Introduction to Malliavin Calculus

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December 9, 2023

Contents

Preliminary

Definition 0.1 (n-th Hermite polynomials)**.** $\forall n \in \mathbb{N}$, $H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$

Property 0.2. $\forall n \in \mathbb{N}$, $(n+1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x)$.

Theorem 0.3 (Doob-Dynkin Lemma)**.**

Proof. See Proposition 3(Page 7) in [**M. M. Rao, 2006**] □

Theorem 0.4. *Given a countable orthogonal set S in a Hilbert space H. The only vector orthogonal to H is the zero vector if and only if S spans H.*

Proof. See [**Young, 1988**]. □

Theorem 0.5. $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ *is dense in* $L^2(\mathbb{R}^n, \mathbb{R})$.

Proof. See Lemma 3.1 (Page 222) in [**Stein and Shakarchi, 2005**]. □

Basic

The following content can be found in [**Nualart, 2006**] and [**David Nualart, 2018**]. For simplicity, we only consider 1-dimensional case.

Definition 1.1 (centered Gaussian family)**.** Given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a subspace $W \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a centered Gaussian family, if it is closed, and all the elements of W are Gaussian random variables with zero mean.

Definition 1.2 (isonormal Gaussian process)**.** A centered Gaussian family W on H is called isonormal Gaussian process, if H is a real and separable Hilbert space, and $W = \{W(h) : h \in H\}$, and $\mathbb{E}[W(f)W(g)] = \langle f, g\rangle_H$, $\forall f, g \in H$.

Example 1.3. Given a Brownian motion B_t with respect to $(\Omega, \mathcal{F}, \mathbb{P})$. $H = L^2([0, T], \mathcal{B}(0, T))$, $W : h \mapsto \int_0^T h(s)dB_s$.

Property 1.4. *The map $h \mapsto W(h)$ is linear.*

Proof. One can verify that $\forall f, g \in H$, $\forall a, b \in \mathbb{R}$, $\mathbb{E}[(W(af + bg) - aW(f) - bW(f))^2] = 0$. $\square$

Chapter 2

Derivative operator

Notation 2.1. $\mathcal{S} \coloneqq \{f(W(h_1), \ldots, W(h_n)) : \forall n \in \mathbb{N}, f \in C_{pol}^{\infty}(\mathbb{R}^n, \mathbb{R}), h_1, \ldots, h_n \in H\}$.

Example 2.2. $e^x \notin C_{pol}^{\infty}(\mathbb{R}, \mathbb{R})$ and $x^3 \in C_{pol}^{\infty}(\mathbb{R}, \mathbb{R})$.

Notation 2.3. $\mathcal{S}_0 \coloneqq \{f(W(h_1), \ldots, W(h_n)) : \forall n \in \mathbb{N}, f \in C_0^{\infty}(\mathbb{R}^n, \mathbb{R}), h_1, \ldots, h_n \in H\}$.

Example 2.4. $x^3 \notin C_0^{\infty}(\mathbb{R}, \mathbb{R})$ and $1 \notin C_0^{\infty}(\mathbb{R}, \mathbb{R})$.

Proposition 2.5. $C_0^{\infty}(\mathbb{R}^n, \mathbb{R}) \neq \emptyset$.

Proof. $\varphi(x) \coloneqq e^{-\frac{1}{1-|x|^2}}$, if $|x| < 1$, and $\coloneqq 0$, otherwise. $\square$

Lemma 2.6. *$\mathcal{S}_0$ is dense in $\mathcal{S}$, with $L^2(\mathcal{G})$, where $\mathcal{G} \coloneqq \sigma(S)$.*

Proof. Using the similar technique as Lemma 4.3.1 (Page 50) in [**Øksendal, 2003**].

Because H is a separable Hilbert space, $\exists \{h_n\}_{n \in \mathbb{N}} \in H$, such that $\{h_n\}_n$ is dense in H. Define $A_n \coloneqq \sigma(W(h_i) : i \leq n)$, then $A_n \subset A_{n+1} \subset \mathcal{G}$ and $\lim_{n \to +\infty} A_n = \mathcal{G}$.

So $\forall F \in \mathcal{S}$ with $L^2(\mathcal{G})$, we have $\mathbb{E}[F|\mathcal{A}_n] \xrightarrow[n \to +\infty]{L^2} \mathbb{E}[F|\mathcal{G}] = F$, from Corollary C.9 (Page 325) in [**Øksendal, 2003**].

From Theorem 0.3, $\forall n \in \mathbb{N}$, $\exists F_n \in L^2(\mathbb{R}^n, \mathbb{R})$, such that $\mathbb{E}[F|\mathcal{A}_n] = F_n(W(h_1), \ldots, W(h_n))$.

In addition, we know that $\forall n \in \mathbb{N}, C_0^{\infty}(\mathbb{R}^n, \mathbb{R})$ is dense in $L^2(\mathbb{R}^n, \mathbb{R})$, from Theorem 0.5.

Finally, F can be approximated by some elements of $\mathcal{S}_0$ in L^2, by Minkowski's inequality. $\square$

Definition 2.7 (derivative)**.** Derivative operator $D : \mathcal{S} \to L^2(\Omega; H)$, $DF = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(W(h_1), \dots, W(h_n))h_i$.

Lemma 2.8. *$\forall F, G \in \mathcal{S}$, $D(FG) = F(DG) + G(DF)$.*

Proof. Multiplication of smooth functions is still a smooth function. Then using the definition. □

Proposition 2.9. *$\forall F \in \mathcal{S}$, $\forall h \in H$, $\mathbb{E}[\langle DF, h\rangle_H] = \mathbb{E}[FW(h)]$.*

Proof. Without loss of generality, we only prove the case when $F = f(W(h), \dots, W(e_n))$ and $\|h\|_H = 1$, where $h, e_1, \dots, e_n$ are orthogonal.

Thus,

$$\begin{aligned}
&\mathbb{E}[\langle DF, h\rangle_H] \\
=&\mathbb{E}[\frac{\partial}{\partial x_1} f(W(h), \dots, W(e_n))] \\
=&\int_{\mathbb{R}^n} \frac{\partial}{\partial x_1} f(x_1, \dots, x_n)\phi(x_1, \dots, x_n) dx_1 \dots dx_n \\
=&\int_{\mathbb{R}^n} f(x_1, \dots, x_n)\phi(x_1, \dots, x_n) x_1 dx_1 \dots dx_n \\
=&\mathbb{E}[f(W(h), \dots, W(e_n))W(h)] \\
=&\mathbb{E}[FW(h)]
\end{aligned}$$

where ϕ is the density of n variable standard normal distribution. □

Proposition 2.10. *$\forall F, G \in \mathcal{S}$, $\forall h \in H$, $\mathbb{E}[G\langle DF, h\rangle_H] + \mathbb{E}[F\langle DG, h\rangle_H] = \mathbb{E}[FGW(h)]$.*

Proof. Define $X \coloneqq FG$, then $X \in \mathcal{S}$. Using Lemma 2.8 and Proposition 2.9. □

Notation 2.11. $\forall f \in \mathcal{S}$, $\|f\|_{1,2}^2 \coloneqq \|f\|_{L^2(\Omega)}^2 + \|Df\|_{L^2(\Omega,H)}^2$

Proposition 2.12. *$\|\cdot\|_{1,2}$ is a norm.*

Proof. To prove sub-additivity, we will apply Minkowski's inequality. □

Notation 2.13. Let $(\mathbb{D}^{1,2}, \|\cdot\|_{1,2})$ be the closure of $(\mathcal{S}, \|\cdot\|_{1,2})$.

Lemma 2.14. *Given $\{X_n\}_n \in \mathcal{S}$, if $X_n \xrightarrow{L^2(\Omega)} 0$, and $DX_n \xrightarrow{L^2(\Omega,H)} U$, then $U = 0$ almost surely.*

Proof. We know that $\lim_{n\to+\infty} \mathbb{E}[X_n^2] = 0$, and $\lim_{n\to+\infty} \mathbb{E}[\langle DX_n - U, DX_n - U\rangle_H] = 0$.

$\forall h \in H$ and $\forall F \in \mathcal{S}_0$,

$$\begin{aligned}
&\mathbb{E}[F\langle U, h\rangle_H] \\
=&\lim_{n\to+\infty} \mathbb{E}[F\langle DX_n, h\rangle_H] && \text{(continuity of inner product)} \\
=&\lim_{n\to+\infty} \mathbb{E}[-X_n\langle DF, h\rangle_H + X_n F W(h)] && \text{(Proposition 2.10)} \\
=&0 && \text{(Cauchy-Schwarz inequality)}
\end{aligned}$$

Because $F \in \mathcal{S}_0$, we know $\langle U, h\rangle_H = 0$, from Theorem 0.4. Also, because $h \in H$, $U = 0$ almost surely. $\square$

Theorem 2.15. *Derivative operator $D : \mathbb{D}^{1,2} \to L^2(\Omega, H)$ is well defined.*

Proof. From Lemma 2.14, we know that $\forall X \in \mathbb{D}^{1,2}$, $\exists\{X_n\} \in \mathcal{S}$, such that $X_n \xrightarrow{L^2(\Omega)} X$, $DX_n \xrightarrow{L^2(\Omega,H)} U$ and $U \in L^2(\Omega, H)$, then $DX \coloneqq \lim_{n\to+\infty} DX_n$. $\square$

Theorem 2.16 (Chain rule)**.** *Given $g \in C^1(\mathbb{R}^d, \mathbb{R})$ with bounded partial derivatives and $F_i \in \mathbb{D}^{1,2}$, $i \in \{1, \dots, d\}$. Then $g(F_1, \dots, F_d) \in \mathbb{D}^{1,2}$ and $D(g(F_1, \dots, F_d)) = \sum_{i=1}^d \frac{\partial}{\partial x_i} g(F_1, \dots, F_d) DF_i$.*

Proof. For simplicity, we only prove the case when $d = 1$.

step 1: When $F \in \mathcal{S}$.

Because the composition of differentiable function and smooth function is still a smooth function, we have $g(F) \in \mathcal{S} \subset \mathbb{D}^{1,2}$ and the chain rule can be obtained easily.

step 2: When $F \notin \mathcal{S}$.

$\exists\{F_k\}_{k\in\mathbb{N}} \in \mathcal{S}$, such that $F_k \xrightarrow{L^2(\Omega)} F$ and $DF_k \xrightarrow{L^2(\Omega,H)} DF$.

In addition, $\forall\varepsilon > 0$, define $\varphi_\varepsilon \coloneqq \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$, where $\varphi(x)$ is the same as Lemma 2.5. Meanwhile define $g_\varepsilon \coloneqq g * \varphi_\varepsilon$, where $*$ is the convolution operator. Clearly, $g_\varepsilon \in C_0^\infty(\mathbb{R}, \mathbb{R})$, and $g_\varepsilon \xrightarrow[\varepsilon\to 0]{pointwise} g$.

So by definition, we have $D(g_\varepsilon(F_k)) = \frac{d}{dx} g_\varepsilon(F_k) DF_k$.

$\forall\varepsilon > 0$ and $\forall k \in \mathbb{N}$, by Minkowski's inequality we have $\|g_\varepsilon(F_k) - g(F)\|_{L^2(\Omega)} \le \|g_\varepsilon(F_k) - g(F_k)\|_{L^2(\Omega)} + \|g(F_k) - g(F)\|_{L^2(\Omega)}$.

In addition, $\|\frac{d}{dx} g(F)DF - \frac{d}{dx} g_\varepsilon(F_k)DF_k\|_{L^2(\Omega,H)} \le \|\frac{d}{dx} g(F)DF - \frac{d}{dx} g(F_k)DF\|_{L^2(\Omega,H)} + \|\frac{d}{dx} g(F_k)DF - \frac{d}{dx} g_\varepsilon(F_k)DF\|_{L^2(\Omega,H)} + \|\frac{d}{dx} g_\varepsilon(F_k)DF - \frac{d}{dx} g_\varepsilon(F_k)DF_k\|_{L^2(\Omega,H)}$,

Because D is well defined, we have $g(F) \in \mathbb{D}^{1,2}$ and the chain rule. $\square$

Chapter 3

Wiener chaos

Definition 3.1 (sigma algebra generated by a set of random variable)**.** $\mathcal{G} = \sigma(\{\sigma(w) : w \in W\})$, where W is a centered Gaussian family.

Lemma 3.2. *The linear span of* $\{e^w : w \in W\}$ *is a dense in* $L^2(\Omega, \mathcal{G}, P)$.

Proof. Using the similar technique as Lemma 4.3.2 (Page 50) in [**Øksendal, 2003**].

Due to Theorem 0.4, it suffices to prove that $\forall X \in L^2(\Omega, \mathcal{G}, P)$, if X is orthogonal to $\{e^w : w \in W\}$, then $X \equiv 0$ almost surely.

Select $X \in L^2(\Omega, \mathcal{G}, P)$, such that $\mathbb{E}[Xe^{W(h)}] = 0, \forall h \in H$. X is well defined, because $X = 0$ meets the requirement.

So $\forall n \in \mathbb{N}$, $G_n(\lambda_1, \ldots, \lambda_n) := \mathbb{E}[Xe^{\sum_{i=1}^n \lambda_i W(h_i)}] = 0$, $\forall \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $\forall h_1, \ldots, h_n \in H$.

Then we can extend G_n to a larger domain, $G_n : \mathbb{C}^n \to \mathbb{C}$. Because G_n is analytic function on $\mathbb{R}^n$, we know that $G \equiv 0$ on $\mathbb{C}^n$.

Meanwhile, $\forall \varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{E}[X\varphi(W(h_i), \ldots, W(h_n))]$
$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{(}\varphi)(y_1, \ldots, y_n) G_n(iy_1, \ldots, iy_n) dy_1 \ldots dy_n = 0$, where $\hat{(}\varphi)$ is Fourier transform of φ.

Because of the above lemma, we know that $X = 0$ only. $\square$

Definition 3.3 (n-th Wiener chaos)**.** Given $L^2(\Omega, \mathcal{G}, P)$, its closed linear subspace $\mathcal{H}_n$ is called n-th Wiener chaos if it is the linear span of $\{H_n(w) : w(h) \in W, \|h\|_H = 1\}$.

Lemma 3.4. *The space* $\mathcal{H}_n$ *and* $\mathcal{H}_m$ *are orthogonal, if* $n \neq m$.

Proof. $\forall X \in \mathcal{H}_n$, $\forall Y \in \mathcal{H}_m$, such that $X = H_n(W(h_1))$ and $Y = H_m(W(h_2))$, where $\|h_1\|_H = 1$ and $\|h_2\|_H = 1$.

Applying moment generating function on (X, Y), $\forall s, t \in \mathbb{R}$, we have $\mathbb{E}[e^{sX - \frac{s^2}{2}} e^{tY - \frac{t^2}{2}}] = e^{st\,\mathbb{E}[XY]}$.

Applying $\frac{\partial^{m+n}}{\partial s^n \partial t^m}$ on both side at $s = 0, t = 0$, we can prove the result. $\square$

Theorem 3.5. *Given a Hilbert space* $L^2(\Omega, \mathcal{F}, P)$*,* $L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$*.*

Proof. Select $X \in L^2(\Omega, \mathcal{F}, P)$, such that $\forall n \in \mathbb{N}$, $\forall Y \in \mathcal{H}_n$, $\mathbb{E}[XY] = 0$. X is well defined, because $X = 0$ meets the requirements. It suffices to prove that $X = 0$ only.

It is known that any polynomial can be treated as a finite linear combination of Hermite polynomial, and Taylor expansion of e^x. So $\mathbb{E}[Xe^{tW(h)}] = 0, \forall t \in \mathbb{R}$ and $\|h\|_H = 1$.

Therefore, from Lemma 3.2, $X = 0$ only. $\square$

Chapter 4

Multiple integrals

Definition 4.1 (elementary functions over T^n)**.**

Notation 4.2. Let $\mathcal{E}_n$ denote the set of all elementary functions over T^n.

Definition 4.3 (symmetric elementary functions)**.**

Definition 4.4 (integral of elementary functions)**.**

Proposition 4.5. *If $f \in \mathcal{E}_n$, then $I_n(f) = I_n(\tilde{f})$.*

Proof. □

Proposition 4.6. *If $f \in \mathcal{E}_n$ and $g \in \mathcal{E}_m$, then $\mathbb{E}[I_n(f)I_m(g)] = \langle \tilde{f}, \tilde{g} \rangle_{L^2}$, when $m = n$, and $= 0$ otherwise.*

Proof. □

Theorem 4.7. *$\mathcal{E}_n$ is dense in $L^2(T^n)$.*

Proof. □

Proposition 4.8. *Given $f \in L^2(T^n)$ and $\{f_k\}_k \in \mathcal{E}_n$, such that $f_k \xrightarrow{L^2(T^n)} f$. Then $\{I_n(f_k)\}_k$ is a Cauchy sequence in $L^2(\Omega)$.*

Proof. □

Definition 4.9 (n-th multiple integrals)**.**

Remark 4.10. Because of Proposition 4.8 the above definition is well defined.

Theorem 4.11. *If $f \in L^2(T^n)$ and $g \in L^2(T^m)$, then $\mathbb{E}[I_n(f)I_m(g)] = \langle \tilde{f}, \tilde{g} \rangle_{L^2}$, when $m = n$, and $= 0$ otherwise.*

Definition 4.12 (iterated Ito integral)**.** $\mathrm{Ito}_n(f)$

Theorem 4.13. *$I_n(f) = Ito_n(f)$.*

Proof. ☐

Theorem 4.14. *Hermite polynomial.*

Proof. ☐

Theorem 4.15. *Surjective*

Proof. ☐

Theorem 4.16. *decomposition.*

Proof. ☐

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