Celestial Mechanics Assignment

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1 Asteroid Orbital Path Model

In this assignment, the asteroid selected for modelling and investigation is 1019 Strackea.

2 Arbitrary Location Calculation

The location chosen to compute the position and velocity vectors of 1019 Strackea was that on April 27, 2019.

The orbital elements of 1019 Strackea as of April 27, 2019 are given as follows by NASA Jet Propulsion Laboratory (JPL):

$e = 0.07122462441687952$

$a = 1.911487883249344$ au

$i = 26.97749056713183°$

$\Omega = 144.4061251926394°$

$\omega = 122.084695721071°$

$M = 254.3168907782681°$

$T = 2458883.873237565665$

$\tau(\text{period}) = 965.2854299508239$ days

The method chosen to compute the position and velocity vectors of the asteroid in this report is pursuant to that explained by van Bemmel (2000).

The following calculations are computed in radians and the output units are in astronomical units (au) and days (d).

The eccentric anomaly, E, according to Kepler's equation, can be computed from M using an iterative solution as follows:

$$M = E - e\sin E$$

Then let

$$f = E - e\sin E - M$$

Hence,

$$\frac{\mathrm{d}f}{\mathrm{d}E} = 1 - e\cos E$$

Then using Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We substitute as:

$$E_{n+1} = E_n - \frac{E_n - e\sin E_n - M}{1 - e\cos E_n}$$

Setting, $E_0 = M$ for the first iteration, we obtain $E = 4.371539933$ after three iterations.
As the asteroid orbits in the x', y' plane, its x' and y' coordinates are given as:

$$x' = a(\cos E - e)$$
$$y' = b\sin E$$

Where b is given as:

$$b = a\sqrt{1 - e^2}$$

Hence, substitution or orbital elements gives:

$$b = 1.90663328$$
$$x' = -0.775131351$$
$$y' = -1.796946916$$

Since the position vector is given as

$$\vec{r} = x'\hat{p} + y'\hat{q}$$

and $\hat{p}$ and $\hat{q}$ are given as

$$\hat{p} = (\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i)\hat{\imath} + (\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i)\hat{\jmath} + (\sin\omega\sin i)\hat{k}$$
$$\hat{q} = (-\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i)\hat{\imath} + (-\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i)\hat{\jmath} + (\cos\omega\sin i)\hat{k}$$

By substitution of orbital elements into above equations we obtain:

$$\hat{p} = (-0.007547531)\hat{\imath} + (-0.923155303)\hat{\jmath} + (0.384353122)\hat{k}$$
$$\hat{q} = (0.964483548)\hat{\imath} + (-0.108208951)\hat{\jmath} + (-0.240961219)\hat{k}$$

Hence,

$$x'\hat{p} = (0.005850328)\hat{\imath} + (0.715566617)\hat{\jmath} + (-0.297924154)\hat{k}$$
$$y'\hat{q} = (-1.733125737)\hat{\imath} + (0.19444574)\hat{\jmath} + (0.43299452)\hat{k}$$

Therefore the position vector, in au, of the asteroid is:

$$\vec{r} = (-1.727275408)\hat{\imath} + (0.910012357)\hat{\jmath} + (0.135070365)\hat{k}$$

To magnitude of the range is given by,

$$r = a(1 - e\cos E)$$

To compute the speeds we need

$$\dot{E} = \frac{1}{r}\sqrt{\frac{\mu}{a}}$$

Where μ is the product of Gm_1m_2

We can find μ using Kepler's third law:

$$\tau^2 = \frac{4\pi^2a^3}{GM}$$

Hence,

$$\mu = GM = \frac{4\pi^2a^3}{\tau^2}$$

Substituting orbital elements we find:

$$\mu = 0.000295912$$
$$r = 1.956999446$$

Thus,

$$\dot{E} = 0.006357772$$

The velocity vector in of the asteroid is given by

$$\vec{v} = \dot{x}'\hat{p} + \dot{y}'\hat{q}$$

We compute the speeds, $\dot{x}'$ and $\dot{y}'$, by:

$$\dot{x}' = -a\dot{E}\sin E$$
$$\dot{y}' = b\dot{E}\cos E$$

Substituting for a, $\dot{E}$, and E gives:

$$\dot{x}' = 0.011453668$$
$$\dot{y}' = -0.004052212$$

Hence,

$$\dot{x}'\hat{p} = (-0.000086447)\hat{\imath} + (-0.010573514)\hat{\jmath} + (0.004402253)\hat{k}$$
$$\dot{y}'\hat{q} = (-0.003908292)\hat{\imath} + (0.000438486)\hat{\jmath} + (0.000976426)\hat{k}$$

Therefore, the velocity vector, in $\mathrm{au} \cdot \mathrm{d}^{-1}$, of the asteroid is:

$$\vec{v} = (-0.003994739)\hat{\imath} + (-0.010135029)\hat{\jmath} + (0.005378679)\hat{k}$$

Thus the following are the position and velocity vectors of 1019 Strackea on April 27, 2019 as computed by hand in au-d units:

$$\vec{r} = (-1.727275408)\hat{\imath} + (0.910012357)\hat{\jmath} + (0.135070365)\hat{k}$$
$$\vec{v} = (-0.003994739)\hat{\imath} + (-0.010135029)\hat{\jmath} + (0.005378679)\hat{k}$$

This compares with the accepted values as given by ephemeris of NASA Jet Propulsion Laboratory in the screenshot below in au-d units:

```
2458600.500000000 = A.D. 2019-Apr-27 00:00:00.0000 TDB
 X =-1.728975110102452E+00 Y = 9.176042997993259E-01 Z = 1.350374170836044E-01
 VX=-4.003085935689917E-03 VY=-1.013427838860611E-02 VZ= 5.378895421300426E-03
```

Figure 1: Position and velocity vectors of 1019 Strackea on April 27, 2019 as provided by NASA Jet Propulsion Laboratory ephemeris.

The following table displays the difference between the values of each component of the position and velocity vectors provided by the NASA Jet Propulsion Laboratory ephemeris and those calculated by hand:

Vector	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{r}$	$-1.6997016425\mathrm{E}{-03}$	$7.5919427014\mathrm{E}{-03}$	$-3.2948035375\mathrm{E}{-05}$
$\vec{v}$	$-8.3471846691\mathrm{E}{-06}$	$7.5038685999\mathrm{E}{-07}$	$2.1640580382\mathrm{E}{-07}$

Figure 2: The value in each cell is obtained from subtracting the hand-computed result from the accepted (JPL) data. In other words, the ($\vec{r}$,$\vec{i}$) cell displays the difference between the value obtained from JPL and that from hand-calculation for the $\vec{i}$-component of the position vector, etc.

3 Calculus Problems

The following questions are completed pursuant to *Anton et al (2009)*.

Problem 1

(a) From (15) we have:

$$\mathbf{v} \times \mathbf{b} = GM\mathbf{u} + \mathbf{C}$$

By setting $t = 0$, we obtain: $\mathbf{v}_0 = v_0\mathbf{j}$ and $\mathbf{u}_0 = 1\mathbf{i}$ since $\theta = 0$.
From (6) we have:

$$\mathbf{b} = r_0 v_0 \mathbf{k}$$

Substituting $\mathbf{v}_0$, $\mathbf{b}$, and $\mathbf{u}_0$ into (15) gives:

$$v_0\mathbf{j} \times r_0 v_0 \mathbf{k} = GM(1\mathbf{i}) + \mathbf{C}$$

Hence,

$$r_0 v_0^2 \mathbf{i} = GM\mathbf{i} + \mathbf{C}$$

Rearranging gives

$$\mathbf{C} = (r_0 v_0^2 - GM)\mathbf{i}$$

This matches (15).

(b) From (7) we have:

$$\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

From (17) we have:

$$\mathbf{v} \times \mathbf{b} = GM\mathbf{u} + (r_0 v_0^2 - GM)\mathbf{i}$$

From (22) we have:

$$e = \frac{r_0 v_0^2}{GM} - 1$$

Hence, substituting (7) into (17) gives:

$$\begin{aligned}\mathbf{v}\times\mathbf{b} &= GM(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})+(r_0v_0^2-GM)\mathbf{i}\\ &=(r_0v_0^2-GM+GM\cos\theta)\mathbf{i}+(GM\sin\theta)\mathbf{j}\\ &=GM\left[\left(\frac{r_0v_0^2}{GM}-1+\cos\theta\right)\mathbf{i}+(\sin\theta)\mathbf{j}\right]\end{aligned}$$

Substituting (22) gives:

$$=GM[(e+\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j}]$$

Q.E.D.

(c) From cross product rule for vector **a** and vector **b**, we have,

$$\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$$

Since vector **v** only lies on the **i-j** plane, and vector **b** only has a **k** component, the θ between **v** and **b** must be $90°$.
Hence,

$$\begin{aligned}\|\mathbf{v}\times\mathbf{b}\| &= \|\mathbf{v}\|\|\mathbf{b}\|\sin(90°)\\ &=\|\mathbf{v}\|\|\mathbf{b}\|\end{aligned}$$

Q.E.D.

(d) From (b) we have:

$$\mathbf{v}\times\mathbf{b}=GM[(e+\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j}]\quad [1]$$

From (c) we have:

$$\|\mathbf{v}\times\mathbf{b}\|=\|\mathbf{v}\|\|\mathbf{b}\|\quad [2]$$

Rearranging [2] we get,

$$\|\mathbf{v}\|=\frac{\|\mathbf{v}\times\mathbf{b}\|}{\|\mathbf{b}\|}\quad [3]$$

Since $\mathbf{b}=r_0v_0\mathbf{k}$, substituting **b** and [1] into [3] gives:

$$v=\frac{\|GM[(e+\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j}]\|}{\|r_0v_0\mathbf{k}\|}$$

Evaluating magnitude of vectors gives:

$$v=\frac{GM\sqrt{(e+\cos\theta)^2+(\sin\theta)^2}}{\sqrt{r_0^2v_0^2}}$$

$$=\frac{GM\sqrt{e^2+2e\cos\theta+\cos^2\theta+\sin^2\theta}}{r_0v_0}$$

$$=\frac{GM}{r_0v_0}\sqrt{e^2+2e\cos\theta+1}$$

Since $e = \frac{r_0 v_0^2}{GM} - 1$, $\frac{GM}{r_0 v_0}$ can be rewritten as $\frac{v_0}{e+1}$. Hence,

$$v = \frac{v_0}{1+e}\sqrt{e^2 + 2e\cos\theta + 1}$$

$$Q.E.D.$$

(e) From (d) we have,

$$v = \frac{v_0}{1+e}\sqrt{e^2 + 2e\cos\theta + 1}$$

Since v_0 and e are constant, v is maximized when $\theta = 0°$ as $\cos 0° = 1$.
The magnitude of force of gravitation is given as

$$\|F_g\| = \frac{GMm}{r^2}$$

.
F_g is at maximum when r is at minimum. Since particle is in an elliptical orbit, particle is at minimum distance from center of force at perigee of orbit because the distance to focus of ellipse is minimized. At this moment, r is at minimum and $\theta = 0°$ hence v is maximized.
Similarly, v is minimized when $\theta = 180°$ because $\cos 180° = -1$. This case only occurs when the particle is at the apogee of the orbit. Hence r is maximized when v is at minimum.

Problem 2

From the definition of an ellipse, we have:

$$c^2 = a^2 - b^2$$

Where:

$c =$ the distance from centre of the ellipse to foci of the ellipse
$a =$ the semi-major axis of the ellipse
$b =$ the semi-minor axis of the ellipse

Hence, when a particle in an elliptical orbit reaches an end of the minor axis, its angle is given as $\theta = 180° - \arctan\left(\frac{b}{c}\right)$.
The speed of the particle in an elliptical orbit is given as,

$$v = \frac{v_0}{1+e}\sqrt{e^2 + 2e\cos\theta + 1}$$

To evaluate $\cos\theta$, we use the cosine identity:

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

With substitution we obtain:

$$\begin{aligned}
\cos\theta &= \cos\left(180^\circ - \arctan\left(\frac{b}{c}\right)\right) \\
&= \cos(180^\circ)\cos\left(\arctan\left(\frac{b}{c}\right)\right) + \sin(180^\circ)\sin\left(\arctan\left(\frac{b}{c}\right)\right) \\
&= -\cos\left(\arctan\left(\frac{b}{c}\right)\right)
\end{aligned}$$

Since a right triangle is made from b, c, and r (distance from particle to focus), Pythagorean theorem gives us

$$b^2 + c^2 = r^2$$

which we can rewrite as

$$c^2 = r^2 - b^2$$

.

However, since the definition of an ellipse gives us

$$c^2 = a^2 - b^2$$

we conclude that $r = a$.

Hence, we can rewrite the previous cosine term:

$$\cos\theta = -\cos\left(\arctan\left(\frac{b}{c}\right)\right) = -\frac{c}{a}$$

Since we have the property of the ellipse,

$$c = ea$$

Where

$c =$ the distance from centre of the ellipse to foci of the ellipse

$a =$ the semi-major axis of the ellipse

$e =$ the eccentricity of the ellipse

The cosine term becomes:

$$\cos\theta = -\frac{c}{a} = -e$$

Substituting this value into the original equation for speed gives:

$$\begin{aligned} v &= \frac{v_0}{1+e}\sqrt{e^2 + 2e(-e) + 1} \\ &= \frac{v_0}{1+e}\sqrt{e^2 - 2e^2 + 1} \\ &= \frac{v_0}{1+e}\sqrt{1 - e^2} \\ &= v_0\sqrt{\frac{(1+e)(1-e)}{(1+e)^2}} \\ &= v_0\sqrt{\frac{1-e}{1+e}} \\ &Q.E.D. \end{aligned}$$

Problem 3

The speed of a particle in an elliptical orbit is given as:

$$v = \frac{v_0}{1+e}\sqrt{e^2 + 2e\cos\theta + 1}$$

Hence the max speed occurs when $\theta = 0$:

$$\begin{aligned} v_{\text{max}} &= \frac{v_0}{1+e}\sqrt{e^2 + 2e\cos(0°) + 1} \\ &= \frac{v_0}{1+e}\sqrt{e^2 + 2e + 1} \\ &= \frac{v_0}{1+e}\sqrt{(1+e)^2} \\ &= v_0 \end{aligned}$$

The minimum speed occurs when $\theta = 180°$:

$$\begin{aligned} v_{\text{min}} &= \frac{v_0}{1+e}\sqrt{e^2 + 2e\cos(-180°) + 1} \\ &= \frac{v_0}{1+e}\sqrt{e^2 - 2e + 1} \\ &= \frac{v_0}{1+e}\sqrt{(1-e)^2} \\ &= v_0\frac{1-e}{1+e} \end{aligned}$$

Hence, v_{min} and v_{max} are related by:

$$v_{\text{min}} = v_{\text{max}}\frac{1-e}{1+e}$$

We can rearrange the above equation to form:

$$v_{\text{max}} = v_{\text{min}}\frac{1+e}{1-e}$$

Q.E.D.

Problem 14

From (22), e is defined as:

$$e = \frac{r_0 v_0^2}{GM} - 1$$

To prove that $e \geq 0$, we do so by contradiction.
Suppose that $e < 0$, from (19) we have:

$$r = \frac{\frac{r_0^2 v_0^2}{GM}}{1 + (\frac{r_0 v_0^2}{GM} - 1)\cos\theta}$$

Since the numerator is positive, we can replace it with a positive constant, k. We can also substitute (22) into (19). Hence we have:

$$r = \frac{k}{1 + e\cos\theta}$$

Since we initially assumed $e < 0$, we can rewrite the above equation as:

$$r = \frac{k}{1 - |e|\cos\theta}$$

Since the only variable on the right side is θ, we can see by inspection that r is minimized when $\theta = 180°$. However, we are constrained by the polar axis as it was chosen so that r is minimized when $\theta = 0°$.
Hence we have reached a contradiction: $\therefore e \geq 0$.

4 Orbiting Bodies

(a) Given:

$$m = 3.23 \text{ solar masses} = 6.42 \times 10^{30}\,\text{kg}$$
$$T = 3.2 \text{ years} = 1.01 \times 10^{8}\,\text{s}$$
$$v = 27\,\text{kms}^{-1} = 27\,000\,\text{ms}^{-1}$$

Let R_2 denote the radius of the star's orbit.
Let R_1 denote the radius of the unseen object's orbit.
Let M denote the mass of the unseen object.

From the relation between speed and period in circular motion we have:

$$v = \frac{2\pi r}{T}$$

Rearranging the equation for r, we get:

$$r = \frac{Tv}{2\pi}$$

Substituting givens gives:

$$R_2 = \frac{(1.01 \times 10^8)(27000)}{2\pi}$$
$$= 4.34 \times 10^{11}\,\text{m}$$

Therefore the radius of star's orbit is 4.34×10^{11} m.

Taking into account of the barycenter of the system, and assuming point-mass we can relate R_1 to R_2 as follows:

$$R_2 = \frac{M}{M+m}(R_1 + R_2)$$

Rearranging for R_1, we get:

$$R_1 = R_2 \left(\frac{M+m}{M} - 1 \right)$$

which simplifies to:

$$R_1 = R_2 \frac{m}{M}$$

Since we assume circular orbits for both objects, we have:

$$F_g = F_c$$

Thus, analyzing the forces acting on the star we have:

$$\frac{GMm}{(R_1 + R_2)^2} = \frac{mv^2}{R_2}$$

Simplifying, we obtain:

$$\frac{GM}{(R_1 + R_2)^2} = \frac{v^2}{R_2}$$

We can now solve for the mass, M, of the unseen object by substituting the expression for R_1 and the given values of constants:

$$\frac{GM}{(R_2 \frac{m}{M} + R_2)^2} = \frac{v^2}{R_2}$$
$$GM = v^2 R_2 \left(\frac{m}{M} + 1 \right)^2$$

Using an algebra engine to evaluate the above expression gives:

$$M = 1.15 \times 10^{31}\,\text{kg}$$

Therefore the mass of the unseen object is $1.150\,64 \times 10^{31}$ kg.
To solve for R_1, the unseen object's orbit radius we evaluate:

$$R_1 = R_2 \frac{m}{M}$$
$$R_1 = (4.34 \times 10^{11}) \frac{6.42 \times 10^{30}}{1.15 \times 10^{31}}$$
$$R_1 = 2.42 \times 10^{11}\,\text{m}$$

Therefore, the radius of the unseen object's orbit about the barycenter of the system is 2.42×10^{11} m.

(b) From the above calculations, the mass of the unseen object is approximately 5.78 $M_\odot$. This falls within the range of stellar black holes, which form following the collapse of a sufficiently massive star. The black hole would effectively be an ideal black body, and would not emit or reflect light like other, similarly massive celestial bodies. By direct observation, it would be invisible to observers on Earth.

The result of this is that detection is only possible through methods such as the analysis conducted in part (a). As such, its presence is observable as the source of the accompanying star's oscillation and must be computed based on its effect on surrounding bodies.

(c) **BONUS:**

i In elliptical orbit, the speed does not remain constant; as the orbit approaches periapsis, the magnitude of gravitational force increases and further accelerates the star. Thus, motion (speed included) would be expressed as a function of the star's distance from the barycenter. Mass and period will both still be given as a constant. As well, some indication must be given of the orbital eccentricity, which will be identical for both bodies. (This cannot be derived from the given information.)

ii It remains that both bodies must be situated opposite one another relative to the barycenter (which locates the focus of both orbits). Kepler's Third Law can be used to derive a relation between period, mass, and the semi-major axis. Thus, this becomes more complex than circular calculations, as determining the semi-major axis is dependent on an unknown mass as well. A similar relation to the one used in part (a) gives:

$$a_1 M_1 = a_2 M_2$$

Where a and M denote semi-major and mass, respectively. Given three unknowns, using these relations in conjunction with a third expression of the gravitational force on the orbiting bodies would generate a solvable system. It should be noted that, while gravitational force is expressed the same way, this third equation would involve a more complicated expression for centripetal force in an ellipse.

5 Lagrangian Nodes

(a) and (b) To calculate the location of L_1 Lagrangian node:

- Let x be the distance between Jupiter and the satellite.
- Let R be the distance between the sun and Jupiter.
- Let M_S be the mass of the sun.
- Let M_J be the mass of the Jupiter.
- Let m be the mass of the satellite.

Analyzing forces acting on the satellite, we see that gravitational force provides centripetal force for the satellite, we get:

$$F_g = F_c$$
$$\frac{GM_S m}{(R-x)^2} - \frac{GM_J m}{x^2} = \frac{mv_{sat}^2}{R-x}$$

Simplifying and substituting the relation $v = \frac{2\pi r}{T}$ for v_{sat} gives:

$$\frac{GM_S}{(R-x)^2} - \frac{GM_J}{x^2} = \frac{4\pi^2(R-x)}{T^2}$$

Rearranging the equation gives:

$$GM_S x^2 - GM_J(R-x)^2 = \frac{4\pi^2(R-x)^3x^2}{T^2}$$

Expanding the equation and grouping like terms gives:

$$GM_S x^2 - GM_J(R^2-2Rx+x^2) = \frac{4\pi^2}{T^2}R^3x^2 - \frac{12\pi^2}{T^2}R^2x^3 + \frac{12\pi^2}{T^2}Rx^4 - \frac{4\pi^2}{T^2}x^5$$

Rearranging terms to one side to form a quintic function gives:

$$\frac{4\pi^2}{T^2}x^5 - \frac{12\pi^2 R}{T^2}x^4 + \frac{12\pi^2 R^2}{T^2}x^3 + \left(GM_S - GM_J - \frac{4\pi^2 R^3}{T^2}\right)x^2 + 2GM_J Rx - GM_J R^2 = 0$$

The mass of the sun is given as: $M_S = 1.989 \times 10^{30}\,\mathrm{kg}$
The mass of the Jupiter is given as: $M_J = 1.898 \times 10^{27}\,\mathrm{kg}$
The distance between Jupiter and the sun is given as: $R = 7.785 \times 10^{11}\,\mathrm{m}$
The period of Jupiter's orbit is given as: $T = 3.78 \times 10^{8}\,\mathrm{s}$
The gravitational constant is given as: $G = 6.674\,08 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$

Plugging in these values and solving the quintic for x using an algebra engine gives:

$$x = 5.054\,680\,755 \times 10^{10}\,\mathrm{m}$$

This compares with the accepted value of 5.205×10^{10} m.

(c) By definition, the Lagrangian nodes locate specific points relative to an orbiting body that orbit at the same rate as it. As such, the calculated L_1 point is given to be orbiting the sun, and not Jupiter. Despite this fixed orbit, it should be noted that, over the course of one Jovian revolution, the satellite also completes one revolution about Jupiter (when considered strictly relative to the planet itself). This follows as a necessary consequence of the fixed nature of the L_1 point.

6 Sources

Anton, H. et al (2009). Calculus: Early Transcendentals (9th ed.). John Wiley & Sons, Inc.

van Bemmel, H.M., Position and Velocity From Classical Elements TR20, DRMO Press (2000)