Equivalence of axiom of choice

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1 Motivation

In this notes, rigorous proofs of some statements, which are equivalent with axiom of choice, will be shown .

2 Axiom of choice

Statement 2.1. (Axiom of choice) *For every non-empty set $\mathcal{A}$, there exists a choice function $f : \mathcal{P}(\mathcal{A}) \to \mathcal{A}$, and $f(a) \in a$ for each $a \in \mathcal{P}(\mathcal{A})$.*

3 Hausdorff maximal principle

The following lemma is a revised version of (Nico Spronk, 2019).

Definition 3.1. Given a non-empty set $\mathcal{A}$ and $\mathcal{F} \subseteq \mathcal{P}(\mathcal{A})$, we define $\mathcal{B}^* = \{x \in \mathcal{A} \setminus \mathcal{B} : \mathcal{B} \cup \{x\} \in \mathcal{F}\}$ for every $\mathcal{B} \in \mathcal{F}$.

Definition 3.2. Given a choice function f in Statement 2.1 and $\mathcal{F}$ in Definition 3.1, we define a function $F : \mathcal{F} \to \mathcal{F}$

$$F(\mathcal{B}) = \begin{cases} \mathcal{B} \cup f(\mathcal{B}^*) & \text{if } \mathcal{B}^* \neq \varnothing \\ \mathcal{B} & \text{otherwise} \end{cases}$$

Definition 3.3. Given a non-empty set X, a sub-collection $\mathcal{F} \subseteq \mathcal{P}(X)$ and function F: $\mathcal{F} \to \mathcal{F}$, a subset $\mathcal{T} \subseteq \mathcal{F}$ is called a *tower* of $\mathcal{F}$ if and only if $\mathcal{T}$ has the following three properties:

$$\varnothing \in \mathcal{T} \tag{3.3.1}$$

$$\text{if } \mathcal{A} \in \mathcal{T}, \text{ then } F(\mathcal{A}) \in \mathcal{T} \tag{3.3.2}$$

$$\text{if } \mathcal{A} \subset \mathcal{T} \text{and } (\mathcal{A}, \subseteq) \text{ is a chain, then } \bigcup_{a \in \mathcal{A}} a \in \mathcal{T} \tag{3.3.3}$$

Lemma 3.4. *Given a non-empty set X, if a sub-collection $\mathcal{F} \subseteq \mathcal{P}(X)$ has the following two properties:*

$$\varnothing \in \mathcal{F} \tag{3.4.1}$$

$$\textit{if } \mathcal{A} \subset \mathcal{F} \textit{and } (\mathcal{A}, \subseteq) \textit{ is a chain, then } \bigcup_{a \in \mathcal{A}} a \in \mathcal{F} \tag{3.4.2}$$

then there is $\mathcal{M} \in \mathcal{F}$ such that $\mathcal{M} \cup \{x\} \notin \mathcal{F}$ for every $x \in X \setminus \mathcal{M}$.

Proof: Given $\mathcal{F}$ and a function F, we define

$$\mathcal{T}_0 = \bigcap \{\mathcal{T} \subseteq \mathcal{F} : \mathcal{T} \text{ is a tower of } \mathcal{F}\}$$

We can verify $\mathcal{T}_0$ is a tower by mathematical induction.
Next our target is to prove $(\mathcal{T}_0, \subseteq)$ is a totally ordered set.
We find an element $C \in \mathcal{T}_0$ such that C is a comparable element, i.e. for each $t \in \mathcal{T}_0 \setminus \{C\}$, $t \subset C$ or $t \supset C$. Indeed C is well defined, because $\varnothing \in \mathcal{T}_0$.
From C, we define

$$\mathcal{T}_C = \{t \in \mathcal{T}_0 : t \subset C\} \bigcup \{C\} \bigcup \{F(C)\} \bigcup \{t \in \mathcal{T}_0 : F(C) \subset t\}$$

It is trivial to see that $\mathcal{T}_C \subseteq \mathcal{T}_0$.
We can prove $\mathcal{T}_C$ is a tower by definition 3.3.
(3.3.1): $\varnothing \subseteq C$, so $\varnothing \in \mathcal{T}_C$.
(3.3.2): For each $x \in \mathcal{T}_C$, $x \in \mathcal{T}_0$ and $F(x) \in \mathcal{T}_0$, because $\mathcal{T}_C \subseteq \mathcal{T}_0$. If $x \subset C$, then $F(x) \subseteq C$, thus $F(x) \in \mathcal{T}_C$; if $x \supset C$, then $F(C) \subseteq F(x)$, thus $F(x) \in \mathcal{T}_C$.
(3.3.3): If there is a chain $(\mathcal{D}, \subset)$ in $\mathcal{T}_C$ and $e = \bigcup_{d \in \mathcal{D}} d$, then $e \in \mathcal{T}_0$, because $\mathcal{T}_C \subseteq \mathcal{T}_0$. If $d \subseteq C$ for each $d \in \mathcal{D}$, then $e \subseteq C$, thus $e \in \mathcal{T}_C$; if $C \subseteq d$ for some $d \in \mathcal{D}$, then $C \subseteq e$, then $F(C) \subseteq e$, thus $e \in \mathcal{T}_C$.
We can get $\mathcal{T}_C = \mathcal{T}_0$ because $\mathcal{T}_0$ is the minimal tower.
Define

$$U = \{C \in \mathcal{T}_0 : C \text{ is a comparable element}\}$$

It is trivial to see that $U \subseteq \mathcal{T}_0$ and $(U, \subset)$ is a well ordered set.
We can prove U is a tower by definition 3.3.

(3.3.1): $\varnothing$ is subset of any set.
(3.3.2): From definition of $\mathcal{T}_C$, we know $F(x)$ is a comparable element in $\mathcal{T}_C$ for any comparable element x in $\mathcal{T}_C$, so $F(x)$ is a comparable element in $\mathcal{T}_0$ as well.
(3.3.3): Given any chain $(\mathcal{D}, \subset)$ in U, $e = \bigcup_{d \in \mathcal{D}} d$, $e \in \{d : d \in \mathcal{D}\} \cup \{F(d) : d \in \mathcal{D}\} \subseteq U$.
Therefore, U is a tower. We finish proving $U = \mathcal{T}_0$ and $(\mathcal{T}_0, \subset)$ is a totally ordered set.
Define

$$\mathcal{M} = \bigcup_{t \in \mathcal{T}_0} t$$

Because $(\mathcal{T}_0, \subset)$ is a chain, $\mathcal{M} \in \mathcal{T}_0$ by definition of $\mathcal{T}_0$. Suppose $\mathcal{M}^* \neq \varnothing$, then $\mathcal{M} \subset F(\mathcal{M}) \in \mathcal{T}_0$, which would violate the definition of $\mathcal{M}$. Therefore, $\mathcal{M}^* = \varnothing$ and $\mathcal{M}$ is what we want. ■

Statement 3.5. (Hausdorff maximal principle) *For every partially ordered set $(\mathcal{S}, \leq)$, there is a chain $\mathcal{M}$, such that $\mathcal{M} \cup \{s\}$ is not a chain for every $s \in \mathcal{S} \setminus \mathcal{M}$.*

Theorem 3.6. *Axiom of choice implies Hausdorff maximal principle.*

Proof: Given a partially ordered set $(\mathcal{S}, \subseteq)$ and let $\mathcal{F}$ denote the set of all chains in $\mathcal{S}$. We can verify that $\mathcal{F}$ has two properties in Lemma 3.4:
(3.4.1): $\varnothing$ is a chain.
(3.4.2): Given any chain $(\mathcal{D}, \subseteq)$ in $\mathcal{F}$ and $e = \bigcup_{d \in \mathcal{D}} d$. $e \in \mathcal{F}$ because e is also a chain.
Therefore, the chain $\mathcal{M}$ can be found like Lemma 3.4. ■

4 Zorn's Lemma

Statement 4.1. (Zorn's Lemma) *Given a partially ordered set $(\mathcal{S}, \subseteq)$, if every chain of $(\mathcal{S}, \subseteq)$ has an upper bound, then there is a maximal element m for $\mathcal{S}$.*

Theorem 4.2. *Hausdorff maximal principle implies Zorn's Lemma.*

Proof: Let $(\mathcal{S}, \subseteq)$ be the partially ordered set, then we can get a maximal chain $\mathcal{M}$ in $(\mathcal{S}, \subseteq)$ by Hausdorff maximal principle. Let m be an upper bound for $\mathcal{M}$. Then $\mathcal{M} \cup \{m\} = \mathcal{M}$. Therefore, m is a maximal element for $\mathcal{S}$. ■

5 Well-ordering principle

To be continued

6 Tychonoff's theorem

To be continued

7 Every vector space has a basis

Statement 7.1. (Every vector space has a basis) *Let K be a field and V be a vector space over K. Then V has a basis.*

Theorem 7.2. *Hausdorff maximal principle implies that every vector space has a basis.*

Proof: Let $\mathcal{A}$ be a collection of all linearly independent subsets of V.

$$\mathcal{A} = \{W \mid W \subseteq V \text{ and } W \text{ is a linearly independent subset}\}$$

It is obvious that $(\mathcal{A}, \subset)$ is a partially order set. From hausdorff maximal principle, we use $\mathcal{M}$ to denote the maximal chain in $\mathcal{A}$.

$$\mathcal{S} = \bigcup_{m \in \mathcal{M}} m$$

We use $\mathcal{N}$ to denote the span of $\mathcal{S}$. It is true that $\mathcal{S} \subseteq \mathcal{N} \subseteq V$.
Suppose $\mathcal{N} \subset V$, then there exists $v \in \mathcal{A}$ and $v \in V$ such that $v \notin \mathcal{S}$ and cannot be a linear combination of the elements in $\mathcal{S}$. So $\mathcal{S} \cup \{v\} \in \mathcal{A}$ and $\mathcal{M} \cup \{\mathcal{S} \cup \{v\}\}$ is a chain in $\mathcal{A}$ such that $\mathcal{M} \subset \mathcal{M} \cup \{\mathcal{S} \cup \{v\}\}$, which violates hausdorff maximal principle. Thus the assumption is wrong.
Therefore, $\mathcal{N} = V$ and $\mathcal{S}$ is the basis of V. ∎

References

Nico Spronk (2019). Axiom of choice et al. `http://www.math.uwaterloo.ca/~nspronk/math453/AofC.pdf`.