Introduction to Density matrices

Introduction to quantum computing using QSim

Jothishwaran C A

Indian Institute of Technology Roorkee

email: jc_a@ece.iitr.ac.in

The Identity Matrix

- Every single qubit state $|\psi\rangle$ can be represented in the computational basis as follows:

$$|\psi\rangle = \langle 0|\psi\rangle|0\rangle + \langle 1|\psi\rangle|1\rangle$$

coordinates

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

"no gate" gate

moving the symbols around a bit it is possible to write:

$$|\psi\rangle = |0\rangle\langle 0|\psi\rangle + |1\rangle\langle 1|\psi\rangle$$

$$= (|0\rangle\langle 0| + |1\rangle\langle 1|)|\psi\rangle$$

- The outer product sum given in parentheses is doing nothing to change $|\psi\rangle$.
- Therefore the object $|0\rangle\langle 0| + |1\rangle\langle 1|$ performs the same role as the Identity matrix.

The Identity Matrix: general results

- Everything done in the previous slide is true for any orthonormal single qubit basis: $\{|a\rangle, |b\rangle\}$

$$\Rightarrow \quad |a\rangle\langle a| + |b\rangle\langle b| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$, for all orthonormal basis.

- This result can be generalized to multi-qubit states as well.
- These discussions will only deal with single qubit states and operators
- Outer products can be used to define transformations that can be applied to qubits.
- More generally outer products between complex vectors can be used to define all kinds of matrices.

Outer product matrices: the Hamiltonian

- It is possible to define matrices of the form:

$$\hat{\mathrm{H}}(\mathrm{t}) = r_0|0\rangle\langle 0| + z(t)|0\rangle\langle 1| + \bar{z}(t)|1\rangle\langle 0| + r_1|1\rangle\langle 1|$$

; $r_0, r_1 \in \mathbb{R}$

$z(t) \in \mathbb{C}$

t - time

$$\hat{\mathrm{H}}(\mathrm{t}) = \begin{bmatrix} r_0 & z(t) \\ \bar{z}(t) & r_1 \end{bmatrix}$$

- This matrix is a Hermitian matrix and has the property:

physical

$$\hat{\mathrm{H}}^{\dagger}(\mathrm{t}) = \hat{\mathrm{H}}(\mathrm{t})$$

- Such matrices ultimately contain all the information about a qubit/quantum system.

From Hermitian to Unitary matrices

$\hat{H}(t)$, the Hamiltonian also defines the gate operation

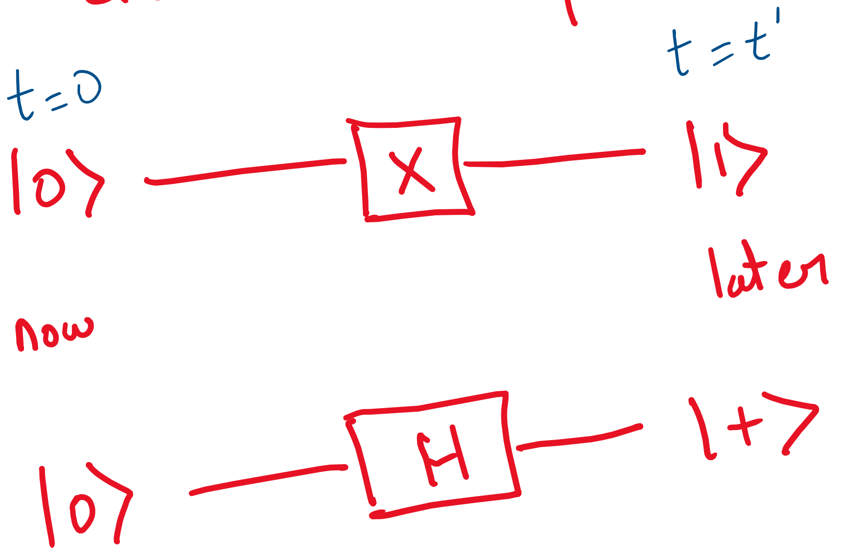
- The time evolution of any state $|\psi\rangle$ is given by:

single-qubit

$$i\hbar \frac{d}{dt}|\psi\rangle = \hat{\mathrm{H}}(t)|\psi\rangle$$ – differential equation

this is one form of the Schrödinger equation

In Quantum computation: the state of the qubit changes over time



Solving the Schrödinger Eqn will give the following relation

$$|\psi(t=t')\rangle = U(0,t')\,|\psi(t=0)\rangle$$

or $\boxed{U(0,t')|\psi(t=0)\rangle = |\psi(t=t')\rangle}$ → form of all gate transformation

$X\underset{\text{now}}{\underline{|0\rangle}} = \underset{\text{later}}{\underline{|1\rangle}}$

$H\underline{|0\rangle} = \underline{|+\rangle}$

If $\hat{H}(t)$ is Hermitian

then, $U(0,t')$ is unitary

$U^\dagger(0,t') = U(t',0)$ → Unitary time evolution operator

$U^\dagger U = U U^\dagger = I$

A Brief comment on Gate sets

Gate set: R_x, R_y, $\sqrt{X}$, S, T, CX $\rightarrow$ for each there is a different $\hat{H}(t)$

$\Rightarrow$ $Z(t)$ is a different

Summary:

qubit is defined by it's Hamiltonian

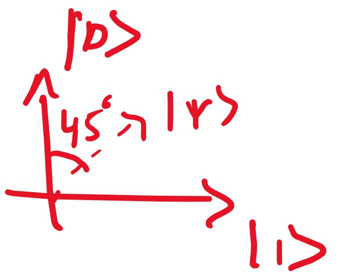
$\downarrow$

U {time evolution operator.}

$U^\dagger U = UU^\dagger = I$; U – we consider this to be the gate

A Question

photon polarization — $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$;



If a photon in $|\psi\rangle$ is incident on a Polarizer oriented along $|0\rangle$, then 50% of the photons are transmitted

The same polarizer, but now a light source that randomly chooses to emit a photon in $|0\rangle$ or $|1\rangle$ with 50% probability. What fraction of the photons are transmitted?

Vertical Polarizer

$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

50% $|0\rangle$ and 50% $|1\rangle$

Transmission Prob. = 50%

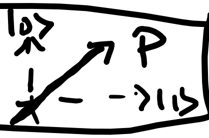
what is the difference??

Ans: different Transmission prob.

Transmission probability = 50%

If different, how to distinguish??

45° Polarizer



Trans. Prob. = 100%

Trans. Prob. = $0.5 \times 0.5 + 0.5 \times 0.5$

$= 0.5$

Polarizer at angle θ

Trans. Prob. $= |\cos(45° - \theta)|^2$

Tran. Prob. $= 0.5 \times |\cos\theta|^2 + 0.5 \times |\sin\theta|^2$

$= 0.5$

$\|\psi\rangle = \frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle$	50% $\|0\rangle$ and 50% $\|1\rangle$
→ Superposition	→ No superposition
→ A physically state from both $\|0\rangle$ & $\|1\rangle$	→ A mixture, each photon is either in $\|0\rangle$ or $\|1\rangle$
→ Every photon has same polarization	→ Mixture of photons with different polarization
→ Transmission Prob. varies with Polarizer angle	→ Transmission probability is const. (for this particular case)

Red is represented by $|\psi\rangle$ – a vector in $\mathbb{C}^2$

What mathematical object represents a mixture? $\longrightarrow$ Density matrix

[Should describe both red & blue cases]

Density Matrices were originally introduced

Jon von Neumann

$|0\rangle$ ———————— $|0\rangle$

no gate is applied

(intended operation)

Quantum analog of the bit-flip error

$|0\rangle$ ——$\boxed{X}$—— $|1\rangle$

X-gate is applied

(due to error)

has a Prob. of occurrence.

Probability of error = 30% = 0.3

$\Rightarrow$ 70% of the time the final state of the qubit is $|0\rangle$

& 30% of the time the final state of the qubit is $|1\rangle$

$$\rho = p_0 |0\rangle\langle 0| + p_1 |1\rangle\langle 1| \quad \rightarrow \quad p_0 - \text{prob. of finding } |0\rangle$$

p_1 - prob of finding $|1\rangle$

$$= 0.7\, |0\rangle\langle 0| + 0.3\, |1\rangle\langle 1|$$

$$= 0.7 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \rho = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.3 \end{bmatrix}$$ - density matrix of a mixed state

In the limit error. prob. $\longrightarrow 0$

p_0 becomes closer $1 \Rightarrow$ the qubit is purely $|0\rangle$

p_1 " " 0

$$\therefore \rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = |0\rangle\langle 0| \rightarrow \text{density matrix for a pure state.}$$

The argument developed here is not only valid for $|0\rangle$ & $|1\rangle$ it is valid any number of states in a statistical mixture

General form of Density operator

(can also be multi-qubit)

If a system is in a mixture of

States: $\{ |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle - - - |\psi_n\rangle \}$

Probabilities $\{ p_1, p_2, p_3 - - - - p_n \}$

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad ; \quad \rho^{+} = \rho \quad ; \quad Tr(\rho) = 1$$

(rho) greek

[sum of diagonal elements]

Pure State

$|\psi\rangle$ —— $\boxed{G}$ —— $\sum_i p_i |\psi_i\rangle\langle\psi_i|$ $\rightarrow$ $2^n \times 2^n$ matrix with complex elements

Noisy Gate

mixed state

$\rho = |\psi\rangle\langle\psi|$

Noiseless gates: Unitary Transformations (U - a single operator)
- reversible

Noisy gates : $\underbrace{\text{Super operators}}_{\text{collection of operators}}$ — Krauss Operators

Lindblad operators

[primary aim of QSim]

Not reversible