Functional Analysis

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Contents

Chapter 0

Preliminary

Definition 0.1 (equivalent norm)**.**

Chapter 1

Function Space

1. Continuous Function Space

Notation 1.1 (space of continuous function)**.** $(C(K), \|\cdot\|_\infty)$, where K is a compact set.

Definition 1.2 (equicontinuity)**.**

Lemma 1.3. *Given* $(C(K), \|\cdot\|_\infty)$ *and* $Y \subset C(K)$. Y *is equicontinuity and uniformly bounded* $\iff$ Y *is totally bounded.*

Theorem 1.4 (Arzela-Ascoli theorem)**.**

Definition 1.5 (Hölder continuity)**.**

Theorem 1.6. *Hölder continuity* $\implies$ *equicontinuity.*

2. L^p Space

Definition 1.7 (L^p space)**.**

Theorem 1.8 (L^2 is a Hilbert space)**.**

Proof. to do □

Theorem 1.9 (L^2 is separable)**.**

Proof. to do □

Chapter 2

Operator

1. Basic

Definition 2.1 (kernel)**.**

Definition 2.2 (range/image)**.**

Definition 2.3 (operator norm)**.**

Theorem 2.4 (equivalent definition of operator norm)**.** *The following statements are equivalent:*

Proof. to do □

Warning 2.5. Not all operator has its norm.

2. Linear Operator

Definition 2.6 (linear operator)**.**

Notation 2.7 (set of linear operator)**.** $\mathcal{L}(X, Y)$

Definition 2.8 (linear bounded operator)**.**

Notation 2.9 (set of linear bounded operator)**.** $\mathcal{B}(X, Y)$

Theorem 2.10. *Given normed vector spaces X, Y and a linear operator $T \colon X \to Y$. $\ker T$ is a linear subspace in X.*

Theorem 2.11. *Given normed vector spaces X, Y and a linear operator $T \colon X \to Y$. $T(X)$ is a linear subspace in Y.*

Theorem 2.12. *Given normed vector spaces X, Y and a linear operator $T \colon X \to Y$. T is injective $\iff \ker T = \{0\}$.*

Theorem 2.13. *Given finite normed vector spaces X, Y and a linear operator $T \colon X \to Y$. T is injective $\iff$ T is surjective.*

Proof. to do □

3. Bounded Operator

Definition 2.14 (bounded operator)**.**

Theorem 2.15 (equivalent definition of linear bounded operator)**.** *Given normed vector spaces X, Y and a linear operator $T: X \to Y$. The following statements are equivalent:*

Proof. to do $\square$

Theorem 2.16. *Given a normed vector spaces X, Y and a linear bounded operator $T: X \to Y$. $\ker T$ is always closed.*

Sketch of the Proof. Because every singleton set in a normed vector space is closed. $\square$

Theorem 2.17. *Given normed vector spaces X, Y, Z and operators $T: X \to Y, S: Y \to Z$. If S, T are bounded, then $\|ST\| \leq \|T\|\|S\|$.*

Corollary 2.18. *Given normed vector spaces X, Y and an operator $T: X \to Y$. $\forall n \in \mathbb{N}$, $\|T^n\| \leq \|T\|^n$.*

Theorem 2.19. *Given any normed vector space X, Y and an operator $T: X \to Y$. If X is finite and T is linear, then T is always bounded.*

Proof. to do $\square$

Theorem 2.20 (set of bounded linear operators is Banach)**.** *Given normed vector spaces X, Y. If Y is Banach, then $\mathcal{B}(X, Y)$ is Banach.*

Proof. We pick any Cauchy sequence $\{T_n\}$ in $\mathcal{B}(X, Y)$. Then, it is obvious that $\forall \varepsilon \in \mathbb{R}^+$, $\exists N(\varepsilon) \in \mathbb{N}$, such that $\|T_i - T_j\| < \varepsilon$, $\forall i, j > N$.

Also, it is true that $\forall x \in X$, $\forall n, m \in \mathbb{N}$, $\|T_n x - T_m x\|_Y = \|(T_n - T_m)x\|_Y \leq \|T_n - T_m\|\|x\|_X$.

So it is true that $\forall x \in X$, $\{T_n x\}$ is a Cauchy sequence in Y. Because Y is complete, $\forall x \in X$, $\exists y \in Y$ such that $y = \lim_{n \to +\infty} T_n x$. In another word, there exists a well defined operator T:

$$\begin{aligned} T \colon X &\to Y \\ x &\mapsto Tx = \lim_{n \to +\infty} T_n x \end{aligned}$$

Define $S \coloneqq \{x \in X : \|x\|_X = 1\}$.

Next, we will prove that $T \in \mathcal{B}(X, Y)$.

Define $M \coloneqq \sup_{n \in \mathbb{N}} \|T_n\|$, and it is true that $M < +\infty$. It is known that $\forall \varepsilon \in \mathbb{R}^+$, $\forall x \in X$, $\exists N(\varepsilon, x) \in \mathbb{N}$, such that $\|Tx - T_n x\|_Y < \varepsilon$, $\forall n > N$. So $\forall \varepsilon \in \mathbb{R}^+$, $\forall x \in S$, $\forall n > N(\varepsilon, x)$, $\|Tx\|_Y = \|Tx - T_n x + T_n x\|_Y \leq \|Tx - T_n x\|_Y + \|T_n x\|_Y \leq \varepsilon + \|T_n\|\|x\|_X \leq M + \varepsilon$. So T is bounded. In addition, it is trivial to show that T is linear.

Next, we will prove that $\lim_{n \to +\infty} \|T - T_n\| = 0$.

It is true that $\forall n \in \mathbb{N}$, $\|T - T_n\| = \sup_{x \in S} \|(T - T_n)x\|_Y$. Also, it is true that $\forall x \in S$, $\lim_{n \to +\infty} \|(T - T_n)x\|_Y = 0$. So, $\lim_{n \to +\infty} \|T - T_n\| = 0$. $\square$

Corollary 2.21. *Dual space of any normed vector spaces is a Banach space.*

Theorem 2.22 (extension theorem)**.**

Proof. to do □

Theorem 2.23 (open mapping theorem)**.**

Proof. to do □

Theorem 2.24 (bounded inverse theorem)**.**

Proof. to do □

Theorem 2.25 (uniform boundedness principle)**.**

Proof. to do □

Theorem 2.26 (等价范数定理)**.**

Proof. to do □

Definition 2.27 (closed graph)**.**

Theorem 2.28 (closed graph theorem)**.**

Proof. to do □

4. Unbounded Operator

5. Compact Operator

Definition 2.29 (compact operator)**.**

Notation 2.30 (set of linear compact operator)**.** $\mathcal{K}(X, Y)$

Theorem 2.31. *$\mathcal{K}(X, Y)$ is closed in $\mathcal{B}(X, Y)$.*

Proof. to do □

Corollary 2.32. *Given a sequence of compact operator $\{K_n\}$, If it is convergent to K. then K is a compact operator.*

Example 2.33 (a linear bounded operator might not be compact)**.**

Theorem 2.34. *Given a compact operator K and a linear bounded operator B. KB and BK are compact operators.*

Proof. to do □

6. Functional

Definition 2.35 (linear functional).

Definition 2.36 (dual space, linear bounded functional).

Notation 2.37 (set of linear functional). $\mathcal{L}(X, \mathbb{R})$ or $\mathcal{L}(X, \mathbb{C})$ or X^*

Notation 2.38 (set of linear bounded functional). $\mathcal{B}(X, \mathbb{R})$ or $\mathcal{B}(X, \mathbb{C})$ or X^*

Definition 2.39 (reflexive space).

Theorem 2.40 (Hahn-Banach theorem).

Proof. to do $\square$

7. Contraction Mapping

Theorem 2.41 (Picard–Lindelöf theorem).

Theorem 2.42 (Peano existence theorem).

Theorem 2.43 (Osgood uniqueness theorem).

Chapter 3

Convergence

1. Uniform Convergence

Definition 3.1 (uniform convergence)**.**

2. Strong Convergence

Definition 3.2 (strong convergence)**.**

3. Weak Convergence

Definition 3.3 (weak convergence)**.**

Theorem 3.4. *Given a normed vector space* $(X, \|\cdot\|_X)$*. If* $\{x_n\}$ *in* X *weakly converges to* $x \in X$*, then* $\|x\|_X \leq \liminf_{n\to+\infty} \|x_n\|_X$*.*

Proof. We pick $f \in X^*$ such that $\|f\| = 1$ and $fx = \|x\|_X$, $\forall x \in X$. Due to Theorem 2.40, f must exist.

Then,

$$\begin{aligned}
\|x\|_X &= fx \\
&= \lim_{n\to+\infty} fx_n \\
&= \liminf_{n\to+\infty} fx_n \\
&\leq \liminf_{n\to+\infty} \|f\|\|x_n\|_X \\
&= \liminf_{n\to+\infty} \|x_n\|_X
\end{aligned}$$

□

Theorem 3.5. *Given a compact operator* $T \colon X \to Y$*. If* $\{x_n\}$ *in* X *weakly converges to* $x \in X$*, then* $\{Tx_n\}$ *converges to* $\{Tx\}$ *in* $\|\cdot\|_Y$*.*

Proof. to do □

Definition 3.6 (Schur's property)**.**

Definition 3.7 (Radon–Riesz property)**.**

Theorem 3.8 (Schur's Lemma)**.**

Proof. to do □

Corollary 3.9. *Every real Hilbert space has the Radon–Riesz property.*

Proof. to do □

4. Weak* Convergence

Bibliography