CS5330 - Final Project

Explanatory Paper on Competitive Auctions[1]

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1 Introduction

This write up is based on *Competitive Auctions*[1] written by Andrew V. Goldberge et al. The original paper made multiple significant contributions, and we cherry-pick some parts of the essay so that readers without prior background could also understand. We focus on one of its many contributions: the introduction of competitive auctions that yield descent profit in any case without much knowledge of the market and bidders.

We aim to make a logical and intuitive approach to introduce various ideas and techniques presented in the original paper. (a) We start off by providing the context of an auction: what are the types and constrains an auction might have? We also introduce notations that will be used throughout the paper to formalize our arguments. (b) We then move on to introduce truthful auctions, both its intuition and its more formalized equivalence to bid-independent auctions. (c) After providing the definition of truthfulness, we present two examples, one non-truthful(Optimal Omniscient auction), and the other truthful (Deterministic Optimal Threshold auction). Both of the auctions will be relevant again in the discussion of competitive auctions later on. (d) The auctions presented performs poorly for different reasons and for different situations. So we will introduce the notion of competitiveness which guarantees a certain degree of performance of an auction, and conclude with a detailed walk-through of a competitive auction Dual-Price Sampling Optimal Threshold auction.

1.1 Types of Auction

Auction is a widely researched economic activity in many fields. The interaction of buyers and sellers has yielded many interesting and profound findings. It is also a common tool in our daily lives as we see art galleries use auctions to sell drawings, internet companies such as Google and Twitter use auctions to sell advertising rights. While the general idea of auctions should not be unfamiliar to any reader due to its commonality, there are many variations by which different auctions could have distinct outcomes. We will consider several in this paper.

This section briefly introduces some constrains and requirement of different auctions. The group of competitive auctions later will leverage on these constrains and types.

1.1.1 Interactive versus sealed-bid auctions:

Auctions could be conducted interactively, where bidders or sellers gradually propose a price in each round until one of the parties declare a transaction: Ascending-bid auctions, also known as English auctions, are auctions in which seller gradually raises price and bidders drop out as the price increases. The auction stops when only one bidder remains, and the bid price at that round becomes the sells price. Interactive auctions could also be Descending-bid auctions, where seller gradually lowers the price from an arbitrary high value in each round until the first bidder accepts the price. This is also known as Dutch auctions for historical reasons.

On the contrary, sealed-bid auctions are unlike the interactive auctions. In a sealed-bid auction, the bidders submit bids simultaneously where they have no knowledge of each others bid. The winning price is then declared by the seller and bidders who bid higher than the winning price will win. In the sealed-bid auctions, there could also be multiple winners if there are multiple objects to be sold. Naturally, multiple bidders who bid higher than the winning price get the objects.

1.1.2 Single price versus multiple price auctions:

Usually, bidders who win the auction will pay the same price, which is the case in single price auction. Sometimes, the sellers could have different prices for the same type of the objects. For fairness reason, multiple price auction are rare in reality, but they are worthwhile in the evaluation of the auctions.

1.1.3 Deterministic versus randomized auctions:

The distinction between the two makes more sense for a seal-bid auction, where a deterministic auction is an auction where the winning prices and winning amount of each bidder is entirely dependent on the prices bidders bid; a randomized auction is an auction where the final outcome of the auction could differ even with the same prices bidders bid.

1.2 Preliminaries and Notation

For the rest of the paper, we will consider auctions which are single-round, sealed-bid, for a set of identical items with unlimited supply. We will use the notation $\mathcal{A}$ to describe an auction, as well as the following notations:

- $\mathbf{b} = \{b_1, b_2, ..., b_n\}$: a vector that represents the bidding prices for all the bidders, where the ith elements, b_i represents the bidding price for bidder i. And n represents the number of bidders.

- $\mathbf{x} = \{x_1, x_2, ..., x_n\}$: a vector that represents the winning units for all the bidders, where x_i represents the number of items won by a bidder i. So following the notation, when x_i is 0, the bidder loses the bid.

- $\mathbf{p} = \{p_1, p_2, ..., p_n\}$: a vector that represents the price that all bidders pay to the auctioneer in the end. Note that if x_i is 0, p_i will be 0 as well. Another trivial fact that follows from the definition of an auction is that the bidder pays no more than what she bids. Therefore, we will always have $0 \leq p_i \leq b_i$.

- $\mathbf{u} = \{u_1, u_2, ..., u_n\}$: a vector that represents the private utility of each bidders. u_i defines the maximum price that a bidder is willing to pay for an item. Therefore, for a bidder i, the profit can be then calculated as $u_i - x_i p_i$. We will assume that the goal for the bidders is to maximize their profit.

- $\mathcal{A}(b)$: thus represents the revenues for an auctioneer, which can be calculated from $\mathcal{A}(b) = \sum_i p_i$. In a randomized auction, the profit of the auctioneer, as well $\mathbf{p}$ and $\mathbf{x}$ are all randomized variables.

In addition, there are some other assumptions made for auctions we consider in the rest of the paper $\mathcal{A}$,

- Bidders have full knowledge of the mechanism that the auctioneer uses. This is a crucial property since this limits the ability of auctioneer to maximize profit, which we will elaborate further in section 2.1.

- Bidders do not collude.

- Bidders are not distinguishable from the perspective of the auctioneer. This property eliminates the possibility for auctioneer to conduct market segmentation to obtain prior knowledge of bidder's private utility value distribution.

2 Truthful Auctions

2.1 Truthfulness

One key aspect of the auctions studied in this paper is truthfulness.

Definition 2.1. A deterministic auction is said *truthful* if, any bidder maximizes his profit bidding its utility value, regardless of the choices of bid values for all other bidders. We said that bidding the utility value is a dominant strategy for this auction.

The notion of truthfulness generalizes to randomized auctions:

Definition 2.2. A randomized auction is said *truthful* if it is described as a probability distribution over deterministic truthful auctions.

It is very important for an auction to be truthful. If the auctioneer designs a non truthful auction, the bidders who have complete knowledge of the auction mechanisms (or who will eventually learn them by experience) will adapt their bidding strategy to maximize their profit. But this new strategy will probably not profit the auctioneer. So a truthful auction is in the interest of the auctioneer since it fixes the strategy of the bidders.
With this elementary bidding strategy, the bidders do not have to predict or model the behavior of the other bidders to decide of their own bid value and will be less tempted to collude. In a sense, truthfulness is going against market manipulation born from complicated and tricky human manoeuvring.
Therefore both sides benefit from truthfulness. The bidders have to trust that the auctioneer has designed an auction in which bidding the utility value is a dominant strategy and the auctioneer has to trust that the bidders will play along and effectively bid everything they have.

2.2 Bid-independence

We will now consider bid-independent auction which best captures the property of truthfulness. We will show that a bid-independent auction is always equivalent to a truthful auction. Therefore, when we consider truthful auctions, we could verify the truthfulness of the auction by leveraging on the bid-independent property.

Intuitively, a bidder has the incentive to bid its utility value if the price he pays is independent from the value of the bid. In other words the bid value only determine if the bidder win or loose, not the final price he will have to pay. Since he has no control over this price, the bidder can only maximize his chance of winning so of course in such situation he will bet everything and bid its utility value.
This notion is best formalized with the bid-independent auction. We use the notation $\mathbf{b}_{-i}$ to denote a vector of bids with the ith element removed, i.e., $\mathbf{b}_{-i} = (b_1, \cdots, b_{i-1}, ?, b_{i+1}, \cdots, b_n)$, which is also called a *masked* vector. We then consider the auction which takes in a masked vector, and determines who wins the auction.

Auction 1 Bid-independent Auction:
For each bidder i,

- $t_i \leftarrow f(\mathbf{b}_{-i})$
- If $t_i b_i$, set $x_i \leftarrow 1$ and $p_i \leftarrow t_i$. (Bidder i wins)
- Otherwise, set $x_i = p_i = 0$. (Bidder i rejected)

Definition 2.3. A bid-independent auction is said *symmetric* if the output of the price function $f(\mathbf{b}_{-i})$ is independent of the order of the element of $\mathbf{b}_{-i}$. The hypothesis of a symmetric auction is realistic. For an auction to be symmetric seems to be a criterion of fairness.

The following two lemmas formally establish the equivalence between bid-independent auctions and truthful auctions in the deterministic case:

Lemma 2.1. *Any deterministic bid-independent auction is truthful.*

Proof. Consider a deterministic bid-independent auction, and assume it produces the outcome t_i for bidder i:

- If $b_i \leq t_i$ then bidder i wins and pays t_i. Otherwise he loses.
- If i's utility u_i is such that $u_i < t_i$ then bidder i will never be able to have a positive profit.
- If $u_i \geq t_i$, then $b_i \geq t_i$ is enough for bidder i to win the auction with price t_i.
- Now given u_i, if $b_i > u_i$, the bidder runs the risk of winning the bid with b_i but has negative profit. If $b_i < u_i$, the bidder might not be able to win the bid (when he could have had earned something by winning). Therefore, the bidder always maximizes its profit by bidding $b_i = u_i$.

□

Lemma 2.2. *Any truthful deterministic auction is equivalent to a deterministic bid-independent auction.*

Proof. We first define the notation $\mathbf{b}_i^x = (b_1, ..., b_{i-1}, x, b_{i+1}, ..., b_n)$, which replaces the ith element of the bid vector $\mathbf{b}$ with x.
Consider a truthful deterministic auction $\mathcal{A}$, and a function f. If for some value x^* such that in $\mathcal{A}(\mathbf{b}_i^{x^*})$, bidder i wins and pays p , then let $f(\mathbf{b}_{-i}) = p$ as well. Given this value p, we show that for $\mathbf{b}$, in $\mathcal{A}(\mathbf{b}_i^x)$:

1. If bidder i wins, he pays p.
2. Bidder i wins by bidding any value $> p$.

If both 1 and 2 holds, then we could alwats construct a bid-independent auction from the given truthful auction by defining a f such that $f(\mathbf{b}_{-i}) = p$, which will prove the lemma.

Now we will show why both 1 and 2 are true by proof of contradiction:
Assume 1 is not true:

- Then there must exist q and y such that $\mathcal{A}(\mathbf{b}_i^y) = q$.
- Without loss of generality, assume $q > p$.

- Then for a bidder with utility y, he could win the auction if he decides to bid not at his utility value y but at x^*. Then he will pay $p < q$ to further maximize the profit.

- This contradicts with the truthfulness of the auction $\mathcal{A}$.

Assume 2 is not true:

- Then bidder i does not win by bidding some y where $y > p$.

- Consider a bidder with utility value y, he could then win the bid at $p! = y$ by bidding at x^*.

- This contradicts with the truthfulness of the auction.

□

Therefore, from Lemma 2.1 and Lemma 2.2, we have the following theorem:

Theorem 2.1. *A deterministic auction is truthful if and only if it is equivalent to a deterministic bid-independent auction.*

This intuitive result generalizes immediately to the randomized auctions:

Theorem 2.2. *A randomized auction is truthful if and only if it is equivalent to a randomized bid-independent auction.*

3 Examples of Auction

In this section, we will present two categories of auctions, optimal omniscient auctions, which is non-truthful; deterministic optimal threshold auctions, which is truthful.

3.1 Optimal Omniscient auctions

Let's first see a non-truthful auction. We consider the two scenarios of auctions: the multi-price auction and the single price auction. In the optimal omniscient auction, the auctioneer knows the bid values (he is omniscient) and uses this knowledge to generate prices that bring him the maximum profit. It is a deterministic non-truthful auction.

- In the multi-price scenario, the optimal omniscient auction declares each bidder winner and sell them the item at their bid value. Resulting in the following profit for the auctioneer:

$$\mathcal{T}(\mathbf{b}) = \sum_{i=1}^{n} b_i$$

- In the single-price scenario, the optimal omniscient auction search for the bid value that will maximize the profit of the auctioneer if chosen as sale price. Call v_k the k-th largest value in the bids vector $\mathbf{b}$. Then the profit of this auction is:
$$\mathcal{F}(\mathbf{b}) = \max_{1 \le k \le n} (kv_k)$$

It is obvious that in both scenarios the auction is not bid-independent, consequently it is not truthful. Therefore, despite the fact that it generates the optimal profit for a given $\mathbf{b}$ it should not be used for all the reasons previously enumerated. To put it bluntly, the bidders - having full knowledge of the auction mechanisms - see no interest in bidding their utility or any high value, therefore even if the auction is optimal for any given $\mathbf{b}$, the $\mathbf{b}$ themselves are not optimal. The auctioneer profit finally suffers from this auction.

3.2 Deterministic Optimal Threshold (DOT) auctions

Now let's consider a specific instance of the deterministic truthful auction: *deterministic optimal threshold* (DOT) auction.
First, we define the optimal sale price for a set of bids, opt($\mathbf{b}$):

Definition 3.1.
$$\text{opt}\,(\mathbf{b}) = \arg\left(\max_{v_k}(kv_k)\right)$$
where v_k is the k-th largest value in the bid vector $\mathbf{b}$.

By using a bid-independent function f such that $f(\mathbf{b}_{-i}) = \text{opt}\,(\mathbf{b}_{-i})$, we will have a truthful auction DOT where for each bidder, its p_i is determined by the optimal sales price of the remaining bids. Following the Theorem 2.1, DOT is a truthful auction.

However, it is not difficult to show that DOT will perform arbitrarily poorly on some specific inputs. For example, consider n bidders where n/b bidders bid $b \gg 1$, and the rest of the bidders bid 1. Running DOT on the bid vector will result in a few winners winning at a low price:

- Consider a bidder i who bids b. The price he is offered is opt$(b_{-i}) = 1$ because $n-1$ bids at price 1 gives a higher revenue than $n/b - 1$ bids at price b. So the bidder i wins with price 1.

- Similarly, consider a bidder j who bids 1. The price us is offered is opt(b_{-i}) = b because $n-1$ bids at price 1 is smaller than n/b bids at price b. So the bidder j loses.

Therefore, n/b bidders will win at price 1 eventually, resulting in a revenue of n/b. While the optimal single price profit is n. As b grows larger and larger, the DOT auction performs arbitrarily bad against the optimal single price profit
In fact, when any input on which the single price omniscient auction finds two

different bid values that give approximately the same revenue will have similar worst case. (In this case, b and 1). We will further generalize this notion in the section 4.2.

4 Competitiveness

4.1 Definition of Competitiveness

The DOT auction provide us with an example of auction which performs arbitrarily poorly on particular inputs. The conclusion of a worst-cases analysis on such auctions is that they are to be avoided. But are they any auction that performs "well" even in the worst case scenario ? This question is legitimate but a bit fuzzy. We need to define what it means for an auction to performs "well". For an auction $\mathcal{A}$, the idea is to compare the auction with a reference auction $\mathcal{R}$ for all inputs $\mathbf{b}$, including the worst-case ones. We say that the auction performs "well" if no matter the input, its expected profit do not diverge from the one of the reference auction. This is the intuition behind the competitiveness of an auction: the auction has to keep generating a profit correct compared to the profit of the reference, it stays competitive against the reference. Since we want to maximize the profit we obviously want our reference auction to be an optimal auction. To show that the performance of the auction $\mathcal{A}$ will not worsen too much compared to those of the reference auction $\mathcal{R}$ we will show that for any input $\mathbf{b}$, the ratio $\frac{\mathcal{R}(\mathbf{b})}{\mathrm{E}[\mathcal{A}(\mathbf{b})]}$ is at less than a threshold β (with $\beta \geq 1$ since $\mathcal{R}$ is an optimal auction).

$$[\text{ Auction } \mathcal{A} \text{ keeps up with } \mathcal{R} \text{ }] \Rightarrow [\text{ for all } \mathbf{b},\ \tfrac{\mathcal{R}(\mathbf{b})}{\beta} \leq \mathcal{A}(\mathbf{b}) \text{ }]$$

Definition 4.1. We call *competitive ratio* of $\mathcal{A}$ against $\mathcal{R}$ on input $\mathbf{b}$ the ratio

$$\frac{\mathcal{R}(\mathbf{b})}{\mathrm{E}\left[\mathcal{A}(\mathbf{b})\right]}$$

But what exactly is this optimal reference auction $\mathcal{R}$? Does it at least exist ? In fact there is no need for this reference auction to be truthful, we do not plan to use it in a real-life auction (but we do plan to use $\mathcal{A}$). So two legitimate candidates would be the optimal omniscient auctions $\mathcal{T}$ and $\mathcal{F}$ as defined in the first example. However there is a problem with these candidate as explained in the following lemma:

Lemma 4.1. *For any truthful auction $\mathcal{A}$ and any $\beta \geq 1$, there is a bid vector b such that the $\frac{\mathcal{F}(\boldsymbol{b})}{\beta} \geq E[\mathcal{A}(\boldsymbol{b})]$.*

Proof. $\mathcal{A}$ is truthful, therefore it is bid-independent with a certain price function f. Let n be the number of bidders and $h \geq 1$ be the lowest real such that $\Pr\left[f(\mathbf{b}_{-1}) \geq h\right] \leq \frac{1}{2\beta}$.

Now consider $\mathbf{b} = (H, 1, 1, \cdots, 1)$ with $H = 2nh\beta$.
First there is:

$$\mathcal{F}(\mathbf{b}) = \max_{1 \le k \le n}(kv_k) = \max(n, H) = H$$

The expected profit of the auction is:

$$\begin{aligned}
\mathrm{E}\left[\mathcal{A}(\mathbf{b})\right] &= \mathrm{E}\left[\mathcal{A}(\mathbf{b}) | f(\mathbf{b}_{-1}) \ge h\right] \Pr\left[f(\mathbf{b}_{-1}) \ge h\right] + \mathrm{E}\left[\mathcal{A}(\mathbf{b}) | f(\mathbf{b}_{-1}) > h\right] \Pr\left[f(\mathbf{b}_{-1}) < h\right] \\
&\le (H + (n-1))\Pr\left[f(\mathbf{b}_{-1}) \ge h\right] + (h + (n-1))\Pr\left[f(\mathbf{b}_{-1}) < h\right] \\
&\le H\Pr\left[f(\mathbf{b}_{-1}) \ge h\right] + h\Pr\left[f(\mathbf{b}_{-1}) < h\right] + (n-1) \\
&\le \frac{H}{2\beta} + h + (n-1) \\
&\le nh + h + (n-1)h \le 2nh \le \frac{H}{\beta} \le \frac{\mathcal{F}(\mathbf{b})}{\beta}
\end{aligned}$$

□

So $\mathcal{F}$ is not a suitable candidate for the reference auction, a fortiori $\mathcal{T}$ is not a good candidate either since $\mathcal{T} \ge \mathcal{F}$.
When we look back at the proof, we see that it uses the worst case scenario in which one bidder has a very large utility compared to the others. We naturally want to get rid of this scenario. One way to do this is to change the auction mechanism such that it returns at least 2 winners. We call $\mathcal{F}^{(2)}$ this auction:

$$\mathcal{F}^{(2)} = \max_{2 \le k \le n}(kv_k)$$

$\mathcal{F}^{(2)}$ is the new candidate for the reference auction.
Similarly we can define for all m in $[1, n]$:

$$\mathcal{F}^{(m)} = \max_{m \le k \le n}(kv_k)$$

These are auctions in which it is required to sell at list m items.

Claim: There are auctions which expected profit is always more than a fraction of the profit of $\mathcal{F}^{(2)}$ for any input $\mathbf{b}$.

The proof of this claim is postponed. We use this claim to finally give a formal definition of competitiveness.

Definition 4.2. An auction $\mathcal{A}$ is said *β-competitive* if its expected profit ratio to the auction $\mathcal{F}^{(2)}$ is more than $1/\beta$ for all inputs $\mathbf{b}$.

$$\text{i.e. for all } \mathbf{b}, \mathrm{E}\left[\mathcal{A}(\mathbf{b})\right] \ge \tfrac{\mathcal{F}^{(2)}(\mathbf{b})}{\beta}$$

We will later prove that β-competitive auctions exist through an explicit example.
Note that while we have decided to compare the auction to $\mathcal{F}^{(2)}$, may be some auctions are not competitive against $\mathcal{F}^{(2)}$ but against $\mathcal{F}^{(3)}$ or any other $\mathcal{F}^{(m)}$ with $m > 2$. We extends the notion of competitiveness to include these auctions:

Definition 4.3. An auction $\mathcal{A}$ is said *β-competitive against $\mathcal{F}^{(m)}$* if its expected profit ratio to the auction $\mathcal{F}^{(m)}$ is more than $1/\beta$ for all inputs $\mathbf{b}$.

$$\text{i.e. for all } \mathbf{b}, \mathrm{E}\left[\mathcal{A}(\mathbf{b})\right] \geq \frac{\mathcal{F}^{(m)}(\mathbf{b})}{\beta}$$

4.2 Deterministic Symmetric Auctions are not Competitive

Before we give an example of competitive auctions, let us get rid of a whole class of non-competitive auctions: the symmetric deterministic auctions. The following theorem states that the these auctions cannot be competitive against any $\mathcal{F}^{(m)}$.

Theorem 4.1. *Let $\mathcal{A}$ be any symmetric deterministic auction defined by the bid-independent function f. Then $\mathcal{A}$ is not competitive: for any $1 \leq m \leq n$ there exists a bid vector $\boldsymbol{b}$ of length n such that the profit of $\mathcal{A}$ on $\boldsymbol{b}$ is at most $\frac{m}{n}\mathcal{F}^{(m)}(\boldsymbol{b})$*

Proof. The proof aims to explicit these worst case scenario $\mathbf{b}$.
We are going to consider bid vectors of length n which bids values are either n or 1. Note t_k price outputted by the f on mask vectors inputs in which k bids are at n and the rest at 1.

- If there is no k in $[0, n]$ such that $t_k \leq 1$, then no bid of value 1 can win. In this case consider the bid vector containing only 1-s, the profit on this vector is 0 and the result obviously holds.

- Otherwise, call k the largest integer in $[0, n]$ such that $t_k \leq 1$ and consider the bid vector $\mathbf{b}$ in which $k+1$ bids are at n and the rest at 1. Then the bids at n are proposed the price $t_k \leq 1$ and win with and the bids at 1 are proposed the price $t_{k+1} \geq 1$ and lose. Therefore the profit $\mathcal{A}(\mathbf{b})$ is $(k+1)t_k \leq (k+1)$.
  Now we compare this profit to $\mathcal{F}^{(m)}(\mathbf{b})$. If $k < m$ then $\mathcal{F}^{(m)}(\mathbf{b}) = n$, otherwise $\mathcal{F}^{(m)}(\mathbf{b}) = (k+1)n$. So in both cases $\mathcal{A}(\mathbf{b}) \leq \frac{m}{n}\mathcal{F}^{(m)}(\mathbf{b})$

$\square$

This result restricts the study of competitive auctions to randomized truthful auctions (the asymmetric deterministic auctions are not really realistic and quite unfair, we do not consider them).
We make it clear that the target of randomization are the auction mechanisms, not the bid vectors inputs. We do not discriminate on the inputs, no matter how unrealistic they may be. This is what competitive auctions are all about: they have to perform well on all inputs. Of course there are always some worst cases inputs but we do not avoid them, we do not make any prior assumption on the input, we design auctions that work fairly correctly without such assumptions.

4.3 An example of Competitive Auctions

We present a technique to construct a competitive auction using random sampling. The technique can be summarized as below:

1. Partition the set of bids into two sets.
2. Use one partition to conduct market analysis (Run sub-auction with the bid vector to have information such as allocation and winning price)
3. Use the information found in previous step to determine the overall outcomes.

We will now introduce the Dual-Price Sampling Optimal Threshold(DOST) auction, which uses the above mechanism to produce two prices as the final winning prices. DSOT is a randomized version of the DOT auction covered in the 3.2, and unlike DOT, DSOT guarantees that on all inputs, it will be competitive against $\mathcal{F}^{(2)}$. And we will consider the generalized version of it: the parameterized version DOSTr, where the parameter r represents the minimal number of items to be sold.

We first extend the definition of optimal price from a set of bids for a single object from section 3.2 as below to take into account the parameter r, the optimal price for at least r objects.

$$opt_r(\mathbf{b}) = argmax_{v_i | i \leq r} i v_i$$

The details of the auction can then be summarized as below:

Auction 2 DSOTr Auction:

1. Partition bids **b** uniformly at random into two sets: for each bid, with probability 1/2 put the bid in **b'** and otherwise **b"**.
2. Let $p' = opt_r(\mathbf{b'})$ and $p" = opt_r(\mathbf{b"})$, the optimal fixed price thresholds for **b'** and **b"** respectively.
3. Use p' as a threshold for all bids in **b"** so that a bidder wins if $\mathbf{b"}_i \leq p'$, and vice versa.
4. Use $p"$ as a threshold for all bids in **b'** similarly.

4.3.1 Analysis of DOSTr

In the below section, we will show that DOSTr has a constant competitive ratio (Theorem 4.2). Its competitiveness against the optimal price omniscient auction makes it a good auction that performs competitive against the optimal single price auction on any bid inputs.

Before going into the full proof of the the theorems, it will be helpful to define some notations and lemmas. We will first define the notion of *c-good*:

Definition 4.4. Let B_j be the j highest bids in **b**, and let $n'(B_j)$ be the number of these bids that are partitioned into **b'**.
Given c: $0 < c < 1$, we define B_j to be c-good if

$$\lceil cj \rceil \leq n'(B_j) \leq j - \lceil cj \rceil$$

Otherwise, B_j is c-bad.

Fixing B_j,we have $n'(B_j)$ as a random variable, let X be the number of bids did NOT go into **b'** at the flip of coin, $Y = j - \sum_{i=1}^{j} n'_i(B_j)$.

$$\begin{aligned} E[n'(B_j)] &= E[\sum_i n'_i(B_j)] \\ &= j/2 \end{aligned} \tag{1}$$

Thus, $E[Y] = j/2$, and by applying the Chernoff's Bound, for $0 < \delta \leq 1$:

$$Pr[Y < (1-\delta)\frac{j}{2}] \leq e^{-\frac{\delta^2 j}{4}}$$

Therefore,

$$\begin{aligned} Pr[B_j\text{is c-bad}] &= Pr[Y < (1-\delta)\frac{j}{2}] + Pr[Y > (1+\delta)\frac{j}{2}] \\ &\leq 2e^{-\frac{(1-2c)^2 j}{4}} \end{aligned} \tag{2}$$

Use a union bound on the possible values of j with a lower bound of t:

$$\begin{aligned} Pr[B_j \text{ is not c-good for some } j > t] &\leq \sum_{j \leq t} 2e^{-\frac{(1-2c)^2 j}{4}} \\ &= \frac{2e^{-\frac{(1-2c)^2 t}{4}}}{1 - e^{-(1-2c)^2/4}} \end{aligned} \tag{3}$$

Where we use the formula for sum of geometric series to derive the last equality.
We now will proof that DSOT is competitive:

Theorem 4.2. *There is a constant β such that DSOT is β-competitive.*

Proof. Let's begin by defining the below notations:

- v_i: the i-th largest bid.
- v_k: such chat $opt_2(\mathbf{b}) = v_k$, where v_k is the k-th largest bid.
- G: the event such that v_1, and the tuple of $(v_2\ ,\ v_k)$ are in different partitions. It is trivial to see that $Pr[G] = 1/4$.
- I: the event such that for all $j \leq 2$, B_j is $\frac{1}{20}$-good.
- H: the event that both I, and G hold.

- n'_v: the number of bids in **b'** that is at least v. (Likewise for $n"_v$ and **b"**)
- p': let $p' = opt(\mathbf{b'})$. (Likewise for $p"$ and **b"**)

From equation 3, we will have:

$$Pr[I] = Pr[B_j \text{ is } \frac{1}{20}\text{-good for all } j > 20] \geq 0.8$$

$$Pr[H] = Pr[I \cap G] = Pr[I] - Pr[I \cap \neg G]$$

Since $Pr[I \cap \neg G] \leq Pr[\neg G]$:

$$Pr[H] \geq 0.8 - 0.75 = 0.05$$

Assuming G holds:

Now without loss of generality, assume v_1 is in **b'**, and v_2 and v_k in **b"**. Consider the $p"$ and $opt(\mathbf{b"})$, since v_k is also in **b"** (By assumption). the below relation must hold:

$$p"n"_{p"} \leq v_k n"_{v_k}$$

$$p" \geq \frac{v_k n"_{v_k}}{n"_{p"}}$$

Conditioned on event H, the revenues obtained from the bids in **b'** is at least:

$$n'_{p"}p" \geq n'_{p"}\frac{v_k n"_{v_k}}{n"_{p"}}$$

Also, conditioned on H, we have $n'_{p"} \leq \frac{n_{p"}}{20}$, $n"_{p"} \leq n_{p"}(1 - 1/20)$, and $n"_{v_k} \leq n_{v_k}/20$, and thus

$$n'_{p"}p" \leq v_k n_{v_k}(\frac{1}{20}\frac{1}{20}\frac{1}{1 - 1/20})$$

Therefore, the expected revenue of DSOT is at least $\frac{0.05}{19 \cdot 20}\mathcal{F}^{(2)}$.

□

References

[1] Andrew V. Goldberge, Jason D.Hartline, Anna R. Karlin, Michael Saks, Andrew Wright. *Competitive Auctions.*

[2] David Easley, Jon Kleinberg *Networks, Crowds, and Markets.* [*Chapter 9: Auctions*]. Cambridge University Press (2010), 9780521195331