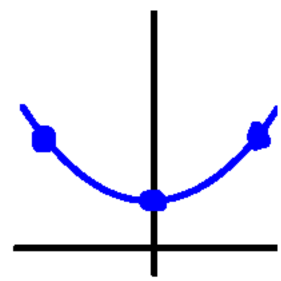
Midterm B1
Section 4

November 7, 2014

Consider the set of points $(-2, 2)$, $(1, 1)$ and $(2, 2)$ and the interpolating polynomial $f(x)$, which is a 2^{nd} degree polynomial that passes through those points.

Problem 1 [15pt]: Write down $f(x)$ as a Lagrangian polynomial.



Before anything, I do a quick *qualitative* sketch to see what's going on.

Since we're told that f is a 2^{nd} degree polynomial, we know that f is either a parabola, a line, or a point. It's clear from my sketch that it has to be a parabola opening upwards. If that's the case, then for $f(x) = \alpha x^2 + \beta x + \gamma$, we must have α strictly greater than 0. It turns out that this quick analysis will not help us to much for *this* problem, but it never hurts to start with an idea of what's going on. That way, if I end up with something like $\alpha = -2$ in the end, I'll know that I messed up somewhere.

Now that we got that out of the way, let's think of how we write a Lagrangian polynomial when given 3 points. For points (x_1, y_1), (x_2, y_2) and (x_3, y_3), we have the formula

$$f(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} \tag{1}$$

Notice how nicely the "$-x_i$" terms look stacked on top of each other like that. In the first fraction we have x_2 right over x_2 and the x_3 over the x_3. The situation is similar for the other two fractions. Also notice that f is a function of x. There are two x's in each numerator. Some people made the mistake of substituting point values in for the x's in (1). That's not good. Suppose we didn't have any x's in our expression. Then (1) would just be a sum of constants, giving you some scalar C. You'd have $f(x) = C$, and there's *no way* that $f(x)$ could go through the 3 points I drew horribly up top.

Also, notice that our sketch implies that if we multiply out (1) to get some expression $f(x) = \alpha x^2 + \beta x + \gamma$, then α better be nonzero and positive. This **can not** happen if we don't have two x's in the numerators. One other thing to note is that we're given 3 points, and we have 3 expressions in (1). Look at where the y_1 and x_1's appear in the first fraction. Look at how similar that is to y_2 and the x_2's in the second fraction.

Let's take a second to think about how we could generalize (1) a little. That is, suppose we were given four points, (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) and we want a corresponding Lagrangian polynomial $g(x)$.

Without knowing "the formula" for a 4^{th} order Lagrangian polynomial, I'm just going to set it up the way (1) is set up:

$$g(x) = y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} + y_4 \frac{(x-?)(x-?)}{(x_4-?)(x_4-?)}$$

If three points determines a 2^{nd} order polynomial, it makes sense that four points determine a 3^{rd} order polynomial. Then we'll need something like this $(x-a)(x-b)(x-c)$ in each numerator.

$$g(x) = y_1 \frac{(x-x_2)(x-x_3)(x-?)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)(x-?)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_2)(x-?)}{(x_3-x_1)(x_3-x_2)} + y_4 \frac{(x-?)(x-?)(x-?)}{(x_4-?)(x_4-?)}$$

By analogy, I'm assuming there must also be extra terms in the denominator. Note that the first x_i in each denominator agrees with the y_i in (1). We'll do the same thing here.

$$g(x) = y_1 \frac{(x-x_2)(x-x_3)(x-?)}{(x_1-x_2)(x_1-x_3)(x_1-?)} + y_2 \frac{(x-x_1)(x-x_3)(x-?)}{(x_2-x_1)(x_2-x_3)(x_2-?)} + y_3 \frac{(x-x_1)(x-x_2)(x-?)}{(x_3-x_1)(x_3-x_2)(x_3-?)} + y_4 \frac{(x-?)(x-?)(x-?)}{(x_4-?)(x_4-?)(x_4-?)}$$

Now we just fill in the ?'s in the obvious way, taking care not to *ever* divide by zero. *Ever.*

$$\begin{aligned} g(x) = & y_1 \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + y_2 \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} + \\ & y_3 \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + y_4 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \end{aligned}$$

And we've just blindly derived the general form of a 4^{th} order Lagrangian polynomial by reason and analogy alone. My point with this is that if you know the formula for a 1^{st} order Lagrangian polynomial given two points, then you have the resources to figure out the formula for a 2^{nd} order Lagrangian polynomial, given 3 points. You just have to be a little creative and notice the patterns.

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