

Origin of heavy tail statistics in equations of the Nonlinear Schrödinger type: an exact result

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We study the formation of extreme events in incoherent systems described by envelope equations, such as the Nonlinear Schrödinger equation. We derive an identity that relates the evolution of the kurtosis (a measure of the relevance of the tails in a probability density function) of the wave amplitude to the rate of change of the width of the Fourier spectrum of the wave field. The result is exact for all dispersive systems characterized by a nonlinear term of the form of the one contained in the Nonlinear Schrödinger equation. Numerical simulations are also performed to confirm our findings. Our work sheds some light on the origin of rogue waves in incoherent dispersive nonlinear media ruled by local cubic nonlinearity.

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Processes that lead to the formation of heavy tails [1–3] in the Probability Density Function (PDF) are of wide interest in many physical contexts [4–10]. It is well known that in homogeneous conditions, if the central limit theorem applies, a wave linear dispersive system characterized by a large number of incoherent waves is described by a Gaussian statistics; in the latter situation extreme events can still appear but they are very rare, and their probability of appearance can be derived exactly, [11, 12].

In the field of ocean waves and nonlinear optics, it has been established that the presence of nonlinearity on top of dispersion can lead to changes in the statistical properties of the system. Often rogue waves and the associated large tails in the PDF can be observed in experiments or numerical simulations from an initially incoherent wave field [13, 14]; the situations characterized by lower probability of extreme events than the Gaussian predictions can also be encountered [15, 16]. In all those cases the nonlinearity plays a key role in creating correlations among modes that ultimately in a deviations from Gaussian statistics. Given a partial differential equation for a dispersive nonlinear system and given an initial condition characterized by some statistical properties, it would be desirable to establish the PDF of the wave field at different times. This is not an easy task, and nowadays, some results can be achieved only by making a number of assumptions, as it is done in the wave turbulence theory [17].

In this Letter we present a very simple identity which can be derived from a family of universal nonlinear dispersive partial differential equations that allows one to relate the changes in the statistical properties of the wave field to the changes of its Fourier spectrum. Specifically, our focus is on the kurtosis, i.e. the fourth-order moment of the PDF which measures the relevance of the tails of

the distribution with respect to the core. Large values of kurtosis imply the presence of heavy tails in the distribution and higher probability of extreme events. We show analytically, without any approximation, that an increase of the spectral bandwidth (typical of a nonlinear evolution) results in an increase/decrease of extreme events in focusing/defocusing regime. Here, we will first discuss the 1D+1 integrable Nonlinear Schrödinger (NLS) equation problem and then we will extend the result to non-integrable NLS type of equation in 2D+1 and confirm our results with extensive numerical simulations.

One-dimensional case. The NLS equation is a universal model for describing nonlinear dispersive waves. For the present discussion, we will consider the NLS equation written as follows:

$$i \frac{\partial A}{\partial x} = \beta \frac{\partial^2 A}{\partial t^2} + \alpha |A|^2 A, \quad (1)$$

where α and β are two constant coefficients that depend on the physical problem considered. If $\alpha\beta > 0$ then the equation is known to be of focusing type, while if $\alpha\beta < 0$ the equation is defocusing. Note that equation (1) is written as an evolution equation in space rather than in time; this notation is common in nonlinear optics and it is also suitable in hydrodynamics for describing the evolution of waves in wave tank experiments. The general problem that one wish to answer is the following: given an incoherent time series characterized by some statistical properties at one boundary of the domain, what is the PDF of the intensity of the wave field along the tank or along the fiber? Will rogue waves appear? We stress that our goal here is not to establish the validity of the NLS equation in a specific field but to highlight a fundamental mechanism that leads to the formation of extreme or rogue waves.

We start by the definition of the kurtosis:

$$\kappa(x, t) = \frac{\langle |A(x, t)|^4 \rangle}{\langle |A(x, t)|^2 \rangle^2}, \quad (2)$$

where $\langle \dots \rangle$ denotes the ensemble average over different realisation of the same process characterized by different phases and Fourier amplitudes. The kurtosis is the fourth order moment of the PDF and it weights the relevance of its tails: heavy tails implies the presence of extreme/rogue waves. In the recent years, some interesting approaches have been developed in order to estimate the kurtosis in NLS type of equations [15, 17–20]; up to our knowledge, they all rely on a closure. Being a non-linear problem with cubic nonlinearity, the evolution of the fourth-order moment is related to the evolution of the sixth-order moment and so on, and a quasi-Gaussian approximation is assumed to get a closed form for the kurtosis. Such a closure is, of course, questionable in the presence of strong nonlinearity. The theoretical development that we present in this Letter is different: it does not require a closure and it is valid for an arbitrary nonlinearity. It is based on the (now trivial) intuition that the nonlinear part of the Hamiltonian is strictly related to the kurtosis. Indeed, the following Hamiltonian density is conserved for equation (1) at every x :

$$H = \frac{1}{T} \int_0^T \beta \left| \frac{\partial A}{\partial t} \right|^2 dt - \frac{1}{T} \int_0^T \frac{\alpha}{2} |A|^4 dt. \quad (3)$$

We now take the ensemble average of the above equation to get:

$$\langle H \rangle = \frac{1}{T} \int_0^T \beta \left\langle \left| \frac{\partial A}{\partial t} \right|^2 \right\rangle dt - \frac{1}{T} \int_0^T \frac{\alpha}{2} \langle |A|^4 \rangle dt. \quad (4)$$

Assuming that the $A(x, t)$ is a statistically stationary process in the interval $[0, T]$, then $\langle |A|^4 \rangle$ is time independent and Eq. (4) can be re-written as follows:

$$\frac{\langle H \rangle}{\langle N \rangle} = \beta \Omega(x)^2 - \frac{\alpha}{2} \langle N \rangle \kappa(x) \quad (5)$$

with $\langle N \rangle$ being the ensemble average of the number density of particles defined as

$$\langle N \rangle = \frac{1}{T} \int_0^T \langle |A(x, t)|^2 \rangle dt = \langle |A(x, t)|^2 \rangle, \quad (6)$$

(the last equality holds for a statistical stationary process) and

$$\Omega(x) = \sqrt{\frac{\sum_n \langle (\frac{2\pi}{T} n)^2 |A_n(x)|^2 \rangle}{\sum_n \langle |A_n(x)|^2 \rangle}}, \quad (7)$$

with $A_n(x)$ being the Fourier coefficients defined as

$$A_n(x) = \frac{1}{T} \int_0^T A(x, t) e^{-i \frac{2\pi}{T} n t} dt. \quad (8)$$

Note that periodic boundary conditions in t have been assumed. The quantity $\Omega(x)$ defined by Eq. (7) is nothing but the definition of the spectral bandwidth. Evaluating the expression (5) at $x = x_0$ (initial condition) and at a generic point x , after eliminating $\langle H \rangle / \langle N \rangle$ from the two resulting equations, we get the following exact relation (note that $\langle H \rangle$ and $\langle N \rangle$ do not depend on space and time):

$$\kappa(x) = \kappa(x_0) + 2 \frac{\beta}{\alpha} \frac{1}{\langle N \rangle} [\Omega(x)^2 - \Omega(x_0)^2]. \quad (9)$$

Equation (9) implies that the variation of the kurtosis is directly related to the variation of the spectral bandwidth. For example, in deep water gravity waves and in the anomalous dispersion regime in optics $\beta/\alpha > 0$ the increase of the spectrum is accompanied by an increase of the kurtosis and, as it will be seen below, to the formation of large tails in the spectrum. On the contrary, when the water depth, h , decreases below a critical value ($k_0 h < 1.36$, with k_0 the wave number of the carrier wave) or when the fiber is characterized by a regular dispersion, then the ratio β/α becomes negative (the NLS is of defocusing type) and an increase of the spectral bandwidth leads to a decrease of the kurtosis (less extreme events than predicted by the Gaussian distribution). If the spectrum is frozen, the kurtosis does not evolve.

In what follows, we consider a few numerical examples; without loss of generality, we solve the NLS equation (1) with $\alpha = \pm 2$ and $\beta = 1$, starting from an initial condition characterized by the following frequency Fourier spectrum:

$$A_n(x=0) = \sqrt{a_0 e^{-\frac{4\pi^2 n^2}{T^2 \sigma^2}}} e^{i\phi_n}, \quad (10)$$

where the phases ϕ_n are distributed uniformly in the $[0, 2\pi)$ interval. The numerical simulations are performed by using a pseudo-spectral method with a step-adaptative algorithm permitting to reach a specified level of numerical accuracy; a box of size $T = 257.36$ has been discretized in 4096 points. The numerical values of a_0 and σ^2 are 1.129 and 10^4 , respectively. The statistical properties of the random wave fields are computed from an ensemble of 10^4 realisations of the random initial condition. Because of the latter choice, the PDF of the real and imaginary part of $A(t, x=0)$ is Gaussian, the PDF of $|A(t, x=0)|$ is distributed according to the Rayleigh distribution having kurtosis equal to 2, and the PDF of the intensity $|A(t, x=0)|^2$ is exponential. In Figures 1 and 2 we show the kurtosis and the spectral bandwidth as a function of the evolution variable x for the focusing, $\alpha = 2$, and the defocusing case, $\alpha = -2$, respectively. It is interesting to note that, regardless of the sign of α , the spectral bandwidth always increases; however, while the kurtosis increases in the focusing case, it decreases in the defocusing one. As was mentioned, high values of kurtosis, implies heavy tails in the PDF. Indeed, in Figure 3 the PDF of the normalized intensity $I = |A(t, x)|^2 / N$ computed after the kurtosis has reached an equilibrium

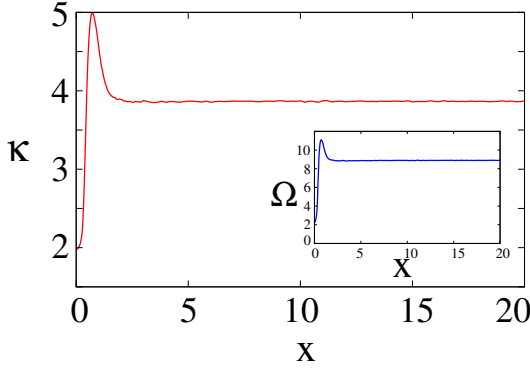


FIG. 1: Evolution of the kurtosis for the focusing NLS equation ($\alpha = 2, \beta = 1$). In the inset the evolution of the spectral bandwidth is shown.

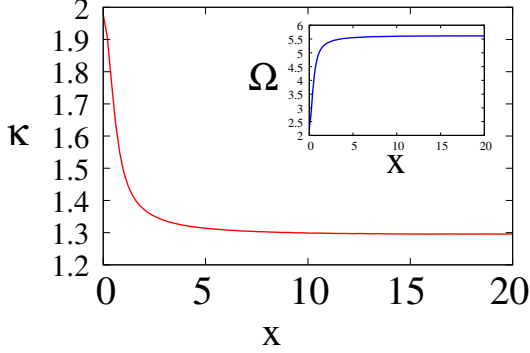


FIG. 2: Evolution of the kurtosis for the defocusing NLS equation. ($\alpha = -1, \beta = 1$). In the inset the evolution of the spectral bandwidth is shown.

state, ($x > 20$), is shown for both the focusing and the defocusing case. Numerical results are compared with the exponential distribution e^{-I} , the expected distribution for a system of linear incoherent waves. Deviations from such distribution are observed for both cases, however, consistently with our derivation, the focusing case shows heavy tails, while the defocusing one, the distribution is below the exponential prediction.

Two-dimensional case. We now consider the NLS in two horizontal dimensions written as an evolution equation in time:

$$i \frac{\partial A}{\partial t} = \left(\beta \frac{\partial^2 A}{\partial x^2} + \gamma \frac{\partial^2 A}{\partial y^2} \right) + \alpha |A|^2 A \quad (11)$$

with $\gamma = \pm 1$. In the water wave context, equation (11) with $\alpha = \beta = 1$ and $\gamma = -1$ arises in the deep water regime and it describes the evolution of the complex wave envelope in weakly nonlinear and narrow band (both in the direction of propagation and in its transverse direction) approximations. The second derivative in the y direction plays the role of diffraction and the equation is known as the Hyperbolic NLS. On the other hand, equation (11) with the choice of $\gamma = \beta = -1$ and $\alpha = 1$, also known as the defocusing Gross-Pitaevskii

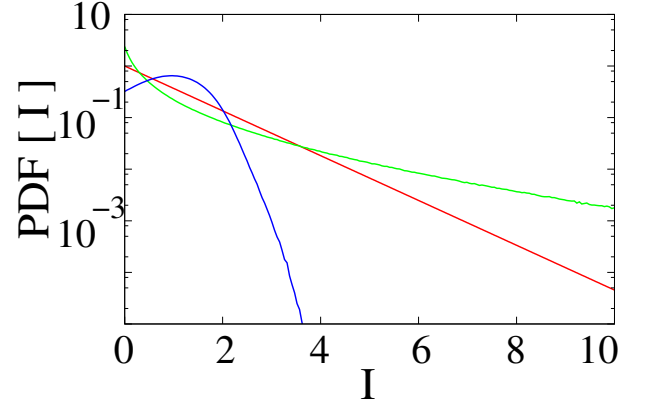


FIG. 3: PDF of the normalized intensity $I = |A(t, x)|^2 / N$ for both focusing (green line) and defocusing (blue line) NLS equation calculated for $x > 20$. The exponential distribution is also shown as a red line.

equation (GPE), describes for instance the dynamics of a two-dimensional Bose-Einstein condensate. We now assume that the system is homogeneous in the domain $L_x \times L_y$ and we follow the same procedure as in the one dimensional case. Keeping in mind that now the kurtosis evolves in time and it is defined as an average over space (rather than time), the same reasoning as before can be applied to get:

$$\kappa(t) = \kappa(t_0) + \frac{2}{\alpha \langle N \rangle} \times \{ \beta [K_x(t)^2 - K_x(t_0)^2] + \gamma [K_y(t)^2 - K_y(t_0)^2] \}, \quad (12)$$

where

$$K_x(t) = \sqrt{\frac{\sum_{k,l} \langle \left(\frac{2\pi}{L_x} k \right)^2 |A_{k,l}(t)|^2 \rangle}{\sum_{k,l} \langle |A_{k,l}(t)|^2 \rangle}}, \quad (13)$$

and $K_y(t)$ is defined in a similar fashion with the only difference that in the brackets L_x is replaced by L_y and k with l .

As in the one dimensional case, we show some instructive numerical simulations of equation (11). For all cases considered, the initial condition is characterized by the following Fourier spectrum:

$$A_{k,l}(t=0) = \sqrt{a_0} e^{-\left(\frac{2\pi}{L}\right)^2 \frac{k^2 + l^2}{\sigma}} e^{i\phi_{k,l}} \quad (14)$$

with $a_0 = 7.8 \times 10^{-5}$, $\sigma = \sqrt{10}$, $L = L_x = L_y = 512$ and phases are taken as randomly distributed. Numerical simulations are performed with a resolution of 1024×1024 with $\Delta x = \Delta y = 0.5$. To improve the statistical convergences, 10 different simulations are performed for each case with different initial random phases.

We start by considering the GPE: generally, given an initial condition localized in Fourier space, the tendency is to observe a broadening of the spectrum, thus, due to

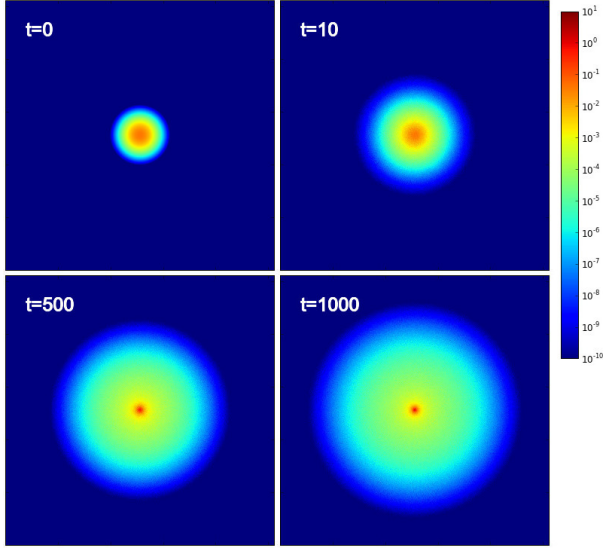


FIG. 4: Evolution of the spectrum for the GPE at different times. The initial conditions are provided in equation (14)

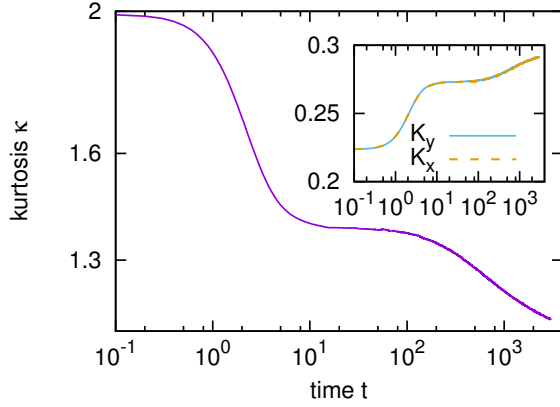


FIG. 5: $\kappa(t)$ as a function of time for numerical simulations of the GPE. The initial conditions are provided in equation (14). In the inset the evolution of the spectral bandwidth is shown.

the fact that $\beta = \gamma = -1$, according to equation (12) we expect to observe a decrease of the kurtosis. Indeed, in Figure 4, we show a density plot of the two dimensional Fourier spectrum at $t = 0, 10, 500, 1000$. It is interesting to observe that the spectrum broadens isotropically and a condensate at the mode $(k, l) = (0, 0)$ forms at large times (red spot in the Figure 4), see [21, 22] for details. The initial kurtosis, shown in Figure 5, decreases from the value of 2: starting from a spectrum characterized by random phases, extreme events are not expected in the defocusing GPE. The situation is different for the hyperbolic NLS where the equation is focusing in x and defocusing in the y direction. Because of the opposite signs in the linear terms, we expect an initial non-isotropic evolution. Indeed, as shown in Figure 6, the spectrum evolves more rapidly in the k_x direction, prob-

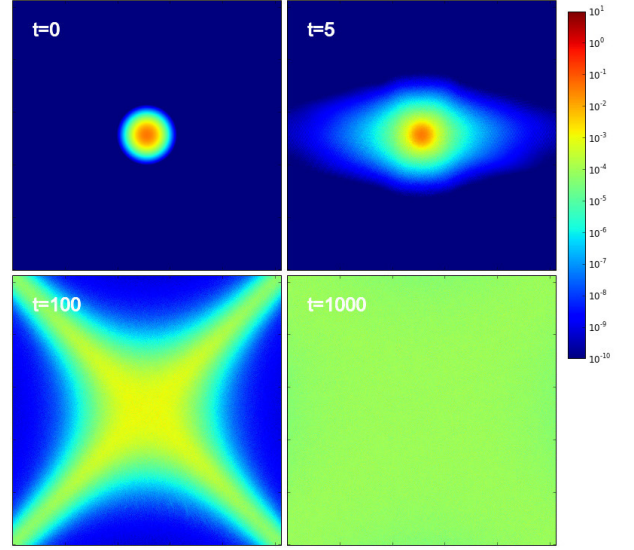


FIG. 6: Evolution of the spectrum for the hyperbolic NLS equation at different times. The initial conditions are provided in equation (14)

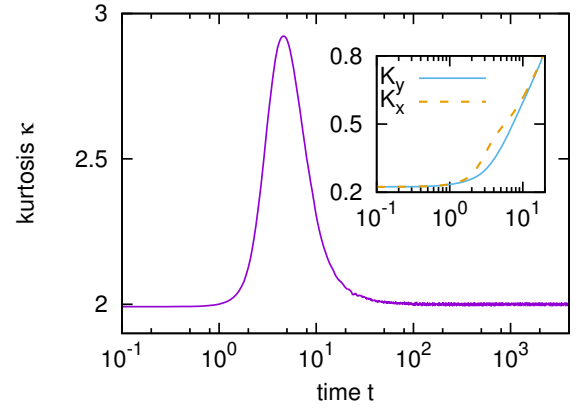


FIG. 7: $\kappa(t)$ as a function of time for numerical simulations of the GPE. The initial conditions are provided in equation (14). In the inset, the evolution of the spectral bandwidth is shown.

ably due to some fast evolution related to an instability of the modulational instability type, see [23]. This results in a fast increase of the kurtosis, see Figure 7 up to $t = 5$. After this initial transient, the spectrum grows also in the transverse direction and the value of the kurtosis reduces accordingly. The signs of β and γ in equation (12) are now opposite, therefore they have a competing effect: an increase of the spectral bandwidth in the k_x direction leads to a increase of the kurtosis, while the opposite happens for an increase of the spectral bandwidth in the k_y direction. After some times the spectrum appears again isotropic and the contribution in (12) from the spectral bandwidth cancels out and the kurtosis tends to its Gaussian value. A snapshot of the intensity of the wave field taken at the time when the kurtosis has a maximum is

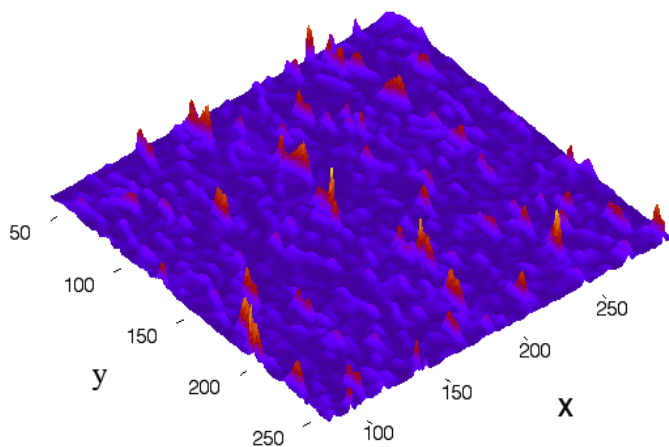


FIG. 8: $|\psi|^2$ as a function of x and y at the time of maximum kurtosis, see Fig. 7.

reported in Fig. 8: clearly, the field is characterized by the presence of a number of rogue waves embedded in an incoherent wave field.

Before concluding, we find opportune to make a comment on the evolution of the spectral bandwidth. So far, we have discussed an analytical result which provides an interesting perspective on the generation of heavy tails. However, equations (9) and (12) are not closed: an evolution equation for the spectrum is still required. The present approach consists in considering the weakly nonlinear limit and derive the standard four-wave *wave kinetic equation* (see [17]) from the (non-integrable) NLS type of equation using the wave turbulence theory. In such an equation the linear energy is a constant of mo-

tion; this is a consequence of the fact that the transfer of energy and number of particle is ruled by exact resonance interactions. Therefore, even though the spectrum may evolve, the spectral bandwidth (related to the quadratic contribution to the Hamiltonian density) remains constant. Thus, if one is interested in studying changes in statistical properties of the wave field, the need of considering non-resonant interactions in the kinetic equation is essential, see [15, 19, 24, 25] for details on the subject.

In conclusion, we have presented an identity for a class of equations characterized by the NLS nonlinearity that relates the variation of the kurtosis with the variation of the spectral bandwidth. In general, given an initial condition characterized by a narrow spectrum with random phases, the spectrum broadens due to the nonlinearity of the problem. This broadening may be accompanied by a change of the statistical properties of the wave field and, in particular, of the kurtosis. The latter weights the tails of the PDF. It should be noted that our approach is rather general as it can be applied whenever a conserved quantity of a partial differential equation contains a moment of the distribution. For example, in the Korteweg-de Vries (KdV) equation, the Hamiltonian is directly connected to the skewness; therefore, given the evolution of the spectrum, a direct information on the asymmetry of the PDF of u is available. **Acknowledgments** M.O. was supported MIUR Grant PRIN 2012BFNWZ2. Dr. B. Giulino is acknowledged for discussions. P.S. and S.R. were partially supported by the Labex CEMPI (ANR-11-LABX-0007-01). The two-dimensional simulations were carried out on the High Performance Computing Cluster at the University of East Anglia.

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