AMATH 301: Extra Credit 1

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Problem 1 Let $A \in \mathbb{R}^{2\times 2}$ be given by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We claim that

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

are eigenvectors of A with corresponding eigenvectors

$$\lambda_1 = i \quad \text{and} \quad \lambda_2 = -i.$$

Proof. Suppose that $A \in \mathbb{R}^{2\times 2}$ is given as above. We will first compute the eigenvalues of A by solving the characteristic equation, $\det(A - \lambda I) = 0$. Observe,

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} \\ &= (-\lambda)(-\lambda) - (-1)(-1) \\ &= \lambda^2 + 1. \end{aligned}$$

By definition of the characteristic equation, we set $\lambda^2 + 1 = 0$ and solve for λ. This gives us two eigenvalues, $\lambda_1 = i$ and $\lambda_2 = -i$. Now we must find nonzero vectors $\vec{v}_1$ and $\vec{v}_2$ which satisfy

$$(A - \lambda I)\,\vec{v}_1 = \vec{0} \quad \text{and} \quad (A - \lambda I)\,\vec{v}_2 = \vec{0}. \tag{1}$$

Which is another way of saying

$$A\vec{v}_1 = \lambda_1\vec{v}_1 \quad \text{and} \quad A\vec{v}_2 = \lambda_2\vec{v}_2. \tag{2}$$

To solve the first equation in (1), we let $\vec{v}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ so that

$$\begin{pmatrix} -\lambda_1 & -1 \\ 1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives us the system of equations

$$\begin{aligned} -\lambda_1 x_1 - x_2 &= 0 \\ x_1 - \lambda_1 x_2 &= 0. \end{aligned}$$

Since we're trying to find a *nonzero* vector which satisfies both of these equations, I'll choose $x_1 = 1$ to make the calculations simple. Substituting $x_1 = 1$ into the first equation in our system gives

$$-\lambda_1 - x_2 = 0$$

implying that

$$x_2 = -\lambda_1.$$

We have to check that this solution works in the second equation before moving on. Substituting $x_1 = 1$ and $x_2 = -\lambda_1$ into the equation $x_1 - \lambda_1 x_2 = 0$ gives

$$\begin{aligned} 1 - \lambda_1(-\lambda_1) &= 0 \\ 1 - i(-i) &= 0 \\ 1 - (i(-i)) &= 0 \\ 1 + i^2 &= 0 \\ 0 &= 0 \end{aligned}$$

so that the second equation is also satisfied. (How do we know that the second equation will always work out? This would be a good exercise if you're interested!) Since both equations work out, then

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad \lambda_1 = i$$

should satisfy $A\vec{v}_1 = \lambda_1 \vec{v}_1$. Let's check.

$$\begin{aligned} A\vec{v}_1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &= \begin{pmatrix} 0 + i \\ 1 + 0 \end{pmatrix} \\ &= \begin{pmatrix} i \\ 1 \end{pmatrix} \\ &= i \begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &= \lambda_1 \vec{v}_1. \end{aligned}$$

Therefore, we can conclude that $\vec{v}_1$ is an eigenvector of A with eigenvalue $\lambda_1 = i$. Using the exact same method as above, we find that

$$\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \lambda_2 = -i$$

are an eigenvector-eigenvalue pair because

$$\begin{aligned} A\vec{v}_2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= \begin{pmatrix} 0 - i \\ 1 + 0 \end{pmatrix} \\ &= \begin{pmatrix} -i \\ 1 \end{pmatrix} \\ &= -i \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= \lambda_2 \vec{v}_2. \end{aligned}$$

This is what we wanted to show. $\square$