Grover's Search Algorithm

Introduction to quantum computing using QSim

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Classical approach to the search problem

- Consider an *n*-variable Boolean function $F: \{0,1\}^n \rightarrow \{0,1\}$ that has the following definition:

$$F(x) = \begin{cases} 1 & ; x = x_0 \\ 0 & ; x \neq x_0 \end{cases}$$

$x_0 \in \{0,1\}^n$

n-bit string

- This function evaluates to 1 if and only if the input value is x_0.
- The problem for this discussion is stated as follows:

"Given oracle access to such a Boolean function, how many times must the oracle of $F(x)$ be queried to find out the value of x_0?"

Oracle access : Only perform input/output queries.

A particular kind of Boolean function

- Given that only oracle access to the function is available, the only option available to us is to check if the function evaluates to 1 at each point.

- This means one would have to query the function a maximum of 2^n times in order to deterministically find out the value of x_0.

- The search problem has a classical computational complexity of $O(N)$, where $N = 2^n$.

linear complexity

Can quantum computing do any better?

Grover's Search Algorithm

What do we have on a Quantum Computer ?

→ Search is on a Boolean function

therefore, $U_F |x\rangle |0\rangle \longrightarrow |x\rangle |F(x)\rangle$: Bit Oracle of F

U_F : unitary transformation

$|x\rangle$: n-qubit

$|0\rangle$: 1-qubit

→ Quantum Parallelism : $U_F (|x_1\rangle + |x_2\rangle) |0\rangle \longrightarrow$

$|x_1\rangle |F(x_1)\rangle + |x_2\rangle |F(x_2)\rangle$

It is possible to evaluate F at multiple points using a single query.

Grover's Search Algorithm

→ Hadamard gate (H): $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$H|0\rangle \rightarrow |+\rangle$

$H|1\rangle \rightarrow |-\rangle$

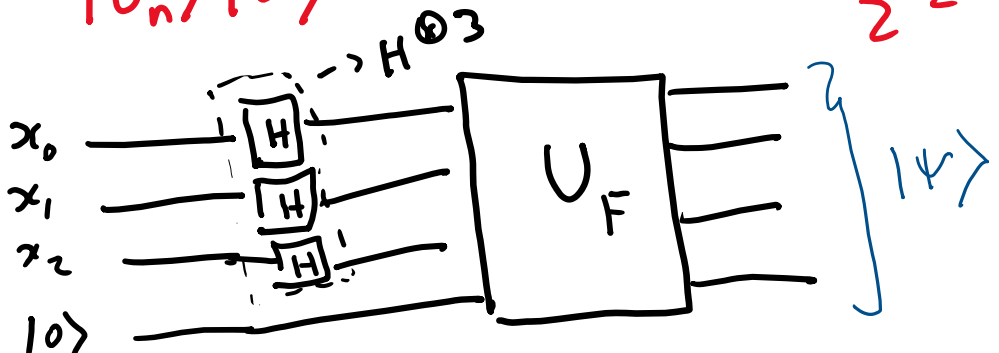
$|0_n\rangle \xrightarrow{H^{\otimes n}} \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle$: Equal weight superposition of all possible values of x

$|0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \Rightarrow |0_n\rangle \equiv |0\rangle^{\otimes n}$

n - times

→ $|0_n\rangle|0\rangle \xrightarrow{H^{\otimes n} \otimes I}$ (I → Identity) $\frac{1}{2^{n/2}} \sum_x |x\rangle |0\rangle \xrightarrow{U_F} \frac{1}{2^{n/2}} \sum_x |x\rangle |F(x)\rangle$

$|\psi\rangle$



$|\psi\rangle$ contains the complete information about F

Since we know about F

$$|\psi\rangle = \frac{1}{2^{n/2}}\left[\sum_{x \neq x_0} |x\rangle|0\rangle + |x_0\rangle|1\rangle\right] \xrightarrow{\text{Measure}}$$

If the last is measured to be in $|1\rangle$ then, the first n qubits will measure to give the value of x_0

What is the probability of Success?

$$\text{Prob}\{\text{success}\} = \left|\frac{1}{2^{n/2}}\right|^2 = \frac{1}{2^n} \longrightarrow$$

This is worse than the classical case.

→ It would be preferable if the coefficient of $|x\rangle|i\rangle$ be increased somehow.

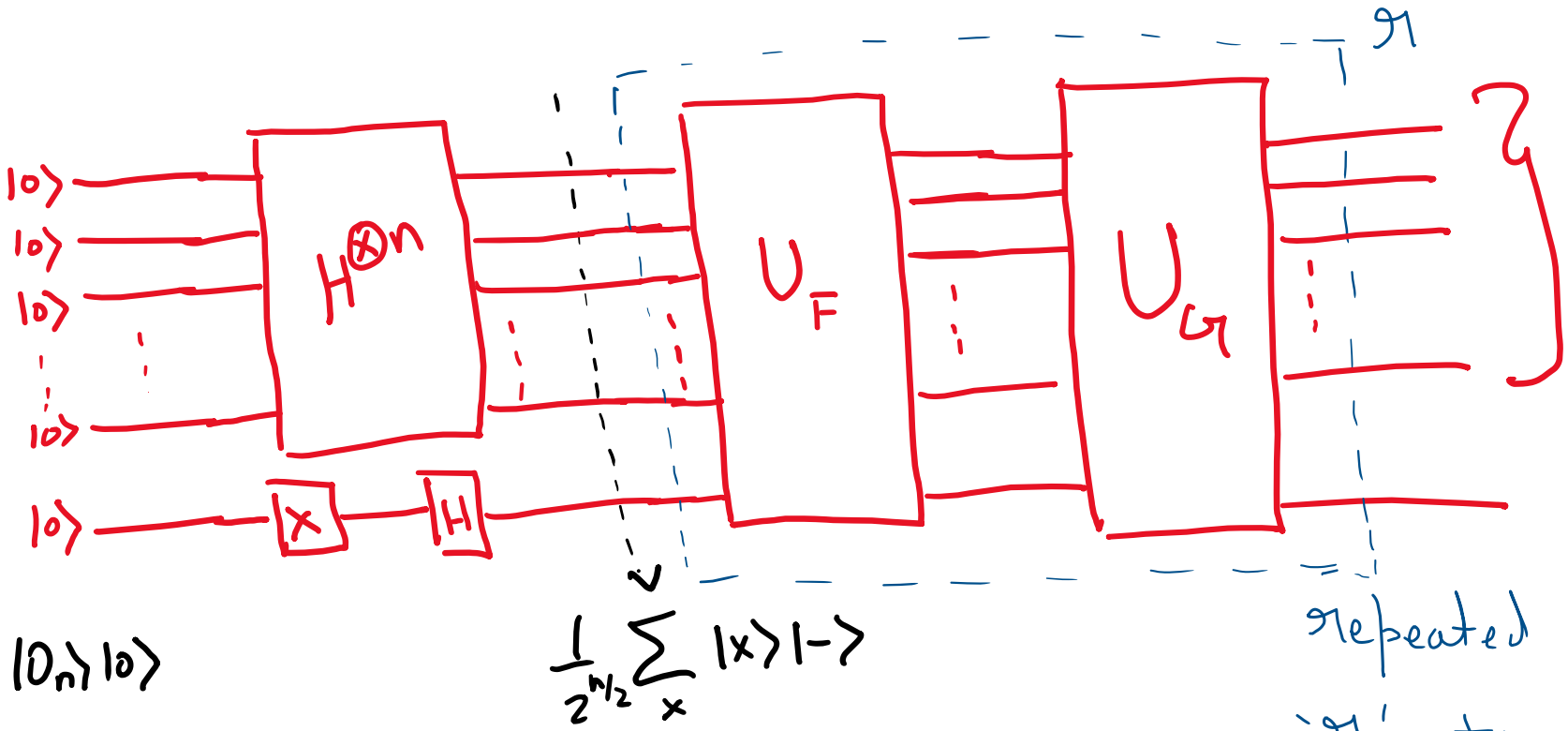
This is made possible using Grover's Rotation : U_G
{Grover Gate}

$U_G = 2|\phi\rangle\langle\phi| - I_n$:- n-qubit Identity matrix ($2^n \times 2^n$)

U_G - $2^n \times 2^n$ matrix : n-qubit transformation

$$|\phi\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle$$

Outline of Grovers Algorithm [n+1 qubit version]



What is the value of r?

: Amplitude Amplification

The measurement on the first n qubits will give the result of X_0 with a high probability

Grover's Algorithm: a brief analysis

i) What is the value of n?

ii) Why did we choose U_{cn}?

$H^{\otimes n}|0_n\rangle \longrightarrow |\phi\rangle$ (as previously defined)

$$\langle\phi| = (H^{\otimes n}|0_n\rangle)^\dagger$$

$$= \langle 0_n|(H^\dagger)^{\otimes n} \quad \because H^\dagger = H$$

$$= \langle 0_n| H^{\otimes n}$$

$$(|\psi\rangle)^\dagger = \langle\psi|$$

$$(|\psi\rangle_1 \otimes |\psi\rangle_2)^\dagger$$

$$= (|\psi\rangle_1)^\dagger \otimes (|\psi\rangle_2)^\dagger$$

$$= \langle\psi_1| \langle\psi_2|$$

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$$

$$(U|\psi\rangle)^\dagger = \langle\psi|U^\dagger$$

Above are some useful results

$H^\dagger = H$; H is unitary $\therefore$ $H^\dagger H = I$

final property of tensor products:

$(A \otimes B) \cdot (C \otimes D)$; $\cdot$ - matrix multiplication

$= (A \cdot C) \otimes (B \cdot D)$

$(H^{\otimes n})^\dagger \cdot (H^{\otimes n}) = I_n$

using these we can redefine U_{CT}

Alternate version of U_G

$$U_G = 2|\phi\rangle\langle\phi| - I_n$$

$$= 2\left[H^{\otimes n}|0_n\rangle\langle 0_n|H^{\otimes n}\right] - H^{\otimes n} \cdot I_n \cdot H^{\otimes n}$$

$$= H^{\otimes n} \cdot \left[2|0_n\rangle\langle 0_n| - I_n\right] \cdot H^{\otimes n}$$

$$|0_n\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{2^n \times 1 \text{ vector}} \Rightarrow |0_n\rangle\langle 0_n| = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & \mathbf{0} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}_{2^n \times 2^n}$$

$$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{2^n \times 2^n} \quad \therefore \; 2|0_n\rangle\langle 0_n| - I_n$$

$$= \begin{pmatrix} 1 & & 0 \\ & -1 & \\ & & \ddots \\ 0 & & & -1 \end{pmatrix}_{2^n \times 2^n} = M$$

$$M|x\rangle = \begin{cases} |0_n\rangle, & \text{if } |x\rangle = |0_n\rangle \\ -|x\rangle, & \text{otherwise} \end{cases}$$

$|x\rangle$ - n-qubit state

$x \in \{0,1\}^n$

$$M|x\rangle = (-1)^{G(x)}|x\rangle$$

$G(x)$ is an n-variable Boolean function

$$G(x) = \begin{cases} 0, & \text{if } x = 0_n \\ 1, & \text{otherwise} \end{cases}$$

$G(x)$ is the n-variable OR gate

$\therefore$ M is the phase oracle of the n-bit OR gate

the $n+1$ qubit version of M

the bit oracle representation of the OR gate $U_{OR} \to n+1$ qubit gate

$$U_{OR}|x\rangle|-\rangle = (-1)^{OR(x)}|x\rangle|-\rangle$$

therefore the $n+1$ qubit version of $U_{Gr} = (H^{\otimes n} \otimes I) \cdot U_{OR} \cdot (H^{\otimes n} \otimes I)$

here, it is understood the the last qubit is in $|-\rangle$ state

What is the value of r ?

when performing Grover's search over an 'n' variable Boolean function

$$r = \left[\frac{\pi}{4} \times \sqrt{2^n} \right] = \left[\frac{\pi}{4} \times 2^{n/2} \right]$$ [presented without proof]

$[m]$ → nearest integer to 'm'

$\therefore$ for $n = 4$; $r = \left[\frac{\pi}{4} \times \sqrt{2^4} \right] = [\pi] = 3$

A 4 variable Grover's Search

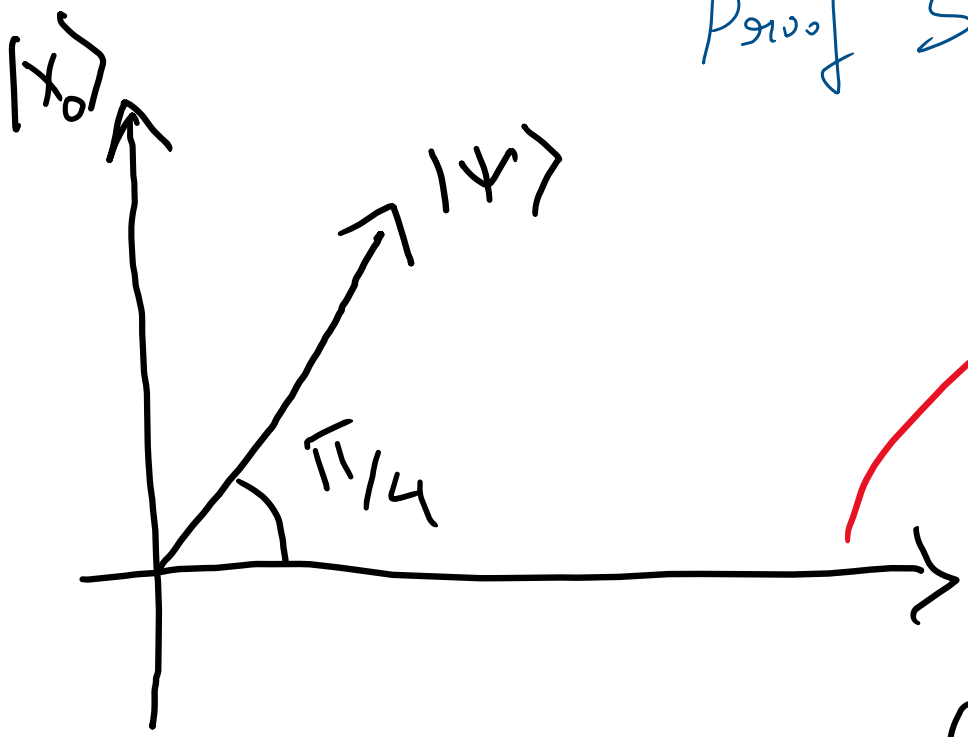
$F(x) = x_3 \wedge x_2 \wedge \bar{x}_1 \wedge x_0$ will evaluate to 1 only at 1101

$\bar{x}_1 = 1 + x_1$; + - XOR operation

$F(x) = x_3 x_2 x_1 x_0 + x_3 x_2 x_0 \rightarrow$ ANF of $F(x)$

$OR(x) = x_3 \vee x_2 \vee x_1 \vee x_0 = [\bar{x}_3 \bar{x}_2 \bar{x}_1 \bar{x}_0]'$

Proof Sketch of Grover iterations:



$U_f U_{Gr}$ causes $|\psi\rangle$ to rotate about the horizontal line

all other vectors

(Hyperplane)

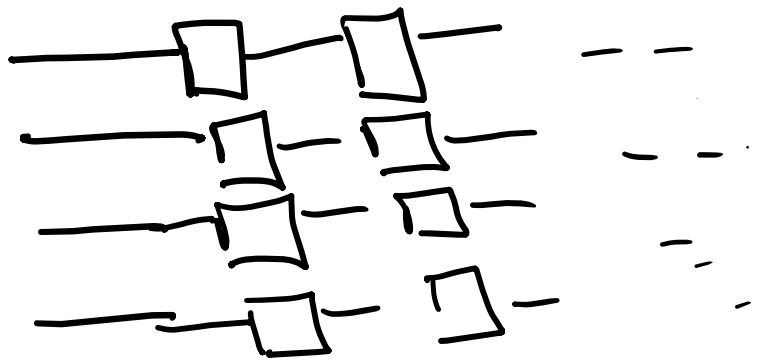
doing "r" rotations $r = \left[\frac{\pi}{4} 2^{n/2}\right]$

will take angle closest to $\pi/2$

A brief note on Quantum Volume (QV)

QV is the size of the largest square circuit that can be run with low error on a Quantum computer.

QV - 16



largest known QV : 2^{16}

What is the required Quantum volume Grover's Algorithm required Q.V. $= O(n^2 2^{n/2})$;

$n = 4 : 16 \times 4 = 64$

$n = 16 : 16 \times 16 \times 2^8 = 2^{16} : N \approx 65{,}000$