Multi-Qubit Transformations

Introduction to Quantum Computing

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Outline

Single qubit transformations

- Consider the single qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$.
- Consider a transformation A that is being applied to $|\psi\rangle$. Let the matrix form of A be given as:

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} ; \; A_{ij} \in \mathbb{C} \; \forall \, i, j \in \{0, 1\}$$

- The action of A on $|\psi\rangle$ is given by $A|\psi\rangle$ which is expressed as:

$$A|\psi\rangle = \underbrace{\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{2\times 1} = \underbrace{\begin{pmatrix} aA_{00} + bA_{01} \\ aA_{10} + bA_{11} \end{pmatrix}}_{2\times 1}$$

'A' is acting on $|\psi\rangle$

- This expression is in its most general form of and requires matrix multiplication.

Linearity: a key property

- Consider the action of A on the computational basis $\{|0\rangle, |1\rangle\}$.

$$A|0\rangle = \begin{pmatrix} \boxed{\begin{matrix} A_{00} \\ A_{10} \end{matrix}} & \begin{matrix} A_{01} \\ A_{11} \end{matrix} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{00} \\ A_{10} \end{pmatrix}$$

similarly,

$$A|1\rangle = \begin{pmatrix} \begin{matrix} A_{00} \\ A_{10} \end{matrix} & \boxed{\begin{matrix} A_{01} \\ A_{11} \end{matrix}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} A_{01} \\ A_{11} \end{pmatrix}$$

- This implies the general result of applying a transformation A on $|\psi\rangle$ can simply be expressed in terms of the *actions* as follows:

$$A|\psi\rangle = aA|0\rangle + bA|1\rangle = \begin{pmatrix} aA_{00} + bA_{01} \\ aA_{10} + bA_{11} \end{pmatrix}$$

- This expression is also a general form but only requires vector addition.

Gates and Basis vectors

- Any state $|\psi\rangle$ can be represented as a superposition of the basis vectors, $|0\rangle$ and $|1\rangle$.
- Therefore, the effect of A on any state $|\psi\rangle$ is defined completely by the *actions* $A|0\rangle$ and $A|1\rangle$.
- Gate operations are unitary transformations (also linear) and so, it is possible to define the *action* of these gates by applying them to the computational basis.
- Some gate operations in their matrix form are shown below:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; ; \; S = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \; ; \; P_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathbf{i}\phi} \end{pmatrix}$$

$i = \sqrt{-1}$

$i^2 = -1$

or

R_ϕ

Gates and their action

earlier it was shown that $Y_s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$Y|0\rangle = -|1\rangle$
$Y|1\rangle = |0\rangle$

in this present^n

$Y_J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Gate (A)	$A\|0\rangle$	$A\|1\rangle$
X	$\|1\rangle$	$\|0\rangle$
Y	$\mathbf{i}\|1\rangle$	$-\mathbf{i}\|0\rangle$
Z	$\|0\rangle$	$-\|1\rangle$
S	$\|0\rangle$	$\mathbf{i}\|1\rangle$
H	$\|+\rangle$	$\|-\rangle$
P_ϕ	$\|0\rangle$	$e^{\mathbf{i}\phi}\|1\rangle$

$Y_s = i\, Y_J$

Table 1: The *actions* of commonly used single qubit gates on the computational basis.

A note on phases

- The actions of gates on the basis states are given with phase factors.
- These factors can be dropped only when they are global phases, for instance consider the state $|0\rangle$ being acted upon by the Y gate.

actual state

$$
\begin{aligned}
Y\,|0\rangle &= \mathbf{i}\,|1\rangle \\
&\equiv |1\rangle
\end{aligned}
$$

global phase

In this discussion, $\equiv$ is used to show equivalence.

- However if the same gate is applied on the $|+\rangle$ state, $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$e^{i\phi} \rightarrow$ phase factor

$$
\begin{aligned}
Y\,|+\rangle &= \frac{1}{\sqrt{2}}\left(Y\,|0\rangle + Y\,|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\mathbf{i}\,|1\rangle - \mathbf{i}\,|0\rangle\right) = \frac{i}{\sqrt{2}}\left(|1\rangle - |0\rangle\right) \\
&\equiv \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)
\end{aligned}
$$

global phase

$\frac{-i}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$

Products of state vectors

$q_1 - |\psi\rangle_1$

$q_2 - |\phi\rangle_2$

1 & 2 are physically distinct qubits

- Consider two qubits labelled "1" and "2", in states $|\psi\rangle_1$ and $|\phi\rangle_2$.
- The two qubit state can now be expressed in terms of the tensor product:

$$|\psi\rangle_1 \otimes |\phi\rangle_2 \equiv |\psi\rangle |\phi\rangle \equiv |\psi\phi\rangle$$

- The labels "1" and "2" are present to denote the order in which the tensor product is performed. They are dropped with the understanding that the ordering is always kept in mind.
- It should also be noted that.

$$|\psi\rangle_1 \otimes |\phi\rangle_2 \not\equiv |\phi\rangle_2 \otimes |\psi\rangle_1$$

Definition of the Tensor Product

2x2

- If the operators are defined as before, $A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$ and $B = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}$, then the tensor product $A \otimes B$ is written as,

2x2

$\otimes$ is defined for any 2 matrices of any size

$$A \otimes B = \begin{pmatrix} A_{00}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} & A_{01}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \\ A_{10}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} & A_{11}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} A_{00}B_{00} & A_{00}B_{01} & A_{01}B_{00} & A_{01}B_{01} \\ A_{00}B_{10} & A_{00}B_{11} & A_{01}B_{10} & A_{01}B_{11} \\ A_{10}B_{00} & A_{10}B_{01} & A_{11}B_{00} & A_{11}B_{01} \\ A_{10}B_{10} & A_{10}B_{11} & A_{11}B_{10} & A_{11}B_{11} \end{pmatrix}$$ 4x4

- This is the complete operator that will act on the state $|\psi\phi\rangle$.
- The tensor product is also referred to as the *direct product* or the *Kronecker product* and is complicated to handle in general.

Tensor Product of Basis vectors

qubit '1'

qubit '2'

- If we are given two single qubit basis $\{|0\rangle_1, |1\rangle_1\}$ and $\{|0\rangle_2, |1\rangle_2\}$

- The two-qubit *tensor product* basis elements are defined as follows.

$|0\rangle_1 \otimes |0\rangle_1$ this is not meaningful

$|0\rangle_1 \otimes |1\rangle_2$ is allowed

$|1\rangle_1 \otimes |1\rangle_2$

$$|0\rangle_1 \otimes |0\rangle_{\cancel{1}\,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 = \begin{pmatrix} 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \\ 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

4×1 vector

- Other basis vectors are similarly calculated.

- Once again the order is **very** important and **should not be changed**.

Operator Products

$|0\rangle_1 \otimes |0\rangle_2$ $|1\rangle_1 \otimes |0\rangle_2$
$|0\rangle_1 \otimes |1\rangle_2$ $|1\rangle_1 \otimes |1\rangle_2$

- The previously defined two-qubit state $|\psi\rangle_1 \otimes |\phi\rangle_2$ can be expressed as a superposition of the 4 two-qubit basis states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.
- Let the transformation A be applied on qubit "1" and B on "2". It would stand to reason that the complete transformation on $|\psi\phi\rangle$ is a 4×4 complex matrix.
- The complete operator is defined as $A_1 \otimes B_2$ and has the same ordering as the tensor product for the qubit states.
- The state labels maybe dropped under the same conditions as with the states.

$$A_1 \otimes B_2 \equiv A \otimes B$$

once again,

$$A \otimes B \not\equiv B \otimes A$$

Utilizing *Actions*

- Let $|\psi\rangle_1 = a\,|0\rangle_1 + b\,|1\rangle_1$ and $|\phi\rangle_2 = c\,|0\rangle_2 + d\,|1\rangle_2$. The effect of $A_1 \otimes B_2$ on this state can be expanded using the actions of A and B on the computational basis and taking the tensor product of the resultant vectors

$$(A \otimes B)\,(|\psi\rangle \otimes |\phi\rangle) \equiv (aA\,|0\rangle + bA\,|1\rangle) \otimes (cB\,|0\rangle + dB\,|1\rangle)$$

- This approach allow operations to be performed without matrix multiplications and will offer significant advantages as the size of the space (no. of qubits) keeps growing.

Example: Single Qubit transformations

- Consider the initial state $|\psi\rangle_1 = |0\rangle_1$ and $|\chi\rangle_2 = |0\rangle_2$.
- Let the X gate be applied only on the first qubit and the second qubit be left as is.
- The combined operation in the two-qubit representation is defined using the identity transformation $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and is defined as:

$$(X \otimes I)\left(|0\rangle \otimes |0\rangle\right) \equiv \left(\underbrace{X|0\rangle}_{|1\rangle} \otimes |0\rangle\right)$$

- The resultant state is therefore $|10\rangle$.

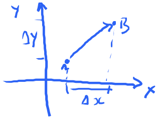
Example: Global phase

- Consider the initial state $|\psi\rangle_1 = |-\rangle_1$ and $|\chi\rangle_2 = |1\rangle_2$.
- Let the Y gate is applied to the first qubit and the P_ϕ gate is applied to the second qubit. (note: P_ϕ is the same as $R_z(\phi)$ in Qiskit)
- The combined operation is defined as:

$$
\begin{aligned}
(Y \otimes P_\phi)(|-\rangle \otimes |1\rangle) &\equiv \frac{1}{\sqrt{2}}(Y|0\rangle - Y|1\rangle) \otimes (P_\phi |1\rangle) \\
&= \frac{1}{\sqrt{2}}(\mathbf{i}|1\rangle + \mathbf{i}|0\rangle) \otimes \left(e^{\mathbf{i}\phi}|1\rangle\right) \\
&= \frac{\mathbf{i}e^{\mathbf{i}\phi}}{\sqrt{2}}(|01\rangle + |11\rangle)
\end{aligned}
$$

- After eliminating the global phases of $\mathbf{i}$ and $e^{\mathbf{i}\phi}$ The resultant state is $\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$.

Example: Hadamard transformation



- Consider the initial state $|\psi\rangle_1 = |0\rangle_1$ and $|\chi\rangle_2 = |0\rangle_2$.

- If the Hadamard transformation H is applied to both qubits, the combined operation may be defined as:

$$
\begin{aligned}
(H \otimes H)(|0\rangle \otimes |0\rangle) &\equiv (H|0\rangle \otimes H|0\rangle) \\
&= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)
\end{aligned}
$$

$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$= 4 \times 1/4 = 1$

- Just like the single qubit Hadamard gate acted on $|0\rangle$ and created equal an superposition of $|0\rangle$ and $|1\rangle$, this transformation acts on $|00\rangle$ to create an equal weight superposition of the four basis states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

Expanding the Product

- Operation $H \otimes H$ in its matrix form looks like this:

$$
\begin{aligned}
H \otimes H &= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}
\end{aligned}
$$

- General state transformations by this operator will require multiplying state vectors by this matrix.