Applied Analysis

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Chapter 1

Metric and Normed Spaces

Exercise 1.1. [Exercise 1.5, Page 30]

Proof. Define

$$\begin{aligned} f\colon \mathbb{R}^+ \cup \{0\} &\to \mathbb{R} \\ x &\mapsto \frac{x}{1+x} \end{aligned}$$

It is obvious that f is an increasing function. □

Exercise 1.2. [Exercise 1.6, Page 30]

Proof. Using $\epsilon - \delta$. □

Proposition 1.3. *Given topology space $(X, \mathcal{T})$ and $A \subseteq X$, if $L(A)$ is the set of all limit points of A, then $L(A)$ is closed.*

Exercise 1.4. [Exercise 1.8, Page 31]

Proof. C is closed by Proposition 1.3. Therefore $\max C = \sup C$ and $\min C = \inf C$.
Because $\mathbb{R}$ is complete, for each $n \in \mathbb{N}$ there exists $M(n) \in \mathbb{R}$ such that $M(n) = \sup_{i>n}\{x_i\}$. It is obvious that $M(n) \geq M(n+1)$ and there must exist $M \in \mathbb{R}$ such that $M = \inf_{n\in\mathbb{N}}\{M(n)\}$.
$\forall c \in C$, $M(n) \geq c$. Thus $M \geq \max C$.
It suffices to show that $M \leq \max C$. We prove it by contradiction. Suppose that $M > \max C$, then $\exists \varepsilon \in \mathbb{R}^+$ such that $M = \max C + 2\varepsilon$. So $\forall n \in \mathbb{N}, M(n) > \max C + \varepsilon$. So we can construct a bounded subsequence $\{y_n\}$ in $\{x_n\}$ such that $\forall n \in \mathbb{N}$, $\max C + \varepsilon \leq y_n \leq M(1)$. By Bolzano-Weierstrass Theorem[Theorem 1.57, Page 23], there must exist a convergent subsequence $\{z_n\}$ in $\{y_n\}$. Define $z \coloneqq \lim_{n\to+\infty} z_n$, then $z \in \mathbb{R}$ and $z > \max C$, which violates the definition of C. Thus $M \leq \max C$.
Therefore $M = \max C$. □

Proposition 1.5. (max–min inequality) *For any function*

$$f\colon A \times B \to \mathbb{R}$$
$$(a,b) \mapsto f(a,b)$$

The following holds:

$$\sup_{a\in A}\inf_{b\in B} f(a,b) \le \inf_{b\in B}\sup_{a\in A} f(a,b)$$

Proof.

$$\begin{aligned} &\forall a \in A, \forall b \in B, \inf_{b\in B} f(a,b) \le f(a,b) \\ \implies &\forall b \in B, \sup_{a\in A}\inf_{b\in B} f(a,b) \le \sup_{a\in A} f(a,b) \\ \implies &\sup_{a\in A}\inf_{b\in B} f(a,b) \le \inf_{b\in B}\sup_{a\in A} f(a,b) \end{aligned}$$

□

Exercise 1.6. [Exercise 1.10, Page 30]

Proof.

$$\begin{aligned} \text{left} &= \limsup_{n\to+\infty}(\inf_{\alpha\in A} x_{n,\alpha}) \\ &= \inf_{i\in\mathbb{N}}\sup_{n>i}\{\inf_{\alpha\in A} x_{n,\alpha}\} && \text{(definition)} \\ &= \inf_{i\in\mathbb{N}}\sup_{n>i}\inf_{\alpha\in A}\{x_{n,\alpha}\}_{n>i} \\ &\le \inf_{i\in\mathbb{N}}\inf_{\alpha\in A}\sup_{n>i}\{x_{n,\alpha}\}_{n>i} && \text{(Proposition 1.5)} \\ &= \inf_{\alpha\in A}\inf_{i\in\mathbb{N}}\sup_{n>i}\{x_{n,\alpha}\}_{n>i} \\ &= \inf_{\alpha\in A}\inf_{i\in\mathbb{N}}\sup_{n>i}\{x_{n,\alpha}\} \\ &= \inf_{\alpha\in A}(\limsup_{n\to+\infty} x_{n,\alpha}) \\ &= \text{right} \end{aligned}$$

□

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