

Equivalence of axiom of choice

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1 Motivation

In this notes, rigorous proofs of some statements, which are equivalent with axiom of choice, will be shown .

2 Axiom of choice

Statement 2.1. (Axiom of choice) *For every non-empty set \mathcal{A} , there exists a choice function $f : \mathcal{P}(\mathcal{A}) \rightarrow \mathcal{A}$, and $f(a) \in a$ for each $a \in \mathcal{P}(\mathcal{A})$.*

3 Hausdorff maximal principle

The following lemma is a revised version of (Nico Spronk, 2019).

Definition 3.1. Given a non-empty set \mathcal{A} and $\mathcal{F} \subseteq \mathcal{P}(\mathcal{A})$, we define $\mathcal{B}^* = \{x \in \mathcal{A} \setminus \mathcal{B} : \mathcal{B} \cup \{x\} \in \mathcal{F}\}$ for every $\mathcal{B} \in \mathcal{F}$.

Definition 3.2. Given a choice function f in [Statement 2.1](#) and \mathcal{F} in [Definition 3.1](#), we define a function $F : \mathcal{F} \rightarrow \mathcal{F}$

$$F(\mathcal{B}) = \begin{cases} \mathcal{B} \cup f(\mathcal{B}^*) & \text{if } \mathcal{B}^* \neq \emptyset \\ \mathcal{B} & \text{otherwise} \end{cases}$$

Definition 3.3. Given a non-empty set X , a sub-collection $\mathcal{F} \subseteq \mathcal{P}(X)$ and [function \$F\$](#) : $\mathcal{F} \rightarrow \mathcal{F}$, a subset $\mathcal{T} \subseteq \mathcal{F}$ is called a *tower* of \mathcal{F} if and only if \mathcal{T} has the following three properties:

$$\emptyset \in \mathcal{T} \tag{3.3.1}$$

$$\text{if } \mathcal{A} \in \mathcal{T}, \text{ then } F(\mathcal{A}) \in \mathcal{T} \quad (3.3.2)$$

$$\text{if } \mathcal{A} \subset \mathcal{T} \text{ and } (\mathcal{A}, \subseteq) \text{ is a chain, then } \bigcup_{a \in \mathcal{A}} a \in \mathcal{T} \quad (3.3.3)$$

Lemma 3.4. *Given a non-empty set X , if a sub-collection $\mathcal{F} \subseteq \mathcal{P}(X)$ has the following two properties:*

$$\emptyset \in \mathcal{F} \quad (3.4.1)$$

$$\text{if } \mathcal{A} \subset \mathcal{F} \text{ and } (\mathcal{A}, \subseteq) \text{ is a chain, then } \bigcup_{a \in \mathcal{A}} a \in \mathcal{F} \quad (3.4.2)$$

then there is $\mathcal{M} \in \mathcal{F}$ such that $\mathcal{M} \cup \{x\} \notin \mathcal{F}$ for every $x \in X \setminus \mathcal{M}$.

Proof: Given \mathcal{F} and a **function** F , we define

$$\mathcal{T}_0 = \bigcap \{ \mathcal{T} \subseteq \mathcal{F} : \mathcal{T} \text{ is a tower of } \mathcal{F} \}$$

We can verify \mathcal{T}_0 is a tower by mathematical induction.

Next our target is to prove $(\mathcal{T}_0, \subseteq)$ is a totally ordered set.

We find an element $C \in \mathcal{T}_0$ such that C is a comparable element, i.e. for each $t \in \mathcal{T}_0 \setminus \{C\}$, $t \subset C$ or $t \supset C$. Indeed C is well defined, because $\emptyset \in \mathcal{T}_0$. From C , we define

$$\mathcal{T}_C = \{t \in \mathcal{T}_0 : t \subset C\} \cup \{C\} \cup \{F(C)\} \cup \{t \in \mathcal{T}_0 : F(C) \subset t\}$$

It is trivial to see that $\mathcal{T}_C \subseteq \mathcal{T}_0$.

We can prove \mathcal{T}_C is a tower by **definition 3.3**.

(3.3.1): $\emptyset \subseteq C$, so $\emptyset \in \mathcal{T}_C$.

(3.3.2): For each $x \in \mathcal{T}_C$, $x \in \mathcal{T}_0$ and $F(x) \in \mathcal{T}_0$, because $\mathcal{T}_C \subseteq \mathcal{T}_0$. If $x \subset C$, then $F(x) \subseteq C$, thus $F(x) \in \mathcal{T}_C$; if $x \supset C$, then $F(C) \subseteq F(x)$, thus $F(x) \in \mathcal{T}_C$.

(3.3.3): If there is a chain (\mathcal{D}, \subset) in \mathcal{T}_C and $e = \bigcup_{d \in \mathcal{D}} d$, then $e \in \mathcal{T}_0$, because $\mathcal{T}_C \subseteq \mathcal{T}_0$. If $d \subseteq C$ for each $d \in \mathcal{D}$, then $e \subseteq C$, thus $e \in \mathcal{T}_C$; if $C \subseteq d$ for some $d \in \mathcal{D}$, then $C \subseteq e$, then $F(C) \subseteq e$, thus $e \in \mathcal{T}_C$.

We can get $\mathcal{T}_C = \mathcal{T}_0$ because \mathcal{T}_0 is the minimal tower.

Define

$$U = \{C \in \mathcal{T}_0 : C \text{ is a comparable element}\}$$

It is trivial to see that $U \subseteq \mathcal{T}_0$ and (U, \subset) is a well ordered set.

We can prove U is a tower by **definition 3.3**.

(3.3.1): \emptyset is subset of any set.

(3.3.2): From definition of \mathcal{T}_C , we know $F(x)$ is a comparable element in \mathcal{T}_C for any comparable element x in \mathcal{T}_C , so $F(x)$ is a comparable element in \mathcal{T}_0 as well.

(3.3.3): Given any chain (\mathcal{D}, \subset) in U , $e = \bigcup_{d \in \mathcal{D}} d$, $e \in \{d : d \in \mathcal{D}\} \cup \{F(d) : d \in \mathcal{D}\} \subseteq U$.

Therefore, U is a tower. We finish proving $U = \mathcal{T}_0$ and (\mathcal{T}_0, \subset) is a totally ordered set.

Define

$$\mathcal{M} = \bigcup_{t \in \mathcal{T}_0} t$$

Because (\mathcal{T}_0, \subset) is a chain, $\mathcal{M} \in \mathcal{T}_0$ by definition of \mathcal{T}_0 . Suppose $\mathcal{M}^* \neq \emptyset$, then $\mathcal{M} \subset F(\mathcal{M}) \in \mathcal{T}_0$, which would violate the definition of \mathcal{M} . Therefore, $\mathcal{M}^* = \emptyset$ and \mathcal{M} is what we want. ■

Statement 3.5. (Hausdorff maximal principle) *For every partially ordered set (\mathcal{S}, \leq) , there is a chain \mathcal{M} , such that $\mathcal{M} \cup \{s\}$ is not a chain for every $s \in \mathcal{S} \setminus \mathcal{M}$.*

Theorem 3.6. *Axiom of choice implies Hausdorff maximal principle.*

Proof: Given a partially ordered set (\mathcal{S}, \subseteq) and let \mathcal{F} denote the set of all chains in \mathcal{S} . We can verify that \mathcal{F} has two properties in Lemma 3.4:

(3.4.1): \emptyset is a chain.

(3.4.2): Given any chain (\mathcal{D}, \subseteq) in \mathcal{F} and $e = \bigcup_{d \in \mathcal{D}} d$. $e \in \mathcal{F}$ because e is also a chain.

Therefore, the chain \mathcal{M} can be found like Lemma 3.4. ■

4 Zorn's Lemma

Statement 4.1. (Zorn's Lemma) *Given a partially ordered set (\mathcal{S}, \subseteq) , if every chain of (\mathcal{S}, \subseteq) has an upper bound, then there is a maximal element m for \mathcal{S} .*

Theorem 4.2. *Hausdorff maximal principle implies Zorn's Lemma.*

Proof: Let (\mathcal{S}, \subseteq) be the partially ordered set, then we can get a maximal chain \mathcal{M} in (\mathcal{S}, \subseteq) by Hausdorff maximal principle. Let m be an upper bound for \mathcal{M} . Then $\mathcal{M} \cup \{m\} = \mathcal{M}$. Therefore, m is a maximal element for \mathcal{S} . ■

5 Well-ordering principle

To be continued

6 Tychonoff's theorem

To be continued

7 Every vector space has a basis

Statement 7.1. (Every vector space has a basis) *Let K be a field and V be a vector space over K . Then V has a basis.*

Theorem 7.2. *Hausdorff maximal principle implies Statement 7.1.*

Proof: Let \mathcal{A} be a collection of all linearly independent subsets of V .

$$\mathcal{A} = \{W \mid W \subseteq V \text{ and } W \text{ is a linearly independent subset}\}$$

It is obvious that (\mathcal{A}, \subset) is a partially order set. From **hausdorff maximal principle**, we use \mathcal{M} to denote the maximal chain in \mathcal{A} .

$$\mathcal{S} = \bigcup_{m \in \mathcal{M}} m$$

We use \mathcal{N} to denote the span of \mathcal{S} . It is true that $\mathcal{S} \subseteq \mathcal{N} \subseteq V$. Suppose $\mathcal{N} \subset V$, then there exists $v \in V$ such that $v \in \mathcal{A}$ and $v \notin \mathcal{S}$. In another word, v cannot be a linear combination of the elements in \mathcal{S} . So $\mathcal{S} \cup \{v\} \in \mathcal{A}$ and $\mathcal{M} \cup \{\mathcal{S} \cup \{v\}\}$ is a chain in \mathcal{A} such that $\mathcal{M} \subset \mathcal{M} \cup \{\mathcal{S} \cup \{v\}\}$, which violates **hausdorff maximal principle**. Thus the assumption is wrong. Therefore, $\mathcal{N} = V$ and \mathcal{S} is the basis of V . ■

References

Nico Spronk (2019). Axiom of choice et al. <http://www.math.uwaterloo.ca/~nspronk/math453/AofC.pdf>.