Introduction to Optimization Problem

Everyone, almost daily, solves optimization problems in informal ways by using mental models.

- If you have ever looked at a map when planning a trip and decided upon a route to your destination to try to minimize either distance or time (or even to maximize scenic or recreational benefits), you have solved an optimization problem.

- If you have ever been faced with too much work like preparing for all assessments & for final examinations and with a constrained in fixed amount of time, you have undoubtedly solved an optimization problem that seeks to allocate the time available so that you would, perhaps, maximize your grades in the various courses.

- Whereas informal optimization is done nearly every day by individuals, organizations carry out formal optimization to assist in such decisions as product mix, pricing, scheduling, routing and logistics, supply chain management, facility location analysis, and financial planning & asset management.

- Optimization as a decision support tool has applications across all functional areas of an organization.

- In an optimization mindset, there is an **objective** you want to either **maximize or minimize**, and there may be **constraints** within which you need to operate. There are also specific quantities, called **decision variables**, over which you have control. Therefore, this is termed a **constrained optimization problem**.

- A **verbal statement of the study time problem** might be that you want to maximize your grade point average. Constraints are a limited total amount of time to study, and a desire to pass every course. Decision variables are the amounts of time allocated to each course. A more structured statement of the problem is

Maximize the objective : Grade point average

Subject to the constraints: i) preparing within available study time
ii) passing each course with a grade of at least A

Decision variables : Amount of time to spend on each course

- Our purpose is not to develop a formal mathematical model of the study time example.
- However, the actual process of thinking about a problem in a structured way often leads to insights you can use to make a better decision.
- You would probably think about your current grade in each course, the remaining requirements of the course, and the

amount of study time needed in order to at least pass each course.

- You might also think about how much time you would need to spend in order to get the best possible grade in each course.
- This structured thinking often leads to a better overall decision, even if a formal mathematical model is never developed.
- The **advantage of formal optimization modeling** is that it can simultaneously consider the effects of alternate decisions to produce the **best overall decision according to the objective.**

Real-Life Application—Frontier Airlines Purchases Fuel Economically

The fueling of an aircraft can take place at any of the stopovers along a flight route. Fuel price varies among the stopovers, and potential savings can be realized by tankering (loading) extra fuel at a cheaper location for use on subsequent flight legs. The disadvantage is that the extra weight of tankered fuel will result in higher burn of gasoline. Linear programming (LP) and heuristics are used to determine the optimum amount of tankering that balances the cost of excess burn against the savings in fuel cost. The study, carried out in 1981, resulted in net savings of about $350,000 per year. With the significant rise in the cost of fuel, many airlines are using LP-based tankering software to purchase fuel.

Simple Example

A furniture dealer deals in only two items–tables and chairs. He has Rs 50,000 to invest and has storage space of at most 60 pieces. A table costs Rs 2500 and a chair Rs 500. He estimates that from the sale of one table, he can make a profit of Rs 250 and that from the sale of one chair a profit of Rs 75. He wants to know how many tables and chairs he should buy from the available money so as to maximize his total profit, assuming that he can sell all the items which he buys.

Such type of problems which seek to maximize (or, minimize) profit (or, cost) form a general class of problems called optimization problems. Thus, an optimization problem may involve finding maximum profit, minimum cost, or minimum use of resources etc.

A special but a very important class of optimization problems is linear programming problem. The above stated optimization problem is an example of linear programming problem. Linear programming problems are of much interest because of their wide applications in industry, commerce, management science etc.

Linear Programming Problem and its Mathematical Formulation

Let us consider the example of furniture dealer and its equivalent mathematical formulation of the problem in two variables

a) The dealer can invest his money in buying tables or chairs or combination thereof. Further he would earn different profits by following different investment strategies.

b) There are certain overriding conditions or constraints viz., his investment is limited to a maximum of Rs 50,000 and so is his storage space which is for a maximum of 60 pieces.

c) Suppose he decides to buy tables only and no chairs, so he can buy 50000 ÷ 2500 = 20 tables. His profit in this case will be Rs (250 × 20 =) Rs 5000. Suppose he chooses to buy chairs only and no tables. With his capital of Rs 50,000, he can buy 50000 ÷ 500 = 100 chairs. But he can store only 60 pieces.

d) Therefore, he is forced to buy only 60 chairs which will give him a total profit of Rs (60 × 75 =) Rs 4500.

e) There are many other possibilities, for instance, he may choose to buy 10 tables and 50 chairs, as he can store only 60 pieces. Total profit in this case would be Rs (10 × 250 + 50 × 75), i.e., Rs 6250 and so on.

f) Thus, the dealer can invest his money in different ways and he would earn different profits by following different investment strategies.

g) Now the problem statement is: How should he invest his money in order to get maximum profit? To answer this question, let us try to formulate the problem mathematically.

Mathematical formulation of the problem

Let x be the number of tables and y be the number of chairs that the dealer buys. Obviously, x and y must be non-negative.

$x \geq 0 \rightarrow$ (1) and $y \geq 0 \rightarrow$ (2) [Non-negative constraints]

The dealer is constrained by the maximum amount he can invest Rs 50,000 and by the maximum number of items he can store is 60. Mathematically stated as

$2500x + 500y \leq 50000$ [investment constraint]

or $5x + y \leq 100 \rightarrow$ (3)

$x + y \leq 60$ [storage constraint] $\rightarrow$ (4)

The dealer wants to invest in such a way so as to maximize his profit, say, Z which stated as a function of x and y is given by

$Z = 250x + 75y$ (called objective function)

Mathematically, the given problem now reduces to:

Maximize $Z = 250x + 75y$

Subject to the constraints

$$5x + y \leq 100$$
$$x + y \leq 60$$
$$x \geq 0, y \geq 0$$

- Here the problem is to maximize the linear function Z subject to certain constraints determined by a set of linear inequalities with variables as non-negative.
- There are also some other problems where we have to minimize a linear function subject to certain constraints determined by a set of linear inequalities with variables as non-negative. Such problems are called Linear Programming Problems.

Thus, a Linear Programming Problem is one that is concerned with finding the optimal value (maximum or minimum value) of a linear function (called objective function) of several variables (say x and y), subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints). The term linear implies that all the mathematical relations used in the problem are linear relations while the term programming refers to the method of determining a particular programme or plan of action.

Here

Objective function: A Linear function $Z = ax + by$, where a, b are constants, which has to be maximized or minimized is called a linear objective function.

In the above example, $Z = 250x + 75y$ is a linear objective function. Variables x and y are called decision variables.

Constraints: The linear inequalities or equations or restrictions on the variables of a linear programming problem are called constraints. The conditions $x \leq 0$, $y \leq 0$ are called non-negative restrictions. In the above example, the set of inequalities (1) to (4) are constraints.

<u>How to find solutions to a linear programming problem?</u>

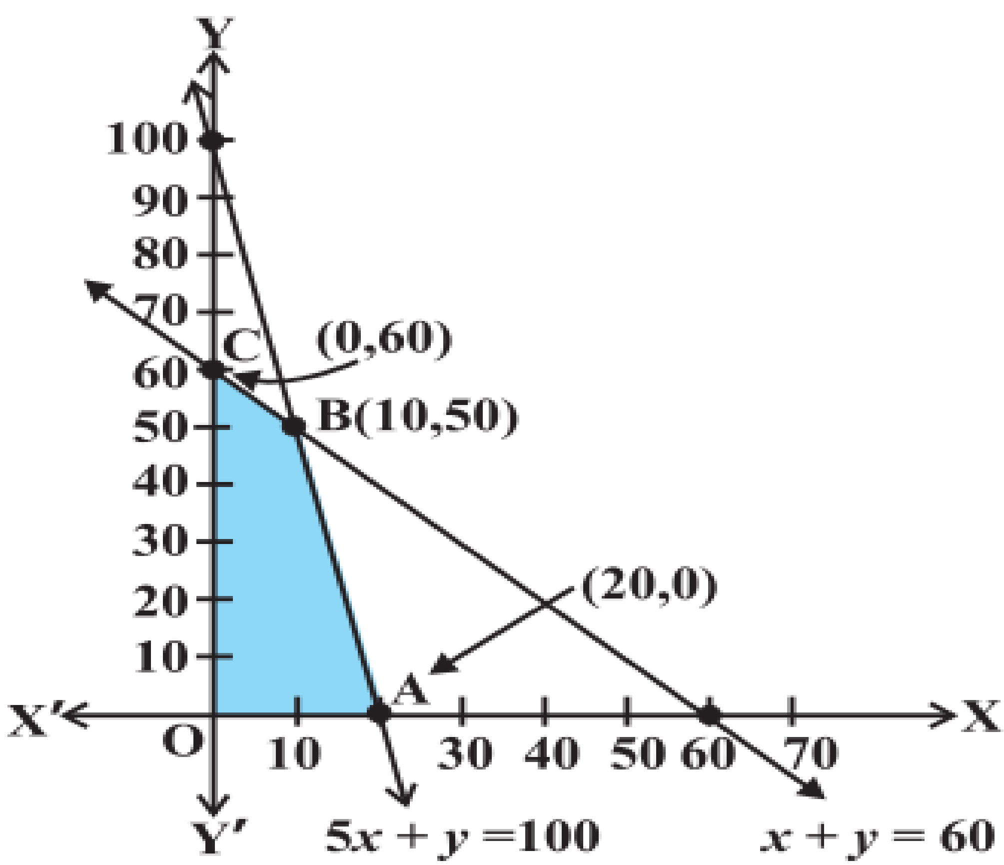
There are many ways to find solutions to a LPP. One of the ways is using Graphically.

Graphical method of solving linear programming problems

Let us discuss the problem of investment in tables and chairs. Now solve this problem graphically. Let us graph the constraints stated as linear inequalities:

$$5x + y \leq 100 \quad \dots (1)$$
$$x + y \leq 60 \quad \dots (2)$$
$$x \geq 0 \quad \dots (3)$$
$$y \geq 0 \quad \dots (4)$$

The graph of this system (shaded region) consists of the points common to all half planes determined by the inequalities (1) to (4) is shown below.



Each point in this region represents a feasible choice open to the dealer for investing in tables and chairs. The region, therefore, is called the feasible region for the problem.

Every point of this region is called a feasible solution to the problem. Thus, we have, Feasible region. The common region determined by all the constraints including non-negative constraints x, $y \geq 0$ of a linear programming problem is called the feasible region (or solution region) for the problem. In the graph, the region OABC (shaded) is the feasible region for the problem. The region other than feasible region is called an infeasible region.

Feasible solutions: Points within and on the boundary of the feasible region represent feasible solutions of the constraints. In the graph, every point within and on the boundary of the feasible region OABC represents feasible solution to the problem. For example, the point (10, 50) is a feasible solution of the problem and so are the points (0, 60), (20, 0) and (0, 0).

Any point outside the feasible region is called an infeasible solution. For example, the point (25, 40) is an infeasible solution of the problem

Optimal (feasible) solution: Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution. Now, we see that every point in the feasible region OABC satisfies all the constraints as given in (1) to (4), and since there are infinitely many points, it is not evident how we should go about finding a point that gives a maximum value of the objective function Z = 250x + 75y.

The following point is are fundamental in solving linear programming problems

- Let R be the feasible region (convex polygon) for a linear programming problem and let Z = $ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point* (vertex) of the feasible region.

  In the above example, the corner points (vertices) of the bounded (feasible) region are: O, A, B and C and it is easy to find their coordinates as (0, 0), (20, 0), (10, 50) and (0, 60) respectively. Let us now compute the values of Z at these points.

Vertex of the feasible region	Corresponding value of Z in Rs.
O (0,0)	0

C (0,60)	4500
B (10,50)	6250← Maximum
A (20,0)	5000

It is observed that the maximum profit to the dealer results from the investment strategy (10, 50), i.e. buying 10 tables and 50 chairs.

This method of solving linear programming problem is referred as Corner Point Method.

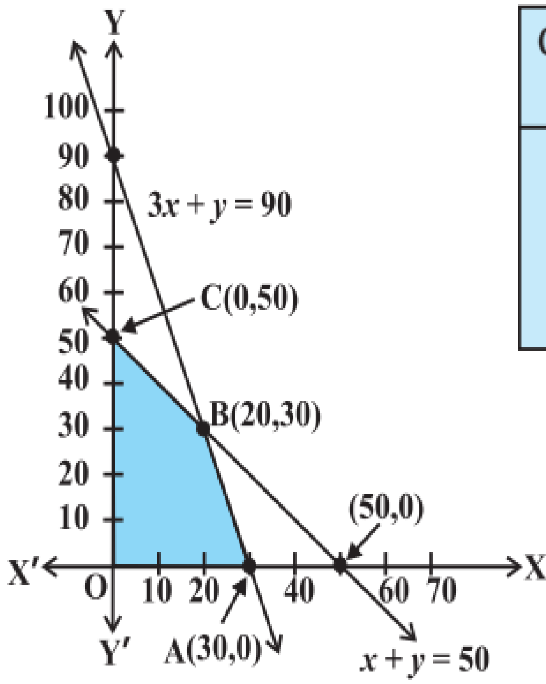
Sample Problem

Solve the following linear programming problem graphically:

$$\text{Maximize } Z = 4x + y \qquad \rightarrow (1)$$

subject to the constraints:

$$x + y \leq 50 \qquad \rightarrow (2)$$
$$3x + y \leq 90 \qquad \rightarrow (3)$$
$$x \geq 0,\ y \geq 0 \qquad \rightarrow (4)$$



Corner Point	Corresponding value of Z
(0, 0)	0
(30, 0)	**120** ← Maximum
(20, 30)	110
(0, 50)	50

Hence, maximum value of Z is 120 at the point (30, 0).

Another Exercise

A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below:

Machine	Time per unit (Minutes)			Machine Capacity (minutes/day)
	Product 1	Product 2	Product 3	
M1	2	3	2	440
M2	4	-	3	470
M3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is Rs. 4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical (L.P.) model that will maximize the daily profit

Maximize $Z = 4x_1 + 3x_2 + 6x_3 \qquad \rightarrow (1)$

x_1, x_2 **and** $x_3 \geq 0$	$\rightarrow$(2)
$2x_1 + 3x_2 + 2x_3 \leq 440$	$\rightarrow$(3)
$4x_1 + 0x_2 + 3x_3 \leq 470$	$\rightarrow$(4)
$2x_1 + 5x_2 + 0x_3 \leq 430$	$\rightarrow$(5)

One More Exercise

PRODUCT MIX PROBLEM

A factory manufactures two products A and B. To manufacture one unit of A, 1.5 machine hours and 2.5 labour hours are required. To manufacture product B, 2.5 machine hours and 1.5 labour hours are required. In a month, 300 machine hours and 240 labour hours are available. Profit per unit for A is Rs. 50 and for B is Rs. 40. Formulate as LPP.

Solution:

Products	Resource/unit	
	Machine	Labour
A	1.5	2.5
B	2.5	1.5
Availability	300 hrs	240 hrs

There will be two constraints. One for machine hours availability and for labour hours availability.

Decision variables:

X_1 = Number of units of A manufactured per month.

X_2 = Number of units of B manufactured per month.

The objective function: Max $Z = 50x_1 + 40x_2$

Subjective Constraints

For machine hours: $1.5x_1 + 2.5x_2 \leq 300$

For labour hours: $2.5x_1 + 1.5x_2 \leq 240$

Non negativity

$x_1, x_2 \geq 0$

Example 2 Solve the following linear programming problem graphically:

Minimise Z = 200 x + 500 y ... (1)

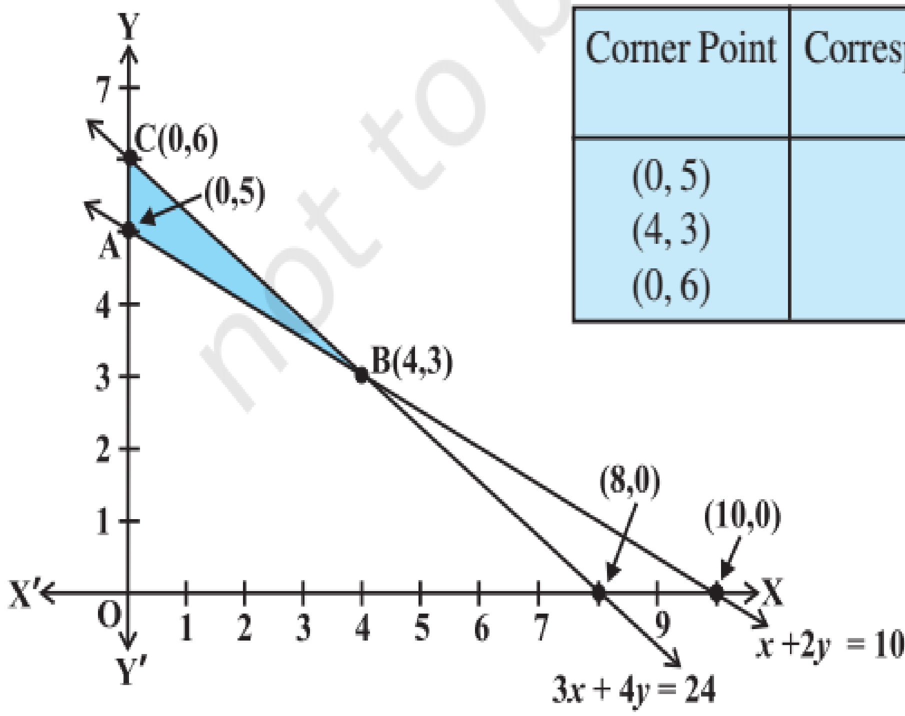
subject to the constraints:

$$x + 2y \geq 10 \quad \text{... (2)}$$

$$3x + 4y \leq 24 \quad \text{... (3)}$$

$$x \geq 0, y \geq 0 \quad \text{... (4)}$$

Solution The shaded region in Fig 12.3 is the feasible region ABC determined by the system of constraints (2) to (4), which is **bounded**. The coordinates of corner points



Corner Point	Corresponding value of Z	
(0, 5)	2500	
(4, 3)	**2300** ←	Minimum
(0, 6)	3000	

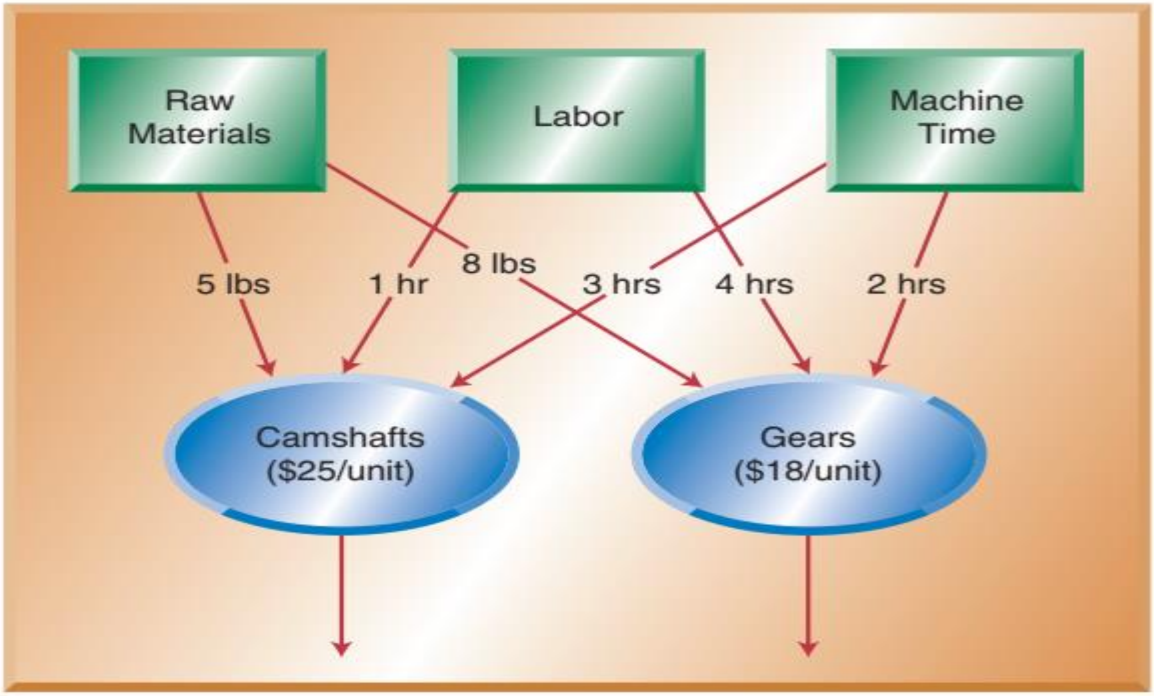
Model the following LPP

DJJ Enterprises manufactures automotive parts. Two of these parts are camshafts and gears. Camshafts earn a profit of $25 per unit and gears earn $18 per unit. Three major resources are utilized in the production process: steel, labor, and machine time. It takes 5 lbs of steel to make a camshaft, and 8 lbs to make a gear. Camshafts require 1 hour of labor; gears require 4 hours. It takes 3 hours machine time per camshaft, and 2 hours per gear. For the current planning period, 5000 lbs steel, 1500 hours labor, and 1000 hours machine time are available. DJJ would like to maximize profit during the current planning period, within allowable resources.

LBS – Pounds $\Longrightarrow$ 1 pound = 453 Grams

5 LBS = 2.26796185 Kgs

Mathematical Model



Decision Variables

C = number of camshafts to make

G = number of gears to make

Objective Function

Maximize Z = 25C + 18G

Subject to the following constraints

$5C + 8G \leq 5000$ [Raw Materials Constraints]

$1C + 4G \leq 1500$ [Labors Constraints]

$3C + 2G \leq 1000$ [Machine time Constraints]

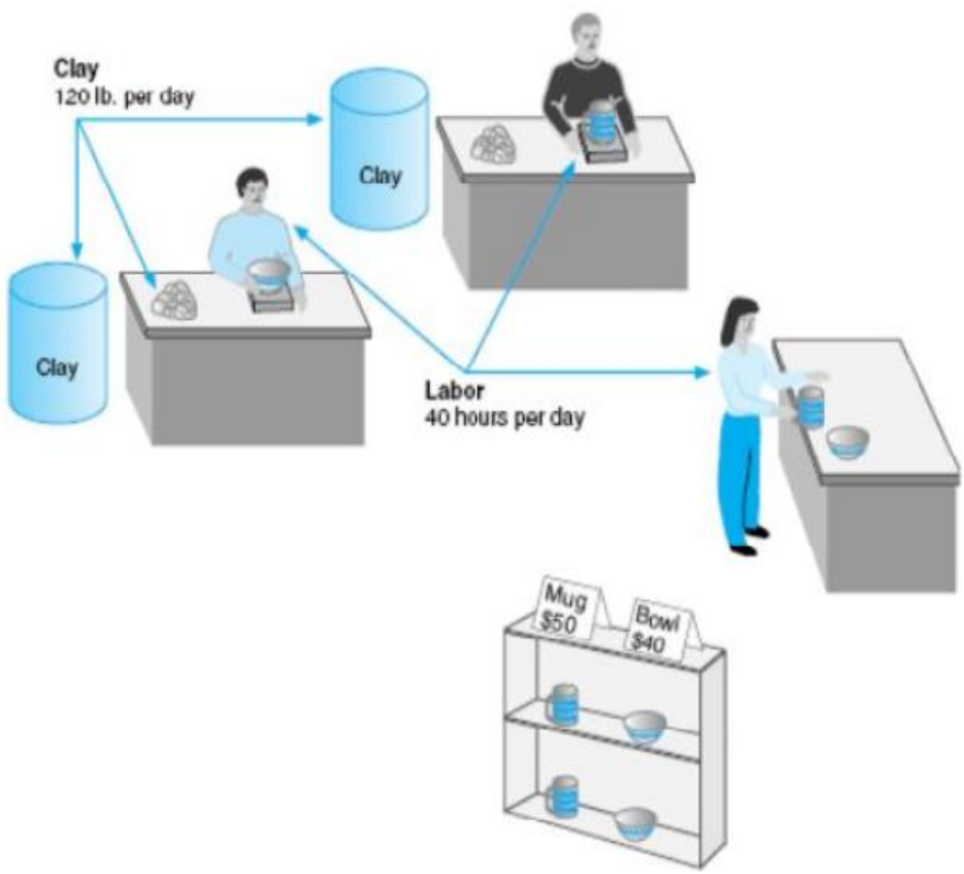
$C \geq 0$ and $G \geq 0$ [Non-Negativity Constraints]

Product MIX problem – Exercise

- Product mix problem - Beaver Creek Pottery Company
- How many bowls and mugs should be produced to maximize profits given labor and materials constraints?
- Product resource requirements and unit profit:

	Resource Requirements		
Product	Labor (Hr./Unit)	Clay (Lb./Unit)	Profit ($/Unit)
Bowl	1	4	40
Mug	2	3	50

Figure 2.1
Beaver Creek Pottery Company



Model the problem

Resource Availability:	40 hrs of labor per day 120 lbs of clay
Decision Variables:	x_1 = number of bowls to produce per day x_2 = number of mugs to produce per day
Objective Function:	Maximize $Z = \$40x_1 + \$50x_2$ Where Z = profit per day
Resource Constraints:	$1x_1 + 2x_2 \leq 40$ hours of labor $4x_1 + 3x_2 \leq 120$ pounds of clay
Non-Negativity Constraints:	$x_1 \geq 0$; $x_2 \geq 0$

Complete Linear Programming Model:

Maximize $\quad Z = \$40x_1 + \$50x_2$

subject to:
$$1x_1 + 2x_2 \leq 40$$
$$4x_2 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$