Project Reading List — Majda 8/8/18

Using Statistical Dynamics to Explain Features of Anomalous Statistics in shallow water waves with Abrupt Depth Changes

A. Majda, N. Moore, (Di Qi?)

I) Experiments (B.S.M., 2018)

(Deterministic)

II) (Model for I) Initial KdV + Scattered KdV after abrupt depth change (ADC)

Ref. R.S. Johnson,... Math Theory of Water Waves Cam. U. Press (1997)

Read pgs 268—277

III) Statistical Dynamics Model for I) based on II)

Read ① Majda + Wang book, Cam. U. Press (2006), Chapter 6

Read ② Abramov, Kovacic, Majda, Comm. Pure Appl. Math (2003) 1–46

Read ③ Bajars, Frank, Leimkuhler, Nonlinearity (2013) 1945–1973

Only Read 1948-1952 and 1956-1957 (Fig. 1)

Plan I)-V Detailed Notes

DM-1

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I Deterministic Dynamics Model with Abrupt Depth Change

Model Assumptions

Assume 1) Nearly unidirectional, periodic narrow channel $2\pi L_0$, $L_0 = \lambda^{-1} << 1$

2) Role of Paddle wheel forcing looking a little downstream

(A) is to generate stat soln with $M = 0$, E_0

where E_0 is function of paddle strength at $t=0$

3) $|k| \le \Lambda$ fixed truncation $u_\Lambda \equiv u$

4) time $t \Leftrightarrow$ downstream coordinate, abrupt depth change at $t = t_0$
(ADC) $t_0 \equiv T_{ADC}$

Use notation from 7/16/18 for non-dim dynamics

B) Dynamics Before ADC, $0 < t \le t_0$, the upstream dynamics

1) $u_t + E_0^{1/2} \lambda^{-3/2} u u_x + \lambda^{-3} u_{xxx} = 0$, $u(x,t)\Big|_{t=0} = u_0(x)$

2) u, 2π periodic, $M = \int_0^{2\pi} u \equiv 0$, $E = \frac{1}{2}\int u^2 \equiv 1$) Three conserved quantities in time

3) Hamiltonian $\boxed{H = E_0^{1/2} \lambda^{-3/2} H_3 - \lambda^{-3} H_2}$ Hamiltonian

4) $H_3 = \int_0^{2\pi} \frac{1}{6} u^3$, $H_2 = \int_0^{2\pi} \frac{1}{2} u_x^2$

C) Dynamics for $t > t_0$ after ADC at t_0, the downstream dynamics

Note use 7/10/18 with 7/16/18 as rough guideline

1) $\tilde{u}_+(x, t_0) = D_+^{1/4} u(x, t_0)$ where u solves B) 1), with depth $0 < D_+ < 1$

2) $(\tilde{u}_+)_t + D_+^{-7/2} E_0^{1/2} \lambda^{-3/2} \tilde{u} \tilde{u}_x + D_+^{3/2} \lambda^{-3} \tilde{u}_{xxx} = 0$; ADC lowers energy, stronger nonlinearity, weaker dispersion

3) Set $u_+(x,t) \equiv D_+^{-1/4} \tilde{u}_+(x,t)$

PM → 2

3) $a_+ = D_+^{\frac{1}{4}} u_+$; $\tilde{u}_t + D_+^{-\frac{7}{4}} \epsilon_0^{\frac{1}{2}} \lambda^{-\frac{3}{2}} \tilde{u}\tilde{u}_x + D_+^{\frac{3}{2}} \lambda^{-3} \tilde{u}_{xxx}$

4) $(u_+)_t + D_+^{-\frac{13}{4}} \epsilon_0^{\frac{1}{2}} \lambda^{-\frac{3}{2}} u_+(u_+)_x + D_+^{\frac{3}{2}} \lambda^{-3} u_{+,xxx} = 0$

$M u_+(t) = 0$, $E(u_+) = 1$ for $t_0 \le t \le +\infty$

5) Hamiltonian H_+ in outgoing state

$H_+ = D_+^{-\frac{13}{4}} \epsilon_0^{\frac{1}{2}} \lambda^{-\frac{3}{2}} H_3 - D_+^{\frac{3}{2}} \lambda^{-3} H_2$ for $t_0 \le t \le +\infty$, Conserved

H_+ more weight on 3rd moment, H_3 by factor $D_+^{-\frac{13}{4}}$ } favors 3rd moment
less weight on H_2 " " $D_+^{\frac{3}{2}}$ } after T_{ADC}.

$0 < D_+ < 1$, par

Statistical Behavior: before ADC

nearly Gaussian Model by $\theta \equiv 0$, zero inv temp in Mixed Ensemble { Fcn of E_0

Gives pdf of H_3, H_2 almost Explicitly at $t = T_{ADC}$; Supplies Stat Initial Data
Note H_3 + H_2 comp separately for H_+ after ADC i.e. $t \ge T_{ADC}$

6) Matching Across T_{ADC}

$u(x,t)\Big|_{t = T_{ADC}} = u_+(x,t)\Big|_{t = T_{ADC}}$

II. Statistical TKdV Perspective

A) Use Micro-Can-Energy - Can Hamiltonian (ASM)

Λ is truncation level, 2Λ DOF, S_Λ is 2Λ dim unit sphere

H_Λ is Hamiltonian, Gibbs measure $\mathcal{G}_\theta$ is prob measure on S_Λ

1) $$\mathcal{G}_\theta = \frac{e^{-\theta H_\Lambda} dS_\Lambda}{\int_{S_\Lambda} e^{-\theta H_\Lambda} dS_\Lambda}$$ $\theta = 0$, zero inv Temp; $\theta < 0$, neg inv temp

Can Gs $\mathcal{G}_\theta$, inv of dynamics invariant dynamics

B) ~~Incoming~~ Stat Dynamics $-\infty < t_0 \le T_{ADC}$

from exp Needs to have nearly Gaussian Statistics, given E_0 = initial energy level obs.

2) From I) B, use TKdV Hamiltonian from B) 3)

$$H_- = E_0^{\frac{1}{2}} \lambda^{-\frac{3}{2}} H_3 - \lambda^{-3} H_2 \equiv H_3^\lambda - H_2^\lambda$$

3) Zero Inv Temp, with H_- in 2) is inv

i.e. $\mathcal{G}_0(H)$ is incoming inv measure

Two Key Prop of 3) as incoming invariant measure

4-1) For above, $\langle H_3^\lambda \rangle_{\mathcal{G}_0}$, $\langle H_2^\lambda \rangle_{\mathcal{G}_0}$ can be calc. by MC and Explicitly (Bessel)

Prop: 4-2) for Λ large or Λ fixed and λ large, $\mathcal{G}_0$ is nearly Gaussian as req in B) 1)

See Fig 1 of B-F-1 and 7/16/18

) Matching Outgoing Initial Stat Measure at $t=T_{ADC}$

From 6) D-M-1 and II, B) above the initial value at $T=T_{ADC}$ for outgoing ~~incoming~~ prob meas, $p_t^+(u_+^*)$ is given by

1) $$\mathcal{G}_0(H) \equiv p_t^+\Big|_{t=T_{ADC}}$$

2) Outgoing Hamiltonian for $t>T_{ADG}$, Use 5) from DM-2

2) $$\boxed{H_+ = D_+^{\frac{-13}{4}} H_3^{\lambda} - D_+^{\frac{3}{2}} H_2^{\lambda}}$$

3) From B(4-1) above and (7-1) above, at $t=T_{ADC}$ we know

3) $$\langle H_+^f \rangle_{p_t^+}\Big|_{t=T_{ADC}} = D_+^{\frac{-13}{4}} \langle H_3^{\lambda} \rangle_{\mathcal{G}_0} + D_+^{\frac{3}{2}} \langle H_2^{\lambda} \rangle_{\mathcal{G}_0}$$

) Transient and long time Outgoing Stat Dynamics

1) Transient ~~lon~~ Outgoing stat dynamics

p_t^+ for $t \ge T_{ADC}$ solves Louisville Eqn with IV in C) 1)

2) H_+ is conserved $\Rightarrow \langle H_+ \rangle_{p_t} = \langle H_+ \rangle_{p_0}$ for all $t \ge T_{ADC}$

3) Long-time Outgoing Dynamics with Ergodicity converging to Outgoing Gibbs Measure

D) 3)-1) $\mathcal{G}_{\theta^+}(H_+)$ where θ^+ is Lagrange multiplier det. by

D) 3-2) $$\langle H_+ \rangle_{\mathcal{G}_{\theta^+}} = \langle H_+ \rangle_{p^+_{TADC}}$$

E, Projects ~~of~~ for Outgoing Stat Solns

1) Short-time PDF expansion and non Gaussian behavior, pos skewness for $f_t(\omega)$

2) Non Gaussian Stat. for outgoing Gibbs measure, $\mathcal{G}_{\sigma}(H^+)$ and stat conv of f_t for β varying

3) Mean field behavior, cent. I — distribution, etc?

Majda 8/10/18

After ADC, (outgoing) KdV eqn
Deterministic KdV ADC Dynamics

1) Not E_0 or L_0 rescaled

$$u(x,\theta) = D_+^{-1/4}\,\tilde{u}(x,\theta) \qquad \text{See pg 2}$$

$$\tilde{u}_t + D_+^{-7/2}\,\tilde{u}\tilde{u}_x + D_+^{3/2}\,\tilde{u}_{xxx} = 0 \qquad \tilde{u}$$

2) E_0 and $L_0 = \lambda^{-1}$, unit energy, 2π periodic, $M=0$

Outgoing TKdV Soln

OTKdV A) $(u_+)_t + D_+^{-13/4} E_0^{1/2} \lambda^{-3/2} \left(\tfrac{1}{2}u_+^2\right)_x + D_+^{3/2}\lambda^{-3} u_{+xxx} = 0$, $t > T_{ADC}$

Incoming TKdV, λ, E_0 prescribed, $M \equiv 0$

B) $(u)_t \equiv$ Same eqn in A) with $D_+ \equiv 1$ $\quad t_0 < t < T_{ADC}$, $t_0 \geq -\infty$

3) Transition Matching Initial Condition

$$\boxed{\tilde{u}_+(x,t)\Big|_{t=T_{ADC}} = D_+^{1/4}\,u(x,t)\Big|_{t=T_{ADC}} \Leftrightarrow u_+\Big|_{t=T_{ADC}} = u\Big|_{t=T_{ADC}}}$$

Terminology TKdV $\leftrightarrow$ Λ fixed

ADC - Abrupt Depth Change T_{ADC}

u or u_- - Incoming TKdV, $-\infty \leq t_0 < t \leq T_{ADC}$

u_+ - Outgoing TKdV $T_{ADC} \leq t < +\infty$

Detailed Steps:

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ADC-KdV to Exact Rescaling

1) $\xi = \frac{1}{\epsilon}\chi(x) - t, \; X = \epsilon x$

2) Before ADC, $\xi = x - t$

3) After ADC, $\chi(x) = \int_0 D_+^{-\frac{1}{2}} x$ so $\xi_+ = D_+^{-\frac{1}{2}} x - t$.

Set $\underline{\tilde{t}_+ = D_+^{-\frac{1}{2}} t}$ $\xi_+ = D_+^{-\frac{1}{2}}(x - \tilde{t})$, $X = \epsilon x$

4) $\underline{\text{pg. } 272}$, $\underline{\text{Johnson}}$, $\boxed{H_0 = D_+^{\frac{1}{4}} \eta_0}$

Original Derivation

$2(H_0)_X + \frac{3}{D_+^{7/4}} H_0 H_{0\xi_+} + \frac{1}{3} D_+^{\frac{1}{2}} H_{0\xi_+\xi_+\xi_+} = 0$ ξ_+ derivative here

5) Change variables: $\boxed{\xi = D_+^{\frac{1}{2}} \xi_+}$

$\frac{\partial}{\partial \xi} = D_+^{\frac{1}{2}} \frac{\partial}{\partial \xi_+} \Leftrightarrow \frac{\partial}{\partial \xi_+} = D_+^{\frac{1}{2}} \frac{\partial}{\partial \xi}$

$2H_0(x)_X + 3 D_+^{-\frac{3}{2}} H_0 H_{0\xi} + \frac{1}{3} D_+^{2} H_{0\xi\xi\xi} = 0$

D^{-3} $\quad D_+^{2}$

$\lambda.\ \xi = x - \tilde{t}_+$, $X = \epsilon x$, $\frac{\partial}{\partial X} = \frac{\partial}{\partial t_+}$ and $\frac{\partial}{\partial t_+} = \frac{\partial t}{\partial t_+}$ $\frac{\partial}{\partial t} = D^{\frac{1}{2}} \frac{\partial}{\partial t}$

Rescaled KdV $\longleftrightarrow$

$D^{-\frac{7}{4}}$ $\quad D^{3/2}$

6) $2(H_0)_t + 3 D_+^{-\frac{3}{2}} H_0 H_{0\xi} + \frac{1}{3} D_+^{3/2} H_{0\xi\xi\xi} = 0$ Time Rescaled KdV

7) $H_0 = \alpha u$, $\frac{3}{2}\alpha \beta^2$; Simple Scaling

NL $\quad \xi$

Comments ADC-Statistics Majda 8/8/18

Downstream after T_{ADC} Note time Rescaling - X rescaling D_{+} factor

0) $t < T_{ADC}$, zero temp steady state, E_0, λ, Nearly Gaussian, IP
1) At $t = T_{ADC}$ Gaussian pdf for $\Lambda_3(D_+, E_0, \lambda)$ $H_3 = N(0, \Pi_0)$, Π_0 ... Mean Field Th

" " for $\Lambda_2(D_+, E_0, \lambda)$ $H_2 = N(-\Pi_0, \Pi_0)$

2) $t \to \infty$ + ergodic Final Stat predicted by TKN

3) Question: Short term stat trans State

$\langle H(D_+) \rangle$ cons Same game as practice notes

Link 2) + 3)

4) If Mean Field Regime still applies

Model 2) pdf by centered I-dist as in MW book

ISB-1

Majda 7/16/18

The ISBS Small Spatial Period Rescaling

1) $u_t + \frac{1}{2}u^2_x + u_{xxx} = 0$

$M = 0, E, H = \frac{1}{6}\int u^3 - \frac{1}{2}\int u_x^2 \boxed{= H_3 - H_2}$ cubic + Quad Parts

$2\pi L_0$ periodic in x, L_0 in expt

2) $\boxed{\tilde{u} = \frac{u(\lambda x, t)\lambda^{1/2}}{E^{1/2}}}$ $\frac{1}{2}\int_0^{2\pi} \tilde{u}^2 = \frac{1}{2}\int_0^{2\pi L\lambda} \frac{u^2(\lambda x, t)\,dx}{E} = 1,$

$\tilde{u}$ E unit energy, $2\pi L\lambda$ periodic

3) Dynamics for $\tilde{u}$, $u(\lambda x, t) = \frac{E^{1/2}}{\lambda}\tilde{u} = \left(\frac{E}{\lambda}\right)^{1/2}\tilde{u}$

u_t

$\tilde{u} = \theta u(\lambda x, t)$

$\tilde{x} = \lambda x$ $u_t = u_t$, $\lambda^{-1}\frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial x}$

$\tilde{u}(\tilde{x}, t) = \theta u(\lambda x, t)$ $\theta = \left(\frac{\lambda}{E}\right)^{1/2}$

$(\lambda = L_0^{-1})$ $\lambda^{-1} = L_0$ $2\pi L_0$ initial period L_0 small, narrow channel

$u_t + \lambda^{-1} u u_x + \lambda^{-3} u_{xxx} = 0$

$\tilde{u}_t + (\lambda\theta)^{-1}\tilde{u}\tilde{u}_x + \lambda^{-3}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$

Rescaled $\longleftrightarrow$ $\boxed{u_t + E^{1/2}\lambda^{-3/2} u u_x + \lambda^{-3} u_{\tilde{x}\tilde{x}\tilde{x}} = 0}$

4) $H = E^{1/2}\lambda^{-9/2} H_3 - \lambda^{-3} H_2$, $\tilde{u}(\tilde{x}, t)$

5) Zero Inverse temp $G_\theta = \frac{e^{\theta H} d\Omega_\lambda}{\int}$, zero inv temp $G_0 = \frac{d\Omega_\lambda}{\text{Area}(\Omega_\lambda)}$ indep of H

No detailed dynamics

A) ① $E^{1/2}\lambda^{-3/2}\left[H_3\right.$ zero mean "Gaussian" for λ large enough $\lambda = 10$,

B) $\lambda^{-3}\left[H_2\right.$ neg. mean " " Gaussian$\left.\right]$ $\boxed{\text{See fig 1-B-F-L}, \beta=0}$

C) Regime, λ large $\Leftrightarrow$ L_0 small pdf is sum of two Gaussian RV

A), zero mean, Var $\cong E\lambda^{-3}$

B) mean neg $= \lambda^{-3}$ and $V \cong \lambda^{-6}$

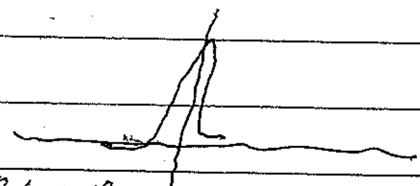
D) pdf(A) + B)) = pdf(A) * pdf(B) $\longleftrightarrow$ Sharply peaked Gaussian with slight neg. mean

SUMMARY: Zero Inv Temp, GAUSSIAN ISBS

5) D = pdf H in 4D for rescaled Hamiltonian in $\langle H \rangle$

$\langle H \rangle = -6\lambda^{3}$, var$(E\lambda^{3})$, nearly Gaussian, small mean

6)



Small VAR, zero mean

Very Close SB-1 with $\langle H \rangle \cong 0 \leftrightarrow$ zero temp. Motivates SB from 6/28/18

Look at in scattered state $\Rightarrow$ neg temp regime