

# AMATH 301: Extra Credit 1

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**Problem 1** Let  $A \in \mathbb{R}^{2 \times 2}$  be given by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We claim that

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

are eigenvectors of  $A$  with corresponding eigenvalues

$$\lambda_1 = i \quad \text{and} \quad \lambda_2 = -i.$$

*Proof.* Suppose that  $A \in \mathbb{R}^{2 \times 2}$  is given as above. We will first compute the eigenvalues of  $A$  by solving the characteristic equation,  $\det(A - \lambda I) = 0$ . Observe,

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} \\ &= (-\lambda)(-\lambda) - (-1)(-1) \\ &= \lambda^2 + 1. \end{aligned}$$

By definition of the characteristic equation, we set  $\lambda^2 + 1 = 0$  and solve for  $\lambda$ . This gives us two eigenvalues,  $\lambda_1 = i$  and  $\lambda_2 = -i$ . Now we must find nonzero vectors  $\vec{v}_1$  and  $\vec{v}_2$  which satisfy

$$(A - \lambda I) \vec{v}_1 = \vec{0} \quad \text{and} \quad (A - \lambda I) \vec{v}_2 = \vec{0}.$$

Which is another way of saying

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \quad \text{and} \quad A\vec{v}_2 = \lambda_2 \vec{v}_2.$$

To solve  $A\vec{v}_1 = \vec{0}$ , we let  $\vec{v}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  so that

$$\begin{pmatrix} -\lambda_1 & -1 \\ 1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives us the system of equations

$$\begin{aligned} -\lambda_1 x_1 - x_2 &= 0 \\ x_1 - \lambda_1 x_2 &= 0. \end{aligned}$$

Since we're trying to find a *nonzero* vector which satisfies both of these equations, I'll choose  $x_1 = 1$  to make the calculations simple. Substituting  $x_1 = 1$  into the first equation in our system gives

$$-\lambda_1 - x_2 = 0$$

implying that

$$x_2 = -\lambda_1.$$

We have to check that this solution works in the second equation before moving on. Substituting  $x_1 = 1$  and  $x_2 = -\lambda_1$  into our second equation,  $x_1 - \lambda_1 x_2 = 0$ , gives

$$1 - \lambda_1 (-\lambda_1) = 0$$

$$1 - i(-i) = 0$$

$$1 - (i(-i)) = 0$$

$$1 + i^2 = 0$$

$$0 = 0$$

so that the second equation is also satisfied. (How could we know that the second equation will always work out? This would be a good exercise if you're interested!) Since both equations work out, then

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad \lambda_1 = i$$

should satisfy  $A\vec{v}_1 = \lambda_1 \vec{v}_1$ . Let's check.

$$\begin{aligned} A\vec{v}_1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &= \begin{pmatrix} 0 + i \\ 1 + 0 \end{pmatrix} \\ &= \begin{pmatrix} i \\ 1 \end{pmatrix} \\ &= i \begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &= \lambda_1 \vec{v}_1. \end{aligned}$$

Therefore, we can conclude that  $\vec{v}_1$  is an eigenvector of  $A$  with eigenvalue  $\lambda_1 = i$ . Using the exact same method as above, we find that

$$\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \lambda_2 = -i$$

are an eigenvector-eigenvalue pair because

$$\begin{aligned} A\vec{v}_2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= \begin{pmatrix} 0 - i \\ 1 + 0 \end{pmatrix} \\ &= \begin{pmatrix} -i \\ 1 \end{pmatrix} \\ &= -i \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= \lambda_2 \vec{v}_2. \end{aligned}$$

This is what we wanted to show. □