Infinite Series

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1 Introduction

Infinite series is widely used in almost all aspects of mathematics. Several technique in infinite series with examples will be discussed.

2 Example

Example 2.1. Given a sequence $\{x_n\}$ with an upper bound, $x_{n+1} > x_n - o(\frac{1}{n})$ for all $n \in \mathbb{N}$, show that $\{x_n\}$ is convergent.

Proof: Since $\{x_n\}$ has an upper bound and least upper bound property, the supremum of $\{x_n\}$ exists. We denote U is supremum of $\{x_n\}$

$$U = \inf_{k \in \mathbb{N}} \sup_{n \geqslant k} \{x_n\}$$

We construct a new sequence $\{y_n\}$

$$\{y_n\} = \{y_k \mid y_k = \sup_{n \geqslant k} \{x_n\} \text{ for all } k \in \mathbb{N}\}$$

It is obvious that $y_n \geqslant x_n$ and $y_n \geqslant U$ for all $k \in \mathbb{N}$.
In addition, $\{y_n\}$ is convergent to U. In another word, for each $\varepsilon > 0$, there exists $N_1(\varepsilon)$ such that $0 \leqslant y_n - U < \varepsilon$ for all $n > N_1(\varepsilon)$.
Therefore, for each $\varepsilon > 0$, there exists $N_1(\varepsilon)$ such that $x_n - U < \varepsilon$ for all $n > N_1(\varepsilon)$.
Since $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent with $\mathrm{Re}(s) > 1$, we have

$$x_n - x_m < O(\frac{1}{n}) \text{ for all } m > n$$

Because U is supremum of $\{x_n\}$, we might have two conditions:

1. If for every $n \in \mathbb{N}$, there exists $m > n$ such that $x_m \geqslant U$. Then for each $\varepsilon > 0$, there exists $n = N_2(\varepsilon) \in \mathbb{N}$ such that $\frac{C}{N_2(\varepsilon)} < \varepsilon$ with constant C and $0 \leqslant x_n - U < \varepsilon$.
   Then for each $\varepsilon > 0$, there exists $n = N_2(\varepsilon)$ such that $U - x_m \leqslant x_n - x_m < O(\frac{1}{n}) < \frac{C}{N_2(\varepsilon)} < \varepsilon$ for all $m > n$.
   Therefore, for each $\varepsilon > 0$, there exists $N_1(\varepsilon)$ and $N_2(\varepsilon)$ such that $|x_n - U| < \varepsilon$ for all $n > \max\{N_1(\varepsilon), N_2(\varepsilon)\}$.

2. If there exists $L \in \mathbb{N}$ such that $x_k < U$ for all $k \geqslant L$. Then we have an infinite sub-sequence $\{z_n\}$

   $$\{z_n\} = \{x_k \mid x_k \in \{x_n\} \text{ and } k \geqslant L\}$$

   and

   $$\inf_{k\in\mathbb{N}} \sup_{n\geqslant k}\{z_n\} = \inf_{k\in\mathbb{N}} \sup_{n\geqslant k}\{x_n\} = U$$

   Thus for each $\varepsilon > 0$, there exists $n = N_3(\varepsilon)$ such that $U - z_n < \varepsilon$ and $z_n - z_m < O(\frac{1}{n}) \leqslant \frac{C}{n}$ with constant C for all $m > n$.
   Therefore for each $\varepsilon > 0$, there exists $n = \max\{\lceil\frac{2C}{\varepsilon}\rceil, N_3(\frac{\varepsilon}{2})\}$ such that C is constant and $U - z_m = U - z_n + z_n - z_m < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $m > n$.
   Therefore for each $\varepsilon > 0$, there exists $n = \max\{\lceil\frac{2C}{\varepsilon}\rceil + L, N_3(\frac{\varepsilon}{2}) + L\}$ such that C is constant and $|U - x_m| < \varepsilon$ for all $m > n$.

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