Tutorial class practice
sheet - 2

1. Let $A = \{1,2,3,4,5,6\}$. Let two relations R and S be defined as follows. $R = \{(x,y) \mid (x<y)\}$ and $S = \{(x,y) \mid (x-y)$ is divisible by $-2\}$. Find ROS using matrix approach.

soln:

$R = \{(1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,3), (2,4), (2,5), (2,6),$
$(3,4), (3,5), (3,6),$
$(4,5), (4,6),$
$(5,6)\}$

$S = \{(1,3), (2,4), (3,5), (4,6)\}$

$$M_R = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 & 0 & 1 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$M_S = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$ROS = M_R * M_S$

$R \circ S = M_R \ast M_S$

$$
= \begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & 1 & 1 & 1 & 1 & 1 \\
2 & 0 & 0 & 1 & 1 & 1 & 1 \\
3 & 0 & 0 & 0 & 1 & 1 & 1 \\
4 & 0 & 0 & 0 & 0 & 1 & 1 \\
5 & 0 & 0 & 0 & 0 & 0 & 1 \\
6 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\ast
\begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$$
= \begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
2 & 0 & 0 & 0 & 0 & 1 & 1 \\
3 & 0 & 0 & 0 & 0 & 0 & 1 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0.
\end{array}
$$

$R \circ S = \{ (1,4), (1,5), (1,6), (2,5), (2,6), (3,6) \}$

2 Let $R = \{ (1,2), (2,3), (3,4) \}$ $S = \{ (1,1), (1,2), (2,1), (2,2),$ $(2,3), (3,1), (3,2), (3,3), (3,4) \}$ be the relations defined on $\{1,2,3\}$ and $\{1,2,3,4\}$. Find the following

(a) $R \cup S$ (b) $R \cap S$ (c) $R - S$ (d) $R \circ S$

soln:

(a) $R \cup S$

$R \cup S = \{ (1,2), (2,3), (3,4), (1,1), (2,1), (2,2), (3,1),$
$(3,2), (3,3), (3,4) \}$

(b) $R \cap S$

$R \cap S = \{ (1,2), (2,3), (3,4) \}$

(c) $R - S$

$R - S = \{ \phi \}$

(d) $R \circ S$

Relation R and S are defined on {1,2,3} and {1,2,3,4} sets.

So R and S forms the Relation matrix of size 3×4 each.

To find composite of R and S ($R \circ S$) we have to multiply M_R and M_S. But matrix multiplication cannot be done on 3×4 and 3×4.

∴ $R \circ S$ cannot able to find.

4. Consider the relation R = {(1,3), (1,4), (3,2), (3,3), (3,4)} on A = {1,2,3,4}

(a) Find the matrix M_R of R

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

(b) Find the domain and Range of R

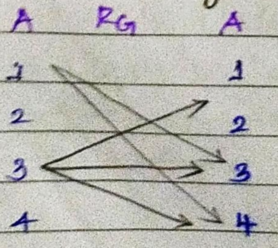
Domain (R) = {1,3}

Range (R) = {2,3,4}

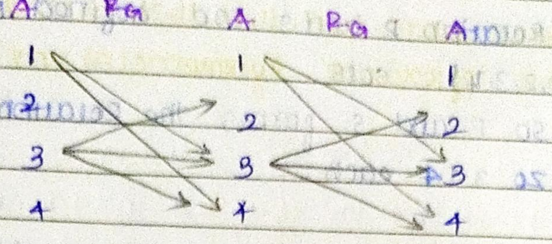
(c) Find R^{-1}.

R^{-1} = {(3,1), (4,1), (2,3), (3,3), (4,3)}

(d) Draw the directed graph of R.



(e). Find the composition relation RoR.

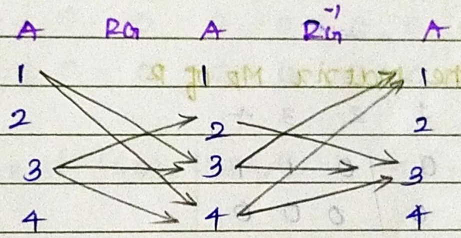


$R \circ R = \{(1,2), (1,3), (1,4), (3,2), (3,3), (3,4)\}$

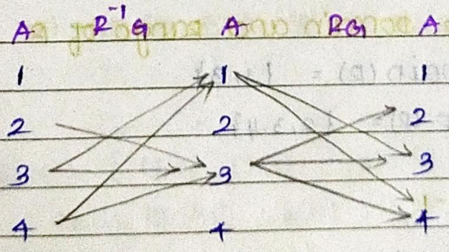
(f) Find $R \circ R^{-1}$ and $R^{-1} \circ R$

$R = \{(1,3), (1,4), (3,2), (3,3), (3,4)\}$

$R^{-1} = \{(3,1), (4,1), (2,3), (3,3), (4,3)\}$



$R \circ R^{-1} = \{(1,1), (1,3), (3,3), (3,1)\}$



$R^{-1} \circ R = \{(2,2), (2,3), (2,4), (3,3), (3,4), (3,2), (4,3), (4,4), (4,2)\}$

6. Let R be the relation on the set {0,1,2,3} containing ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), (3,0). Find Reflexive(R), symmetric(R) and Transitive(R).

Let $A = \{0,1,2,3\}$
$R = \{(0,1), (1,1), (1,2), (2,0), (2,2), (3,0)\}$

Reflexive:

R is not reflexive because $(3,3) \notin R$

Reflexive closure:

$\text{Reflexive}(R) = R \cup \Delta_A$
$= \{(0,1),(1,1),(1,2),(2,0),(2,2),(3,0)\} \cup \{(0,0),(1,1),(2,2),(3,3)\}$

$\text{Reflexive}(R) = \{(0,0),(0,1),(1,1),(1,2),(2,0),(2,2),(3,0),(3,3)\}$

Symmetric:

R is not symmetric because $(0,1) \in R$ but $(1,0) \notin R$

Symmetric closure:

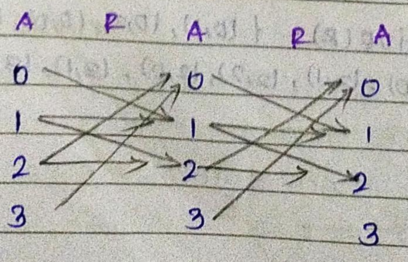
$\text{Symmetric}(R) = R \cup R^{-1}$
$= \{(0,1),(1,1),(1,2),(2,0),(2,2),(3,0)\} \cup \{(1,0),(1,1),(2,1),(0,2),(2,2),(0,3)\}$

$\text{Symmetric}(R) = \{(0,1),(1,0),(1,1),(1,2),(2,1),(2,0),(0,2),(2,2),(3,0),(0,3)\}$

Transitive:

R is not transitive because $(0,1) \in R$ and $(1,2) \in R$ but $(0,2) \notin R$.

Transitive closure:



$R^2 = R \circ R$

$= \{(0,1), (0,2), (1,1), (1,2), (1,0), (2,0), (2,1), (2,2), (3,1)\}$

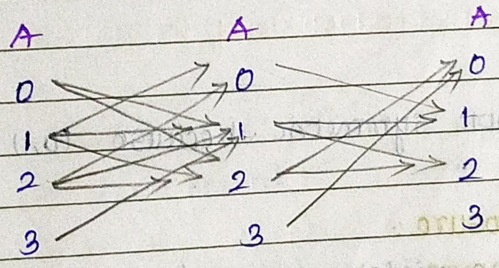
$R^* = R^2 \cup R$

$= \{(0,1), (0,2), (1,1), (1,2), (1,0), (2,0), (2,1), (2,2), (3,1)\} \cup$
$\{(0,1), (1,1), (1,2), (2,0), (2,2), (3,0)\}$

$R^* = \{(0,1), (1,1), (0,2), (1,2), (1,0), (2,0), (2,2), (3,0), (2,1), (2,2), (3,1)\}$

R^* is not transitive because $(0,1) \in R^*$ and $(1,0) \in R^*$ but $(0,0) \notin R^*$.

$R^3 \Rightarrow$



$R^3 = R^2 \circ R$
$= \{(0,1), (0,2), (0,0), (1,1), (1,2), (1,0), (2,1), (2,2), (2,0), (3,1), (3,2)\}$

$R^* = R^3 \cup R$
$= \{(0,1), (0,2), (0,0), (1,1), (1,2), (1,0), (2,1), (2,2), (2,0), (3,1), (3,2), (3,0)\}$

R^* is transitive.

$\therefore$ transitive$(R) = \{(0,1), (0,2), (0,0), (1,1), (1,2), (1,0), (2,1), (2,2), (2,0), (3,1), (3,2), (3,0)\}$

9. consider the relation $R = \{(a,a),(a,b),(b,c),(c,c)\}$ on the set $A = \{a,b,c\}$. Find (i) reflexive(R) (ii) symmetric(R) (iii) transitive(R).

Let $A = \{a,b,c\}$

$R = \{(a,a),(a,b),(b,c),(c,c)\}$

Reflexive :

R is not reflexive because $(b,b) \notin R$

Reflexive closure :

$\text{Reflexive}(R) = R \cup \Delta_A$

$= \{(a,a),(a,b),(b,c),(c,c)\} \cup \{(a,a),(b,b),(c,c)\}$

$\text{Reflexive}(R) = \{(a,a),(a,b),(b,b),(b,c),(c,c)\}$

symmetric :

R is not symmetric because $(a,b) \in R$ but $(b,a) \notin R$.

symmetric closure :

$\text{symmetric}(R) = R \cup R^{-1}$

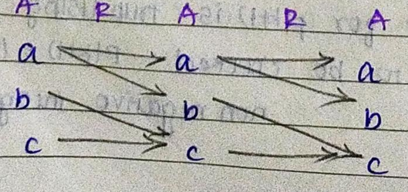
$= \{(a,a),(a,b),(b,c),(c,c)\} \cup \{(a,a),(b,a),(c,b),(c,c)\}$

$\text{symmetric}(R) = \{(a,a),(a,b),(b,a),(b,c),(c,b),(c,c)\}$

transitive :

R is not transitive because $(a,b) \in R$ and $(b,c) \in R$ but $(a,c) \notin R$.

Transitive closure :



$R^2 = R \circ R$

$= \{(a,a), (a,b), (a,c), (b,c), (c,c)\}$

$R^* = R^2 \cup R$

$= \{(a,a), (a,b), (a,c), (b,c), (c,c)\} \cup$

$\{(a,a), (a,b), (b,c), (c,c)\}$

$R^* = \{(a,a), (a,b), (a,c), (b,c), (c,c)\}$

R^* is transitive.

Reflexive $(R) = [(a,a), (a,b), (a,c), (b,c), (c,c)]$

14. Use Mathematical Induction to show that $n^2 - 7n + 12$ is non-negative whenever n is an integer greater than 3.

Let $P(n) = n^2 - 7n + 12$ is non negative $\forall\, n > 3$.

Step 1: Basis Step

check for $n = 4$

$P(4) = 4^2 - 7(4) + 12$

$= 16 - 28 + 12$

$= 0 \rightarrow 0$ is non negative Integer.

True for $n = 4$

Step 2: Induction Step

Let us assume that $P(n)$ is true for $n = k$.

i.e., $P(k)$ is true.

$P(k) = k^2 - 7k + 12$ is non negative integer $\forall\, k > 3$

Step 3 ::

check for $P(k+1)$ is true using $P(k)$.

statement to be checked $P(k+1) = (k+1)^2 - 7(k+1) + 12$ is non negative integer $\forall\, k > 3$

$P(k+1) = (k+1)^2 - 7(k+1) + 12$

$= k^2 + 1 + 2k - 7k - 7 + 12$

$= \underbrace{(k^2 - 7k + 12)} + (2k + 1 - 7)$

from step (ii) it

is non negative $\forall k > 3$

$= (k^2 - 7k + 12) + \underbrace{(2k - 6)}$

it is also non negative $\forall k > 3$

sum of two non negative number is also a non negative number

$P(k+1) = (k^2 - 7k + 12) + (2k - 6)$

$P(k+1)$ is true for $n = k+1$

$\therefore$ $P(n)$ is true for all values of n $\forall n > 3$

15 Use Mathematical Induction to show that $8^n - 3^n$ is a multiple of 5.

Let $P(n) = 8^n - 3^n$ is a multiple of 5

Step 1: Basis Step :

check for $n = 1$

$P(1) = 8^1 - 3^1 = 8 - 3 = 5$

Statement : 5 is a multiple of 5

$P(1)$ is true. true for $n = 1$

Step 2: Induction Step :

Let us assume that $P(n)$ is true for $n = k$

i.e., $P(k)$ is true.

$P(k) = 8^k - 3^k$ is a multiple of 5

Step 3 :

check for P(K+1) is true using P(K)

statement to be checked $P(k+1) \Rightarrow 8^{k+1} - 3^{k+1}$ is a multiple of 5

$$P(k+1) = 8^{k+1} - 3^{k+1}$$
$$= 8^k . 8 - 3^k . 3$$
$$= 8^k (5+3) - 3^k . 3$$
$$= 5(8^k) + 3(8^k) - 3^{k+1}$$
$$= \underbrace{5(8^k)}_{\text{a number multiplied by 5 is a multiple of 5}} + 3(\underbrace{8^k - 3^k}_{\text{From step (ii) multiple of 5}})$$

any number multiplied with the multiple of 5 is also a multiple of 5

sum of two numbers which are the multiple of 5 is also a multiple of 5.

$P(k+1) = 5(8^k) + 3(8^k - 3^k)$

$P(k+1)$ is true for $n = k+1$

$\therefore$ $P(n)$ is true for all values of n.

16. Use Mathematical Induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Let $P(n) = n^3 - n$ is divisible by 3.

Step 1 : Basis step.

check for $n = 1$

$P(1) = 1^3 - 1 = 0 \Rightarrow 0$ is divisible by 3

$n = 1$ is true

$P(1)$ is true

Step 2: Induction step:

Let us assume that P(n) is true for $n=k$

P(k) is true.

$P(k) = k^3 - k$ is divisible by 3.

Step 3:

Check for P(k+1) is true using P(k).

statement to be checked $P(k+1) = (k+1)^3 - (k+1)$ is divisible by 3.

$(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$$P(k+1) = (k+1)^3 - (k+1)$$
$$= k^3 + 1 + 3k^2 + 3k - k - 1$$
$$= k^3 - k + 3k^2 + 3k$$
$$= \underbrace{k^3 - k}_{\text{from step (ii) divisible by 3}} + \underbrace{3(k^2 + k)}_{\text{any number multiplied by 3 is also divisible by 3}}$$

sum of 2 numbers divisible by 3 is also divisible by 3

$$P(k+1) = k^3 - k + 3(k^2 + k)$$

P(k+1) is true for $n = k+1$

$\therefore$ P(n) is true for all values of n.

17. Use Mathematical Induction to prove that $n^3 + 2n$ is a multiple of 3

Let $P(n) = n^3 + 2n$ is a multiple of 3

Step 1: Basis step

Check for $n=1$

$$P(1) = 1^3 + 2(1)$$
$$= 1 + 2 = 3$$

Statement: 3 is multiple of 3

$n=1$ is true P(1) is true.

Step 2: Induction step

Let us assume that P(n) is true for n = k

i.e., P(k) is true

$P(k) = k^3 + 2k$ is a multiple of 3

Step 3:

Check for P(k+1) is true using P(k).

statement to be checked $P(k+1) = (k+1)^3 + 2(k+1)$ is a multiple of 3.

$$P(k+1) = (k+1)^3 + 2(k+1)$$
$$= k^3 + 1 + 3k^2 + 3k + 2k + 2$$
$$= (k^3 + 2k) + 3k^2 + 3k + 3$$
$$= \underbrace{(k^3 + 2k)}_{\text{from step (i) multiple of 3}} + \underbrace{3(k^2 + k + 1)}_{\text{any number multiplied by 3 is a multiple of 3.}}$$

sum of 2 numbers multiple of 3 is also a multiple of 3.

$P(k+1) = (k^3 + 2k) + 3(k^2 + k + 1)$

P(k+1) is true for n = k+1

∴ P(n) is true for all values of n.