DATE : __/__/__

PAGE :

Unit-2 **Set Theory**

$N = \{1, 2, 3, \ldots\}$

$W = \{0, 1, 2, 3, \ldots\}$

$\mathbb{Z} = \{\ldots -2, -1, 0, 1, \ldots\}$

$Q = \{x : x = p/q \; ; q \neq 0, \; p, q \in \mathbb{Z}\}$

$\mathbb{R} - Q = \{\sqrt{2}, \sqrt{3}, \ldots, e, \pi, \ldots\}$

$R = Q \cup (R - Q)$

$C = \{z : z = a + ib \quad i = \sqrt{-1}, \; a, b \in \mathbb{R}\}$

Set theory:- Set is a well-defined collection of distinct objects.

$A = \{a, b, c, d\}$

$e \notin A$ {It means e is not member of set A}

$d \in A$

$z \notin A$

finite Set:-

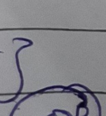
$\{a, e, i, o, u\}$ This is a finite set.

Subset:- Let A & B be two sets. then we say that A is a subset of B. if Every element of A is also a elements of B.

$A \subset B$

$A = \{1, 2, 3\}$

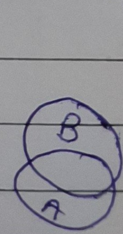
$B = \{1, 2, 3, 4, 5\}$



$A \not\subset B$

$A = \{1, 2, 3\}$

$B = \{1, 2, 4, 5, 6\}$



Equal Set :-

$A = \{1,2,3\}$, $B = \{1,3,2\}$

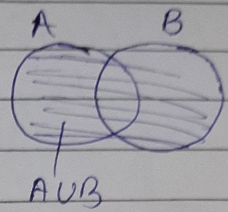
$A \subseteq B$ and $B \subseteq A$

$\boxed{A = B}$

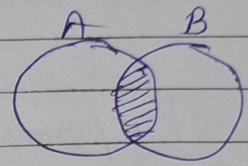
Union of two set :-

$A = \{1,2,3\}$ $B = \{3,4,5\}$

$A \cup B = \{1,2,3,4,5\}$



Intersection of two set :-



$A = \{1,2,3\}$, $B = \{3,4,5\}$

$A \cap B = \{3\}$

Difference of two set :-

$A = \{1,2,3,4\}$, $B = \{3,4,5,6\}$

$A - B = \{1,2\}$

$B - A = \{5,6\}$

Fields :- let F be a non-empty set with two operations • and + then set F is called a field if the following properties are satisfied.

① Closure property :-

$a+b \in F \quad \forall\, a,b \in F$

② Associative property :-

$a+(b+c) = (a+b)+c \quad \forall\, a,b,c \in F$

③ Existence of additive identity:→

$\forall\, a \in F \;\exists\; 0 \in F$ s.t

$$a+0 = a = 0+a$$

④ Additive Inverse:→

For each $a \in F \;\exists\; -a \in F$ s.t

$$a+(-a) = 0 = (-a)+a$$

⑤ Commutative Property:→

$$a+b = b+a \quad \forall\, a, b \in F$$

Properties of Multiplications:-

⑥ Closure Property:→

$$a \cdot b \in F \quad \forall\, a, b \in F$$

⑦ Associative Property:→

$$a\cdot(b\cdot c) = (a\cdot b)\cdot c \quad \forall\, a, b, c \in F$$

⑧ ~~F $\forall a, b$~~

$\forall\, a \in F \;\exists\; 1 \in F$ s.t

$$a \cdot 1 = a = 1 \cdot a$$

⑨ For each $(a \neq 0) \in F \;\exists\; \frac{1}{a} \in F$ s.t

$$a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a$$

⑩ Commutative Property:-

$$a \cdot b = b \cdot a \quad \forall\, a, b \in F$$

DATE : / /
PAGE :

(11) Distributive Property:-

$\forall \ a, b, c \in F$

$$a \cdot (b+c) = a \cdot b = a \cdot c$$

$$(b+c) \cdot a = b \cdot a + c \cdot a$$

If any set satisfy the above 11 properties is called a field.

* Set of Rational Number, Real Number and Complex Number form field.
Number is the member of field.

Internal Binary Composition:-

Let A be a non-empty set then a mapping $f: A \times A \to A$ is called an internal binary composition.

Ex:-> $f: N \times N \to N$

$$f(x, y) = x^2 + y^2$$

$$f(1, 2) = 1^2 + 2^2 = 5 \in N$$

This mapping is an internal binary composition

Ex:- $f(x, y) = \frac{x}{y}$

$$f(1,2) = 1/2 \notin N$$

This mapping is not an internal binary composition

$f: N \times N \to N$

Ex:- $f(x, y) = x^2 - y^2$

$$f(1, 2) = 1^2 - 2^2 = -3 \notin N$$

External Binary Composition

let A and F be two non-empty sets then the mapping
$f : A \times F \longrightarrow A$ is called an external binary composition.

Eg:-

$\alpha \in F$ $\quad$ $A \in$ Set of 2×2 matrix

$\alpha \in F$ $\quad$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\alpha A = \alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix} \in \text{Set of } 2\times2 \text{ matrix.}$$

Ex:- $\alpha \in F$ $\quad$ $v = (x_1, x_2, x_3)$

$$\alpha . v = \alpha (x_1, x_2, x_3)$$
$$= (\alpha x_1, \alpha x_2, \alpha x_3) \in V$$

$$R = \{x : x \in R\}$$

$V_2(R) = R^2 = R \times R = \{(x, y) : x, y \in \mathbb{R}\}$

let $v_1 \in R^2$

$v_1 = (x_1, y_1)$ $\quad$ $x_1, y_1 \in \mathbb{R}$

$v_1, v_2 \in R^2$

$v_1 = (x_1, y_1)$, $\quad$ $v_2 = (x_2, y_2)$ $\quad$ $x_1, x_2, y_1, y_2 \in \mathbb{R}$

$V_2(C) = \{(x_1, iy_1), (x_2 + iy_2) : x_1, x_2, y_1, y_2 \in \mathbb{R}\}$

DATE: / /
PAGE:

$V_3(R) = R^3 = \{(x, y, z) : x, y, z \in R\}$

$\vdots$

$V_n(R) = R^n = \{(x_1, x_2 \ldots x_n) : x_1, x_2, x_3 \ldots x_n \in R\}$

Definition of Vector Space:-

Let V be a non-empty set and F be a field then the set $(V(F), +, \cdot)$ is called a vector space if the following properties are satisfied.

(i) Closure Property:-

$v_1, v_2 \in V \Rightarrow v_1 + v_2 \in V \quad \forall\ v_1, v_2 \in V$

(ii) Associative Property:-

$\forall\ v_1, v_2, v_3 \in V$

$v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$

(iii) Existence of identity:-

$\forall\ v_1 \in V \quad \exists\ 0 \in V$

s.t $\quad v_1 + 0 = v_1 = 0 + v_1$

(iv) Eistence of Inverse:-

for each $v_1 \in V \quad \exists -v_1 \in V$

s.t $\quad v_1 + (-v_1) = 0 = (-v_1) + v_1$

(v) Commutative Property:-

$\forall\ v_1, v_2 \in V$

$v_1 + v_2 = v_2 + v_1$

~~properties~~

DATE : / /
PAGE :

Properties of scalar multiplication :-

(vi) $\forall\ \alpha \in F$ and $\forall\ v_1 \in V \Rightarrow \alpha v_1 \in V$

(vii) $\forall\ \alpha \in F$ and $v_1 + v_2 \in V \Rightarrow \alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$

(viii) $\forall\ \alpha, \beta \in F$ and $v_1 \in V$ $(\alpha+\beta) v_1 = \alpha v_1 + \beta v_1$

(ix) $\forall\ \alpha, \beta \in F$ and $v_1 \in V \Rightarrow (\alpha\beta) v_1 = \alpha(\beta v_1)$

(x) $\forall\ v_1 \in V\ \exists\ 1 \in F$ $1 \cdot v = v$

Que:-> Prove that the set of all matrices is not a vector space.

Sol:-> $A_{2\times2} \in M$, $B_{3\times3} \in M$

set of Matrices

The sum of these two matrix is not possible.

$A_{2\times2} + B_{3\times3} \notin M$

$\Rightarrow$ Set of all matrices is not a vector space.

Que:- Prove that the set of polynomial of degree three is not a vector space.

$P = \{ f(x) : f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad a_i \in R \}$

let $f(x), g(x) \in P$

$f(x) = -x^3 + x^2 + x + 1$

$g(x) = x^3 + 2x + 1$

$f(x) + g(x) = x^2 + 3x + 2 \notin P$

This is not a vector space.

Que Prove that F(F) forms a vector space.

(i) $\forall u, v \in F \Rightarrow u+v \in F$ ($\because$ F is a field)

(ii) $\forall u, v, w \in F$
$u+(v+w) = (u+v)+w$ (Associative property holds in field)

(iii) $\forall u \in F \quad \exists \; 0 \in F$ s.t (such that)
$u+0 = u = 0+u$

(iv) For each $u \in F \quad \exists \; -u \in F$ s.t
$u+(-u) = 0 = (-u)+u$

(v) $\forall u, v \in F$
$u+v = v+u$
Commutative property also holds in F.

(vi) $\forall u \in F \qquad \forall \alpha \in F \qquad \Rightarrow \alpha u \in F$

(vii) $\forall u \in F \qquad \forall \alpha, \beta \in F \qquad \Rightarrow (\alpha+\beta)u = \alpha u + \beta u$

(viii) $\forall u, v \in F \quad \forall \alpha \in F \Rightarrow \quad \alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$.

(ix) $\forall u \in F \quad \forall \alpha, \beta \in F$
$(\alpha\beta)u = \alpha(\beta u)$

(x) $\forall u \in F \quad \exists \; 1 \in F$ s.t
$1 \cdot u = u = u \cdot 1$

F(F) forms a vector space.

DATE : / /
PAGE :

Note:- $Q(Q)$, $R(R)$, $C(C)$ are also vector space.
$R(Q)$, $C(Q)$, $C(R)$ are also vector space.

Q:- Prove that $Q(R)$ is not a vector space

$u \in Q$, $\alpha \in R \Rightarrow \alpha u \notin Q$
i.e. why $Q(R)$ is not a vector space.

Note:- $R(C)$, $Q(C)$ these are not vector space.

Q:- Prove that $V_n(R) = R^n = \{u : u = (x_1, x_2 \ldots x_n)$, $x_i \in R\}$
(Euclidean space)

prove that under usual addition & scalar multiplication defined by $u+v = (x_1, x_2, \ldots x_n) + (y_1, y_2, y_3 \ldots y_n) = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$
where $x_i+y_i \in \mathbb{R}$, $x_i, y_i \in \mathbb{R}$ $1 \le i \le n$

$\alpha u = \alpha(x_1, x_2 \ldots x_n)$
$= (\alpha x_1, \alpha x_2 \ldots \alpha x_n)$ $\alpha x_i \in R$ $x_i \in R$
forms a vector space. $1 \le i \le n$

Sol ① $\forall$ $u, v \in V_n(R)$
$u = (x_1, x_2 \ldots x_n)$
$v = (y_1, y_2 \ldots y_n)$ $x_i, y_i \in \mathbb{R}$ $1 \le i \le n$

$u+v = (x_1, x_2, \ldots x_n) + (y_1, y_2 \ldots y_n)$
$= (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$

$\left\{ \begin{array}{ll} x_i, y_i \in \mathbb{R} & 1 \le i \le n \\ x_i + y_i \in \mathbb{R} & 1 \le i \le n \end{array} \right\}$
Sum of two real no's is a real no.

DATE: __/__/__
PAGE: ____

(ii) $\forall\ u, v, w \in V_n(R)$

$u = (x_1, x_2 \ldots, x_n)$
$v = (y_1, y_2 \ldots, y_n)$
$w = (w_1, w_2 \ldots, w_n)$

$u+(v+w) = (x_1, x_2, \ldots, x_n) + ((y_1, y_2, \ldots y_n) + (w_1, w_2, \ldots w_n))$

$= (x_1, x_2 \ldots x_n) + ((y_1+w_1, y_2+w_2 \ldots, y_n+w_n))$

$= (x_1+(y_1+w_1), x_2+(y_2+w_2), \ldots, x_n+(y_n+w_n))$

$= ((x_1+y_1)+w_1, (x_2+y_2)+w_2, \ldots, (x_n+y_n)+w_n)$

(Associative property holds in field)

$= ((x_1+y_1, x_2+y_2, \ldots, x_n+y_n)) + (w_1, w_2 \ldots w_n)$
$= ((x_1, x_2, \ldots x_n) + (y_1, y_2 \ldots y_n)) + (w_1, w_2 \ldots w_n)$
$= (u+v)+w.$

(iii) $\forall\ u \in V_n(R) \quad \exists\ 0 \in V_n(R)$
$u = (x_1, x_2 \ldots x_n) \qquad 0 = (0, 0, \ldots, 0)$
$u+0 = (x_1, x_2 \ldots x_n) + (0, 0, \ldots 0)$
$= (x_1+0, x_2+0, \ldots x_n+0)$
$= (x_1, x_2, \ldots, x_n) = u$

$0+u = u$
$u+0 = u = 0+u$

PAGE :

(iv) Foreach $u \in V_n(R)$ $x_i \in \mathbb{R}$ $1 \leq i \leq n$

$u = (x_1, x_2 \ldots x_n)$ s.t.

$\exists -u = (-x_1, -x_2 \ldots, -x_n)$

there exist $u + (-u) = (x_1, x_2, x_3 \ldots x_n) + (-x_1, -x_2, \ldots, -x_n)$

$= (x_1 - x_1, x_2 - x_2, \ldots x_n - x_n)$

$= (0, 0, 0 \ldots 0) = 0$

$-u + u = 0$

$u + (-u) = 0 = (-u) + u$

(v) let $u, v \in V_n(R)$

$u = (x_1, x_2 \ldots x_n)$

$v = (y_1, y_2, \ldots y_n)$ $x_i, y_i \in \mathbb{R}$ $1 \leq i \leq n$

$u + v = (x_1, x_2, \ldots, x_n) + (y_1, y_2 \ldots y_n)$

$= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

$= (y_1, y_2 \ldots y_n) + (x_1, x_2, x_3 \ldots x_n)$

$\boxed{u + v = v + u}$

(vi) $\forall \alpha \in R$ and $\forall u \in V_n(R)$

Here $u = (x_1, x_2 \ldots x_n)$ $\forall x_i \in R$ $1 \leq i \leq n$

$\alpha u = \alpha(x_1, x_2 \ldots x_n)$

$= (\alpha x_1, \alpha x_2, \ldots \alpha x_n)$ [each $x_i \in R$, $1 \leq i \leq n$, $\alpha \in R$
$\alpha x_i \in R$, $1 \leq i \leq n$]

(vii) $\forall \alpha \in \mathbb{R}$ and $\forall u, v \in V_n(R)$

$u = (x_1, x_2 \ldots x_n)$

$v = (y_1, y_2 \ldots y_n)$ $x_i, y_i \in R$ $1 \leq i \leq n$

$\alpha(u+v) = \alpha[(x_1, x_2 \ldots x_n) + (y_1, y_2, \ldots y_n)]$

$= \alpha[(x_1 + y_1), (x_2 + y_2), \ldots]$

$= \alpha(x_1 + y_1), \alpha(x_2, y_2) \ldots \alpha(x_n, y_n))$

$= \alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, \ldots \ldots, \alpha x_n + \alpha y_n$

$= (\alpha x_1, \alpha x_2, \ldots \alpha x_n) + (\alpha y_1, \alpha y_2, \ldots, \alpha y_n)$

$= \alpha (x_1, x_2 \ldots \ldots x_n) + \alpha (y_1, y_2, \ldots \ldots y_n)$

$= \alpha u + \alpha v$

(viii) $\forall\ \alpha, \beta \in R$ and $\forall\ u \in V_n(R)$

$(\alpha + \beta) u.$

$u = x_1, x_2, x_3 \ldots \ldots x_n \quad x_i \in R \quad 1 \le i \le n$

$(\alpha + \beta) u = (\alpha + \beta)(x_1, x_2, x_3 \ldots \ldots x_n)$

$= (\alpha + \beta) x_1, (\alpha + \beta) x_2, (\alpha + \beta) x_3 \ldots \ldots, (\alpha + \beta) x_n$

$= (\alpha x_1, \alpha x_2, \alpha x_3 \ldots \alpha x_n) + (\beta x_1, \beta x_2, \ldots \beta x_n)$

$= \alpha (x_1, x_2 \ldots \ldots x_n) + \beta(x_1, x_2, \ldots x_n)$

$= \alpha u + \beta u$

(ix) $\forall\ \alpha, \beta \in R$ and $\forall\ u \in V_n(R)$

$(\alpha\beta) u = \alpha(\beta u)$ (to show)

$u = x_1, x_2 \ldots \ldots x_n \quad x_i \in R \quad 1 \le i \le n$

$(\alpha\beta) u = (\alpha\beta)(x_1, x_2, \ldots \ldots x_n)$

$= ((\alpha\beta) x_1, (\alpha\beta) x_2 \ldots \ldots, (\alpha\beta) x_n)$

$= \alpha(\beta x_1, \beta x_2, \ldots \ldots \beta x_n)$

$= \alpha(\beta(x_1, x_2 \ldots \ldots x_n))$

$= \alpha(\beta u)$

(x) $\forall\ u \in V_n(R)$ and $1 \in R$

$u = (x_1, x_2 \ldots \ldots x_n)$

$1 \cdot u = 1 \cdot (x_1, x_2 \ldots \ldots x_n)$

$= (1 \cdot x_1, 1 \cdot x_2 \ldots \ldots 1 \cdot x_n)$

$= (x_1, x_2 \ldots \ldots x_n)$

$\boxed{1 \cdot u = u}$

Hence $(V_n(R), +, \cdot)$ is a vector space.

PAGE :

Q:- Consider the set $M_{n\times n}(R) = \{A : A = [a_{ij}]_{n\times n} \ \forall a_{ij} \in R, 1 \le i,j \le n\}$

Show that $M_{n\times n}(R)$ is a vector space.

Sol:- (i) $\forall A, B \in M_{n\times n}(R)$

$A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$ $\forall a_{ij}, b_{ij} \in R$, $1 \le i,j \le n$

$$A+B = [a_{ij}]_{n\times n} + [b_{ij}]_{n\times n}$$

$$= [a_{ij} + b_{ij}]_{n\times n} \in M_{n\times n}(R) \left[\begin{array}{l} a_{ij}, b_{ij} \in R \ \ 1 \le i, j \le n \\ a_{ij} + b_{ij} \in R ; 1 \le i, j \le n \end{array}\right]$$

=

(ii) $\forall A, B, C \in M_{n\times n}(R)$

$A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $C = [c_{ij}]_{n\times n}$

$a_{ij}, b_{ij}, c_{ij} \in R$ $1 \le i,j \le n$

$$A+(B+C) = [a_{ij}]_{n\times n} + \left([b_{ij}]_{n\times n} + [c_{ij}]_{n\times n}\right)$$

$$= [a_{ij}]_{n\times n} + \cancel{[b_{ij}]} \left([b_{ij} + c_{ij}]_{n\times n}\right)$$

$$= [a_{ij} + (b_{ij} + c_{ij})]_{n\times n}$$

As associative property hold in case of real no.

$$= [(a_{ij} + b_{ij}) + c_{ij}]_{n\times n}$$

$$= [a_{ij} + b_{ij}]_{n\times n} + [c_{ij}]_{n\times n}$$

$$= ([a_{ij}]_{n\times n} + [b_{ij}]_{n\times n}) + [c_{ij}]_{n\times n}$$

$$= (A+B) + C$$

(iii)

(iv)

DATE: / /
PAGE:

(iii) $\forall A \in M_{n\times n}(R)\ \exists\ \underset{\text{zero}}{\underset{\uparrow}{O}} = [O_{ij}]_{n\times n}$ s.t.

$$A + O = [a_{ij} + O_{ij}]_{n\times n}$$
$$= [a_{ij}]_{n\times n}$$
$$= A.$$

Similarly, $O + A = A$

$$\boxed{A + O = A = O + A}$$

(iv) for each $A = [a_{ij}]_{n\times n} \in M_{n\times n}(R)\ \exists\ -A = [-a_{ij}]_{n\times n} \in M_{n\times n}(R)$ s.t. $A + (-A) = [a_{ij}]_{n\times n} + [-a_{ij}]_{n\times n}$

$$= [a_{ij} - a_{ij}]_{n\times n}$$
$$= [O_{ij}]_{n\times n} = O$$

Similarly $(-A) + A = O$

$$A + (-A) = O = (-A) + A$$

(v) $\forall A, B \in M_{n\times n}(R)$

$A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$ where $a_{ij}, b_{ij} \in R$

$1 \le i, j \le n$

$$A + B = [a_{ij}]_{n\times n} + [b_{ij}]_{n\times n}$$
$$= [a_{ij} + b_{ij}]_{n\times n}$$

In case of real no.s commutative property also holds.

$$= [b_{ij} + a_{ij}]_{n\times n} = [b_{ij}]_{n\times n} + [a_{ij}]_{n\times n}$$
$$= B + A.$$

$\forall A, B \in M_{n\times n}(R) \Rightarrow A + B = B + A$

DATE : __/__/__
PAGE : ____

(vi) $\forall \alpha \in R$ and $\forall A \in M_{n\times n}(R)$

$A = [a_{ij}]_{n\times n}$ $a_{ij} \in R$ & $1 \leq i,j \leq n$

$\alpha A = \alpha [a_{ij}]_{n\times n}$ [as $a_{ij} \in R$, $\alpha \in R$, $1 \leq i,j \leq n$

$= [\alpha a_{ij}]_{n\times n}$ [$\alpha a_{ij} \in R$ $\forall$ $1 \leq i,j \leq n$

$\in M_{n\times n}(R)$

(vii) $\forall \alpha \in R$ and $\forall A, B \in M_{n\times n}(R)$

$A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$ $a_{ij}, b_{ij} \in R$ $1 \leq i,j \leq n$

$\alpha(A+B) = \alpha[[a_{ij}]_{n\times n} + [b_{ij}]_{n\times n}]_{n\times n}$

$= \alpha[[a_{ij}+b_{ij}]_{n\times n}]_{n\times n}$

$= [\alpha(a_{ij}+b_{ij})]_{n\times n}$

$= [\alpha a_{ij} + \alpha b_{ij}]_{n\times n}$

$= [\alpha a_{ij}]_{n\times n} + [\alpha b_{ij}]_{n\times n}$

$= \alpha[a_{ij}]_{n\times n} + \alpha[b_{ij}]_{n\times n}$

$= \alpha A + \alpha B$

(viii) $\forall \alpha, \beta \in R$ and $\forall A \in M_{n\times n}(R)$

$A = [a_{ij}]_{n\times n}$ $a_{ij} \in R$ $1 \leq i,j \leq n$

$(\alpha+\beta)A = (\alpha+\beta)[a_{ij}]_{n\times n}$

$= [(\alpha+\beta)a_{ij}]_{n\times n}$

$= [\alpha a_{ij} + \beta a_{ij}]_{n\times n}$

$= \alpha[a_{ij}]_{n\times n} + \beta[a_{ij}]_{n\times n}$

$= \alpha A + \beta A.$

(ix) $\forall\ \alpha, \beta \in R$ and $A \in M_{n\times n}(R)$

$A = [a_{ij}]_{n\times n}$ $\quad a_{ij} \in R \quad 0 \le i,j \le n$

$$
\begin{aligned}
(\alpha\beta)A &= (\alpha\beta)\ [a_{ij}]_{n\times n} \\
&= [\alpha\beta a_{ij}]_{n\times n} \\
&= \alpha\ [\beta a_{ij}]_{n\times n} \\
&= \alpha(\beta\ [a_{ij}]_{n\times n}) \\
&= \alpha\ (\beta A)
\end{aligned}
$$

(x) $\forall\ A \in M_{n\times n}(R)$ and $1 \in R$

$A_{ij} = [a_{ij}]$ $\quad a_{ij} \in R$

$1 \le i,j \le n$

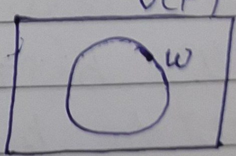
$$
\begin{aligned}
1.A &= 1.\ [a_{ij}]_{n\times n} \\
&= [1.a_{ij}]_{n\times n} \\
&= [a_{ij}]_{n\times n} = A
\end{aligned}
$$

$(M_{n\times n}(R), +, \cdot)$ is a vector space.

Subspace :- Let W be a non-empty subset of V(F).

If W is itself a vectorspace then we say that W is a subspace of V(F) under the same operation.

V(F)



(i) $W \neq \phi$ (Non-empty)

(ii) $W \subseteq V(F)$

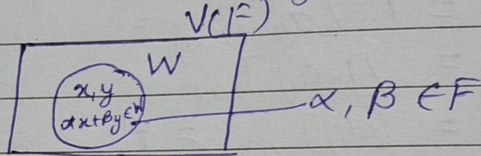
(iii) $0 \in W$

DATE : __/__/__
PAGE : ____

Theorem 1:- Let W be a non-empty subset of the vector space $V(F)$ then this W is called the subspace of $V(F)$.
if (i) $\forall\ x, y \in W \Rightarrow x+y \in W$
(ii) $\forall\ x \in W$ and $\alpha \in F \Rightarrow \alpha x \in W$

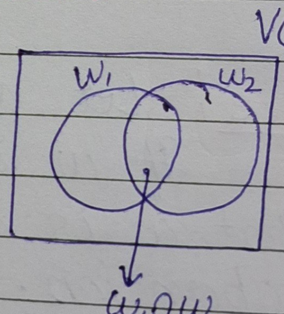
Theorem 2:-> Above are same only change in (i) condition
(i) $\forall\ x, y \in W \Rightarrow x - y \in W$

Theorem 3:-> Let W be a non-empty subset of $V(F)$ then W is called a subspace of $V(F)$.
if $\forall\ \alpha, \beta \in F$ and $\forall\ x, y \in W \Rightarrow \alpha x + \beta y \in W$



Imp

Q:- Prove that intersection of two subspaces is again a subspace of $V(F)$.



Sol:- Let W_1 & W_2 be two subspaces of $V(F)$.
We want to prove that $W_1 \cap W_2$ is also a subspace of $V(F)$.
Let $x, y \in W_1 \cap W_2$ and $\alpha, \beta \in F$
$\Rightarrow x, y \in W_1$ & $x, y \in W_2$
$\Rightarrow$ As $x, y \in W_1$ and $\alpha, \beta \in F$
and W_1 is a subspace of $V(F)$.
$\alpha x + \beta y \in W_1$ ——— (1)

DATE: / /
PAGE:

Again $x, y \in W_2$ and $\alpha, \beta \in F$ and W_2 is a subspace of $V(F)$

$$\alpha x + \beta y \in W_2 \quad — (2)$$

from (i) & (ii)

$$\alpha x + \beta y \in W_1 \cap W_2$$

$\Rightarrow \forall \; x, y \in W_1 \cap W_2$ and $\alpha, \beta \in F$

$\Rightarrow \alpha x + \beta y \in W_1 \cap W_2$

$\Rightarrow W_1 \cap W_2$ is a subspace of $V(F)$.

Note:- $V_3(R) = \{(x, y, z) : x, y, z \in R\}$

$W_1 = \{(0, y, z) : x, z \in R\}$

$W_2 = \{(x, 0, z) : x, z \in R\}$

$W_1 \cap W_2 : \{(x, 0, z) : x, z \in R\}$

Imp.

Q1:- Prove that union of two subspace is not a subspace.

Let $V(F)$ be the vector space.

$W_1 = \{(x, 0, 0) : x \in R\}$

$W_2 = \{(0, y, 0) : y \in R\}$

We want to show that $W_1 \cup W_2$ is not a subspace

$W_1 \cup W_2 = \{(x, 0, 0), (0, y, 0) : x, y \in R\}$

Let $u, v \in W_1 \cup W_2$

Suppose $u \in W_1$ and $v \in W_2$

$u = (x, 0, 0)$, $v = (0, y, 0)$ where $x, y \in R$

$u + v = (x, 0, 0) + (0, y, 0)$

$= (x, y, 0) \notin W_1 \cup W_2$

$W_1 \cup W_2$ is not a subspace of $V(F)$.

Q:- Prove that $W = \{(x,y,z) : x+y+z=5\}$ is not a subspace of $V_3(F)$.

Sol:- As $(0,0,0) \notin W$
we can say that W is not a subspace of $V_3(F)$.

Q:- Let V be the vector space over R.

$W = \{X : AX = B\}$ where B is non-zero element. Prove that it is not a subspace of $V(F)$.

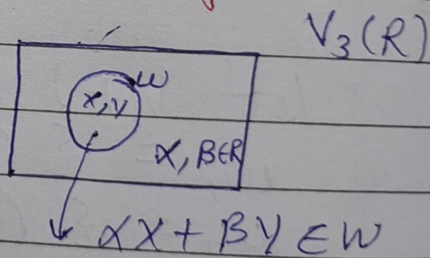
Sol:- clearly $0 \notin W$
we can say that W is not a subspace of $V(F)$

Q:- Let $V_3(R)$ be a vector space over R.

$$W = \{X : AX = 0\}$$

Prove that W is a subspace of $V_3(R)$

Sol:- Let $x, y \in W$ and $\alpha, \beta \in R$



$\therefore AX = 0$ —— (1)
$AY = 0$ —— (2)

consider $A(\alpha X + \beta Y) = \alpha AX + \beta BY$
$= \alpha(0) + \beta(0)$
$= 0$

$\alpha X + \beta Y \in W$

$\Rightarrow \forall X, Y \in W$ and $\forall \alpha, \beta \in R \Rightarrow \alpha X + \beta Y \in W$

$\Rightarrow W$ is a subspace of $V(F)$.

DATE: / /
PAGE:

Q:- Let a, b, c be a fixed element of a field. Show that $W = \{(x,y,z) : ax+by+cz=0\}$ is a subspace of $V_3(F)$.

Sol:→ $(0,0,0) \in W$

$W \neq \phi$

Also, $W \subseteq V$

$\forall\ u, v \in W$ and $\forall\ \alpha, \beta \in F$

$u = (x_1, y_1, z_1)$ & $v = (x_2, y_2, z_2)$ where $x_1, x_2, y_1, y_2, z_1, z_2 \in F$

$ax_1 + by_1 + cz_1 = 0$ —— ①

$ax_2 + by_2 + cz_2 = 0$ —— ②

$$\alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)$$
$$= (\alpha x_1, \alpha y_1, \alpha \cancel{y_1}) + (\beta x_2, \beta y_2, \beta z_2)$$
$$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \alpha z_2)$$

consider

$$a(\alpha x_1 + \beta x_2) + b(\alpha y_1 + \beta y_2) + c(\alpha z_1 + \beta z_2)$$
$$= \alpha(ax_1 + by_1 + cz_1) + \beta(ax_2 + by_2 + cz_2) \quad \{\text{from eq. ①}\}$$
$$= \alpha(0) + \beta(0)$$
$$= 0$$

$\alpha u + \beta v \in W$

$\Rightarrow \forall\ \alpha, \beta \in F$ and $\forall\ u, v \in W \Rightarrow \alpha u + \beta v \in W$

$\Rightarrow W$ is a subspace of $V_3(R)$

Q:- Let
Show that
$W = \{f(x) \in V : y = f(x)$ is the solution of the differential eq. $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$

Sol.⇒ $0 = 0(x) \in W$

$W \neq \phi$

clearly $W \subseteq V$

$\forall f(x), g(x) \in W$ and $\forall \alpha, \beta \in F$

$$\frac{d^2 f(x)}{dx^2} + a\frac{d f(x)}{dx} + b f(x) = 0 \quad \text{——— (1)}$$

$$\frac{d^2 g(x)}{dx^2} + a\frac{d g(x)}{dx} + b g(x) = 0 \quad \text{——— (2)}$$

Consider

$$\frac{d^2(\alpha f(x) + \beta g(x))}{dx^2} + a\frac{d}{dx}(\alpha f(x) + \beta g(x)) + b(\alpha f(x) + \beta g(x))$$

$$\alpha\frac{d^2}{dx^2}f(x) + \beta\frac{d^2(g(x))}{dx^2} + a\alpha\frac{d}{dx}f(x) + a\beta\frac{d(g(x))}{dx} + b\alpha f(x) + b\beta g(x)$$

$$= \alpha\left(\frac{d^2}{dx^2}f(x) + a\frac{d}{dx}f(x) + bf(x)\right) + \beta\left(\frac{d^2 g(x)}{dx^2} + a\frac{d(g(x))}{dx} + bg(x)\right)$$

putting value from eq (1) & (2)

$= \alpha(0) + \beta(0)$

$= 0$

⇒ $\alpha f(x) + \beta g(x) \in W$

⇒ $\forall \alpha, \beta \in F$ and $\forall f(x), g(x) \in W \Rightarrow \alpha f(x) + \beta g(x) \in W$

⇒ W is a subspace of $V(F)$.

Q:- Let V be the vector space of 2×2 matrices over real. ~~Prove that~~

$$W = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \ \alpha, \beta \in R \right\}$$

Show that W is a subspace of $V(F)$.

DATE: / /
PAGE:

Sol:- $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

$\Rightarrow W \neq \phi$

clearly $W \subseteq V$

$\forall A, B \in W$ & $\forall \alpha, \beta \in R$

$A = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{bmatrix}$, $B = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{bmatrix}$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$

$$\alpha A + \beta B = \alpha \begin{bmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{bmatrix} + \beta \begin{bmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha\alpha_1 & 0 \\ 0 & \alpha\beta_1 \end{bmatrix} + \begin{bmatrix} \beta\alpha_2 & 0 \\ 0 & \beta\beta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha\alpha_1 + \beta\alpha_2 & 0 \\ 0 & \alpha\beta_1 + \beta\beta_2 \end{bmatrix}$$

$\because \left[\begin{matrix} \alpha_1, \beta_1, \alpha_2, \beta_2 \in R \text{ \& } \alpha, \beta \in R \\ \alpha\alpha_1 + \beta\alpha_2 \, , \, \alpha\beta_1 + \beta\beta_2 \in R \end{matrix} \right]$

$$= \begin{bmatrix} \alpha\alpha_1 + \beta\alpha_2 & 0 \\ 0 & \alpha\beta_1 + \beta\beta_2 \end{bmatrix} \in R$$

W is a subspace of V(F).

Q:- Let V be a vector space over R. Examine ~~whether whether~~ W is vector space or not.

$W = \{(a, b, c) : c \text{ is integer}\}$

Sol:- let $u \in W$ and $\sqrt{2} \in R$

let $u \in W$ and $\sqrt{2} \in R$ where $c \in \mathbb{Z}$

$u = (a, b, c)$ & $\sqrt{2} \in R$

DATE : __/__/__
PAGE : ______

$\sqrt{2}\, u = \sqrt{2}(a,b,c)$
$= (\sqrt{2}a, \sqrt{2}b, \sqrt{2}c) \notin W$

$\Rightarrow$ W is not a subspace of the vector space.

Q:- Above statement is same.
$W = \{(a,b,c) : a \geq b \geq c\}$

Sol:- let $u \in W$ & $-1 \in R$
$u = (a,b,c)$ and $-1 \in R$ $\quad a \geq b \geq c$
$-u = -1(a,b,c)$
$= (-a,-b,-c) \notin W \quad -a \leq -b \leq -c$

$\Rightarrow$ W is not a subspace of V(F).

Linear Span:-

let $S = \{v_1, v_2, \dots, v_n\}$ be a non-empty set.
$L(S) = \{x : x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n\}$

Ex:- $S = \{(1,0), (0,1)\}$
$L(S) = \{x : x = \alpha(1,0) + \beta(0,1)\}$

$= \{(1,1), (1,2), (3,2)$

$\alpha = 1, \beta = 1$
$x = 1(1,0) + 2(0,1)$
$= (1,0) + (0,2)$
$= (1,2)$

$\alpha = 3, \beta = 2$
$x = 3(1,0) + 2(0,1)$
$= (3,0) + (0,2)$
$= (3,2)$

let S be a non-empty subset of n vectors.
$S = \{v_1, v_2 \dots v_n\}$
then the linear span of S is denoted by L(S) and it is given by $L(S) = \{x : x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n\}$

Ex;
Imp
Theor
Sol

Ex:- $S = \{(2,3), (1,3)\}$

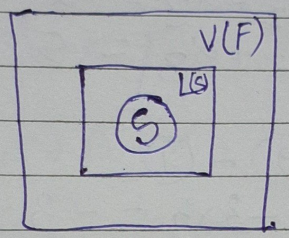
$L(S) = \{x : x = \alpha(2,3) + \beta(1,3)\}$

* $\alpha = 1, \ \beta = 2$

$x = 1(2,3) + 2(1,3)$
$= (2,3) + (2,6)$
$= (4,9)$

* $\alpha = 3, \ \beta = 2$

$x = 3(2,3) + 2(1,3)$
$= (6,9) + (2,6)$
$= (8,15)$



Imp

Theorem:- Prove that $L(S)$ is a subspace of $V(F)$.

Sol:-> let $S = \{v_1, v_2, \ldots v_n\}$

$L(S) = \{x : x = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n : \alpha_i\text{'s} \in F\}$

$\forall \ x, y \in L(S)$ and $\forall \ (\alpha, \beta) \in F$

$x = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ where $\alpha_i\text{'s} \in F$, $1 \le i \le n$

$y = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$ where $\beta_i\text{'s} \in F$, $1 \le i \le n$

Consider,

$\alpha x + \beta y = \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) + \beta(\beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n)$

$= v_1(\alpha\alpha_1 + \beta\beta_1) + v_2(\alpha\alpha_2 + \beta\beta_2) + \ldots + v_n(\alpha\alpha_n + \beta\beta_n) \in L(S)$

$\left\{ \begin{array}{l} \alpha_i\text{'s} \in F \text{ and } \beta_i\text{'s} \in F \quad 1 \le i \le n \text{ also } \alpha, \beta \in F \\ \alpha\alpha_i + \beta\beta_i \in F \quad 1 \le i \le n \end{array} \right\}$

$\alpha x + \beta y \in L(S)$

$\forall \ x, y \in L(S)$ & $\alpha, \beta \in F \Rightarrow \alpha x + \beta y \in L(S)$

$L(S)$ is a subspace of $V(F)$.

DATE : __/__/__
PAGE : ______

Que → Express $v = (1,-2,5)$ as a linear combination of v_1, v_2, v_3.

$v_1 = (1,1,1)$, $v_2 = (1,2,3)$, $v_3 = (2,-1,1)$

Sol:- $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$(1,-2,5) = \alpha_1(1,1,1) + \alpha_2(1,2,3) + \alpha_3(2,-1,1)$

$(1,-2,5) = (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2, 3\alpha_2) + (2\alpha_3, -\alpha_3, \alpha_3)$

$(1,-2,5) = (\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 - \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3)$

$\alpha_1 + \alpha_2 + 2\alpha_3 = 1$ —— (1)

$\alpha_1 + 2\alpha_2 - \alpha_3 = -2$ —— (2)

$\alpha_1 + 3\alpha_2 + \alpha_3 = 5$ —— (3)

from (2) - (1)

$\alpha_2 - 3\alpha_3 = -3$ —— (4)

from (3) - (2)

$\alpha_2 + 2\alpha_3 = 7$ —— (5)

from (4) & (5)

$\cancel{\alpha_2} - 3\alpha_3 = -3$

$\underline{\cancel{\alpha_2} + 2\alpha_3 = 7}$
$\quad - \quad - \quad\quad -$

$-5\alpha_3 = -10$

$\boxed{\alpha_3 = 10/5} = 2$ $\quad \boxed{\alpha_3 = 2}$

$\alpha_2 + 2\alpha_3 = 7$

$\alpha_2 + 4 = 7$

$\boxed{\alpha_2 = 3}$

$\alpha_1 + \alpha_2 + 2\alpha_3 = 1$

$\alpha_1 + 3 + 4 = 1$

$\boxed{\alpha_1 = -6}$

DATE: ___/___/___
PAGE: ______

$$\boxed{v = -6v_1 + 3v_2 + 2v_3}$$

Que→ Find the condition of A, B & C such that the matrix $E = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ is a linear combination of $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$$E = \alpha A + \beta B + \gamma C$$

$$\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = \begin{bmatrix} \alpha & \alpha \\ 0 & -\alpha \end{bmatrix} + \begin{bmatrix} \beta & \beta \\ -\beta & 0 \end{bmatrix} + \begin{bmatrix} \gamma & -\gamma \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = \begin{bmatrix} \alpha+\beta+\gamma & \alpha+\beta-\gamma \\ -\beta & -\alpha \end{bmatrix}$$

$$\alpha+\beta+\gamma = a \quad \text{—— (1)}$$
$$\alpha+\beta-\gamma = b \quad \text{—— (2)}$$
$$-\beta = -b \quad \text{—— (3)}$$
$$-\alpha = c \quad \text{—— (4)}$$

$$\boxed{\beta = b} \qquad \boxed{\alpha = -c}$$

$$\alpha+\beta+\gamma = a$$
$$-c+b+\gamma = a$$
$$\boxed{\gamma = a-b+c}$$

$$\alpha+\beta-\gamma = b$$
$$-c+b-\gamma = b$$
$$-c+b-b = \gamma$$
$$\boxed{\gamma = -c}$$

$$\boxed{E = -cA + bB - cC}$$

DATE : __/__/__
PAGE : ______

Que:- Examine whether $(1,-3,5)$ belong to $L(S)$ where $S = \{(1,2,1), (1,1,-1), (4,5,-2)\}$ or not?

Sol:- If suppose that

$$(1,-3,5) \in L(S)$$

$$(1,-3,5) = \alpha(1,2,1) + \beta(1,1,-1) + \gamma(4,5,-2)$$
$$(1,-3,5) = (\alpha+\beta+4\gamma,\ 2\alpha+\beta+5\gamma,\ \alpha-\beta-2\gamma)$$

$$\alpha+\beta+4\gamma = 1 \quad \text{--- } (1)$$
$$2\alpha+\beta+5\gamma = -3 \quad \text{--- } (2)$$
$$\alpha-\beta-2\gamma = 5 \quad \text{--- } (3)$$

from (2) − (1)
$$\alpha+\gamma = -4 \quad \text{--- } (4)$$

from (1) + (3)
$$2\alpha+2\gamma = 6$$
$$\alpha+\gamma = 3 \quad \text{--- } (5)$$

~~from~~ This is impossible.
$$(1,-3,5) \notin L(S).$$

Q:- What is the condition on V such that $v = (a,b,c)$ is a linear combination of $v_1 = (2,1,0)$, $v_2 = (1,-1,2)$, $v_3 = (0,3,-4)$

Sol:- $v = \alpha v_1 + \beta v_2 + \gamma v_3$

$$(a,b,c) = \alpha(2,1,0) + \beta(1,-1,2) + \gamma(0,3,-4)$$
$$= (2\alpha+\beta,\ \alpha-\beta+3\gamma,\ 2\beta-4\gamma)$$

$$2\alpha+\beta = a \quad \text{--- } (1)$$
$$\alpha-\beta+3\gamma = b \quad \text{--- } (2)$$
$$2\beta-4\gamma = c \quad \text{--- } (3)$$

Adding ① + ②

$$3\alpha + 3\gamma = a + b \quad \text{—} ④$$
$$\alpha + \gamma = (a+b)/3$$

$2 \times ① - ③$

$$4\alpha + 4\gamma = 2a - c$$
$$\alpha + \gamma = \frac{2a-c}{4} \quad \text{—} ⑤$$

from ④ & ⑤

$$\frac{a+b}{3} = \frac{2a-c}{4}$$
$$4(a+b) = 3(2a-c)$$
$$\boxed{2a - 4b - 3c = 0}$$

Que: Express $f = x^2 + 4x - 3$ as a linear combination of $f_1 = x^2 - 2x + 5$, $f_2 = x + 3$, $f_3 = 2x^2 - 3x$

Sol.→
$$f = c_1 f_1 + c_2 f_2 + c_3 f_3$$
$$(x^2 + 4x - 3) = c_1(x^2 - 2x + 5) + c_2(x+3) + c_3(2x^2 - 3x)$$
$$x^2 + 4x - 3 = x^2(c_1 + 2c_3) + x(-2c_1 + c_2 - 3c_3) + 5c_1 + 3c_2$$

$$c_1 + 2c_3 = 1 \quad \text{—} ①$$
$$-2c_1 + c_2 - 3c_3 = 4 \quad \text{—} ②$$
$$5c_1 + 3c_2 = -3 \quad \text{—} ③$$

{comparing coefficient}

from ①
$$1 - c_1 = 2c_3$$
$$c_3 = \frac{1 - c_1}{2} \quad \text{—} ④$$

from ③
$$-3 - 5c_1 = 3c_2$$
$$c_2 = -\frac{(3 + 5c_1)}{3} \quad \text{—} ⑤$$

DATE : __/__/__
PAGE : ____

from (2)

$$4 = -2c_1 + c_2 - 3c_3$$
$$= -2c_1 - \frac{(3+5c_1)}{3} - 3\frac{(1-c_1)}{2}$$
$$4 = \frac{-12c_1 - 6 - 10c_1 - 9 + 9c_1}{6}$$
$$24 = -22c_1 + 9c_1 - 9 - 6$$
$$24 + 15 = -13c_1$$
$$c_1 = \frac{39}{-13} = -3 \quad , \quad \boxed{c_1 = -3}$$

$$c_1 + 2c_3 = 1$$
$$-3 + 2c_3 = 1$$
$$\boxed{c_3 = 2}$$

$$c_2 = -\frac{(3+5c_1)}{3} = -\left(\frac{3+(-15)}{3}\right)$$
$$= -\left(\frac{-12}{3}\right) = 4$$
$$\boxed{c_2 = 4}$$

$$\boxed{f = -3f_1 + 4f_2 + 2f_3}$$

Linearly dependent and Independent:-

$v_1, v_2 \ldots v_n$

$c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$

Ex:- $c_1v_1 + c_2v_2$

$c_1(2,4) + c_2(1,2) = 0$

$1(2,4) + (-2)(1,2) = 0$

$(2,4) + (-2,-4) = 0$

$(2-2, 4-4) = 0$

$(0,0) = 0$

DATE: / /
PAGE:

Linearly dependent Vector :→

Let $v_1, v_2, ----, v_n$ be n vectors.
if $\exists$ non-zero scalars.
$\alpha_1, \alpha_2, ----, \alpha_n \in F$ such that
$\alpha_1 v_1 + \alpha_2 v_2 + ---- + \alpha_n v_n = 0$
then vectors are called L.D (Linearly dependent vector).

Linearly independent Vector :→

Let $v_1, v_2 ----- v_n$ be n vectors.
such that $\alpha_1 v_1 + \alpha_2 v_2 + ----- + \alpha_n v_n = 0$
$\alpha_1 = \alpha_2 = ----- = \alpha_n = 0$

Q:- Prove that the vectors $u = (1,2,-3)$, $v = (1,-3,2)$ and $w = (2,-1,5)$ are linearly independents.

Sol:- $\alpha u + \beta v + \gamma w = 0$

$\alpha(1,2,-3) + \beta(1,-3,2) + \gamma(2,-1,5) = (0,0,0)$

$(\alpha, 2\alpha, -3\alpha) + (\beta, -3\beta, 2\beta) + (2\gamma, -\gamma, 5\gamma) = (0,0,0)$

$\alpha + \beta + 2\gamma = 0$ — (1)

$2\alpha - 3\beta - \gamma = 0$ — (2)

$-3\alpha + 2\beta + 5\gamma = 0$ — (3)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$AX = 0$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{vmatrix} = 1(-15+2) - 1(10-3) + 2(4-9)$$

$= -13 - 7 - 10$

$= -30 \neq 0$

$\alpha = \beta = \gamma = 0$

The given vectors are linearly independent.

DATE : __/__/__
PAGE : ____

Note ① if $|A| \neq 0$ then all scalars are zero. The vectors are L.I.

② if $|A| = 0$ then all scalar are not zero the vectors are L.D.

imp

Que. If the vectors $(b, 1, 0)$, $(1, b, 1)$, $(0, 1, b)$ are linear dependent then find the value of b.

As vectors are L.D. $|A| = 0$

$$|A| = \begin{vmatrix} b & 1 & 0 \\ 1 & b & 1 \\ 0 & 1 & b \end{vmatrix}$$

$$\begin{aligned} 0 &= b(b^2-1) - 1(b) \\ &= b^3 - b - b \\ &= b^3 - 2b \\ &= b(b^2-2) \end{aligned}$$

$b = 0$, $b^2 - 2 = 0$

$b^2 = 2$

$\boxed{b = \pm\sqrt{2}}$

$$\boxed{b = 0, +\sqrt{2}, -\sqrt{2}}$$

Que. Find the value of K so that the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} K \\ 0 \\ 1 \end{bmatrix}$$ are L.D.

$$\begin{vmatrix} 1 & 1 & K \\ -1 & 2 & 0 \\ 2 & -2 & 1 \end{vmatrix} = 0$$

$1(2) - 1(-1) + K(2-4) = 0$

$2 + 1 - 2K = 0 \Rightarrow \boxed{K = 3/2}$

DATE: __/__/__
PAGE: ____

Linear Sum of Subspaces :-

let $W_1, W_2, \ldots W_n$ be n subspace of V(F) then
$W = W_1 + W_2 + \ldots + W_n$ is called the linear sum of the subspaces.

Disjoint Subspace :-

let W_1 and W_2 be two subspaces of V(F) if $W_1 \cap W_2 = \phi$ then W_1 & W_2 are called disjoint subspace.

eg :- $V_3(R) = \{(x, y, z) : x, y, R \in R\}$ is a vector space

$W_1 = \{(x, 0, 0) : x \in R\}$

$W_2 = \{(0, y, 0) : x \in R\}$