Dense

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1 Motivation

Dense is a crucial concept in topology and analysis. Several related theorems will be discussed.

2 Separable

Theorem 2.1. *If a normed vector space has a schauder basis, then it is separable.*

Proof: Given a normed vector space $(X, \|\cdot\|_X)$ on field $\mathbb{F}$, we have to construct a subtle subspace $(Y, \|\cdot\|_X)$ and show that $Y \subset X$ is countable and dense.
We use $\{e_n\}$ to denote the schauder basis of $(X, \|\cdot\|_X)$. It is true that there exists a dense and countable subset $\mathbb{G} \subset \mathbb{F}$.
Given basis $\{e_n\}$, we define the subset Y

$$Y = \bigcup_{n\in\mathbb{N}} \{y_n \mid y_n = \sum_{k=1}^{n} g_k e_k \text{ and } g_k \in \mathbb{G}\}$$

The function $\bigcup_{n\in\mathbb{N}} G^n \to Y$ is bijective and G^n is countable for all $n \in \mathbb{N}$. Thus Y is countable.
Next, it suffices to show Y is dense.
Due to schauder basis $\{e_n\}$, for each $x \in X$ and $\varepsilon > 0$, there exist $N_1(\varepsilon)$ and $\{f_k(\varepsilon)\}$ such that $\|x - \sum_{k=1}^{n} f_k e_k\|_X < \varepsilon$ and $f_k \in \mathbb{F}$ if $n > N_1(\varepsilon)$ for n.
Moreover, because $\mathbb{G}$ is dense, for each $f_k \in \mathbb{F}$, $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $g_k(\varepsilon) \in \mathbb{G}$ such that $|f_k - g_k| < \varepsilon$.

Given x and $\{e_n\}$, for each $\varepsilon > 0$, there exist $N_1(\frac{\varepsilon}{2})$, $\{\frac{\varepsilon}{2N_1(\frac{\varepsilon}{2})\|e_k\|_X}\}$ and $\{f_k(\frac{\varepsilon}{2})\}$ such that

$$\begin{aligned}
\|x - \sum_{k=1}^{n} g_k e_k\|_X &= \|x - \sum_{k=1}^{n} f_k e_k + \sum_{k=1}^{n} f_k e_k - \sum_{k=1}^{n} g_k e_k\|_X \\
&\leqslant \|x - \sum_{k=1}^{n} f_k e_k\|_X + \|\sum_{k=1}^{n} f_k e_k - \sum_{k=1}^{n} g_k e_k\|_X \\
&< \frac{\varepsilon}{2} + \sum_{k=1}^{n} |f_k - g_k| \|e_k\|_X \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

if $n > N_1(\frac{\varepsilon}{2})$ and $|f_k - g_k| < \frac{\varepsilon}{2N_1(\frac{\varepsilon}{2})\|e_k\|_X}$ for n and $\{g_k\}$.
Therefore Y is a dense and countable subset and $(X, \|\cdot\|_X)$ is separable. ■

Warning 2.2. The above normed vector space $(X, \|\cdot\|_X)$ is not necessary to be *complete*.

Example 2.3. Given a normed vector space $(X, \|\cdot\|_X)$, $X = C^0[0,1]$ and $\|f\|_X = \|f\|_2 = (\int_0^1 f(t)^2\, dt)^{\frac{1}{2}}$. Then $(X, \|\cdot\|_X)$ is not complete.

Proof: We have the sequence $\{f_n\} \subset X$

$$f_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{2} - \frac{1}{2^n}) \\ (x - \frac{1}{2})2^{n-1} + \frac{1}{2} & x \in [\frac{1}{2} - \frac{1}{2^n}, \frac{1}{2} + \frac{1}{2^n}] \\ 1 & x \in (\frac{1}{2^n} + 2^{-n}, 1] \end{cases}$$

It is obvious that $\{f_n\}$ is convergent to f

$$f(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}) \\ \frac{1}{2} & x = \frac{1}{2} \\ 1 & x \in (\frac{1}{2}, 1] \end{cases}$$

and $f \notin X$. ■