Equivalence of definitions of closure of topological subspace

Sibo WANG

1 Motivation

Closure in topology is an essential concept and it has various definitions. In this note, equivalence of those definitions will be proved.

2 Equivalence of definitions of closure of topological subspace

Lemma 2.1. *Given a topology space $(X, \mathcal{T})$. Infinite intersection of closed set in X is a closed set.*

Proof: (De Morgan's laws) $\blacksquare$

Definition 2.2. Given a topology space $(X, \mathcal{T})$, for $A \subseteq X$, a point $x \in X$ is called *a limit point of A* if for each $B \in \{B : x \in B \in \mathcal{T}\}, B \cap A \neq \varnothing$.

Lemma 2.3. *Given a topology space $(X, \mathcal{T})$ and $A \subseteq X$. If A' is the set that only contains all limit points of A, then $A \cup A'$ is closed.*

Proof: To be continued $\blacksquare$

Notation 2.4. $\overline{A}$: closure of A in a topology space $(X, \mathcal{T})$.

Theorem 2.5. *The following definitions are equivalent:*

1. *Let A' denote the set of all limit points of A, then $\overline{A} = A \bigcup A'$.*
2. $\overline{A} = \bigcap\{A \subseteq B : B \text{ is closed in } X\}$

3. *$\overline{A}$ is the smallest closed set containing A.*

Proof:

$1 \Rightarrow 2$: On the one hand, given A, for every $B \in \{A \subseteq B : \text{B is closed in } X\}$, $\overline{A} \subseteq \overline{B}$. Because B is closed, $\overline{A} \subseteq B$ by lemma 2.6. Thus $\overline{A} \subseteq \bigcap\{A \subseteq B : \text{B is closed in } X\}$.
On the other hand, A' is a closed set containing A by lemma 2.3, so $\bigcap\{A \subseteq B : \text{B is closed in } X\} \subseteq \overline{A}$ by 2.
Therefore, $\overline{A} = \bigcap\{A \subseteq B : \text{B is closed in } X\}$.

$2 \Rightarrow 1$: On the one hand, A' is a closed set containing A by lemma 2.3, so $\bigcap\{A \subseteq B : \text{B is closed in } X\} \subseteq \overline{A}$ by 2.
On the other hand, proof by contradiction. We assume $\bigcap\{A \subseteq B : \text{B is closed in } X\} \subset \overline{A}$, then by lemma 2.1 there exists a closed set $B \in \{A \subseteq B : \text{B is closed in } X\}$ such that $A \subseteq B \subset \overline{A}$. Let C denote $\overline{A} \setminus B$, so for each $c \in C$, they are called a limit point of A. However, there exists an open set $X \setminus B$ such that $c \in X \setminus B$ for each $c \in C$ and $(X \setminus B) \cap A = \varnothing$, which violates definition 2.2. Thus the assumption is wrong and $\bigcap\{A \subseteq B : \text{B is closed in } X\} \supseteq \overline{A}$.
Therefore, $\overline{A} = \bigcap\{A \subseteq B : \text{B is closed in } X\}$.

$2 \Rightarrow 3$: lemma 2.1

$3 \Rightarrow 2$: lemma 2.1

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Lemma 2.6. *Given 1, if $A \subseteq B$ in a topology space $(X, \mathcal{T})$, then $\overline{A} \subseteq \overline{B}$.*

Proof: (using lemma 2.3, proof by contradiction) ∎