Quiz Assignment 3 Solutions

Note: All multi-qubit state representations are big endian, unless specified otherwise

Multi-Qubit States and Operations, Quantum Entanglement

1. Consider the following bases:
   a. $\{ |0\rangle|0\rangle, |1\rangle|1\rangle\}$
   b. $\{ |+\rangle| +\rangle, |-\rangle|-\rangle\}$
      A. a. and b. both are orthonormal bases.
      B. a. is an orthonormal basis, but b. is not.
      C. a. is not an orthonormal basis, but b. is an orthonormal basis.
      D. Neither a. nor b. is an orthonormal basis.

   Solution: D
   While these sets are certainly orthonormal, they do not span the entire two-qubit state space, meaning not all two qubit states can be written using only these vectors. Two more basis vectors would be needed for that. So, the correct option is D.

2. Consider the following bases:
   a. $\{ |0\rangle| +\rangle, |1\rangle|-\rangle\}$
   b. $\{ |0\rangle|i\rangle, |1\rangle| -i\rangle\}$
      A. a. and b. both are orthonormal bases.
      B. a. is an orthonormal basis, but b. is not.
      C. a. is not an orthonormal basis, but b. is an orthonormal basis.
      D. Neither a. nor b. is an orthonormal basis.

   Solution: D
   While these sets are certainly orthonormal, they do not *span* the entire two-qubit state space, meaning not all two qubit states can be written using only these vectors. Two more basis vectors would be needed for that. So, the correct option is D.

3. Is the operation $(X \otimes I) \circ (I \otimes X)$ equivalent to $X \otimes X$?
      A. True
      B. False

   Solution: A
   This can be seen from matrix multiplication or by evaluating the action of both operations on a general two-qubit state. $(X \otimes I)$ is a bit flip on the first qubit only. $(I \otimes X)$ is a bit flip on the second qubit only. This is equivalent to a bit flip on both qubits separately, which is $X \otimes X$.

4. The operation $H \otimes H$ is self-adjoint
      A. True
      B. False

   Solution: A
   The easiest way to see this is to note that the Hadamard transformation on a single qubit is self-adjoint. Thus, $(H \otimes H)^\dagger = H^\dagger \otimes H^\dagger = H \otimes H$. Alternatively, we can also see that $(H \otimes H) \circ (H \otimes H) = I$. Since $(H \otimes H)$ is a unitary operator, this means that $(H \otimes H)^\dagger = H \otimes H$.

5. A general multi-qubit state can be visualised on one or more Bloch spheres
   - A. True
   - **B. False**

Solution: B

Only product states (or separable states) can be visualized on one or more Bloch spheres. For entangled states, the individual qubits are not in a definite single-qubit state and hence they cannot be visualised thus.

6. How many real numbers are required to completely describe a two-qubit state?
   - A. 4
   - **B. 6**
   - C. 8
   - D. 5

Solution: B

Two-qubit states can be represented using column vectors for 4 complex numbers. 4 complex numbers can be described by 8 real numbers (real part and imaginary part for each). However, there is also the added constraint that the vectors must be normalised. There is another constraint that two states are equivalent if they differ only up to a global phase. This brings down the number of real parameters to 6. For a more detailed description, please refer to the slides for multi-qubit states.

7. If we apply the operations $CNOT_2^1 \circ (I \otimes H)$ to the $|0\rangle|0\rangle$ state, the resultant state is:
   - A. Entangled
   - **B. Separable**

Solution: B

$(I \otimes H)|0\rangle|0\rangle = |0\rangle|+\rangle$. Since the first qubit is still in the $|0\rangle$ state, the controlled-NOT operation will not change anything and acts like an identity transformation. The final state is separable.

8. In which representation would a multi-qubit state be written as $|q_0, q_1, \cdots, q_{n-2}, q_{n-1}\rangle$?
   - **A. Big Endian**
   - B. Little Endian

Solution: A

Big endian states end in the MSB on the right. That is the case in the state above.

9. If the Bell rotation $CNOT \circ (H \otimes I)$ is applied to the initial state $|B_{00}\rangle$, the resultant state is:
   - **A. Entangled**
   - B. Separable

Solution: A

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \xrightarrow{(H\otimes I)} \frac{1}{\sqrt{2}}(|+\rangle|0\rangle + |-\rangle|1\rangle) = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle - |11\rangle)$$

$$\xrightarrow{CNOT} \frac{1}{2}(|00\rangle + |11\rangle + |01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|-\rangle|0\rangle + |+\rangle|1\rangle)$$

This is not a separable state. We can prove this by contradiction. Assuming that the above state can be separated into single qubit states, the following decomposition is possible

$$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$$

From the definition of the tensor product, this gives us the following 4 equations:

$ac = 1$

$ad = 1$

$bc = -1$

$bd = 1$

Dividing the first equation by the second gives $\frac{c}{d} = 1$, whereas dividing the third equation by the fourth gives $\frac{c}{d} = -1$. Clearly this is a contradiction and hence, these equations do not have a solution. We can safely conclude that the state is not separable.

10. If the Bell rotation $CNOT \circ (H \otimes I)$ is applied to the initial state $|0\rangle|+\rangle$, the resultant state is:
    A. Entangled
    B. Separable

Solution: B

$$|0\rangle|+\rangle \xrightarrow{(H\otimes I)} |+\rangle|+\rangle = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle + |11\rangle) \xrightarrow{CNOT} \frac{1}{2}(|00\rangle + |11\rangle + |01\rangle + |10\rangle) = |+\rangle|+\rangle$$

The state is separable.

11. The quantum states $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$ and $\frac{1}{\sqrt{2}}(|+\rangle|+\rangle + |-\rangle|-\rangle)$ refer to the same quantum state.
    A. True
    B. False

Solution: A

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$\frac{1}{\sqrt{2}}(|+\rangle|+\rangle + |-\rangle|-\rangle) = \frac{1}{\sqrt{2}}\left[\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) + \left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)\right]$$
$$= \frac{1}{2\sqrt{2}}\left[\big((|0\rangle + |1\rangle)(|0\rangle + |1\rangle)\big) + \big((|0\rangle - |1\rangle)(|0\rangle - |1\rangle)\big)\right]$$
$$= \frac{1}{2\sqrt{2}}[(|00\rangle + |01\rangle + |10\rangle + |11\rangle) + (|00\rangle - |01\rangle - |10\rangle + |11\rangle)] = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

The two states are the same

12. The quantum states $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle)$ and $\frac{1}{\sqrt{2}}(|\mathbf{i}\rangle|\mathbf{i}\rangle + |-\mathbf{i}\rangle|-\mathbf{i}\rangle)$ refer to the same quantum state.
    A. True
    B. False

Solution: A

$|\mathbf{i}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $|-\mathbf{i}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

$$\frac{1}{\sqrt{2}}(|\mathbf{i}\rangle|\mathbf{i}\rangle + |-\mathbf{i}\rangle|-\mathbf{i}\rangle)$$
$$= \frac{1}{\sqrt{2}}\left[\left(\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\right) + \left(\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\right)\right]$$
$$= \frac{1}{2\sqrt{2}}\left[\left((|0\rangle + i|1\rangle)(|0\rangle + i|1\rangle)\right) + \left((|0\rangle - i|1\rangle)(|0\rangle - i|1\rangle)\right)\right]$$
$$= \frac{1}{2\sqrt{2}}[(|00\rangle + i|01\rangle + i|10\rangle - |11\rangle) + (|00\rangle - i|01\rangle - i|10\rangle - |11\rangle)] = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
The two states are the same

13. The quantum state $\frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle)$ is not entangled.
    A. True
    B. False

Solution: B

This is not a separable state since neither of the qubits is in a definite single qubit state. Refer to Q9.

14. The representation of the state $|0\rangle|0\rangle$ in terms of Bell basis is
    A. $|00\rangle = \frac{1}{\sqrt{2}}(|B_{00}\rangle - |B_{10}\rangle)$
    B. $|00\rangle = \frac{1}{\sqrt{2}}(|B_{10}\rangle + |B_{11}\rangle)$
    C. $\boldsymbol{|00\rangle = \frac{1}{\sqrt{2}}(|B_{00}\rangle + |B_{10}\rangle)}$
    D. $|00\rangle = \frac{1}{\sqrt{2}}(|B_{10}\rangle - |B_{11}\rangle)$

Solution: C

As defined in the lectures, $|B_{00}\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$ and $|B_{10}\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle)$. It is easily seen that $|00\rangle = \frac{1}{\sqrt{2}}(|B_{00}\rangle + |B_{10}\rangle)$

15. Consider a two-qubit system with Qubit "1" being in the state $|\psi\rangle_1 = |+\rangle$ and Qubit "2" is in the state $|\phi\rangle_2 = |-\rangle$. The vector form of the state $|\psi\rangle|\phi\rangle$ is:

A. $\boldsymbol{\frac{1}{2}\begin{pmatrix}+1\\-1\\+1\\-1\end{pmatrix}}$ B. $\frac{1}{2}\begin{pmatrix}+1\\+1\\-1\\-1\end{pmatrix}$ C. $\frac{1}{2}\begin{pmatrix}+i\\-1\\+i\\-1\end{pmatrix}$ D. $\frac{1}{2}\begin{pmatrix}+1\\-i\\+1\\-i\end{pmatrix}$

Solution: A

$$|\psi\rangle|\phi\rangle = |+\rangle|-\rangle = \frac{1}{2}[(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)] = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$
$$= \frac{1}{2}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}+1\\-1\\+1\\-1\end{pmatrix}$$

16. The matrix form of the operator $X \otimes H$ is:

A $\boldsymbol{\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1&1\\0&0&1&-1\\1&1&0&0\\1&-1&0&0\end{pmatrix}}$ B $\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&-1\\1&0&-1&0\end{pmatrix}$ C $\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&-1&-1\\0&0&-1&1\\1&1&0&0\\1&-1&0&0\end{pmatrix}$ D $\frac{1}{\sqrt{2}}\begin{pmatrix}0&-1&0&1\\-1&0&1&0\\0&1&0&-1\\1&0&-1&0\end{pmatrix}$

Solution: A

$$X \otimes H = \begin{pmatrix}0&1\\1&0\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1&1\\0&0&1&-1\\1&1&0&0\\1&-1&0&0\end{pmatrix}$$

17. In a system of two qubits, Y gate is applied to the second qubit and the X gate is applied to the first qubit. The action of this operation on basis element $|00\rangle$ and the resultant state when this operation is applied to the initial state $|01\rangle$ are respectively
    A. $\boldsymbol{i|11\rangle}$ and $\boldsymbol{|10\rangle}$
    B. $i|10\rangle$ and $|11\rangle$
    C. $-|11\rangle$ and $|10\rangle$
    D. $i|11\rangle$ and $-i|10\rangle$

Solution: A

The complete operator is $X \otimes Y$. Now, the action (as defined in lecture slides on multi-qubit operations)

$$X \otimes Y\,|00\rangle = X|0\rangle \otimes Y\,|0\rangle = |1\rangle \otimes i|1\rangle = i|11\rangle$$

$$X \otimes Y\,|01\rangle = X|0\rangle \otimes Y\,|1\rangle = |1\rangle \otimes -i|0\rangle = -i|10\rangle \approx |10\rangle \text{ (upto a global phase)}$$

In the last step we have also calculated the resultant state, which we get by removing the global phase after evaluating the action.

18. The matrix form of the operator $Z \otimes Y$ is:

A $\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$ B $\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$ C $\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ D $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

Solution: A

$$Z \otimes Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

19. The action of the transformation $(H \otimes X) \circ CNOT$ on the basis element $|11\rangle$ is:
    A. $\boldsymbol{\frac{|01\rangle - |11\rangle}{\sqrt{2}}}$ B. $\frac{|01\rangle - i|10\rangle}{\sqrt{2}}$ C. $\frac{|00\rangle + i|11\rangle}{\sqrt{2}}$ D. $\frac{|01\rangle + |11\rangle}{\sqrt{2}}$

Solution: A

$$|11\rangle \xrightarrow{CNOT} |10\rangle \xrightarrow{(H\otimes X)} |-\rangle|1\rangle = \frac{|01\rangle - |11\rangle}{\sqrt{2}}$$

20. The state $\frac{|+\rangle|-\rangle + |-\rangle|+\rangle}{\sqrt{2}}$ is an entangled state
    A. True
    B. False

Solution: A

Neither of the qubits is in a separable single-qubit state. This can also be seen by writing the state in the standard basis.

$$\frac{|+\rangle|-\rangle + |-\rangle|+\rangle}{\sqrt{2}} = \frac{(|0\rangle + |1\rangle)(|0\rangle - |1\rangle) + (|0\rangle - |1\rangle)(|0\rangle + |1\rangle)}{2\sqrt{2}}$$

$$= \frac{(|00\rangle - |01\rangle + |10\rangle - |11\rangle) + (|00\rangle + |01\rangle - |10\rangle - |11\rangle)}{2\sqrt{2}} = \frac{|0\rangle|0\rangle - |1\rangle|1\rangle}{\sqrt{2}} = |B_{10}\rangle$$

This is an entangled state.