EXPERIMENT - 1

VECTORS AND LINEAR COMBINATIONS

Vector Space: A vector space V over a field F consists of a set on which two operations (addition and scalar multiplication) are defined so that for each pair of elements x, y in V there is a unique element $x+y$ in V and for each element a in F and x in V, there is a unique element ax in V such that the following conditions hold:

Vector Addition Properties:

→ Commutativity: $\forall x, y \in V$, $x+y = y+x$

→ Associativity: $\forall x, y, z \in V$, $x+(y+z) = (x+y)+z$

→ Identity: $\exists 0 \in V \ni x+0 = x$, $\forall x \in V$

→ Inverse: $\forall x \in V$, $\exists y \in V \ni x+y = 0$

Scalar Multiplication Properties:

→ Unitary Law: $\forall x \in V$, $1 \cdot x = x$

→ Associative Law: $\forall a, b \in F$ & $\forall x \in V$, $(ab)x = a(bx)$

→ Distributive Law: $\forall a, b \in F$ & $\forall x, y \in V$, $(a+b)x = ax+bx$
$a(x+y) = ax+ay$

Linear Combination: For a given non-empty subset S of V, we call an element $v \in V$ is a linear combination of S if there exists a finite number of vectors $v_1, v_2 \dots v_n$ in S and scalars $\alpha_1, \alpha_2 \dots \alpha_n$ in F such that
$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

Span: The span of a non empty subset S of V denoted by span (S) is the set containing all linear combinations of the vector in S.

Determine whether the given vector is in the span of S. If yes, give the corresponding scalars for the linear combination. Also, write a python program for the same.

1. $V = (2,-1,1)$, $S = \{(1,0,2), (-1,1,1)\}$

$(2,-1,1) = \alpha_1(1,0,2) + \alpha_2(-1,1,1)$

$\Rightarrow \alpha_1 - \alpha_2 = 2$

$\alpha_2 = -1$

$2\alpha_1 + \alpha_2 = 1$

$\Rightarrow \alpha_1 = 1, \quad \alpha_2 = -1$

$(2,-1,1) = 1(1,0,2) - 1(-1,1,1)$

hence the given vector is in the span of S

Python Program:

```
import numpy as np
import sympy as sp
def isp(S,V):
    n= len(S)
    var= sp.symbols(f'a:{n}')
    m=-V
    for i,v in enumerate(var):
        m=m+v*S[i]
    sca= sp.solve(m,var)
    if len(sca)==0:
        print("No, the given vector is not in the given span")
        return False
    else:
        print("The given vector is in the span of S")
        print("The scalars are",sca)
```

```
v = np.array ((2, -1, 1))
s = np.array ([(1,0,2), (-1,1,1)])
isp(s,v)
```

2. $v = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} 1 &= \alpha_1 + \alpha_3 \\ 2 &= \alpha_2 + \alpha_3 \\ -3 &= -\alpha_1 \\ 4 &= \alpha_2 \end{aligned}$$

$\Rightarrow \alpha_2 = 4, \ \alpha_1 = 3, \ \alpha_3 = -2$

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = 3 \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + -2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$\therefore$ hence the given vector is in the span of S

<u>Python Program:</u>

```
v = sp.Matrix ( [[1,2], [-3,4]] )
v1 = sp.Matrix ( [[1,0], [-1,0]] )
v2 = sp.Matrix ([[1,1], [0,0]])
v3 = sp.Matrix ( [[0,1], [0,1]] )
s = np.array ([v1, v2, v3])
isp( s,v)
```

EXPERIMENT-2

LINEAR DEPENDENCE, INDEPENDENCE, BASIS AND DIMENSION

Linearly dependent vectors : A set of vectors $\{V_1, V_2 \ldots V_k\}$ are said to be linearly dependent if $\exists$ some scalars $\alpha_1, \alpha_2, \ldots \alpha_k$ not all zero such that

$$\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_k V_k = 0$$

Linearly independent vectors: A set of vectors $\{V_1, V_2 \ldots V_k\}$ are said to be linearly independent if $\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_k V_k = 0$
$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$

Basis: A basis B of a vector space V is a linearly independent subset of V that spans V

Dimension: The number of elements of a basis of the vectorspace V is called the dimension of V

THEOREM: If V is a finite dimensional vectorspace of dimension n, any linearly independent subset of V that contains exactly n vectors is a basis of V

Check whether the following vectors are linearly dependent or independent

$S = \{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1)\}$

$\Rightarrow \alpha_1(1,0,0,-1) + \alpha_2(0,1,0,-1) + \alpha_3(0,0,1,-1) + \alpha_4(0,0,0,1) = 0$

$\Rightarrow \alpha_1 + 0\alpha_2 + 0\alpha_3 + 0\alpha_4 = 0 \Rightarrow \alpha_1 = 0$

$0\alpha_1 + \alpha_2 + 0\alpha_3 + 0\alpha_4 = 0 \Rightarrow \alpha_2 = 0$

$0\alpha_1 + 0\alpha_2 + \alpha_3 + 0\alpha_4 = 0 \Rightarrow \alpha_3 = 0$

$-\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 0 \Rightarrow \alpha_4 = 0$

$\therefore$ since $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

The vectors are linearly independent

2. $S = \{1 + x + x^2,\ x + x^2,\ x^2\}$

$\Rightarrow \alpha_1(1 + x + x^2) + \alpha_2(x + x^2) + \alpha_3(x^2) = 0$

$\alpha_1 + x(\alpha_1 + \alpha_2) + x^2(\alpha_1 + \alpha_2 + \alpha_3) = 0$

on comparing LHS and RHS

$\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0$

$\therefore$ since $\alpha_1 = \alpha_2 = \alpha_3 = 0$

The vectors are linearly independent

II <u>Verify whether B is the basis of V</u>

1. $B = \{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1)\}$

$V = \mathbb{R}^4$

$\Rightarrow \alpha_1(1,0,0,-1) + \alpha_2(0,1,0,-1) + \alpha_3(0,0,1,-1) + \alpha_4(0,0,0,1) = 0$

$\alpha_1 + 0\alpha_2 + 0\alpha_3 + 0\alpha_4 = 0 \Rightarrow \alpha_1 = 0$

$0\alpha_2 + \alpha_2 + 0\alpha_3 + 0\alpha_4 = 0 \Rightarrow \alpha_2 = 0$

$0\alpha_1 + 0\alpha_2 + \alpha_3 + 0\alpha_4 = 0 \Rightarrow \alpha_3 = 0$

$-\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 0 \Rightarrow \alpha_4 = 0$

$\therefore$ Since $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

The vectors are linearly independent

$|B| = 4$, dimension $(\mathbb{R}^4) = 4$

$\Rightarrow$ B is a basis of $\mathbb{R}^4$

Python Program

```
import numpy as np
import sympy as sp
def is_ind(S):
    alphas = sp.symbols(f'a: {len(S)}')
    M = None
    For i, var in enumerate(alphas):
        if M is None:
            M = var * S[i]
        else:
            M = M + var * S[i]
    soln = sp.solve(M, alphas, manual = True)
    if np.abs(soln[0]).sum() == 0:
        print("Yes, the vectors are linearly independent")
        return True
    else:
        print("The vectors are linearly dependent")
        print("The scalars are: ", soln)
        return False
def is_basis(B, dim):
    if is_ind(B):
        if len(B) == int(dim):
            print(f'and the dimension of v={dim}=number of elements in R')
            print("The given set is a basis")
            return True
        else:
            print("dimension is not matched hence it is not a basis")
            return False
```

```
else:
    print ("The given set is not a basis")
    return False
S = np.array([1,0,0,-1],[0,1,0,-1],[0,0,1,-1],[0,0,0,1]])
dim = 4
is_basis (S, dim)
```

2. $B = \{1+x+x^2, x+x^2, x^2\}$, $V = P_3(\mathbb{R})$

$\Rightarrow \alpha_1(1+x+x^2) + \alpha_2(x+x^2) + \alpha_3(x^2) = 0$

$\alpha_1 + x(\alpha_1+\alpha_2) + x^2(\alpha_1+\alpha_2+\alpha_3) = 0$

comparing LHS and RHS

$\alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0$

$\therefore$ since $\alpha_1 = \alpha_2 = \alpha_3 = 0$

The vectors are linearly independent

$|B| = 3$

dimension $(P_3(\mathbb{R})) = 4$

$\therefore$ B is not a basis of $P_3(\mathbb{R})$

<u>Python program</u>

```
S = np.array([1+x+x**2, x**2+x, x**2])
dim = 4
is_basis (S, dim)
```

EXPERIMENT-3
LINEAR TRANSFORMATIONS

Linear Transformation: Let V & W be two vector spaces over a field F, then $T: V \to W$ is a linear transformation if
$T(x_1 + x_2) = T(x_1) + T(x_2)$
$T(cx_1) = cT(x_1) \quad \forall x_1, x_2 \in V$ & $c \in F$

THEOREM: Let V & W be two vector spaces over a field F then $T: V \to W$ is linear if & only if
$T(x_1 + x_2) = cT(x_1) + T(x_2) \quad \forall x_1, x_2 \in V, c \in F$

Verify whether the following transformations are linear or not

1. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, 2x_1 + x_2)$

$T(0,0) = (0, 2.0 + 0) = (0,0)$
$x = (x_1, x_2)$
$y = (y_1, y_2)$

$$\begin{aligned} T(cx+y) &= T(c(x_1, x_2) + (y_1, y_2)) \\ &= T((cx_1, cx_2) + (y_1, y_2)) \\ &= T(cx_1 + y_1, cx_2 + y_2) \\ &= (cx_1 + y_1, 2cx_1 + 2y_1 + cx_2 + y_2) \end{aligned}$$

$$\begin{aligned} cT(x) + T(y) &= cT(x_1, x_2) + T(y_1, y_2) \\ &= (cx_1 + y_1, 2cx_1 + 2y_1 + cx_2 + y_2) \end{aligned}$$

$T(cx + y) = cT(x) + T(y)$
$\therefore$ T is a linear transformation

Python Program

1.
```
import sympy as sp
import numpy as np
def T(v):
    return np.array([v[0], 2*v[0]+v[1]])
def is_linear(T, domain_dim):
    x = np.array(sp.symbols(f'x:{domain_dim}'))
    y = np.array(sp.symbols(f'y:{domain_dim}'))
    c = sp.symbol('c')
    lhs = T(c*x+y)
    rhs = c*T(x) + T(y)
    lhs = [sp.expand(i) for i in lhs]
    print(f'T(c*x+y) = {lhs}')
    rhs = [sp.expand(i) for i in rhs]
    print(f'c*T(x) + T(y) = {rhs}')
    if lhs == rhs:
        print("The given map is a linear transformation")
        return True
    else:
        print("The given map is not a linear transformation")
        return False
is_linear(T, 2)
```

2. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = ((x_1 - x_2), 2x_3, 1)$

$T(0,0,0) = (0,0,1) \neq (0,0,0)$

$\therefore$ <u>T is not a linear transformation</u>

Python Program

2. def T(V):
    return np.array (V[0]-V[1], 2*V[2], 1)
is_linear (T, 3)

3. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_2+x_3, x_1+x_3, x_1+x_2)$

$T(0,0,0) = (0,0,0)$
$x = (x_1, x_2, x_3)$
$y = (y_1, y_2, y_3)$

$$\begin{aligned} T(cx+y) &= T(c(x_1,x_2,x_3) + (y_1,y_2,y_3)) \\ &= T((cx_1+cx_2+cx_3) + (y_1,y_2,y_3)) \\ &= T(cx_1+y_1, cx_2+y_2, cx_3+y_3) \\ &= (cx_2+y_2+cx_3+y_3, cx_1+y_1+cx_3+y_3, cx_1+y_1+cx_2+y_2) \end{aligned}$$

$$\begin{aligned} cT(x)+T(y) &= cT(x_1,x_2,x_3) + T(y_1,y_2,y_3) \\ &= c(x_2+x_3, x_1+x_3, x_1+x_2) + (y_2+y_3, y_1+y_3, y_1+y_2) \\ &= (cx_2+cx_3, cx_1+cx_3, cx_1+cx_2) + (y_2+y_3, y_1+y_3, y_1+y_2) \\ &= (cx_2+y_2+cx_3+y_3, cx_1+y_1+cx_3+y_3, cx_1+y_1+cx_2+y_2) \end{aligned}$$

$T(cx+y) = cT(x) + T(y)$
$\therefore$ T is a linear transformation

Python Program

def T(V):
    return np.array(V[1]+V[2], V[0]+V[2], V[0]+V[1])
is_linear (T, 3)

EXPERIMENT-4
MATRIX OF LINEAR TRANSFORMATION

(A) FINDING THE MATRIX OF LINEAR TRANSFORMATION

If T is any linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ there is always an $m \times n$ matrix A with the property that $T(x) = Ax \quad \forall x \in \mathbb{R}^n$

How to find the matrix of linear transformation

Example: $T: \mathbb{R}^4 \to \mathbb{R}^3$

$B_1 = \{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$

& $B_2 = \{w_1, w_2, w_3\}$ is a basis of $\mathbb{R}^3$

Then $T(v_1), T(v_2), T(v_3), T(v_4) \in \mathbb{R}^3$ and can be expressed as a linear combination of w_1, w_2, w_3

$$T(v_1) = a_{11}(w_1) + a_{21}(w_2) + a_{31}(w_3)$$
$$T(v_2) = a_{12}(w_1) + a_{22}(w_2) + a_{32}(w_3)$$
$$T(v_3) = a_{13}(w_1) + a_{23}(w_2) + a_{33}(w_3)$$
$$T(v_4) = a_{14}(w_1) + a_{24}(w_2) + a_{34}(w_3)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = [T]_{B_1}^{B_2}$$

A is a matrix of T with respect to the basis B_1 & B_2

1. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x-y+z,\ 2x+3y-z/2,\ x+y-2z)$ with respect to the basis $B_1 = \{(-1,1,0),\ (5,-1,2),\ (1,2,1)\}$
$B_2 = \{(1,1,0),\ (0,0,1),\ (1,5,2)\}$

$T(-1,1,0) = (-2,1,0)$
$T(5,-1,2) = (8,6,0)$
$T(1,2,1) = (0,\ 15/2,\ 1)$

$(-2,1,0) = a_{11}(1,1,0) + a_{21}(0,0,1) + a_{31}(1,5,2)$
$\Rightarrow a_{11} + a_{31} = -2$
$a_{11} + 5a_{31} = 1$
$a_{21} + 2a_{31} = 0$

$\Rightarrow a_{31} = \frac{3}{4},\quad a_{21} = -\frac{3}{2},\quad a_{11} = \frac{-11}{4}$

$(8,6,0) = a_{12}(1,1,0) + a_{22}(0,0,1) + a_{32}(1,5,2)$
$\Rightarrow a_{12} + a_{32} = 8$
$a_{12} + 5a_{32} = 6$
$a_{22} + 2a_{32} = 0$
$\Rightarrow a_{32} = -\frac{1}{2},\quad a_{12} = \frac{17}{2},\quad a_{22} = 1$

$(0,\ 15/2,\ 1) = a_{13}(1,1,0) + a_{23}(0,0,1) + a_{33}(1,5,2)$
$\Rightarrow a_{13} + a_{33} = 0$
$a_{13} + 5a_{33} = 15/2$
$a_{23} + 2a_{33} = 1$
$\Rightarrow a_{33} = 15/8,\quad a_{13} = -15/8,\quad a_{23} = -11/4$

$$\therefore A = \begin{bmatrix} -11/4 & 17/2 & -15/8 \\ -3/2 & 1 & -11/4 \\ 3/4 & -1/2 & 15/8 \end{bmatrix}$$

Python Program

```
1. import numpy as np
   from numpy import linalg as la
   def T(x, y, z):
       return [x-y+z, 2*x+3*y+(1/2)*z, x+y-2*z]
   B1 = np.array([[-1, 1, 0], [5, -1, 2], [1, 2, 1]])
   print("The specified basis for the domain space is", B1)
   B2 = np.array([[1, 1, 0], [0, 0, 1], [1, 5, 2]])
   print("The specified basis for the co-domain space is", B2)
   w1 = T(-1, 1, 0)
   w2 = T(5, -1, 2)
   w3 = T(1, 2, 1)
   print("Evaluating T at the basis B1: ", w1, w2, w3)
   v1 = la.solve(B2.T, w1)
   v2 = la.solve(B2.T, w2)
   v3 = la.solve(B2.T, w3)
   M = np.array([v1, v2, v3])
   print("The matrix of the linear transformation is: \n", M)
```

(B) FINDING THE LINEAR TRANSFORMATION WHEN THE MATRIX & BASIS ARE GIVEN

Procedure: we need to find $T: \mathbb{R}^2 \to \mathbb{R}^2$

where matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ of T

with respect to B_1 & B_2 is given

Find $T(v_1) = a_1 w_1 + a_2 w_2$, $T(v_2) = b_1 w_1 + b_2 w_2$

let $x, y \in \mathbb{R}^2$, Find c_1 & $c_2 \Rightarrow (x, y) = c_1 v_1 + c_2 v_2$

$T(x, y) = T(c_1 v_1) + T(c_2 v_2) = c_1 T(v_1) + c_2 T(v_2)$

2. Find the linear transformation for the matrix $\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}$ with respect to the basis $B_1 = \{(1,-1), (1,1)\}$
$B_2 = \{(1,0), (0,1)\}$

$T(V_1) = 2(1,0) + 4(0,1) = (2,4)$
$T(V_2) = 3(1,0) - 5(0,1) = (3,-5)$

let $(x,y) = C_1 V_1 + C_2 V_2$
$= C_1(1,-1) + C_2(1,1)$
$\Rightarrow C_1 + C_2 = x$
$-C_1 + C_2 = y$
$\Rightarrow 2C_2 = x+y$
$C_2 = \frac{x+y}{2}$, $C_1 = \frac{x-y}{2}$

$(x,y) = \left(\frac{x-y}{2}\right) T(1,-1) + \left(\frac{x+y}{2}\right) T(1,1)$
$= \left(\frac{x-y}{2}\right)(2,4) + \left(\frac{x+y}{2}\right)(3,5)$
$= (x-y, 2x\ 2y) + \left(\frac{3x+3y}{2}, \frac{5x+5y}{2}\right)$
$= \left(\frac{5x+y}{2}, \frac{-x-9y}{2}\right)$

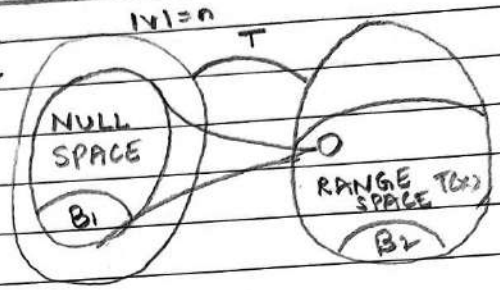
Python Program

2.
```
import sympy as sp
import numpy as np
def lt_of_matrix(B1, B2, M):
    k, m = len(B1), len(B2)
    T = [sum(M[j,i] * B2[j] for j in range(m)) for i in range(k)]
print("The images of basis elements are")
```

```
for i in range (k):
    print (f'T({B1[i]}) = {T[i]}')
x = sp.symbols (f'x : {k}')
c = sp.symbols (f'c: {k}')
for i in range (k):
    x = x - c[i] * B1[i]
s = sp.solve (x, c)
transform = 0
for i, v in enumerate (s):
    transform += s[v] * T[i]
print ("The required linear transformation is ", tuple (transform))
B1 = np.array ([[1, -1], [1, 1]])
B2 = np.array ([[1, 0], [0, 1]])
M = sp.Matrix ([[2, 3], [4, -5]])
lt_of_matrix (B1, B2, M)
```

P.B.
03/02/2024

EXPERIMENT - 5
RANK NULLITY THEOREM

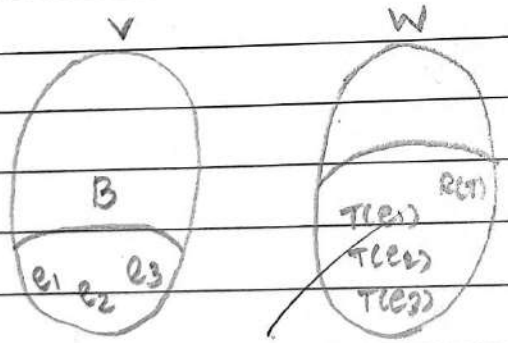


Range Space: Let $T: V \to W$ be a linear transformation, then the range space of T denoted by $R(T)$ is the set $\{T(x) \mid x \in V\}$
The dimension of Range space is called as the rank of T denoted by R

Null Space: The null space of T denoted by $N(T)$ is the set $\{x \in V \mid T(x) = 0_W\}$. The dimension of the null space is called the nullity of T denoted by N

Rank Nullity Theorem: Let V and W be vector spaces and let $T: V \to W$ be a linear transformation. If V is finite dimensional then Nullity (T) + Rank (T) = dimension (V)
$\Rightarrow N + R = n$

note:



The images of a basis span the range space

Find the range space, rank, null space and nullity for the linear transformations given below and hence verify rank-nullity theorem. Also write a python program to verify the same.

1. $T: \mathbb{R}^3 \to \mathbb{R}^3$
$T(e_1) = (1,-1,0)$, $T(e_2) = (2,0,1)$, $T(e_3) = (1,1,1)$

To prove $N + R = 3$

STEP 1: FINDING RANK

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_1 - R_3 \\ R_2 \to 2R_1 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \quad R_3 \to R_2 - R_3$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

span$\{(1,-1,0),(2,0,1),(1,1,1)\}$ = span$\{(1,-1,0),(0,-2,-1)\}$ = range space

since $\{(1,-1,0),(0,-2,-1)\}$ are linearly independent and spans $R(T)$, it forms a basis of $R(T)$

$\therefore$ hence dimension of $R(T) = R = 2$

STEP 2: FINDING NULLITY

$$\text{Null space} = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0,0)\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid T(x(1,0,0)+y(0,1,0)+z(0,0,1)) = (0,0,0)\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid xT(1,0,0) + yT(0,1,0) + zT(0,0,1) = (0,0,0)\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid x(1,-1,0) + y(2,0,1) + z(1,1,1) = (0,0,0)\}$$
$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x+2y+z=0 \\ -x+0y+z=0 \\ 0x+y+z=0 \end{array} \right\}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad R_3 \to 2R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} 2y+2z=0 \\ x+2y+z=0 \\ \Rightarrow y=-z,\ x=z \end{array}$$

$$\Rightarrow \text{Null space} = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y,z) = z(1,-1,1)\}$$

Dimension of Null space = 1

$R + N = 2 + 1 = 3 = |\mathbb{R}^3|$

$\therefore$ Rank Nullity theorem is verified

Python Program

1.
```
import sympy as sp
def rnt(n, RS):
    rank = RS.rank()
    B = RS.rref()
    print("The range space is spanned by the non zero row of: \n", B[0])
    N = RS.transpose().nullspace()
    print("The null space is spanned by columns of ", N)
    nullity = len(N)
    if rank + nullity == n:
        print("The rank nullity theorem is verified")
        return True
    else:
        print("You made some mistake because theorem has to be true")
        return False
RS = sp.matrix([[1,-1,0], [2,0,1], [1,1,1]])
rnt(3, RS)
```

2. $T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(e_1) = (1,1,0)$, $T(e_2) = (0,1,1)$, $T(e_3) = (1,2,1)$

To prove $N+R = 3$

STEP 1: FINDING RANK

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank} = 2$$

$\text{Span}\{(1,1,0),(0,1,1),(1,2,1)\} = \text{Span}\{(1,1,0),(0,1,1)\}$
$= \text{Range Space}$

Since $\{(1,1,0),(0,1,1)\}$ are linearly independent and spans $R(T)$, it forms a basis of $R(T)$
hence dimension of $R(T) = 2$

STEP 2: To find nullity

$$\begin{aligned}
\text{Null Space} &= \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0,0)\} \\
&= \{(x,y,z) \in \mathbb{R}^3 \mid T(x(1,0,0) + y(0,1,0) + z(0,0,1)) = (0,0,0)\} \\
&= \{(x,y,z) \in \mathbb{R}^3 \mid xT(1,0,0) + yT(0,1,0) + zT(0,0,1) = (0,0,0)\} \\
&= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x + 0y + z = 0 \\ x + y + 2z = 0 \\ 0x + y + z = 0 \end{array} \right\}
\end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x+z=0 \Rightarrow x=y=-z$
$y+z=0$

Null space $= \{ (x,y,z) \in \mathbb{R}^3 \mid (x,y,z) = z(-1,-1,1) \}$

Dimension of null space is $= 1$
$\Rightarrow R+N = 2+1 = 3 = |\mathbb{R}^3|$
$\therefore$ Rank Nullity theorem is verified

Python Program

```
2. RS= sp.Matrix([[1,1,0], [0,1,1], [1,2,1]])
rnk(3,RS)
```

23/14/V

EXPERIMENT-6

SOLUTIONS TO THE PROBLEM ON DIFFERENT TYPES OF PARTIAL DIFFERENTIAL EQUATIONS

TYPE - I

The general form of the equation is

$F(p, q) = 0$ —①

Let the solution of the equation be

$z = ax + by + c$ —②

differentiate partially wrt x and y in eq ②

we get $\frac{\partial z}{\partial x} = p = a$

$\frac{\partial z}{\partial y} = q = b$

Now substitute these values in eq ①

$F(a, b) = 0$

In order to find the complete solution we have to reduce one constant in eq ②

Let $b = \phi(a)$

$z = ax + \phi(a)\, y + c$

1. Solve $p^2 + q^2 = 1$

The general solution of the type of equation is $z = ax + by + c$

$p = a$ and $q = b$

$\Rightarrow a^2 + b^2 = 1$

$b^2 = 1 - a^2$

$b^2 = \pm\sqrt{1 - a^2}$

$\therefore$ The complete solution of the equation is

$z = ax + (\sqrt{1 - a^2})\, y + c$

1. Python Program

```
from sympy import *
from sympy.abc import x,y,a,b,c
f= Function ('F')
u= f(x,y)
p= u.diff (x)
q= u.diff (y)
eq= p**2+ q**2-1
Eq = eq.subs(p, a) .subs (q,b)
print ("Equation becomes", Eq)
b_val = solve (Eq ,b)
print ("b = ", b_val)
z= a*x +b*y + c
ans = z.subs (b, b_val [0])
print ("The required solution is ", ans)
```

2. Solve $pq = 5$

The general solution is $z = ax + by + c$

$p = a,\ q = b$

$\Rightarrow ab = 5$

$\Rightarrow b = \frac{5}{a}$

$\therefore$ The complete solution is

$z = ax + \frac{5y}{a} + c$

Python Program

```
eq = p*q - 5
```

3. Solve $pq + p + q = 0$

The general solution is $z = ax + by + c$

$p = a, \; q = b$

$\Rightarrow ab + a + b = 0$

$b(1+a) = -a$

$b = \frac{-a}{(1+a)}$

The complete solution is

$z = ax + \left(\frac{-a}{(1+a)}\right) y + c$

Python Program

```
eq = p*q + p + q
```

TYPE-II

$F(z, p, q) = 0$

let $u = x + ay$

$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \frac{\partial u}{\partial x} \Rightarrow p = \frac{dz}{du}$

$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \frac{\partial u}{\partial y} \Rightarrow q = a \frac{dz}{du}$

$\therefore$ The complete solution of the form $F(p, q, z)$ is

$F\left(\frac{dz}{du}, a\frac{dz}{du}, z\right) = 0$

1. Solve $p(1-q^2) = q(1-z)$

we know that $p = \frac{dz}{dv}$, $q = a\frac{dz}{dv}$

$\Rightarrow \frac{dz}{dv}\left(1-\left(\frac{a\,dz}{dv}\right)^2\right) = a\frac{dz}{dv}(1-z)$

$\Rightarrow 1 - a^2\left(\frac{dz}{dv}\right)^2 = a - az$

$\Rightarrow 1 - a + az = a^2\left(\frac{dz}{dv}\right)^2$

$\Rightarrow \sqrt{1-a+az} = a\frac{dz}{dv}$

$\Rightarrow \int dv = \int \frac{a}{\sqrt{1-a+az}}\,dz$ let $t = 1-a+az$, $dt = a\,dz$

$\Rightarrow v = \int \frac{1}{\sqrt{t}}\,dt$

$\Rightarrow v = 2\sqrt{t} + c$

$\therefore \underline{\underline{x + ay = 2\sqrt{1-a+az}}}$

Python Program

```
from sympy import Function, diff, Eq, solve, dsolve
from sympy import x, y, v, p, a, b
z= Function ("z") (v)
eqn = p*(1-q**2) - q*(1-z)
eqn1 = eqn.subs(p, diff(z,v)).subs(q, a*diff(z,v))
print (eqn1)
h1= solve (eqn1, diff(z,v))
print (h1)
sol= dsolve (h1[0] - diff(z,v))
```

```
print (sol)
ans = sol.subs (u, x+a*y)
print ("Required answer is")
print (ans)
```

2. solve $p(1+q) = zq$

we know that $p = \frac{dz}{du}$, $q = a\frac{dz}{du}$

$\Rightarrow \frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = z\,a\frac{dz}{du}$

$\Rightarrow 1 + a\frac{dz}{du} = za$

$\Rightarrow a\frac{dz}{du} - za = -1$

$\Rightarrow \frac{dz}{du} - z = -\frac{1}{a}$

$\Rightarrow \int \frac{dz}{za-1} = \int \frac{du}{a}$

$\frac{u}{a} = \frac{1}{a}\log(za-1)$

$\Rightarrow u = \log(az-1)$

$\therefore x + ay + c = \log(az-1)$

$e^{x+ay+c} = az-1$

$z = \frac{1 + e^{x+ay+c}}{a}$

Python Program

```
eqn = p*(1+q) - z*q
```

3. Solve $p = qz$

wkt $p = \frac{dz}{dv}$, $q = a\frac{dz}{dv}$

$\Rightarrow \frac{dz}{dv} = a\frac{dz}{dv} z$

$\therefore \underline{\underline{z = \frac{1}{a}}}$

<u>Python Program</u>

```
eqn = p - q*z
```

<u>TYPE III</u>

The general form is:

$F(x, y, p, q) = 0$

(i) We have to separate $F_1(x,p)$ and $F_2(y,q)$

(ii) $F_1(x,p) = F_2(y,q) = k$ where k is a non zero constant

(iii) Find the values of p and q

(iv) Substitute the values in $dz = p\,dx + q\,dy$

(v) Integrate the above to get the general solution

1. Solve $p + x = q + y$

$p + x = q + y = k$

$\Rightarrow p + x = k$ & $q + y = k$

$\Rightarrow p = k - x$ & $q = k - y$

Substituting the values in $dz = p\,dx + q\,dy$

$$\int dz = \int (k-x)\,dx + (k-y)\,dy$$

$$z = kx - \frac{x^2}{2} + ky - \frac{y^2}{2}$$

$$\therefore \underline{z = k(x+y) - \frac{1}{2}(x^2+y^2) + c}$$

<u>Python Program:</u>

```
from sympy.abc import x, y, a, b, c, k, p, q
from sympy import *
lhs = p+x
rhs = q+y
r1 = Eq(lhs, k)
r2 = Eq(rhs, k)
h1 = solve(r1, p)
h2 = solve(r2, p)
print("p=", h1, "and q=", h2)
z = integrate(h1[0], x) + integrate(h2[0], y)
print("The solution is ", z)
```

2. Solve $p - \cos x - \frac{\cos y}{q} = 0$

$$p - \cos x = \frac{\cos y}{q} = k$$

$$p = k + \cos x \quad \& \quad q = \frac{\cos y}{k}$$

$$\int dz = \int (k + \cos x)\,dx + \frac{\cos y}{k}\,dy$$

$$\therefore \underline{z = kx + \sin x + \frac{\sin y}{k} + c}$$

Python program
lhs = p - cos(x)
rhs = cos(y) / q

3. solve $p+q = \sin x + \sin y$

$p - \sin x = \sin y - q = k$

$\Rightarrow p = k + \sin x \quad , \quad q = \sin y - k$

$\Rightarrow \int dz = \int (k+\sin x)dx + (\sin y - k)\, dy$

$\Rightarrow z = kx - \cos x - ky - \cos y$

$\therefore \underline{\underline{z = k(x-y) - (\cos x + \cos y) + c}}$

Python Program
lhs = p - sin(x)
rhs = sin(y) - q

TYPE - IV - LAIRAUT'S EQUATION

The general form is
$F(x,y,z,p,q) = 0$ or $px + qy + F(p,y) = z$

The solution of the equation will be $z = ax + by + c$

$p = \frac{\partial z}{\partial x} = a \quad , \quad q = \frac{\partial z}{\partial y} = b$

Solution of the equation is $ax + by + F(a,b) = 0$

1. Solve $z = px + qy + (p^2+q^2)$

WKT $p = \frac{\partial z}{\partial x} = a$

$q = \frac{\partial z}{\partial y} = b$

$\therefore$ The solution is : $z = ax + by + (a^2+b^2)$

Python Program:

```
from sympy.abc import x,y,a,b,p,q
from sympy import *
z = p*x + q*y + (p**2 + q**2)
sol = z.subs([(p,a), (q,b)])
print("general solution is ",sol)
```

2. Solve $z = px + qy + \frac{pq}{p-q}$

WKT $p = \frac{\partial z}{\partial x} = a$, $q = \frac{\partial z}{\partial y} = b$

$\therefore$ The solution is: $z = ax + by + \frac{ab}{a-b}$

Python Program:

```
z = p*x + q*y + p*q / (p-q)
```

3. Solve $z = px + qy + \log(pq)$

WKT $p = \frac{\partial z}{\partial x} = a$, $q = \frac{\partial z}{\partial y} = b$

$\therefore$ The solution is: $z = ax + by + \log(ab)$

Python Program:

```
z = p*x + q*y + log(p*q)
```

EXPERIMENT- 7

SOLUTIONS TO THE PROBLEMS OF TOTAL DIFFERENTIAL EQUATIONS

The general form of Total Differential Equation is

$Pdx + Qdy + Rdz = 0$ — (1)

where P, Q and R are the functions of x, y and z

Theorem: The differential $Pdx + Qdy + Rdz = 0$ is said to satisfy the integrability condition if

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

OR

$$\begin{vmatrix} P & Q & R \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = 0$$

The general solution to the method:

i) check the integrability condition

ii) Assume any one of the variables to be a constant say $z = a \Rightarrow dz = 0 \Rightarrow Pdx + Qdy = 0$

(iii) Differentiate Totally

$Pdx + Qdy = f(z)$ — (2)

(iv) Solve the equation (2)

v) Compare with equation (1)

vi) on solving we will get the general solution

1. Solve: $(y+z)\,dx + (z+x)\,dy + (x+y)\,dz = 0$

$P = y+z, \quad Q = z+x, \quad R = x+y$

Checking integrability

$$(y+z)\left(\frac{\partial(z+x)}{\partial z} - \frac{\partial(x+y)}{\partial y}\right) + (z+x)\left(\frac{\partial(x+y)}{\partial x} - \frac{\partial(y+z)}{\partial z}\right) + (x+y)\left(\frac{\partial(y+z)}{\partial y} - \frac{\partial(z+x)}{\partial x}\right)$$

$$= (y+z)(1-1) + (z+x)(1-1) + (x+y)(1-1) = 0$$

$\therefore$ It is integrable

Solving: $(y\,dx + z\,dx) + (z\,dy + x\,dy) + (x\,dz + y\,dz) = 0$

$$d(xy) + d(zx) + d(zy) = 0$$

$\therefore$ solution is $xy + yz + zx = c$

Python Program

```
from sympy import symbols, simplify, diff
x,y,z = symbols('x, y, z')
P = y+z
Q = z+x
R = x+y
F = simplify(P*(diff(Q,z) - diff(R,y)) + Q*(diff(R,x) - diff(P,z))
            + R*(diff(P,y) - diff(Q,x)))
if F == 0:
    print("It is integrable")
else:
    print("It is not integrable")
```

2 Solve: $yz\,dx + 2xz\,dy - 3xy\,dz = 0$

Checking integrability:

$P = yz, \quad Q = 2xz, \quad R = -3xy$

$$yz\left(\frac{\partial(2xz)}{\partial z} - \frac{\partial(-3xy)}{\partial y}\right) + 2xz\left(\frac{\partial(2xz)}{\partial x} - \frac{\partial(yz)}{\partial z}\right)$$

$$+ (-3xy)\left(\frac{\partial(yz)}{\partial y} - \frac{\partial(-3xy)}{\partial x}\right)$$

$= 2xyz + 3xyz - 6xyz - 2xyz - 2xyz - 6xyz + 3xyz$

$= 5xyz - 8xyz + 3xyz$

$= 0$

$\therefore$ <u>It is integrable</u>

Solving: Lets assume $z = a$

$\Rightarrow dz = 0$

$ya\,dx + 2xa\,dy = 0$

$y\,dx = -2x\,dy$

$$\Rightarrow \int \frac{dx}{x} = -2\int \frac{dy}{y}$$

$\Rightarrow \log x = -2\log y + \log c$

$\Rightarrow \log x + 2\log y = \log c$

$\Rightarrow \log x + 2\log y = \log c$

$\Rightarrow \log(xy^2) = \log c$

$\Rightarrow xy^2 = c = f(z)$ — ②

differentiating totally in equation ②

$y^2 dx + x\,2y\,dy = f'(z)\,dz$ — ③

$$③ \times \frac{z}{y} \Rightarrow yz\,dx + 2xz\,dy = f'(z)\,\frac{z}{y}$$

$3xy = F'(z)\,\frac{z}{y}$

$\Rightarrow 3xy^2 = F'(z)\,z$

$\Rightarrow 3F(z) = \frac{dF(z)}{dz}\,z$

$\Rightarrow 3\int \frac{dz}{z} = \int \frac{dF(z)}{F(z)}$

$\Rightarrow 3\log z = \log F(z) + \log c$

$\Rightarrow 3\log z - \log F(z) = \log c$

$\Rightarrow \log z^3 - \log F(z) = \log c$

$\Rightarrow \log\left(\frac{z^3}{F(z)}\right) = \log c$

$\Rightarrow z^3 = cF(z)$

$\Rightarrow z^3 = cxy^2$

<u>Solution is : $z^3 = cxy^2$</u>

<u>Python Program</u>

```
P = y*z
Q = 2*x*z
R = -3*x*y
```

3. Solve: $(x+z)^2 dy + y^2(dx + dz) = 0$

Checking integrability

$P = y^2$, $Q = (x+z)^2$, $R = y^2$

$$y^2\left(\frac{\partial(x+z)^2}{\partial z} - \frac{\partial(y^2)}{\partial y}\right) + (x+z)^2\left(\frac{\partial(y^2)}{\partial x} - \frac{\partial(y^2)}{\partial z}\right) + y^2\left(\frac{\partial(y^2)}{\partial y} - \frac{\partial(x+z)^2}{\partial x}\right)$$

$= 2y^2(x+z) - 4y^3 + 4y^3 - 2y^2(x+z) = 0$

$\therefore$ <u>It is integrable</u>

Solving: Lets assume $y = a$

$\Rightarrow dy = 0$

$y^2 dx + y^2 dz = 0$

$\Rightarrow dx + dz = 0$

$\Rightarrow x + z = F(y)$ — ②

Applying total differentiation in equation ②

$dx + dz = F'(y) dy$

$y^2(dx + dz) - y^2 F'(y) dy = 0$

$(x+z)^2 = -y^2 F'(y)$

$(F(y))^2 = -y^2 F'(y)$

$\int \frac{dy}{y^2} = -\int \frac{d(F(y))}{F(y)^2}$

$\Rightarrow -\frac{1}{y} = \frac{1}{F(y)} + c$

$\Rightarrow -\frac{1}{y} = \frac{1}{x+z} + c$

$\Rightarrow -\frac{1}{y} - \frac{1}{x+z} = c$

$\Rightarrow \frac{1}{y} + \frac{1}{x+z} = c$

$\Rightarrow \frac{x+y+z}{y(x+z)} = c$

$\therefore$ The solution is: $x + y + z = c\,y(x+z)$

Python Program

```
P = y**2
Q = (x+z) **2
R = y**2
```

4. Solve: $3x^2\,dx + 3y^2\,dy - (x^3+y^3-e^{2z})\,dz = 0$

Checking integrability:

$P = 3x^2$, $Q = 3y^2$, $R = -(x^3+y^3-e^{2z})$

$$3x^2\left(\frac{\partial(3y^2)}{\partial z} - \frac{\partial(-(x^3+y^3-e^{2z}))}{\partial y}\right) + 3y^2\left(\frac{\partial(-(x^3+y^3-e^{2z}))}{\partial x} - \frac{\partial(3x^2)}{\partial z}\right)$$

$$+ -(x^3+y^3-e^{2z})\left(\frac{\partial(3x^2)}{\partial y} - \frac{\partial(3y^2)}{\partial x}\right)$$

$= 3x^2(0+3y^2) + 3y^2(-3x^2-0) + (-(x^3+y^3-e^{2z})(0-0))$

$= 3x^2y^2 - 3x^2y^2 = 0$

$\therefore$ <u>It is integrable</u>

Solving: Lets assume $z = a \Rightarrow dz = 0$

$3x^2\,dx + 3y^2\,dy = 0$

$\int 3x^2\,dx = -\int 3y^2\,dy$

$x^3 + y^3 = c$

$\Rightarrow x^3 + y^3 = F(z)$

Differentiating totally

$3x^2\,dx + 3y^2\,dy = F'(z)\,dz$

$3x^2\,dx + 3y^2\,dy - F'(z)\,dz = 0$

$\Rightarrow F'(z) = x^3 + y^3 + e^{2z}$

$F'(z) = F(z) + e^{2z}$

$F'(z) - F(z) = e^{2z}$

$IF = e^{-\int dz} = e^{-z}$

Solution: $F(z) \cdot IF = \int e^{2z}(IF)\,dz + c$

$F(z)\,e^{-z} = \int e^{2z}e^{-z}\,dz$

$F(z)\,e^{z} = e^{z} + c$

$\Rightarrow (x^3+y^3)e^{-z} = e^2+c$

$\therefore$ The solution is : $\dfrac{x^3+y^3}{e^z} - e^z = c$

Python Program

```
P= 3**x**2
Q= 3*y**2
R= -(x***3 + y**3 + exp(2*z))
```

5. Solve: $(2x^2+2yx+2xz^2+1)\,dx + dy + 2z\,dz = 0$

checking Integrability:

$P = (2x^2+2yx+2xz^2+1)$, $Q=1$, $R=2z$

$$(2x^2+2yx+2xz^2+1)\left(\frac{\partial(1)}{\partial z} - \frac{\partial(2z)}{\partial y}\right) + 1\left(\frac{\partial(2z)}{\partial x} - \frac{\partial(2x^2+2yx+2xz^2+1)}{\partial z}\right)$$

$$+ 2z\left(\frac{\partial(2x^2+2yx+2xz^2+1)}{\partial y} - \frac{\partial(1)}{\partial x}\right)$$

$= (2x^2+2yx+2xz^2+1)(0-0) + 1(0-4xz) + 2z(2x-0)$

$= -4xz + 4xz = 0$

$\therefore$ It is integrable

Solving: Lets assume $x = a \Rightarrow dx = 0$

$\Rightarrow dy + 2z\,dz = 0$

$\int dy = -\int 2z\,dz$

$\Rightarrow y + z^2 = c = f(x)$

Total differentiation:

$F'(x)dx = dy + 2z\,dz$

$-F'(x)\,dx + dy + 2z\,dz = 0$

$F'(x) = -2x^2 - 2xy - 2xz^2 - 1$

$= -2x^2 - 2x(y^2+z^2) - 1$

$= -2x^2 - 2x\,F(x) - 1$

$\frac{\partial F}{\partial x} = -2x^2 - 2x\,F(x) - 1$

$F'(x) + 2xF(x) = -2x^2 - 1$

$IF = e^{\int 2x\,dx} = e^{x^2}$

Solution : $F \cdot IF = \int Q \cdot IF\,dx + c$

$F \cdot e^{x^2} = -\int 2x^2 + 1\, e^{x^2} dx + c$

Let $x^2 = t \Rightarrow 2x\,dx = dt$

$F \cdot e^{x^2} = -\int \sqrt{t}\, e^t - \int e^t \frac{1}{2\sqrt{t}}\, dt + c$

$\Rightarrow F e^{x^2} = -\sqrt{t}\, e^t + c$

$\therefore$ <u>solution is : $(y+z)e^{x^2} = -x e x^2 + c$</u>

<u>Python Program</u>

```
P= 2*x**2 +2*x*y + 2*z**2 +1
Q = 1
R = 2*z
```

EXPERIMENT-8

SOLUTIONS OF ONE DIMENSIONAL WAVE EQUATIONS USING FOURIER SERIES WITH DIRCHLET CONDITIONS

In general, the one dimensional wave equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{--- (1)}$$

where $u(x,t)$ is the deflection of the string subject to boundary conditions

$$\left.\begin{aligned} &\text{i) } u(0,t) = 0 \\ &\text{ii) } u(l,t) = 0 \end{aligned}\right\} \quad \text{--- (2)}$$

and Initial conditions

$$\left.\begin{aligned} &\text{iii) } u_t(x,0) = 0 \\ &\text{or } \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \end{aligned}\right\} \quad \text{--- (3)}$$

$$\text{iv) } u(x,0) = f(x), \quad 0 \le x \le 1$$

Let us assume the solution of eq (1) is of the form

$$u(x,t) = X(x)\,T(t) \quad \text{--- (4)}$$

use eq (4) in eq (1)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\left(X(x)\,T(t)\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(X(x)\,T(t)\right)$$

$$T(t)\frac{\partial^2}{\partial x^2}\left(X(x)\right) = \frac{X(x)}{c^2}\frac{\partial^2}{\partial t^2}T(t)$$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{c^2 T(t)}\frac{\partial^2 T(t)}{\partial t^2}$$

$$\Rightarrow \frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{c^2 T(t)} \frac{d^2 T(t)}{dt^2} = k$$

$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k$	$\frac{1}{T(t)\, c^2} \frac{d^2 T(t)}{dt^2} = k$
$\frac{d^2 X(x)}{dx^2} - X(x)k = 0$	$\frac{d^2 T(t)}{dt^2} - T(t)\, c^2 k = 0$
(i)	(ii)

$u(0,t) = 0 \Rightarrow u(x,t) = X(x)\,T(t)$

$= X(0)\,T(t) = 0$

$\Rightarrow X(0) = 0$

$u(l,t) = 0 \Rightarrow X(l)\,T(t) = 0$

$X(l) = 0$

<u>CASE 1</u>: $\frac{d^2 X}{dx^2} - kX = 0$

when $k = 0$

$\frac{d^2 X}{dx^2} = 0$

$\Rightarrow D^2 X = 0$

$\Rightarrow m^2 = 0$

$X(x) = (C_1 + C_2 x)\, e^{mx}$

$\Rightarrow X(x) = (A + Bx)$

$A = 0 \Rightarrow X(x) = Bx$

$X(l) = 0 \Rightarrow Bl = 0$

$\Rightarrow B = 0 \Rightarrow X(x) = 0$

since $X(x) = 0$

Initial conditions will not satisfy

CASE 2: $k = \lambda^2$, $\lambda \neq 0$

$$\frac{d^2X}{dx^2} - \lambda^2 x = 0$$

$(D^2 - \lambda^2)x = 0$

$m^2 - \lambda^2 = 0$

$\Rightarrow m^2 = \lambda^2$

$\Rightarrow m = \pm\lambda$

$x(x) = Ae^{-\lambda x} + Be^{\lambda x}$

$x = 0 \Rightarrow X(0) = 0$

$A + B = 0$

$\Rightarrow A = -B$

$X(x) = B(-e^{\lambda x} + e^{\lambda x})$

$x = l \Rightarrow x(l) = 0$

$B(-e^{\lambda l} + e^{\lambda l}) = 0$

$\Rightarrow A = 0,\ B = 0$

$X(x) = 0$ which will not satisfy initial conditions

CASE 3: $k = -\lambda^2$, $\lambda \neq 0$

$$\frac{d^2x}{dx^2} + \lambda^2 x = 0$$

$(D^2 + \lambda^2)x = 0$

$\Rightarrow m^2 + \lambda^2 = 0$

$\Rightarrow m^2 = -\lambda^2$

$\Rightarrow m = \pm\lambda i$

$X(x) = A\cos\lambda x + B\sin\lambda x$

$X(0) = 0 \Rightarrow A = 0$

$\Rightarrow X(x) = B\sin\lambda x$

$X(l) = 0 \Rightarrow B\sin\lambda l = 0$

$\Rightarrow \sin\lambda l = \sin n\pi$

$\Rightarrow \lambda = \frac{n\pi}{l}$

$X(x) = B \sin\left(\frac{n\pi}{l}\right) x$

general form $X(x) = B_n \sin\left(\frac{n\pi}{l}\right) x$ — (6)

$\frac{d^2T}{dt^2} - c^2 T k = 0$

$\frac{d^2T}{dt^2} + c^2 T \lambda^2 = 0$

$\frac{d^2T}{dt^2} + \left(\frac{c^2 n^2 \pi^2}{l^2}\right) T = 0$

$m^2 = \frac{c n^2 \pi^2}{l^2} = 0$

$\Rightarrow m = \pm\left(\frac{cn\pi}{l}\right)$

$T(t) = C \cos\left(\frac{cn\pi}{l}\right) t + D \sin\left(\frac{cn\pi}{l}\right) t$

$T_n(t) = C_n \cos\left(\frac{cn\pi}{l}\right) t + D_n \sin\left(\frac{cn\pi}{l}\right) t$

$U(x,t) = X(x)\, T(t)$

$U_n(x,t) = X_n(x)\, T_n(t)$

$= B_n \sin\left(\frac{n\pi}{l}\right) \left[C_n \cos\left(\frac{cn\pi}{l}\right) t + D_n \sin\left(\frac{cn\pi}{l}\right) t \right]$

$= \sin\left(\frac{n\pi}{l}\right) x \left[E_n \cos\left(\frac{cn\pi}{l}\right) t + F_n \sin\left(\frac{cn\pi}{l}\right) t \right]$ — (7)

$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t)$

$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l}\right) x \left[E_n \cos\left(\frac{cn\pi}{l}\right) + F_n \sin\left(\frac{cn\pi}{l}\right) \right]$

$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l}\right) x \left[-E_n \sin\left(\frac{cn\pi}{l}\right) + F_n \cos\left(\frac{cn\pi}{l}\right) \right]$

$\frac{\partial u}{\partial t} = 0 \Rightarrow \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l}\right) \times F_n\left(\frac{cn\pi}{l}\right) = 0 \quad \Rightarrow F_n = 0$

$u(x,t) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{l}\right) \times E_n \cos\left(\frac{cn\pi}{l}\right)t$

$u(x,0) = f(x)$

$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l}\right) \times G f(x)$

1. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0 \le x \le \pi, \quad t > 0$

i) $u_t(x,0) = 0$

ii) $u(x,0) = \sin x$

$u(0,t) = u(\pi,t) = 0$

$c = 1, \; l = \pi$

$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left[E_n \cos \frac{n\pi ct}{l} + F_n \sin \frac{n\pi ct}{l}\right]$ —①

$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{\pi} \left[E_n \cos \frac{n\pi ct}{\pi} + F_n \sin \frac{n\pi ct}{\pi}\right]$

$u(x,t) = \sum_{n=1}^{\infty} \sin nx \left[E_n \cos nt + F_n \sin nt\right]$ —②

By (i) $u_t(x,0) = 0$

$u(x,t) = \sum_{n=1}^{\infty} \sin nx \left[-E_n \sin nt + F_n \cos nt\right]$

$0 = u_t(x,0) = \sum_{n=1}^{\infty} \sin nx \cdot F_n$

$\Rightarrow \underline{F_n = 0}$

By (ii) $u(x,0) = \sin x$, substitute (ii) in ②

$u(x,0) = \sum_{n=1}^{\infty} \sin nx \; E_n$

$\sin x = E_1 \sin x + E_2 \sin 2x + \cdots$

$E_1 = 1, \quad E_2 = E_3 = E_4 \cdots = 0$

$u(x,t) = \sum_{n=1}^{\infty} \sin nx \, E_n \cos nt$

$= \sin x \, E_1 \cos t$

$\therefore \; u(x,t) = \sin x \cos t$

Python Program:

```
from sympy import pi,sin,cos,integrate,pprint
from sympy.abc import x,t
def wave(c,l,f,g):
    sol=0
    for n in range(1,4):
        en = (2/l)*integrate(f*sin(n*pi*x/l),(x,0,l))
        fn = (2/(n*pi*c))*integrate(g*sin(n*pi*x/l),(x,0,l))
        sol= sol+(en*cos(n*pi*c*t/l)+fn*sin(n*pi*c*t/l)
                                    *sin(n*pi*x/l)
    print("The solution of the given wave equation is
    pprint(sol)
wave(1,pi, sin(x),0)
```

2. A tightly stretched elastic string of length l with fixed points $x=0$, $x=2$ is initially in the position given by $u(x,0) = \sin^3\left(\frac{\pi x}{2}\right)$ it is released from position of rest. Find displacement $u(x,t)$

(i) $u_t(x,0) = 0$, (ii) $u(x,0) = \sin^3(\pi x/2)$

$l = 2$

$$U(x,t) = \sum_{n=0}^{\infty} \sin \frac{n\pi x}{l} \left[E_n \cos \frac{n\pi c t}{2} + F_n \sin \frac{n\pi c t}{l} \right] \quad \text{--- (1)}$$

$$U(x,t) = = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{2} \left[E_n \cos \frac{n\pi c t}{2} + F_n \sin \frac{n\pi c t}{2} \right]$$

By (i) $U_t(x,0) = 0$

$$U_t(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{2} \left[-E_n \sin \frac{n\pi c t}{2} + F_n \frac{n\pi c}{2} \cos \frac{n\pi c t}{2} \right]$$

$$\Rightarrow \sum_{n=1}^{\infty} \sin \left(\frac{n\pi x}{2} \right) \frac{n\pi c}{2} F_n = 0$$

$$\Rightarrow \underline{F_n = 0}$$

From (ii) $U(x,0) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{2} \left[E_n (1) + 0 \right]$

$$= \sin^3 \left(\frac{\pi x}{2} \right) = \frac{3}{4} \sin \frac{\pi x}{2} - \frac{1}{4} \sin \frac{3\pi x}{2}$$

$$= \sum_{n=1}^{\infty} E_n \sin \left(\frac{n\pi x}{2} \right) = \frac{3}{4} \sin \frac{\pi x}{2} - \frac{1}{4} \sin \frac{3\pi x}{2}$$

$$= E_1 \sin \frac{\pi x}{2} + E_2 \sin \frac{2\pi x}{2} + \dots = \frac{3}{4} \sin \pi x - \frac{1}{4} \sin \frac{3\pi x}{2}$$

By comparision

$$E_1 = \frac{3}{4}, \quad E_3 = -\frac{1}{4}, \quad E_2 = E_4 = \dots E_5 = 0$$

$$\underline{U(x,t)} = \frac{3}{4} \sin \frac{\pi x}{2} \cos \left(\frac{c\pi t}{2} \right) - \frac{1}{4} \sin \frac{3\pi x}{2} \cos \left(\frac{3c\pi t}{2} \right)$$

<u>Python Program</u>

```
wave(c, 2, sin(x*pi/2)**3, 0)
```

22/03/2023