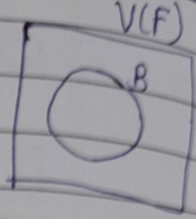
Unit: 3

Basis of a vector space:-

A set $B = \{v_1, v_2 \ldots \ldots v_n\}$ is called a basis of vector space $V(F)$ if

(i) B is a linearly independent set.

(ii) $L(B) = V$.



e.g.:→ $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$(3,4,5) = 3(1,0,0) + 4(0,1,0) + 5(0,0,1)$

$(-3,5,4) = -3(1,0,0) + 5(0,1,0) + 4(0,0,1)$

(i) $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$L(B) = \{\alpha(1,0,0), \beta(0,1,0) + \gamma(0,0,1)\}$ $\alpha, \beta, \gamma \in R$

Next we want to show that

$L(B) = V$

$L(B) \subseteq V$ ——— (1)

Next we want to show that

$V \subseteq L(B)$ ⟶ (1)

Let $x \in V$

$x = (\alpha, \beta, \gamma) = \alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) \in L(B)$

$V \subseteq L(B)$ ⟶ (2)

from (1) & (2)

$\boxed{L(B) = V}$

Que-1:→ Prove that the set $B = \{e_1, e_2, \ldots, e_i, \ldots, e_n\}$ is a basis of $V_n(R)$ where $e_i = (0, 0 \ldots, 1, 0, \ldots, 0)$

Sol:→ In order to prove that B is a basis of $V_n(R)$ We want to show that (i) B is a L.I. set

(ii) $L(B) = V_n(R)$

(i) B is a L.I set

$\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n = 0$

$\alpha_1 (1,0,0\ldots) + \alpha_2 (0,1,0,\ldots 0) + \ldots + \alpha_n (0,0\ldots,1) = (0,0\ldots0)$

$(\alpha_1, 0, 0 \ldots 0) + (0, \alpha_2, 0, \ldots, 0) + \ldots + (0, 0, \ldots \alpha_n) = (0,0,0)$

$\alpha_1, \alpha_2, \ldots \alpha_n = 0$

$\alpha_1 = 0, \alpha_2 = 0, \ldots \alpha_n = 0$

$\Rightarrow B$ is a Linear Independent Set.

(ii) Next we want to show that $L(B) = V_n(R)$

$L(B) \subseteq V_n(R) \longrightarrow (1)$

Next we want to show that

$V_n(R) \subseteq L(B)$

let $x \in V_n(R)$

$x = (\alpha_1, \alpha_2 \ldots \alpha_n)$ where $\alpha_i \in R$ $1 \le i \le n$

$x = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n \in L(B)$

Now $x \in V \Rightarrow x \in L(B)$

$V \subseteq L(B) \longrightarrow (2)$

from (1) & (2)

$L(B) = V_n(R)$

$\Rightarrow B$ is a basis of $V_n(R)$.

Imp (10 Marks)

Q:- Show that the set $B = \{1, x, x^2, \ldots x^m\}$ of (m+1) polynomial is a basis for the vector space $P_m(R)$ of all polynomial of degree m over R.

Sol:- $P_m(R) = \{f(x) : f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m, a_i \in R, 0 \le i \le m\}$

we want to show that (i) B is a L.I set

(ii) $L(B) = P_m(R)$

(i) TP: B is a L.I. set

$\alpha_0(1) + \alpha_1 x + \alpha_2 x^2 + \alpha_m x^m = 0$

$\alpha_0(1) + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_m x^m = 0.1 + 0.x + \ldots + 0x^m$

$\alpha_0(1) + \alpha_1 x + \alpha_2 x^2 + \ldots$

$\alpha_0 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0, \ldots\ldots, \alpha_m = 0$

Set B is a linear independent set.

(ii) Next we want to show that $L(B) = P_m(R)$

$$L(B) \subseteq P_m(R) \longrightarrow (1)$$

Next we want to show that

$$P_m(R) \subseteq L(B)$$

Let $f(x) \in P_m(R)$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

$$= a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_m(x^m) \in L(B)$$

$f(x) \in P_m(R) \Rightarrow f(x) \in L(B)$

$$P_m(R) \subseteq L(B) \quad \text{---} (2)$$

from (1) & (2) we get

$$P_m(R) = L(B)$$

$\Rightarrow B$ is a Basis of $P_m(R)$.

Q:- Let V be the vector space ~~for~~ $\forall$ 2x2 symmetric matrices of B. Find the basis & dimension of B.

$$V = \left\{ A : A = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} ; \alpha, \beta, \gamma \in R \right.$$

Consider set $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Now we show that B is a Basis set.

(i) To prove: B is a L.I set

$$\alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\alpha = 0, \beta = 0, \gamma = 0$

B is a L.I Set.

(ii) Next we want to show that $L(B) = V$

As we know that $L(B) \subseteq V \longrightarrow ①$

Next we want to show that

$$V \subseteq L(B)$$

let $A \in V$

$$A = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in L(B)$$

$A \in V \Rightarrow A \in L(B)$

$V \subseteq L(B) \longrightarrow ②$

from ① & ② we get

$$V = L(B)$$

$\Rightarrow$ B is a basis of $V(R)$.

Disjoint Subspaces:- Two subsapaces W_1 & W_2 are called disjoint subspaces if $W_1 \cap W_2 = \{0\}$

Finitely Generated Vector Spaces:- let $V(F)$ be a vector space over the field F. & S be any finite subset of $V(F)$.

if $L(S) = V(F)$

Then we say that $V(F)$ is a Finitely generated vector space.

DATE : / /
PAGE :

Dimension of a vector space :- Number of element in basis of a vector space is called the dimension of a vector space.

Q :- Determine wheather and not the following forms is a basis

(i) $(1,1,1)$, $(1,2,3)$, $(2,-1,1)$

In order to prove that it is a basis or not only we need to check the given vectors are linearly independent or not.

$$\alpha(1,1,1)+\beta(1,2,3)+\gamma(2,-1,1)=(0,0,0)$$

$$\cancel{(\alpha+\beta+2\gamma)+(\alpha,2\beta,2}$$

$$(\alpha+\beta+2\gamma,\ \alpha+2\beta-\gamma,\ \alpha+3\beta+\gamma)=(0,0,0)$$

$$\alpha+\beta+2\gamma=0$$
$$\alpha+2\beta-\gamma=0$$
$$\alpha+3\beta+\gamma=0$$

$$\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & -1\\ 1 & 3 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$|A| = 1(2+3)-1(1+1)+2(3-2)$$
$$= 5-2+2$$
$$= 5 \neq 0$$

$\alpha=\beta=\gamma=0 \Rightarrow$ The given vector are L.I.

$\{(1,1,1),(1,2,3),(2,-1,1)\}$ is forms a basis of $V_3(R)$.

(ii) $(1,2,3)$, $(1,0,-1)$, $(3,-1,0)$

In order to prove that it is a basis or not only we need to check the given vectors are L.I. or not.

$$\alpha(1,2,3)+\beta(1,0,-1)+\gamma(3,-1,0)=(0,0,0)$$

$$(\alpha+\beta+3\gamma,\ 2\alpha-\gamma,\ 3\alpha-\beta)=(0,0,0)$$

$$\alpha+\beta+3\gamma=0$$
$$2\alpha-\gamma=0$$
$$3\alpha-\beta=0$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|A| = 1(0-1)-1(0+3)+3(-2-0)$$
$$= -1-3-6$$
$$= -10 \neq 0$$

$\alpha=\beta=\gamma=0 \Rightarrow$ The given vector are L.I. $\{(1,2,3),(1,0,-1),(3,-1,0)\}$ is form a basis of $V_3(R)$.

(iii) $(1,2,3)$, $(0,1,2)$, $(0,0,3)$

$$\alpha(1,2,3)+\beta(0,1,2)+\gamma(0,0,3)=(0,0,0)$$

$$(\alpha,\ 2\alpha+\beta,\ 3\alpha+2\beta+3\gamma)=(0,0,0)$$

$\alpha=0$, $\quad 2\alpha+\beta=0 \quad \beta=0$, $\quad 3\alpha+2\beta+3\gamma=0 \quad \gamma=0$

PAGE :

Que → Extend the given set $\{(3,-1,2)\}$ to two different basis of R^3.

Sol^n → We know that the usual basis of R^3 is $\{(1,0,0), (0,1,0), (0,0,1)\}$

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, Make the above matrix in echlon form. {but here first Row remain Same}

$3R_2$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_1$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 - R_2$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{1}{2}R_3$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_4 - R_3$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\{(3,-1,2), (0,1,-2), (0,0,1)\}$$

This is the first basis.

$$\{(3,-1,2), (0,1,0), (0,0,1)\}$$

This is the second basis.

(ii) $\{(-1,2,5)\}$ extend its two different basis.

We know that the usual basis of R^3 is

$$\{(1,0,0), (0,1,0), (0,0,1)\}.$$

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 + R_1$

$$\sim \begin{bmatrix} -1 & 2 & 5 \\ 0 & 2 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

do not write like this b/c first operation w.r.t. is $\rightarrow$ $(2R_3 - R_2)$ $(R_3(2) - R_2)$

$$\sim \begin{bmatrix} -1 & 2 & 5 \\ 0 & 2 & 5 \\ 0 & 0 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$5R_4 + R_3$

$$\sim \begin{bmatrix} -1 & 2 & 5 \\ 0 & 2 & 5 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\{(-1,2,5), (0,2,5), (0,0,-5)\}$
is a first basis.

$\{(-1,2,5), (0,1,0), (0,0,1)\}$
is a second basis.

Q:- Let W_1 and W_2 be a two subspaces of R^3
where $W_1 = \{(a,b,c) : b = 2a, c = a+b\}$
$W_2 = \{(a,b,c) : 2a+b-3c = 0\}$
Find the basis and dimensions of $W_1, W_2, W_1 \cap W_2$

Sol:- $W_1 = \{(a,b,c) : b = 2a, c = a+b\}$

$b = 2a, \quad c = a+b$
$c = a+2a$
$c = 3a$

$(a,b,c) = (a, 2a, 3a)$
$= a(1,2,3)$

Basis of W_1 is $\{(1,2,3)\}$

$\boxed{\dim(W_1) = 1}$

$W_2 = \{(a,b,c) : 2a+b-3c = 0\}$
$2a+b-3c = 0$
$b = 3c - 2a$ ~~, c~~

$(a,b,c) = (a, 3c-2a, c)$
$= (a, -2a, 0) + (0, 3c, c)$
$= a(1,-2,0) + c(0,3,1)$.

Basis of $W_2 = \{(1,-2,0), (0,3,1)\}$

$\boxed{\dim(W_2) = 2}$

$W_1 \cap W_2 = \{(a,b,c) : b = 2a, c = a+b,$
$2a+b-3c = 0\}$

DATE : __/__/__
PAGE : ______

$b = 2a$, $c = a+b$
$= a+2a$
$= 3a$

$2a+b-3c=0$
$2a+2a-3(3a)=0$
$4a-9a=0$
$5a=0$
$\boxed{a=0}$
$\boxed{b=0}$
$\boxed{c=0}$

$W_1 \cap W_2 = \{(0,0,0)\}$
$\dim(W_1 \cap W_2) = 0$

Q:- Let W_1 be the subspace of R^4 where
$W_1 = \{(a,b,c,d) ; \ b+c+d=0\}$
find the basis and dimension of W_1.

Sol:→ $b+c+d=0$
$b = -c-d$
$(a,b,c,d) = (a, -c-d, c, d)$
$= (a,0,0,0) + (0,-c,c,0) + (0,-d,0,d)$
$= a(1,0,0,0) + c(0,-1,1,0) + d(0,-1,0,1)$

Basis of $W_1 = \{(1,0,0,0), (0,-1,1,0), (0,-1,0,1)\}$
$\boxed{\dim W_1 = 3}$

Q:- $W_2 = \{(a,b,c,d) : \ a+b=0, \ c=2d\}$
find the basis and dimension of W_2

Sol:- $a+b=0$
$a=-b$
$c=2d$
$(a,b,c,d) = (-b, b, 2d, d)$
$= (-b,b,0,0) + (0,0,2d,d)$
$= b(-1,1,0,0) + d(0,0,2,1)$

DATE : / /
PAGE :

basis of $W_2 = \{(-1,1,0,0), (0,0,2,1)\}$

$\dim W_2 = 2$

Q:→ Let V be a vector space of 2×2 Matrices over R. Show that the matrices

$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ -1 & -5 \end{bmatrix}$$

form a basis of V over R. Hence find the coordinate of the matrix $\begin{bmatrix} 4 & -11 \\ -11 & -7 \end{bmatrix}$ relative to this basis.

Sol:→ In order to show that $A=\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, B=\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, C=\begin{bmatrix} 4 & -1 \\ -1 & -5 \end{bmatrix}$

forms a basis of V over R. it is sufficient to prove that these matrices are L.I.

$$\alpha A + \beta B + \gamma C = 0$$

$$\alpha\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} + \beta\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + \gamma\begin{bmatrix} 4 & -1 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\alpha + 2\beta + 4\gamma = 0$$
$$-2\alpha + \beta - \gamma = 0$$
$$-2\alpha + \beta - \gamma = 0$$
$$\alpha + 3\beta - 5\gamma = 0$$

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & -1 \\ 1 & 3 & -5 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 + 2R_1, \; R_3 - R_1$

DATE: PAGE:

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 7 \\ 0 & 1 & -9 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -9 \\ 0 & 5 & 7 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 5R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -9 \\ 0 & 0 & 52 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$52\gamma = 0$

$\boxed{\gamma = 0}$

$\beta - 9\gamma = 0$

$\boxed{\beta = 0}$

$\alpha + 2\beta + 4\gamma = 0$

$\boxed{\alpha = 0}$

The given matrices A, B, C forms a basis of V over R.

$D = \alpha A + \beta B + \gamma C$

$$\begin{bmatrix} 4 & -11 \\ -11 & -7 \end{bmatrix} = \alpha\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} + \beta\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + \gamma\begin{bmatrix} 4 & -1 \\ -1 & -5 \end{bmatrix}$$

$\alpha + 2\beta + 4\gamma = 4$

$-2\alpha + \beta - \gamma = -11$

$\alpha + 3\beta - 5\gamma = -7$

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & -1 \\ 1 & 3 & -5 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \nu \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -7 \end{bmatrix}$$

$R_2 + 2R_1 \quad , \quad R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 7 \\ 0 & 1 & -9 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \nu \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -11 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & -9 \\ 0 & 5 & 7 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \nu \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -3 \end{bmatrix}$$

$D = \alpha A + \beta B + \nu C$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -9 \\ 0 & 0 & 52 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \nu \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ 52 \end{bmatrix}$$

$52\nu = 52$

$\boxed{\nu = 1}$

$\beta - 9\nu = -11$

$\boxed{\beta = -2}$

$\alpha + 2\beta + 4\nu = 4$

$\boxed{\alpha = 4}$

$\boxed{D = 4A - 2B + C}$

Definition of coset :- let V be a vector space over the field F. Let H be a subset of V, the coset of H in V is given by

$$H+a = \{ \cancel{H}+a : a \in V \}$$
$$a+H = \{ a+H : a \in V \}$$

Quotient Space :- let V(F) be a vector space over the field F and H be a subset of V.

$\frac{V}{H} = \{ x+H : x \in V \}$ is called the quotient space.