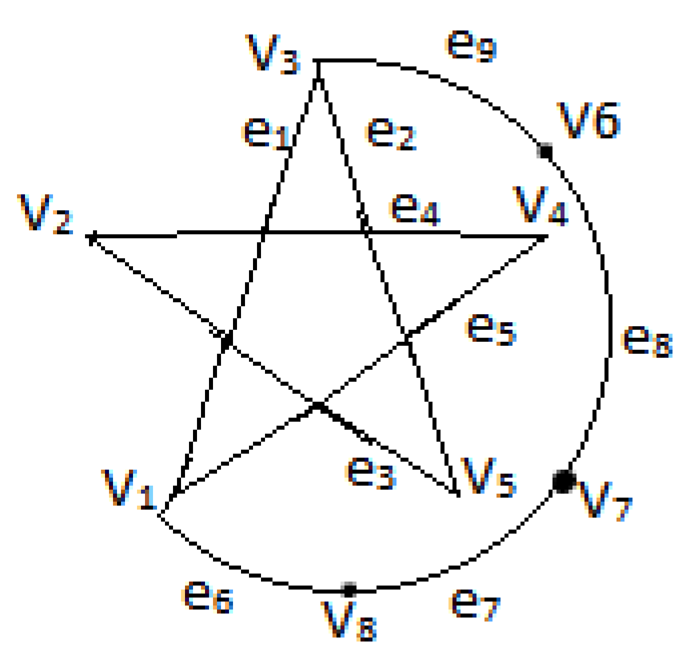
In any graph, if the degree of a vertex is one , then that vertex is called pendant vertex.

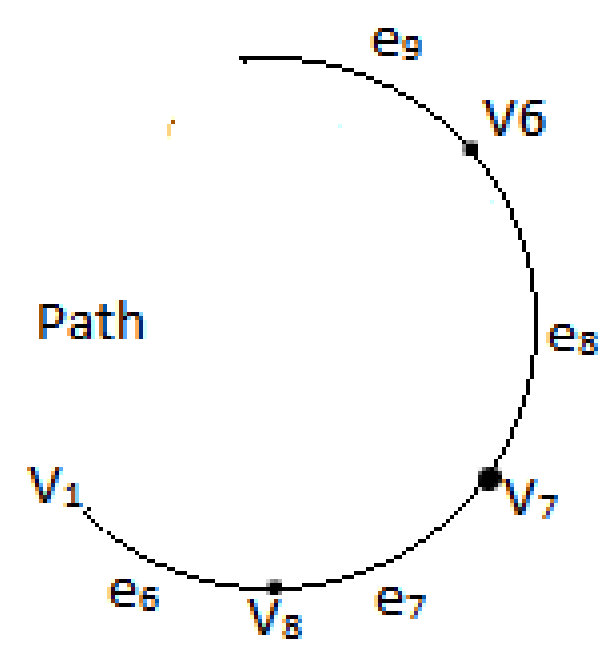
Walk

A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices such that each edge is incident with the preceding and the following vertices.

Path

In an open walk, if all the vertices appear only once, then it is called a path. In a path the end vertices are of degree 1 and other vertices are of degree 2. A path cannot have a loop or self loop.

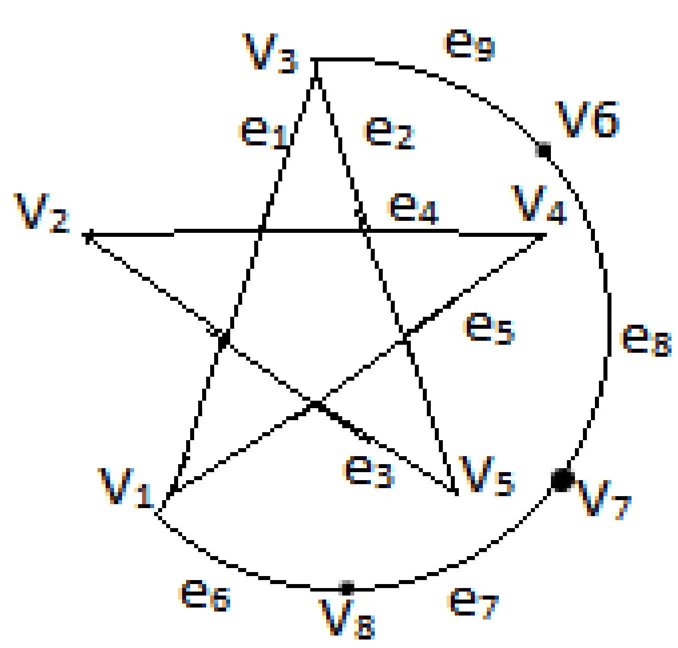


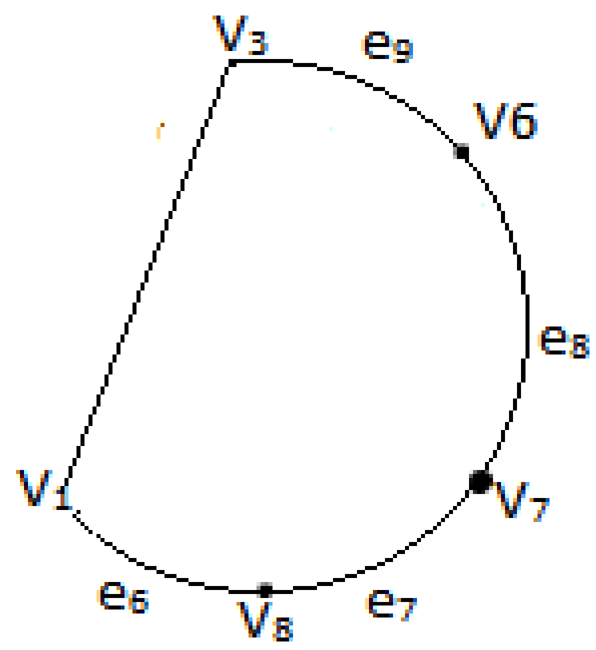


Path : $V_3\ e_9\ V_6\ e_8\ V_7\ e_7\ V_8\ e_6\ V_1$

Circuit

A closed walk in which no vertex appear more than once, except the end vertices is called as a circuit.

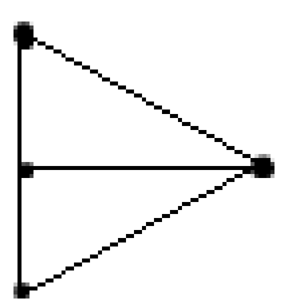




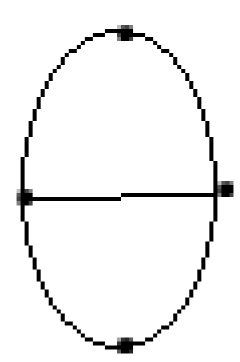
In a circuit all vertices are of degree 2.

Isomorphism in graph

Two graph G and G' are said to be isomorphic if there is a one – to – one correspondence between their vertices and edges, sch that their incidence relationship is preserved.



G



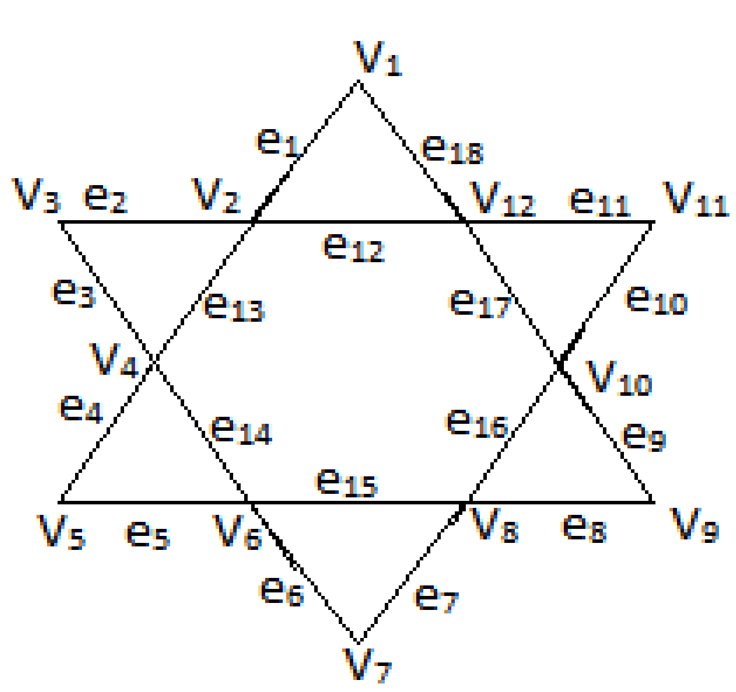
G'

Euler graph

A graph G is said to be Euler graph if it has a Euler circuit.

Euler circuit (Euler Tour) is a circuit in G covering all the edges of G exactly once.

Example

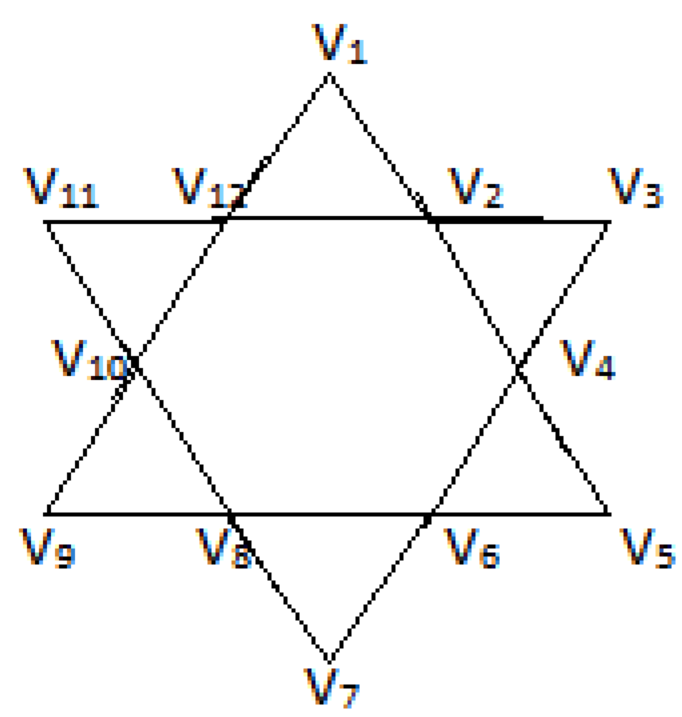


Hamiltonian circuit

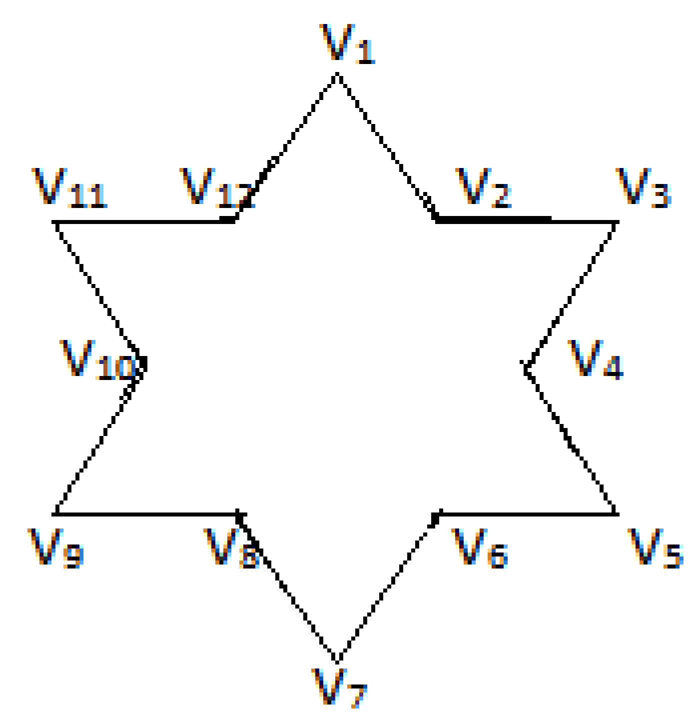
A circuit which covers all the vertices of the graph exactly once is called Hamilton circuit.

Hamiltonian graph

A graph which has Hamiltonian circuit is called Hamiltonian graph.

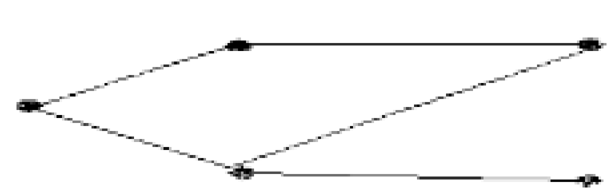


Hamiltonian Graph



Hamiltonian Circuit

Example A Graph which is neither Euler nor Hamiltonian.

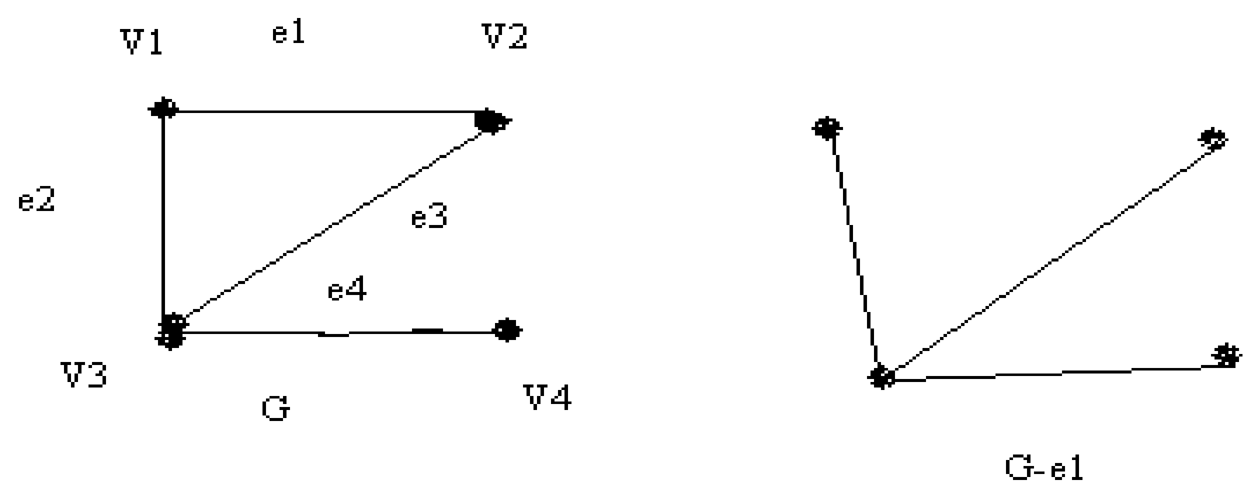


Cut edge

An edge ‘ e’ of a graph G is said to be cut edge of G if w(G-e) > w(G).

That is by deleting an edge e in G it will increase the number of components of the graph G.

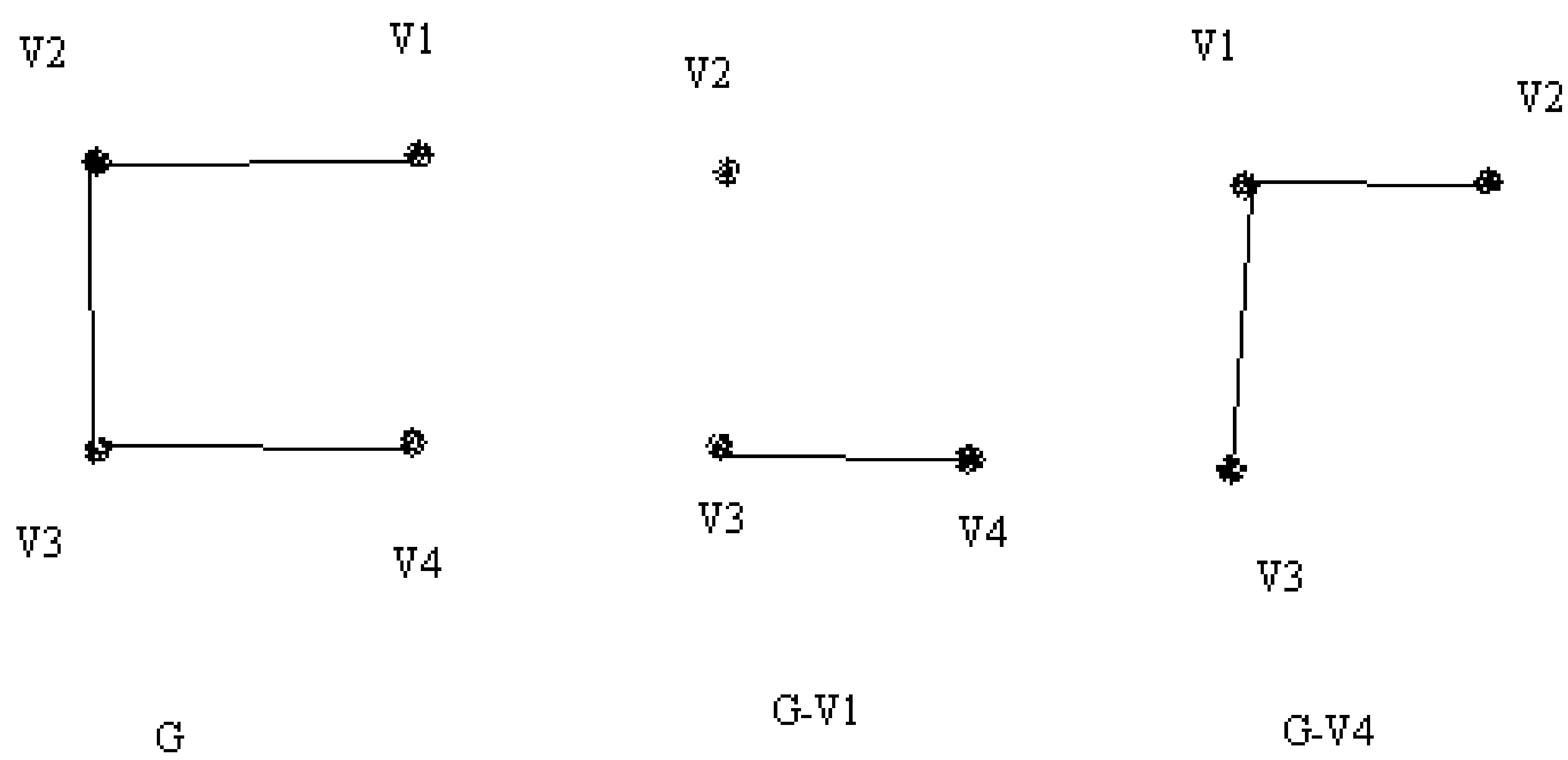
Example



Cut vertex

A vertex V of a graph G is a cut vertex if w(G-V) > w(G)

V2 is cut vertex but V4 is not.



Theorem 1

Let G =(V,E) is a graph with vertices $\{V_1,V_2,V_3,\dots\dots V_n\}$ and edges then prove that $\sum d(V_i)=2p$.

Given

Let G = (V,E) graph

$V=\{V_1,V_2,V_3,\dots\dots V_n\}$

$|V(G)|=n$ and $|E(G)|=p$

To Prove

$\sum d(V_i) = 2p$

Proof

Consider $\sum d(V_i) = d(V_1) + d(V_2) + \ldots + d(V_3)$

= the sum of the degree of all vertices

= the sum of end point of all the edges

= 2p

Theorem 2

Prove that the number of vertices of odd degree in a graph is always even.

Given

Let G =(V,E) is a graph.

To prove

The number of odd degree vertices is even.

Proof

Consider $V(G) = V_1 + V_2$

Where V_1 is the set of odd degree vertices

V_2 is the set of even vertices

$\sum dV(G) = d(V_1) + d(V_2)$

We know that

$\sum dV(G) = 2p$ (p is the number of edges)

= even

Even number = $\sum d(V_1) + \sum d(V_2)$

Even number = $\sum d(V_1)$ + even number

$\sum d(V_1)$ = even number – even number

= even number

$\sum d(V_1)$= even

Hence the number of odd degree vertices is even.

Theorem 3

A graph G is disconnected if and only if its vertex set V can be partition into two non-empty disjoint sets V_1 and V_2 such that there exist no edges in G whose one end vertex in V_1 and other end is in V_2 .

To prove

Assume that G is disconnected.

The vertex set of G can be partition into two non empty sub set of V_1 , V_2 such that there is no edge one end in V_1 and other end in V_2 . That is to prove

1.) $V_1 \neq \emptyset$, $V_2 \neq \emptyset$
2.) $V_1 \cap V_2 = \emptyset$
3.) There is no edge one end in V_1 other end in V_2

Proof

Since G is disconnected there exist a pair of vertices a and b such that there is no path in between them.

Select the component C_a such that the set of all edges connected with a and the neighbouring vertices .

Let V_1 denotes the set of all vertices in Ca . Therefore V_1 is non empty since a € V_1.

Let us define $V_2 = V - V_1$. This implies V_2 is non empty since b € V_2 . Clearly $V_1 \cap V_2 = \emptyset$

Assume that there is an edge x.y . Such that x € V_1 and y € V_2 . Since x € V_1 there is a path between a and x and y € V_2 there is a path between y and b . This implies that we get a path between a and b which is a contradiction to our assumption that G is disconnected .

Therefore there is no edge end in V_1 and the other end in V_2 .

converse part

Assume that the vertex set of G can be partition into two non empty disjoint subsets V_1 and V_2 such that there is no edge one end in V_1 and other end in V_2.

Clearly we have two components G_1 and G_2 whose vertex set is V_1 and V_2 . This implies G is disconnected.

Theorem 4

If G is a graph (connected or disconnected) with exactly two vertices of odd degree then there is a path joining these two vertices.

Given

Let G be a graph . Let V_1 and V_2 be the two odd degree vertices.

To prove

There is a path between V_1 and V_2 .

Proof

Case 1

Let G be connected clearly there is a path between V_1 and V_2

Case 2

Let G be disconnected therefore G have at least two component

We know that the number of odd degree vertices in graph is always even. Therefore V_1 and V_2 must belong to the same component . Since each component is a connected graph therefore we have a path between V_1 and V_2 .

Theorem 5

In a simple graph with n vertices the maximum number of edges is $\frac{n(n-1)}{2}$

Given

Let G be a simple graph with n vertices

To prove

$|E(G)| \leq \frac{n(n-1)}{2}$

Proof

The proof is by method of induction on n

Let n =1

Then G be a isolated vertex

$|E(G)| = 0 = \frac{1(1-1)}{2}$

Let n =2

Then G be single edge

$|E(G)| = 1 = \frac{2(2-1)}{2}$

Therefore the result is true for n=1 and n=2

Assume that the result is true for n=k

Let us prove the result for n=k+1. Consider the graph G with k vertices we know that

$|E(G)| = \frac{k(k-1)}{2}$

Construct the graph G' such that adding a new vertex V' in G now draw the edges from all the vertices in G to V'. Therefore we have

$$|E(G)| = \frac{k(k-1)}{2} + k$$

$$= \frac{k^2 - k + 2k}{2}$$

$$= \frac{k^2 + k}{2}$$

$$= \frac{k(k+1)}{2}$$

$$= \frac{(k+1)(k+1-1)}{2}$$

K+1= n from the above equation

$$\boxed{|E(G)| = \frac{n(n-1)}{2}}$$

Therefore the result is true for n=k+1. Hence the result is true for all n.

Theorem 6

Let G be a simple graph with n vertices and k components then the maximum number of edges of G is $\frac{(n-k)(n-k+1)}{2}$

Proof

Let G be a simple graph.

$|V(G)| = n$ and $|\omega(G)| = k$

To Prove $|E(G)| \leq \frac{(n-k)(n-k+1)}{2}$

Let $G_1, G_2, \ldots, G_k$, are the components of G and $n_1, n_2, \ldots, n_k$ are the number of vertices in $G_1, G_2, \ldots, G_k$, respectively.

Therefore $n_1 + n_2 + \cdots + n_k = n$

$$\sum_{i=1}^{k} n_i = n \rightarrow 1$$

We know that A simple graph with n vertices have the maximum number of edges is $\frac{n(n-1)}{2}$

G_1 have almost $\frac{n_1(n_1-1)}{2}$ edges

That is $|E(G_1)| = \frac{n_1(n_1-1)}{2}$

Similarly, $\left|E\left(G_2\right)\right| = \dfrac{n_2\left(n_2 - 1\right)}{2}, \cdots, \left|E\left(G_k\right)\right| = \dfrac{n_k\left(n_k - 1\right)}{2}$

The maximum number of edges of G are

$$\left|E(G)\right| = \left|E\left(G_1\right)\right| + \left|E\left(G_2\right)\right| + \cdots + \left|E\left(G_k\right)\right|$$

$$\left|E(G)\right| = \frac{n_1\left(n_1 - 1\right)}{2} + \frac{n_2\left(n_2 - 1\right)}{2} + \cdots + \frac{n_k\left(n_k - 1\right)}{2}$$

$$\left|E(G)\right| = \frac{1}{2}\left[n_1^2 - n_1 + n_2^2 - n_2 + \cdots + n_k^2 - n_k\right]$$

$$\left|E(G)\right| = \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \left(n_1 + n_2 + \cdots + n_k\right)\right]$$

$$\left|E(G)\right| = \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$\left|E(G)\right| = \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right] \rightarrow 2$$

From equation 1, we have

$$n_1 + n_2 + \cdots + n_k = n$$

$$n_1 - 1 + n_2 - 1 + \cdots + n_k - 1 = n - k$$

Squaring on both sides

$$\left[n_1 - 1 + n_2 - 1 + \cdots + n_k - 1\right]^2 = (n - k)^2$$

$$\left(n_1 - 1\right)^2 + \left(n_2 - 1\right)^2 + \cdots + \left(n_k - 1\right)^2 + \text{Product terms}(PT) = (n - k)^2$$

$$n_1^2 + n_2^2 + \cdots + n_k^2 - 2\left(n_1 + n_2 + \cdots + n_k\right) + \left(1 + 1 + \cdots + 1\right) + PT = (n - k)^2$$

$$\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + k + PT = (n - k)^2$$

$$\sum_{i=1}^{k} n_i^2 - 2n + k + \text{PT} = (n - k)^2$$

$$\sum_{i=1}^{k} n_i^2 + PT = (n-k)^2 - k + 2n$$

$$\sum_{i=1}^{k} n_i^2 \le (n-k)^2 - k + 2n \rightarrow 3$$

Substitute equation 3 in 2

$$|E(G)| = \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right] \rightarrow 2$$

$$|E(G)| \le \frac{1}{2}\left[(n-k)^2 - k + 2n - n\right]$$

$$|E(G)| \le \frac{1}{2}\left[(n-k)^2 - k + n\right]$$

$$|E(G)| \le \frac{1}{2}\left[(n-k)^2 + n - k\right]$$

$$\boxed{\therefore |E(G)| \le \frac{1}{2}(n-k)\left[(n-k+1)\right]}$$

Theorem 7

Prove that in a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Proof

Let G be a simple graph.

$$|V(G)| = n$$

$$|E(G)| \ge \frac{(n-1)(n-2)}{2}$$

To prove G is connected

Proof is by the method of contradiction.

Let G is disconnected.

Then G has at least two components G_1 and G_2.

Let V_1 and V_2 be the set of vertices in G_1 and G_2.

Let $|V_1| = m$ and $|V_2| = n - m$

Note that $1 \leq m \leq n - 1$ and there is no edge such that one end in V_1 and other end in V_2

The maximum number of edges of G is

$$|E(G)| = |E(G_1)| + |E(G_2)|$$

$$|E(G)| = \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2}$$

$$|E(G)| = \frac{1}{2}\left[m^2 - m + n^2 - nm - n - nm + m^2 + m\right]$$

$$|E(G)| = \frac{1}{2}\left[2m^2 - 2nm + n^2 - n\right]$$

$$|E(G)| = \frac{1}{2}\left[2m^2 - 2nm + n(n-1)\right]$$

$$|E(G)| = \frac{1}{2}\left[2m^2 - 2nm + (n+2-2)(n-1)\right]$$

$$|E(G)| = \frac{1}{2}\left[2m^2 - 2nm + (n-2)(n-1) + 2(n-1)\right]$$

$$|E(G)| = \frac{1}{2}\left[(n-2)(n-1) + 2m^2 - 2nm + 2n - 2\right]$$

$$|E(G)| = \frac{1}{2}\left[(n-2)(n-1) + 2(m^2-1) - 2n(m-1)\right]$$

$$|E(G)| = \frac{1}{2}\left[(n-2)(n-1) - 2(m-1)(n-m-1)\right]$$

$$|E(G)| \leq \frac{1}{2}\left[(n-2)(n-1)\right]$$

Which is a contradiction to $\left|E(G)\right| \geq \frac{(n-1)(n-2)}{2}$

Therefore G must be connected.

Theorem 8

A connected graph is a Euler graph if and only if all the vertices of G are of even degree.

Given

Let G be a connected Euler graph.

To prove

Every vertex of G is of even degree.

Proof

Since G is a Euler graph it has Euler circuit. That G has a closed walk C that covers all the edges of G exactly once. Choose a arbitrary vertex V of G. Clearly there are two edges incident on V . This implies degree of V is at least 2.

Suppose the vertex V is incident with another circuit means the degree of v is 4. Preceding in this way every vertex of G is of degree 2 or multiple of 2. Therefore every vertex of G is of even degree.

Converse part

Assume that G is a connected graph and every vertex of G is of even degree

To prove

G is Euler

Proof

Choose a arbitrary vertex in G .since degree of G is even, there are at least two edges incident to V . Starting from V trace a walk as possible covering the maximum edges of G and come back to V.

If this closed walk (C) covers all the edges of G then G has a Euler circuit C . Suppose c doesn' t cover all the edges then proceeding as follows.

Let C' =G-C

Since G is connected there exist at least one vertex is common to C and C' .From that vertex find a closed walk.

Let this closed walk be C'' . If C∪ C'' gives a Euler circuit then G is a Euler graph. Otherwise proceeding the above step until have to covering all the edges of G . Hence G is a Euler graph.

Theorem 9

In a connected graph G exactly 2K odd degree vertices then there exist K edge disjoint sub graphs such that they together contain all the edges of G and each sub graph is a unicursal graph.

Proof

Let G be a graph with 2K odd degree vertices $V_1, V_2, \ldots \ldots .. V_k$ and $W_1, W_2, \ldots \ldots \ldots . W_K$ each V_i

Add K edges between the pair of vertices (V_1, W_1), $(V_2, W_2) \ldots \ldots \ldots .. (V_k, W_k)$.

Let us denote these edges as $f_1, f_2, \ldots \ldots f_k$ and the remaining edges of G be $e_1, e_2, \ldots .. e_k$.

Clearly the addition of the edges $f_1, f_2 \ldots \ldots f_k$ contribute degree 1 to each Vi . This implies $d(V_i)$ is even for every i .

We know that in a connected graph if every vertices is of even degree then it is Euler graph. Therefore G is Euler we can find the closed walk C covering all the edges of G

Delete f_1 from walk C it gives a universal line similarly deletion of $f_1, f_2 \ldots .. f_k$ will give a edge disjoint unicursal line . Therefore we get a K edge disjoint sub graph of G and each sub graph is a unicursal line

Theorem 10

A connected graph G is Euler graph if and only if it can be decomposed into circuits.

Proof

Let G be a connected Euler graph and V can be decomposed into circuits.

Let V_1 be any vertex of G . Since G is Euler, $d(V_i)$ is even for every i.

Therefore there are two edges incident with V_1 . Let V_2 be the adjacent vertex of V_1 and V_2 is also even degree therefore V_2 is adjacent to V_3 .

Continuing in this way we get a circuit arriving at V_1 . Let us denote this circuit be C_1 . Remove C_1 from the graph G. Consider $G - C_1$ all the vertices in $G\text{-}C_1$ are of even degrees. Proceeding the above steps repeatedly we will get a edge disjoint circuits $C_1, C_2 \ldots . C_n$ hence G can be decomposed into circuit.

Converse part

Let G be a connected graph which is decomposed into circuits.

To prove G is Euler.

Choose any arbitrary vertex V (G) clearly V is in any of the circuit . If V is in only one circuit the degree of V is 2. If V is in more than one circuit then degree of vertex V is multiple of two.

This argument is true for every vertex of G. Therefore every vertex of G are of even degree. Hence G is Euler graph.

Theorem 11

Let G be a complete graph with n vertices then it has $\frac{n(n-1)}{2}$ edges

Proof

Let G be a graph with n vertices $V_1, V_2 \ldots .. V_n$.

And G be a complete graph and every vertex is of degree (n-1) . There V_1 is incident with (n-1) edges and V_2 is incident with remaining (n-2) edges and V_3 is incident with remaining (n-3) edges.

The number of edges of G = (n-1) + (n-2) + (n-3) +… . 2 +1

$$= 1+2+3+\ldots ..+(n-1) \quad [\text{This is a A.P}(1+2+..+n= n(n+1)/2)]$$

$$= \frac{(n-1)(n-1+1)}{2}$$

The number of edges of G $= \frac{n(n-1)}{2}$

Theorem 12

In a complete graph with n vertices there are $\frac{(n-1)}{2}$ edges disjoint Hamilton circuit if n is odd (n ≥3).

Given:

Let G is a complete graph and |E(G)|=n and n is odd (n ≥3)

Proof:

We know that the complete graph with n vertices have $\frac{n(n-1)}{2}$ edges

Therefore |E(G)| = $\frac{n(n-1)}{2}$

In the graph G any Hamilton circuit with n vertices have exactly n edges .

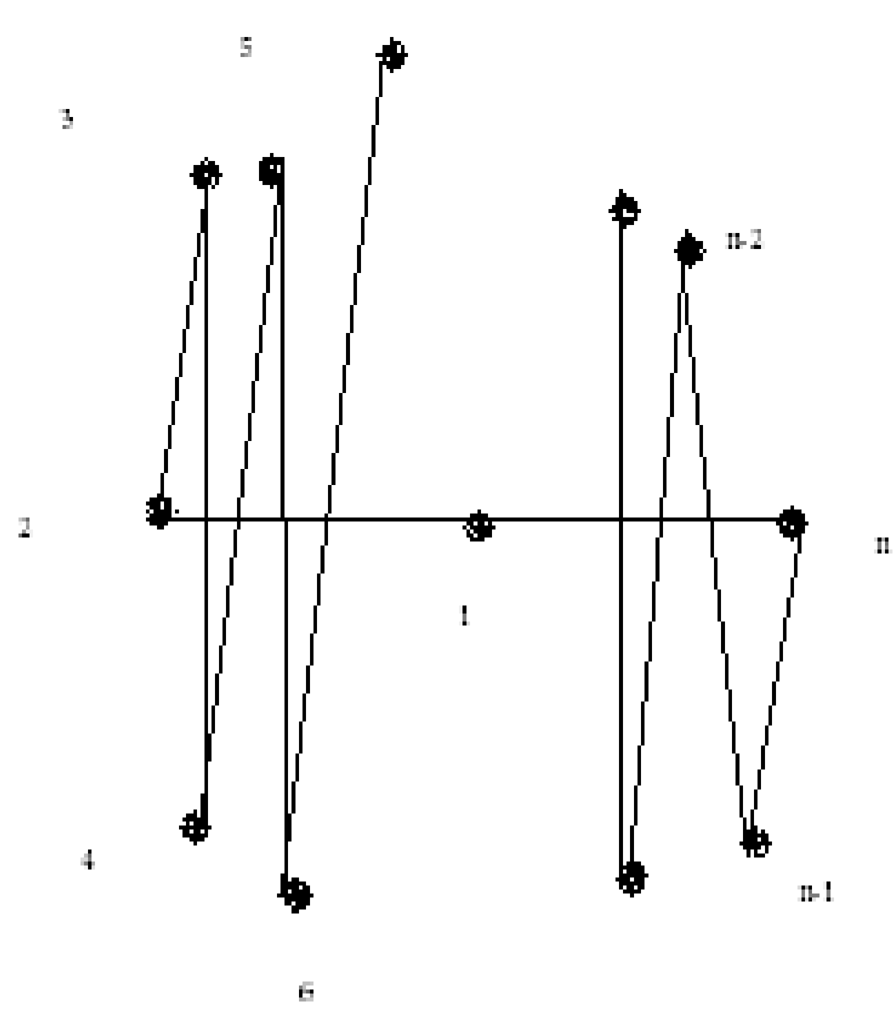
The number of Hamilton circuit in G is equal to $= \dfrac{\frac{n(n-1)}{2}}{n} = \frac{(n-1)}{2}$

It remains to show that these $\frac{(n-1)}{2}$ Hamilton circuit are edge disjoint

Arrange the vertices 1,2,... .n and the edges to form a Hamilton circuit .

Keep the vertex 1 , fixed on the centre of the circle and the vertices 2 and n on the ends of diameter passing through the center.

Now the remaining (n-3) vertices alternatively equally spaced with an angle $\frac{n-3}{360^\circ}$



Arrange the vertices 3, 5,7, ... n-2 on the upper half of the circle and the remaining vertices 4,6,8,... . n-3,n-1 on the lower half of the circle.

Clearly we have $\frac{(n-3)}{2}$ vertices on the upper half and the lower half of the circle.

Consider the circuit 1,2,3,4,... n-2,n-1,n. Now rotate the circle as 3,1,n-1 as diameter .

Similarly we get $\frac{(n-3)}{2}$ Hamilton edge disjoint Hamilton circuit therefore the edge circuit

Hamilton circuits $= \frac{(n-3)}{2} + 1$

$$= \frac{n-3+2}{2}$$

$$= \frac{n-1}{2}$$

Hence a complete graph with n vertices has $\frac{n-1}{2}$ edges disjoint Hamilton circuits.

Theorem 13

Prove that any 2 simple connected graph with n vertices, all of degree 2 are isomorphic

Proof

Let G_1,G_2 are simple, connected graphs with n vertices.

Given that each vertex in both G_1 and G_2 are of degree 2.

Let $V_1, V_2, \ldots, V_n$ be vertices of G_1.

We know that $\sum d(V_i) = 2p = 2 \cdot$ the number of edges

$d(V_1) + d(V_2) + \ldots d(V_n) = 2 \cdot$ the number of edges

$2+2+\ldots+2=2 \cdot$ the number of edges.

$2n=2 \cdot$ the number of edges.

Therefore Number of edges in G_1 =n.

Similarly we can find number of edges in G_2 as n, Where $u_1, u_2, \ldots .u_n$ are vertices of G_2.

So we can find a one to one mapping from vertices of G_1 into vertices of G_2.

Hence G_1 and G_2 are isomorphic. Hence any 2 simple connected graph with n vertices are of degree 2 are isomorphic

UNIT-V

TREES

Tree is a connected acyclic graph.

In a Tree if there is a vertex v distinguished from all other vertex then it is said to be **root** of the tree.

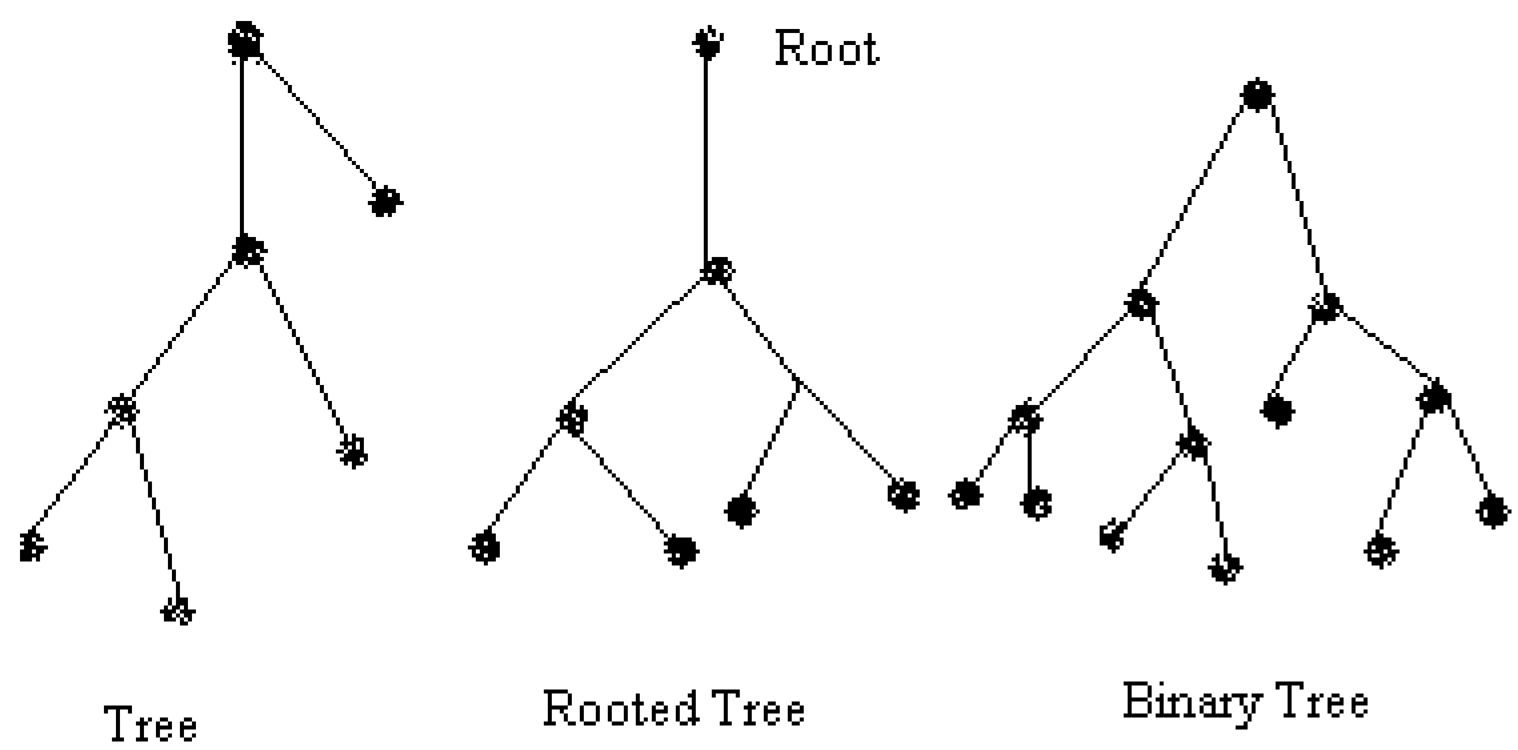
In a Tree if there is a root then it is said to be **rooted tree**.

A tree is said to be **binary tree** if there is only one vertex is of degree two.

Every binary tree is rooted tree but every rooted tree is not binary tree.

A graph G is said to be **minimally connected** if removal of any edge *e* in G disconnects the graph.

That is $W(G-e) > W(G)$



Properties of trees:

A graph is a tree if and only if there is one and only one path between every pair of vertices.
A graph is a tree if and only if it is minimally connected.
In any tree (with 2 or more vertices) there are at least two pendant vertices

Properties of binary tree:

Every binary tree is a rooted tree.
The number of vertices n is a binary tree is always odd.

In a binary tree T with n vertices, the number of pendant vertices is n+1/2.

Rank and nullity

Let G be a graph with n vertices and e edges and k components then Rank and Nullity are defined by,

Rank = n-k

Nullity = e- (n-k)=e-n+k

Theorem

Prove that Rank and Nullity is equal to number of edges of G.

Proof

By defined of Rank and Nullity

Rank = n-k

Nullity = e- (n-k)

Rank + Nullity =e-(n-k) + (n-k)

=e

Rank + Nullity = the number of edges of G

Theorem 1

Any graph G is a tree if and only if every pair of vertices is connected by one and only one path.

Proof

Assume G is a tree.

To prove Every pair of vertices are connected by exactly one path.

Let V_1 and V_2 are any two vertices in G , assume that the vertices V_1 and V_2 are connected by two different paths P_1 and P_2.

Clearly $P_1 \cup P_2$ will form a circuit in G. This is a contradiction to our assumption that G is a tree. Therefore for every pair of vertices in G are connected by exactly one path.

Converse Part

Assume that in G every pair of vertices are connected by exactly one path.

To prove G is a tree.

Assume that G is not a tree and G has a circuit C.

We know that in any circuit every pair of vertices are connected by two different paths, which is a contradiction to our assumption. Therefore G is a tree.

Theorem 2

A connected graph G with n vertices is a tree if and only if it has n-1 edges

Proof

Assume that G is a connected graph with n vertices and n-1 edges.

To prove G is a tree.

Suppose if G is not a tree then G has circuit C.

Let p be the number of vertices in the circuit C. Since G is a connected graph the remaining n-p vertices are incident with at least n-p edges.

Therefore the total number of edges in G= p+(n-p);

G= n

This is a contradiction for assumption. Therefore G is a tree.

Converse Part

Assume that G is a tree with n vertices.

To prove G has n-1 edges.

The proof is by Method of induction on n.

Let n=1 being an isolated vertex.

Let n=2 , be a single edge. Therefore theorem is true for n =1,2.

Let us assume that the theorem is true for less than n vertices (upto n-1). Since G is a tree every pair of vertices are connected by exactly one path.

Let V_1 and V_2 be any two vertices in G connected by a Single edge $e_{k.}$

Clearly G – e_k becomes disconnected. G - e_k have at least two components G_1 and G_2.

Let K_1 be the number of vertices in G_1 and K_2 be the number of vertices in $G_{2.}$

Clearly G_1 and G_2 are trees with less than n vertices.

The number of edges in $G_1 = K_1 - 1$.

The number of edges in $G_2 = K_2 - 1$.

The number of edges in G = $K_1 - 1 + K_2 -1+1$

$= K_1 + K_2 - 1$.

= n-1.

Therefore the Theorem is true for n vertices.

Theorem 3

A graph is a tree if and only if it is minimally connected.

Proof

Assume that G is a tree.

To prove G is minimally connected.

If G is not minimally connected then there exists an edge e (let V_1 and V_2 are end vertices of the edge e) such that G- e is connected.

Consider G- e , there exists a path connecting V_1 and V_2 in G- e. The union of this path and edge e form a circuit in G which is a contradiction for assumption G is a tree. Therefore G is minimally connected.

Converse Part

Assume that G is minimally connected.

To prove G is a tree.

If G is not a tree then G has a circuit.

We know that the removal of any edge from the circuit leaves the graph is connected, which is a contradiction for our assumption that G is minimally connected. Therefore G is s tree.

Theorem 4

A graph with n vertices, n-1 edges and no circuits is a connected graph.

Proof

Assume that G is a graph with n vertices, n-1 edges and there are no circuits.

To prove G is a connected graph.

Assume that G is a not connected. If G is not connected then, G have at least two components G_1 and G_2, such that there exist a pair of vertices V_1 in G_1 and V_2 in G_2.

Clearly $G \cup e$ $(e = V_1V_2)$ is a circuit less connected graph. Hence it is a tree.

The number of edges in G = n-1+1 = n

This is a contradiction to our assumption. Therefore G is a connected Graph.

Theorem 5

In any tree with two or more vertices there exist at least two pendent vertices.

Proof

Let G be a tree with n vertices $n \geq 2$.

If n = 2 being a single edge have two pendent vertices. Let n>2 .

Since G is a tree with n vertices (n> 2), G has n-1 edges.

Each edge incident with at least two vertices.

The total number of degrees of G = $\sum d(V)$ = 2(n-1) = 2n – 2.

These 2n-2 degrees have to be divided by n vertices equally without leaving any vertex as isolated vertex. This is possible only if we give degree one for at least two vertices which are pendent. Hence G has at least 2 pendent vertices.

Theorem 6

Prove that every Binary tree is a rooted tree.

Proof

Since in a binary tree there is only one vertex of degree two which can be distinguished from all the other vertices.

Hence the vertex of degree 2 can be taken as a root vertex.

Therefore, it is a rooted tree.

Theorem 7

Prove that the number of vertices n in a binary tree is always odd.

Proof

Let n be the number of vertices in a binary tree.

Since there is only one vertex of degree 2. Except the degree 2 vertex, the remaining (n-1) vertices are of odd degree.

Since the number of odd vertices in a graph is always even, therefore (n-1) is an even number.

This implies that n is an odd number.

Hence the number of vertices n in a binary tree is always odd.

Theorem 8

Prove that in a binary tree T with n vertices, the number of pendent vertices is (n+1)/2.

Proof

To prove The number of pendent vertices in a tree is $\frac{n+1}{2}$.

Let p be the number of pendent vertices in a binary tree T .

T has only one vertex of degree 2 and we have the number of vertices of degree 3 is n-1-p.

In a graph we have ,

$\sum_{i=1}^{n} d(V_i) = 2e$.

$\sum_{i=1}^{n} d(V_i) = 2(n\text{-}1)$ (1)

In a binary tree , there are p vertices of degree 1 , 1 vertex of degree 2 , and n-1-p vertices of degree 3.

Equation (1) becomes

1(p) + 2(1) + 3(n-1-p) = 2(n-1)

p + 2 + 3n- 3- 3p = 2n – 2

-2p + 3n – 1 = 2n – 2.

-2p = 2n – 2 – 3n + 1

-2p = -n – 1

2p = n+1

$p = \frac{n+1}{2}$

Hence the number of pendent vertices in a binary tree is $\frac{n+1}{2}$.

Theorem 9

Every tree has either one or two centres.

Proof

Let us start with a tree T having more than 2 vertices.

The maximum distance max[d(v,vi)] occurs only when V_i is a pendent vertex.

T will have atleast 2 pendent vertices.

Remove the pendent vertices from T. Let the resulting grapg be named as T_1.

T_1 is also a tree but the eccentricity of T_1 will be reduced by 1. Since T_1 is a tree it will have pendent vertices.

We can again remove all pendent vertices from T_1 and obtain a tree $T_{2.}$

Continue this process until there is a vertex or edge left.

If there is a vertex left, then there is one centre.

If there is an edge left then there are 2 centres.

Theorem 10

Every connected graph G has atleast one spanning tree.

Proof

Let G be a connected graph.

Case(i)

If G has no circuit, then G will be a tree covering all the vertices of G, hence G itself is a spanning tree.

Case(ii)

Assume that G has a circuit .

Remove an edge from the circuit such that the graph is connected.

Then we get a spanning tree. If G has more than one circuit then continue to removing process of edges from G, without affecting the connectedness, till we get a spanning tree.

Theorem 11

Let G be a connected graph with n vertices and e edges. Then prove that with respect to any spanning tree T has (n-1) branches and (e-n+1) chords.

Proof

Let G be a connected graph with n vertices and e edges.

To prove

T has (n-1) branches. Let T be a spanning tree of G. Since T is a spanning tree it has n vertices and hence it has (n-1) edges.(ie) (n-1) branches.

To prove that T has (e-n+1) chords.

If e is the number of edges in a connected graph.

e = number of chords of G + number of branches of G.

e = number of chords +(n-1)

Number of chords of T = e - (n-1)

Number of chords of T = (e-n+1)

Thus T has (e-n+1) chords

Kruskal' s algorithm

It is used to find a shortest spanning for a given weighted graph.

i) Arrange the edges of a graph with the increasing order of weights.

ii) The edge with least weight is included in the spanning tree T, which is to be constructed.

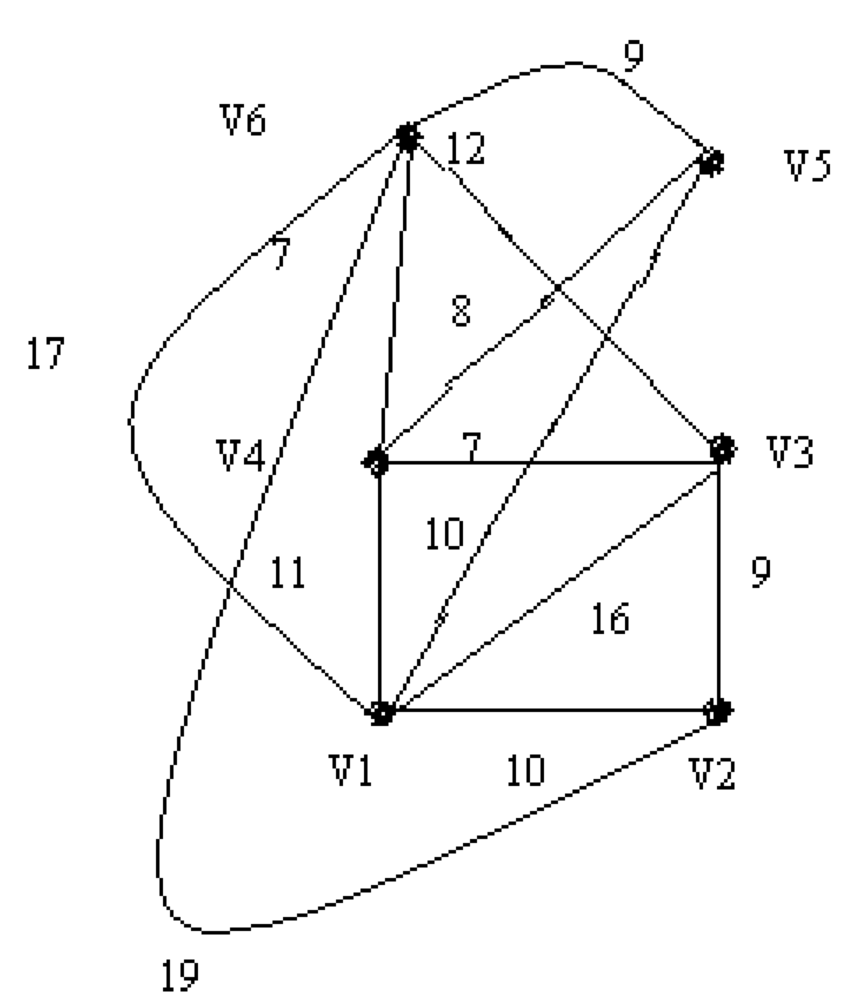
iii) Examine the next edge with least weight and include this edge in T, if it does not form a circuit with previously chosen edges.

iv) This procedure is continued until there are no edges to be included.

v) We get a spanning tree.

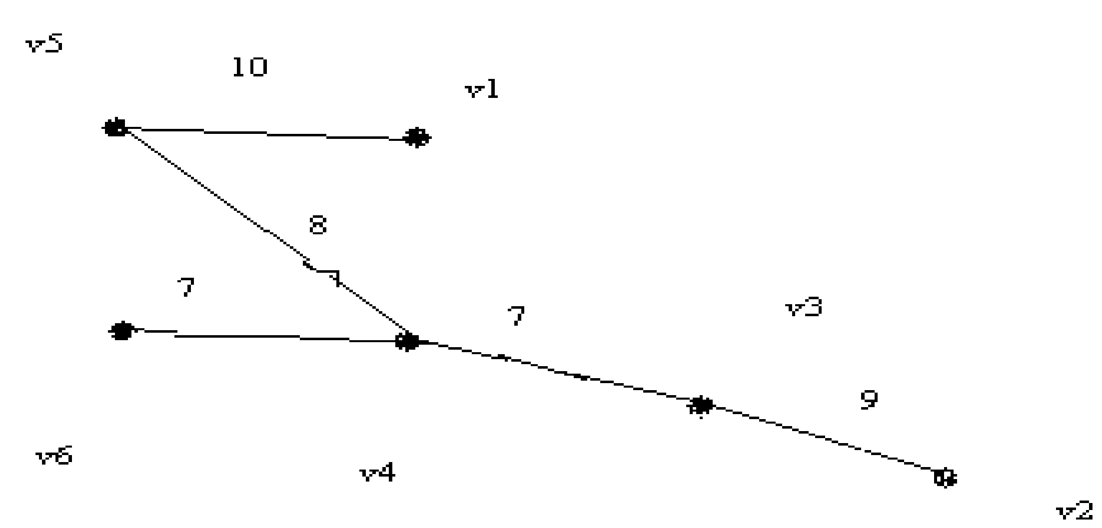
Example.

Find the Spanning tree for the following graph



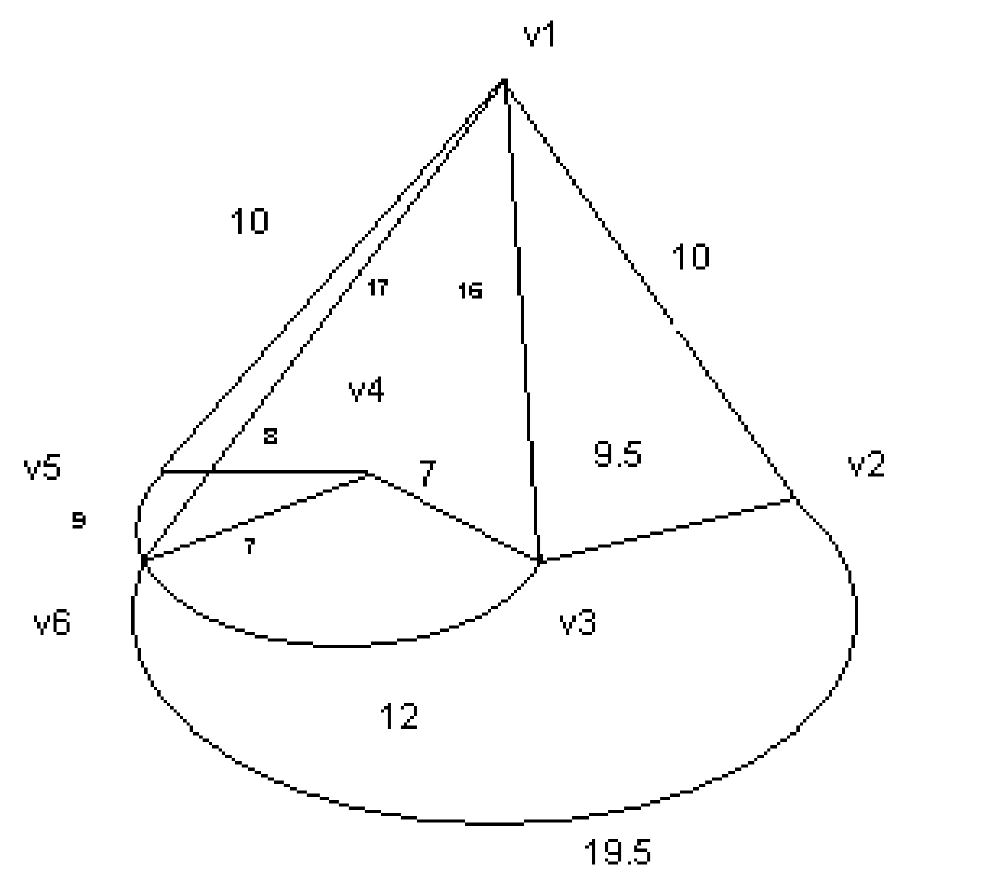
S.No	edges	weight	Ascending order	
1	(v_1,v_2)	10	(v_4,v_6)	7
2	(v_2,v_3)	9	(v_3,v_4)	7
3	(v_6,v_2)	19	(v_5,v_4)	8
4	(v_3,v_4)	7	(v_5,v_6)	9
5	(v_3,v_6)	12	(v_2,v_3)	9
6	(v_4,v_6)	7	(v_1,v_5)	10
7	(v_4,v_1)	11	(v_1,v_2)	10
8	(v_1,v_3)	16	(v_4,v_1)	11
9	(v_1,v_6)	17	(v_3,v_6)	12
10	(v_5,v_4)	8	(v_1,v_3)	16
11	(v_5,v_6)	9	(v_1,v_6)	17
12	(v_1,v_5)	10	(v_6,v_2)	19

Shortest spanning tree

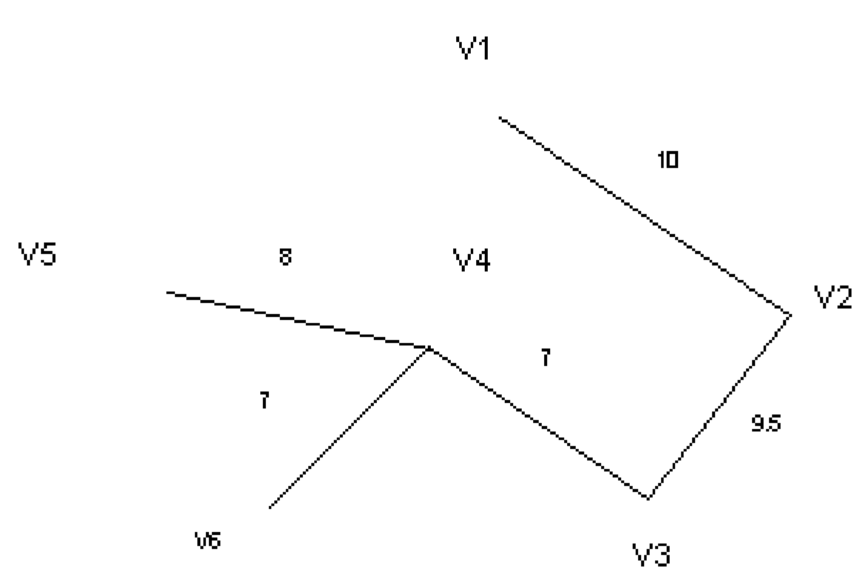


[Weight of the spanning tree=41]

2.Find the shortest spanning tree in the graph using kruskal' s algorithm.
The numbers shown on the edges are the corresponding weights.

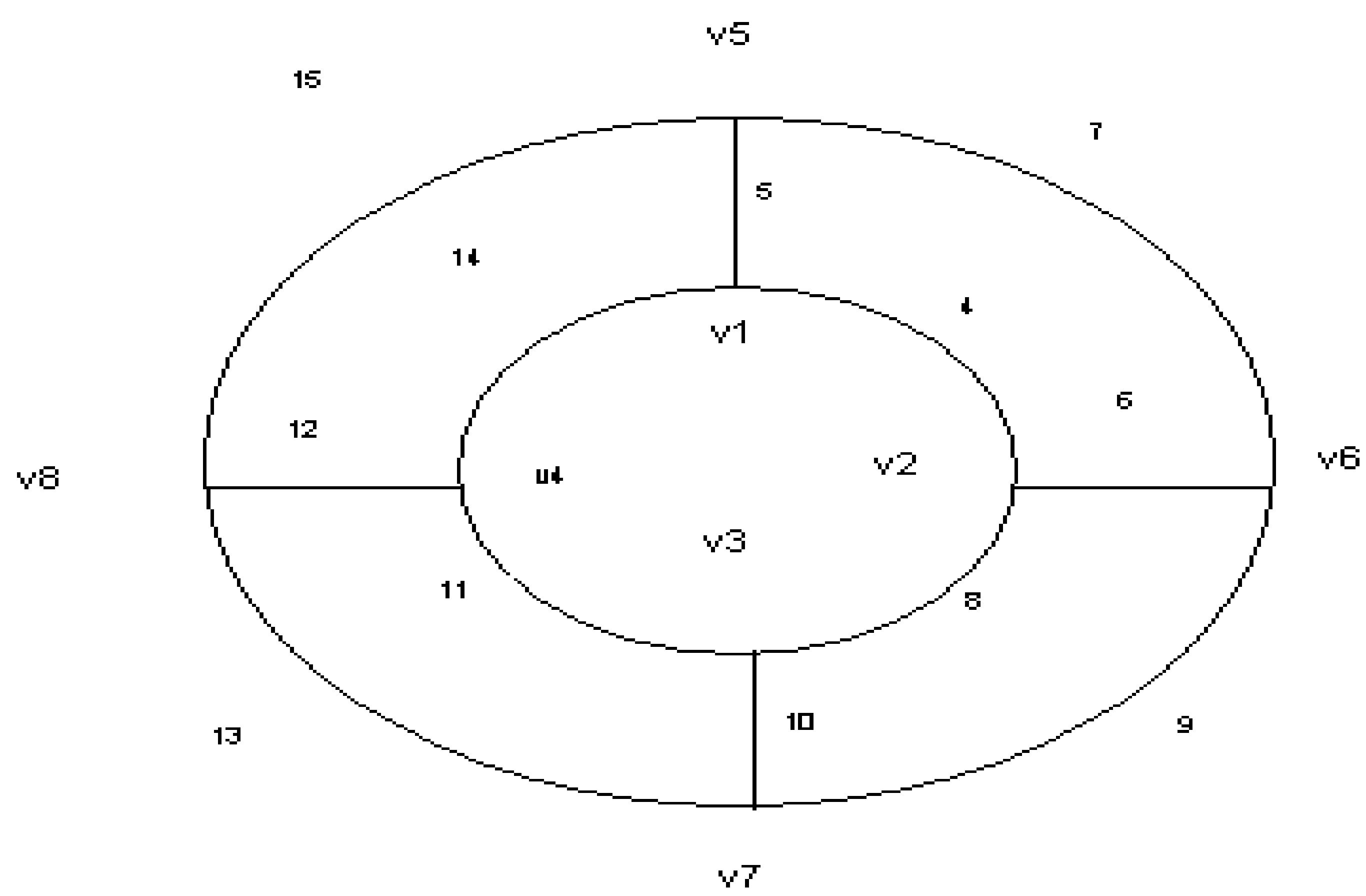


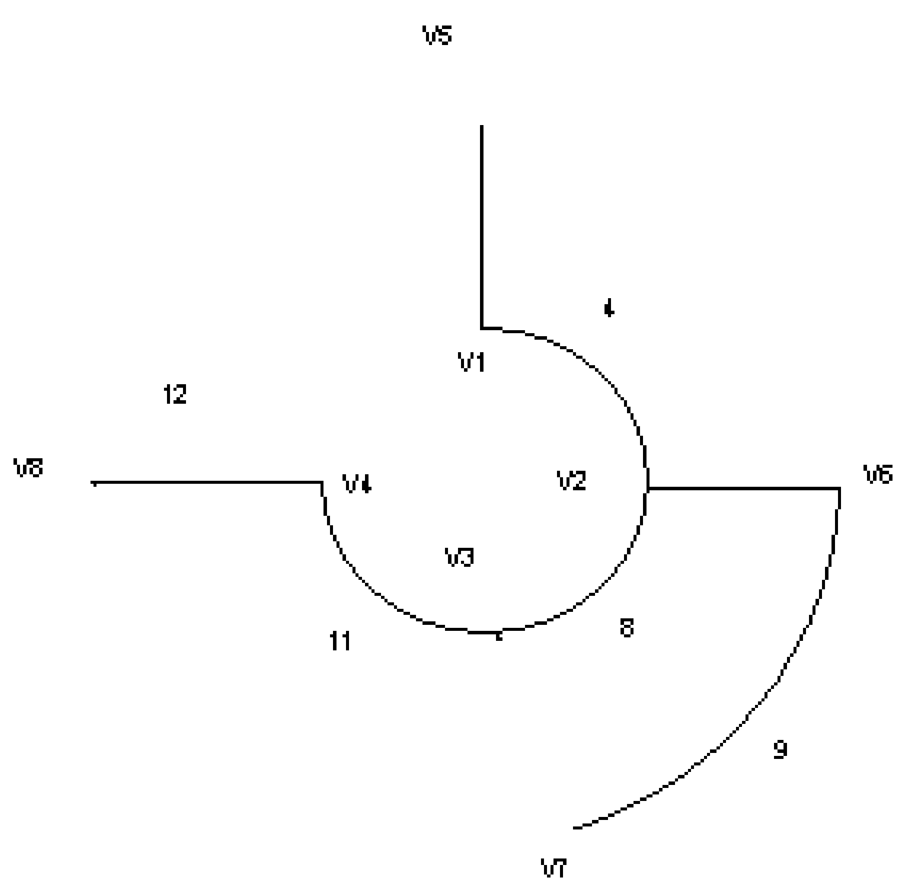
EDGES	WEIGHTS
(V_3,V_4)	7
(V_4,V_5)	7
(V_4,V_5)	8
(V_5,V_6)	9
(V_2,V_3)	9.5
(V_1,V_2)	10
(V_1,V_5)	10
(V_3,V_6)	12
(V_1,V_3)	16
(V_1,V_6)	17
(V_2,V_6)	19.5



Weight of the spanning tree = 8+7+7+9.5+10=41.5

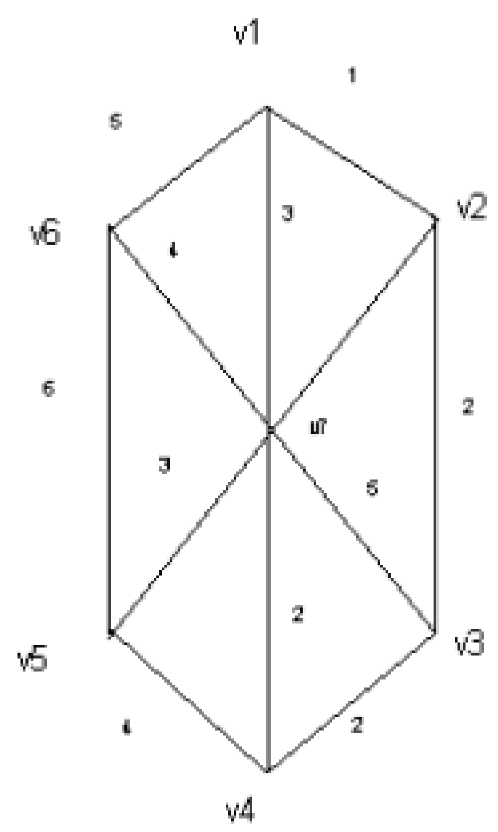
3.Find a minimal spanning tree of a weighted graph G using kruskal algorithm.



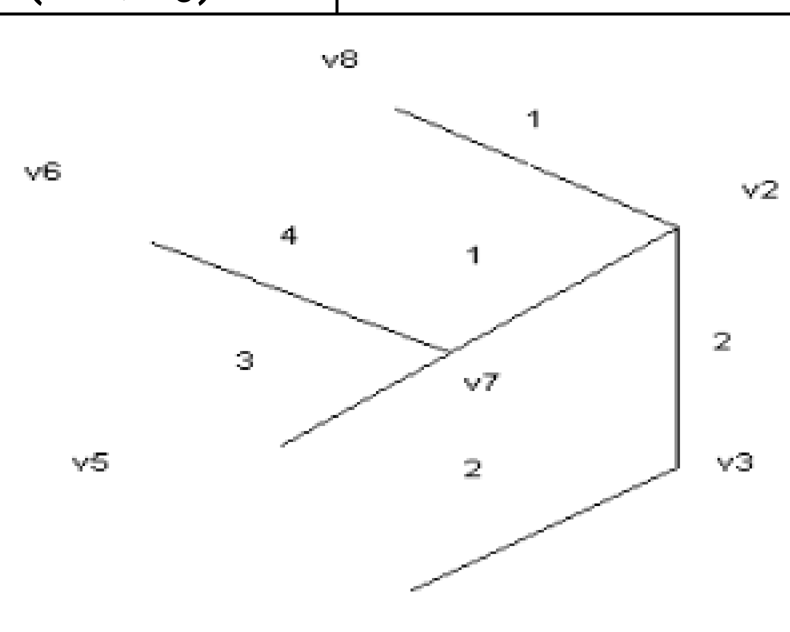


Weight=12+11+8+4+9+5+6=55

4.Find a numerical spanning tree of a weighted graph G using kruskal algorithm.

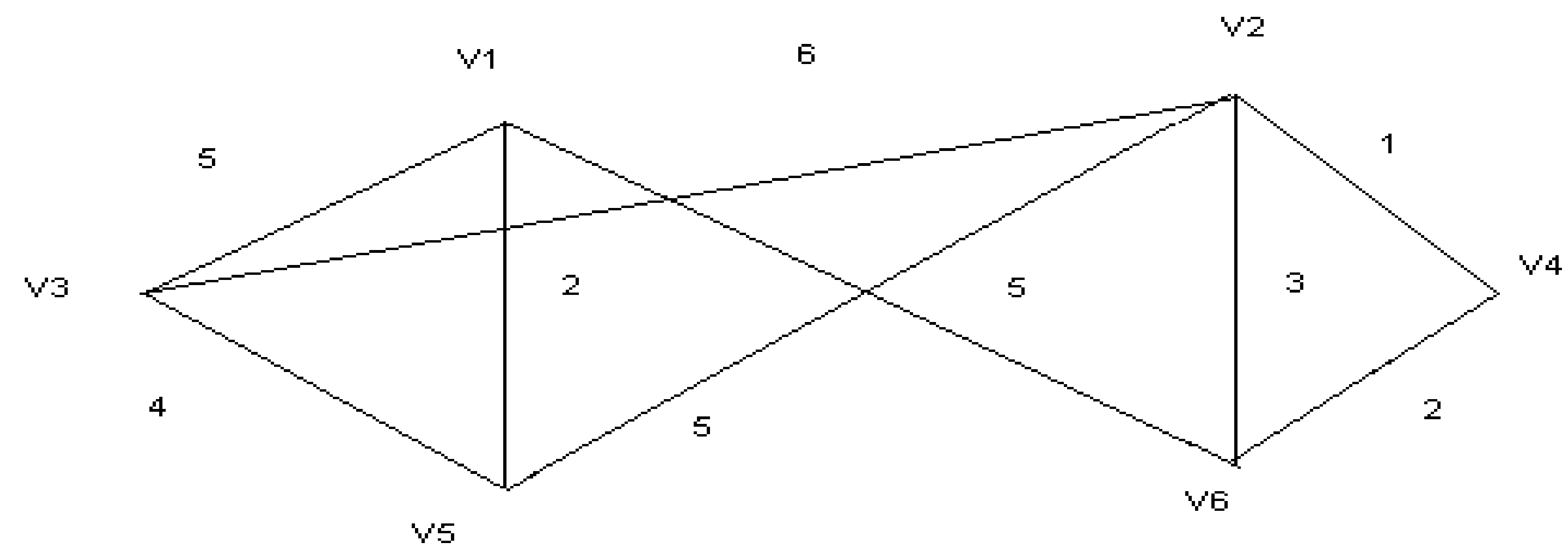


EDGES	WEIGHTS
(V_1,V_2)	1
(V_2,V_7)	1
(V_2,V_3)	2
(V_3,V_2)	2
(V_4,V_7)	2
(V_1,V_7)	3
(V_5,V_7)	3
(V_5,V_4)	4
(V_6,V_7)	4
(V_1,V_6)	5
(V_3,V_7)	5
(V_6,V_5)	6
(V_1,V_2)	1
(V_2,V_7)	1
$(V2,V_3)$	2
$(V3,V_4)$	2
$(V7,V_5)$	3
$(V7,V_6)$	4

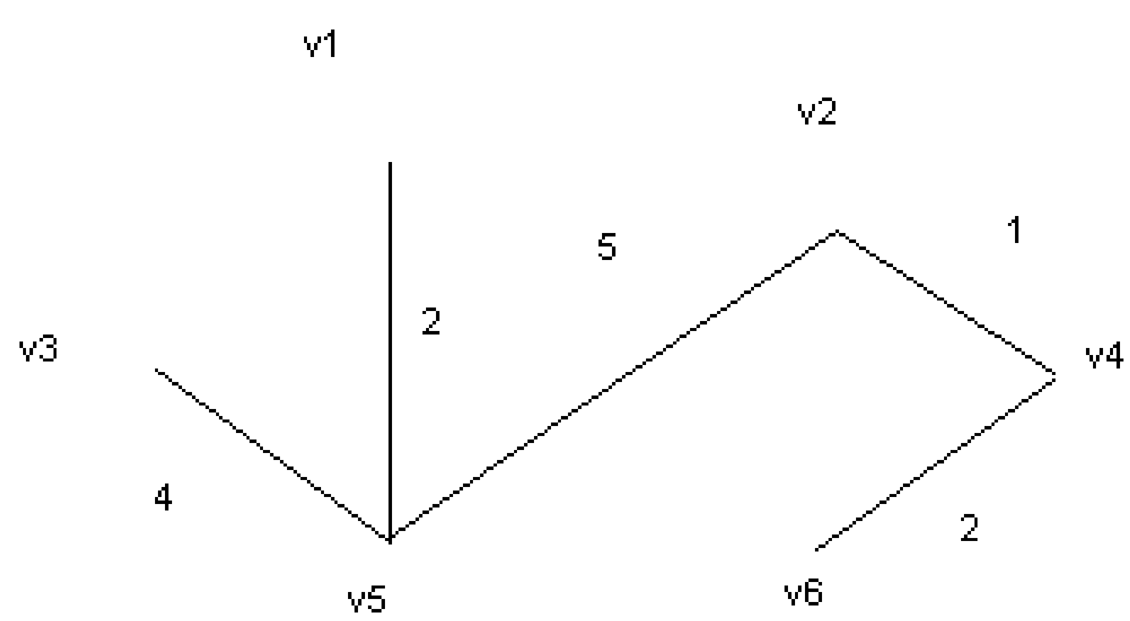


Weight of spanning tree=4+3+1+1+2+2+2=15

5.Find a numerical spanning tree of a weighted graph G using kruskal algorithm.



EDGES	WEIGHTS
(V,V_4)	1
(V_1,V_5)	2
(V_4,V_6)	2
(V_2,V_6)	3
(V_3,V_5)	4
(V_1,V_3)	5
(V_3,V_2)	5
(V_1,V_6)	5
(V_3,V_2)	6
(V_2,V_4)	1
(V_4,V_6)	2
(V_5,V_1)	2
(V_5,V_3)	4
(V_5,V_2)	5



Weight of spanning tree=4+2+5+1+2=14

Draw a tree with one ,two and three leafs