General Topology

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Contents

Chapter 0

Preliminary

1. Map

Theorem 0.1. *There is a bijective map between* $(0,1]$ *and* $[0,1]$.

Chapter 1

Topology Space

1. Topology Space

Definition 1.1. (open set)

Definition 1.2. (closed set) Complement of open set.

Definition 1.3. (topology space)

Definition 1.4 (topological subspace)**.**

Theorem 1.5 (subspace generated by a subset)**.**

Definition 1.6. (basis)

Definition 1.7. (neighborhood)

2. Set

Definition 1.8. (limit point of set)

Definition 1.9. (derived set)

Definition 1.10. (adherent point)

Definition 1.11. (isolation point)

Definition 1.12. (interior point)

Definition 1.13. (boundary point)

Warning 1.14. limit point $\implies$ adherent point, but adherent point $\not\Longrightarrow$ limit point.

Example 1.15. to-do

Theorem 1.16. *Given topology space $(X, \mathcal{T})$ and $Y \subseteq X$. Y is closed if and only if Y is derived set.*

Proof. Proof by contradiction. □

Definition 1.17. (closure)

Theorem 1.18. *Given topology space $(X, \mathcal{T})$, the following statements are equivalent:*

Definition 1.19 (dense set)**.**

Definition 1.20 (separable space)**.**

3. Open Cover

Definition 1.21. (open cover)

Definition 1.22. (compact space)

Definition 1.23. (precompact space)

Theorem 1.24. *Given a topology space $(X, \mathcal{T})$ and $Y, K \subset X$. If Y is closed and K is compact, then $Y \cap K$ is compact.*

Theorem 1.25 (Cantor's intersection theorem)**.**

Proof. to do □

4. Sequence

Definition 1.26. (sequence)

Definition 1.27. (subsequence)

Definition 1.28. (limit point of sequence)

Warning 1.29. may not be the same.

Example 1.30. to do

Definition 1.31. (limit of sequence)

Warning 1.32. may not be the same.

Example 1.33. to do

Warning 1.34. In some topology space, the limit of a sequence may not be unique.

Example 1.35. to do

Theorem 1.36. *In any metric space, the limit of any sequence is unique.*

Proof. Proof by contradiction. □

Theorem 1.37. *Given a topology space $(X, \mathcal{T})$ and the sequence $\{x_n\} \in X$. If $\{x_n\}$ converges to $x \in X$, then any subsequence $\{y_n\}$ in $\{x_n\}$ converges to x as well.*

Corollary 1.38. *Given a topology space $(X, \mathcal{T})$ and the sequence $\{x_n\} \in X$. Let $\{y_n\}$ and $\{z_n\}$ denote two different subsequence in $\{x_n\}$. If $\{y_n\}$ and $\{z_n\}$ converge to $y \in X$ and $z \in X$ respectively with $y \neq z$, then $\{x_n\}$ is not a convergent sequence.*

5. Metric Space

Definition 1.39. (metric)

Definition 1.40. (metric space)

Theorem 1.41. *Every metric space (X, d) can generate a topology space $(X, \mathcal{T}_d)$.*

Theorem 1.42. *Any compact metric space is separable.*

Theorem 1.43. *Given a compact metric space (X, d) and $Y \subseteq X$. If Y is closed, then Y is compact.*

Definition 1.44. (Cauchy sequence)

Definition 1.45. (convergence of sequence)

Theorem 1.46. *Given a Cauchy sequence. If it has a convergent subsequence, then the Cauchy sequence is convergent.*

Proof. □

Definition 1.47. (complete space)

Theorem 1.48. *Given a complete metric space (X, d) and a subset $Y \subset X$, Y is complete if and only if Y is closed.*

Warning 1.49. Given a non-complete metric space (X, d) and a subset $Y \subset X$, Y is complete implies Y is closed, but Y is closed cannot imply Y is complete.

Example 1.50. to do

Definition 1.51 (sequential compact)**.**

Definition 1.52. (bounded space)

Definition 1.53. (totally bounded space)

Warning 1.54. Not every bounded space is a totally bounded space.

Example 1.55. to do

Theorem 1.56. *totally bounded $\Longrightarrow$ separable.*

Theorem 1.57. *Given a metric space (X, d), the following statements are equivalent:*

Lemma 1.58. *sequential compact $\Longrightarrow$ totally bounded.*

Proof. We prove it by contradiction. Suppose it is not totally bounded, then $\exists \varepsilon$ such that X cannot be covered by finite open balls.

So we can find an infinite sequence $\{x_i\}$ in X such that $d(x_i, x_j) \geq \varepsilon$, $\forall i, j \in \mathbb{N}$ and $i \neq j$. Otherwise, X is totally bounded.

Hence $\{x_i\}$ has no convergent subsequence. So X is not sequential compact, which is a contradiction. □

Lemma 1.59. *sequential compact $\Longrightarrow$ complete.*

Proof. □

Lemma 1.60. *Given a metric space (X, d) and $Y \subseteq X$. Y is compact $\iff$ Y is sequential compact.*

Proof. □

Lemma 1.61. *Given a metric space (X, d) and $Y \subseteq X$. Y is compact $\iff$ Y is complete and totally bounded.*

Proof. □

Theorem 1.62. *Given a metric space (X, d) and $Y \subseteq X$. Y compact space $\iff$ Y is sequential compact $\iff$ Y is complete and totally bounded.*

Proof. □

6. Normed Vector Space

Definition 1.63 (norm)**.**

Definition 1.64 (normed vector space)**.**

Theorem 1.65. *Every normed vector space $(X, \|\cdot\|)$ can generate a metric space $(X, \|\cdot\|_d)$.*

Definition 1.66 (linear subspace, vector subspace)**.**

Corollary 1.67. *Given a normed vector space $(X, \|\cdot\|)$. Every singleton set in X is closed.*

Definition 1.68 (equivalent norm)**.**

Corollary 1.69 (equivalent norm is an equivalent relation)**.**

Lemma 1.70 ($\|\cdot\|_1$ is a norm)**.**

Notation 1.71 (L_1 norm)**.** $\|\cdot\|_1$

Lemma 1.72. *In any finite normed vector space, any norm $\|\cdot\|_a$ is continuous under $\|\cdot\|_1$.*

Proof. It suffice to prove that $\forall \varepsilon \in \mathbb{R}^+, \exists \delta(\varepsilon) \in \mathbb{R}^+$, such that $\|x_1 - x_2\|_1 < \delta \implies |\|x_1\|_a - \|x_2\|_a| < \varepsilon$, $\forall x_1, x_2 \in X$.

Firstly, it is obvious that $\forall x_1, x_2 \in X$, $|\|x_1\|_a - \|x_2\|_a| < \|x_1 - x_2\|_a$.

Then $\forall x_1, x_2 \in X$:

$$\begin{aligned} \|x_1 - x_2\|_a &\le \sum_{i=1}^{n} \left|\alpha_1^i - \alpha_2^i\right| \left\|e^i\right\|_a \\ &\le \max_i \{\left\|e^i\right\|_a\} \sum_{i=1}^{n} \left|\alpha_1^i - \alpha_2^i\right| \\ &\le \|x_1 - x_2\|_1 \max_i \{\left\|e^i\right\|_a\} \end{aligned}$$

Define $\delta \coloneqq \frac{\varepsilon}{\max_i\{\|e^i\|_a\}}$. It is well defined, due to finite normed vector space.

Therefore, $\forall \varepsilon \in \mathbb{R}^+$, $\exists \delta \in \mathbb{R}^+$, such that $\|x_1 - x_2\|_1 < \delta \implies |\|x_1\|_a - \|x_2\|_a| < \varepsilon$, $\forall x_1, x_2 \in X$. $\square$

Theorem 1.73. *In every finite normed vector space, all norms are equivalent.*

Proof. Define $S \coloneqq \{x \in X\colon \|x\|_1 = 1\}$. Because S is closed and bounded in finite normed vector space X, S is compact.

It suffices to prove that $\exists C_{min}, C_{max} \in \mathbb{R}^+$ such that $\forall x \in S$, $C_{min} \leq \|x\|_a \leq C_{max}$.

Due to Lemma 1.72, $\|\cdot\|_a$ is a continuous function. Because of Extreme Value Theorem, it has minimum A_{min} and maximum A_{max} on compact set S. In another word, $A_{min} \leq \|\cdot\|_a \leq A_{max}$. $\square$

Definition 1.74. (Banach space)

7. Inner Product Space

Definition 1.75 (inner product space)**.**

Definition 1.76 (Hilbert space)**.**

Chapter 2

Continuous and Homeomorphism

Definition 2.1. (continuous map)

Definition 2.2. (homeomorphism)

Definition 2.3. (pointwise continuity)

Definition 2.4. (uniformly continuity)

Theorem 2.5. *Uniformly continuity implies pointwise continuity.*

Theorem 2.6. *(Dini's theorem)*

Chapter 3

Separation Axiom

Chapter 4

Product Space

1. Finite Product Space

Definition 4.1. (product topology)

2. Countable Product Space

Chapter 5

Quotient Space

Definition 5.1. (quotient map)

Definition 5.2. (quotient space)

Bibliography