Escola Secundária de Paredes

Proposta de Resolução da Ficha de Trabalho 8

Matemática A

12.º Ano de Escolaridade • Turma: $B+C+H$

Aula de Apoio

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1. .

1.1. .

$$-1 \le \sin\left(2x - \frac{\pi}{4}\right) \le 1, \forall x \in \mathbb{R}$$

$$\therefore -\frac{1}{2} \le \frac{1}{2}\sin\left(2x - \frac{\pi}{4}\right) \le \frac{1}{2}, \forall x \in \mathbb{R}$$

$$\therefore \frac{1}{4} - \frac{1}{2} \le \frac{1}{4} + \frac{1}{2}\sin\left(2x - \frac{\pi}{4}\right) \le \frac{1}{2} + \frac{1}{4}, \forall x \in \mathbb{R}$$

$$\therefore -\frac{1}{4} \le \frac{1}{4} + \frac{1}{2}\sin\left(2x - \frac{\pi}{4}\right) \le \frac{3}{4}, \forall x \in \mathbb{R}$$

$$\therefore -\frac{1}{4} \le f(x) \le \frac{3}{4}, \forall x \in D_f$$

Portanto, $D'_f = \left[-\frac{1}{4}; \frac{3}{4}\right]$

1.2. Pretende-se determinar a expressão geral das soluções da equação $f(x) = -\frac{1}{4}$

$$f(x) = -\frac{1}{4} \Leftrightarrow \frac{1}{4} + \frac{1}{2}\sin\left(2x - \frac{\pi}{4}\right) = -\frac{1}{4} \Leftrightarrow \frac{1}{2}\sin\left(2x - \frac{\pi}{4}\right) = -\frac{1}{4} - \frac{1}{4} \Leftrightarrow \frac{1}{2}\sin\left(2x - \frac{\pi}{4}\right) = -\frac{2}{4} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\sin\left(2x - \frac{\pi}{4}\right) = -\frac{1}{2} \Leftrightarrow \sin\left(2x - \frac{\pi}{4}\right) = -1 \Leftrightarrow \sin\left(2x - \frac{\pi}{4}\right) = \sin\left(\frac{3\pi}{2}\right) \Leftrightarrow$$

$$\Leftrightarrow 2x - \frac{\pi}{4} = \frac{3\pi}{2} + k2\pi, k \in \mathbb{Z} \Leftrightarrow 2x = \frac{3\pi}{2} + \frac{\pi}{4} + k2\pi, k \in \mathbb{Z} \Leftrightarrow 2x = \frac{7\pi}{4} + k2\pi, k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{\frac{7\pi}{4}}{2} + \frac{k2\pi}{2}, k \in \mathbb{Z} \Leftrightarrow x = \frac{7\pi}{8} + k\pi, k \in \mathbb{Z}$$

A expressão algébrica dos minimizantes da função f, $x = \frac{7\pi}{8} + k\pi, k \in \mathbb{Z}$

1.3. Pretende-se determinar a expressão geral das soluções da equação $f(x) = 0$

$$f(x) = 0 \Leftrightarrow \frac{1}{4} + \frac{1}{2}\sin\left(2x - \frac{\pi}{4}\right) = 0 \Leftrightarrow \frac{1}{2}\sin\left(2x - \frac{\pi}{4}\right) = -\frac{1}{4} \Leftrightarrow \sin\left(2x - \frac{\pi}{4}\right) = \frac{-\frac{1}{4}}{\frac{1}{2}} \Leftrightarrow$$

$$\Leftrightarrow \sin\left(2x - \frac{\pi}{4}\right) = -\frac{1}{2} \Leftrightarrow \sin\left(2x - \frac{\pi}{4}\right) = \sin\left(-\frac{\pi}{6}\right) \Leftrightarrow$$

$$\Leftrightarrow 2x - \frac{\pi}{4} = -\frac{\pi}{6} + k2\pi \vee 2x - \frac{\pi}{4} = \pi + \frac{\pi}{6} + k2\pi, k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow 2x = -\frac{\pi}{6} + \frac{\pi}{4} + k2\pi \vee 2x = \pi + \frac{\pi}{6} + \frac{\pi}{4} + k2\pi, k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow 2x = -\frac{2\pi}{12} + \frac{3\pi}{12} + k2\pi \vee 2x = \frac{12\pi}{12} + \frac{2\pi}{12} + \frac{3\pi}{12} + k2\pi, k \in \mathbb{Z} \Leftrightarrow$$

$\Leftrightarrow 2x=\frac{\pi}{12}+k2\pi \vee 2x=\frac{17\pi}{12}+k2\pi, k\in\mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x=\frac{\frac{\pi}{12}}{2}+\frac{k2\pi}{2} \vee x=\frac{\frac{17\pi}{12}}{2}+\frac{k2\pi}{2}, k\in\mathbb{Z} \Leftrightarrow x=\frac{\pi}{24}+k\pi \vee x=\frac{17\pi}{24}+k\pi, k\in\mathbb{Z}$

As expressões gerais dos zeros da função f são $x=\frac{\pi}{24}+k\pi \vee x=\frac{17\pi}{24}+k\pi, k\in\mathbb{Z}$

2. .

2.1. Ora,

$g\left(\frac{\pi}{3}\right)=2\sqrt{3}-4\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=2\sqrt{3}-4\cos\left(\frac{2\pi}{6}+\frac{\pi}{6}\right)=2\sqrt{3}-4\cos\left(\frac{3\pi}{6}\right)=2\sqrt{3}-4\cos\left(\frac{\pi}{2}\right)=$

$=2\sqrt{3}-4\times 0=2\sqrt{3}-0=2\sqrt{3}$

$g(0)=2\sqrt{3}-4\cos\left(0+\frac{\pi}{6}\right)=2\sqrt{3}-4\cos\left(\frac{\pi}{6}\right)=2\sqrt{3}-4\times\frac{\sqrt{3}}{2}=2\sqrt{3}-2\sqrt{3}=0$

Assim,

$g\left(\frac{\pi}{3}\right)+g(0)=2\sqrt{3}+0=2\sqrt{3}$

2.2. Determinemos o contradomínio da função g

$-1\leq\cos\left(x+\frac{\pi}{6}\right)\leq 1, \forall x\in\mathbb{R}$

$\therefore 4\geq -4\cos\left(x+\frac{\pi}{6}\right)\geq -4, \forall x\in\mathbb{R}$

$\therefore -4\leq -4\cos\left(x+\frac{\pi}{6}\right)\leq 4, \forall x\in\mathbb{R}$

$\therefore 2\sqrt{3}-4\leq 2\sqrt{3}-4\cos\left(x+\frac{\pi}{6}\right)\leq 2\sqrt{3}+4, \forall x\in\mathbb{R}$

$\therefore 2\sqrt{3}-4\leq g(x)\leq 2\sqrt{3}+4, \forall x\in D_g$

Portanto, $D'_g=\left[2\sqrt{3}-4; 2\sqrt{3}+4\right]$

Pretende-se determinar a expressão geral das soluções da equação $g(x)=2\sqrt{3}+4$

$g(x)=2\sqrt{3}+4 \Leftrightarrow 2\sqrt{3}-4\cos\left(x+\frac{\pi}{6}\right)=2\sqrt{3}+4 \Leftrightarrow -4\cos\left(x+\frac{\pi}{6}\right)=4 \Leftrightarrow \cos\left(x+\frac{\pi}{6}\right)=\frac{4}{-4} \Leftrightarrow$

$\Leftrightarrow \cos\left(x+\frac{\pi}{6}\right)=-1 \Leftrightarrow \cos\left(x+\frac{\pi}{6}\right)=\cos(\pi) \Leftrightarrow$

$\Leftrightarrow x+\frac{\pi}{6}=\pi+k2\pi, k\in\mathbb{Z} \Leftrightarrow x=\pi-\frac{\pi}{6}+k2\pi, k\in\mathbb{Z} \Leftrightarrow x=\frac{6\pi}{6}-\frac{\pi}{6}+k2\pi, k\in\mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x=\frac{5\pi}{6}, k\in\mathbb{Z}$

A expressão algébrica dos maximizantes da função g, é $x=\frac{5\pi}{6}, k\in\mathbb{Z}$

2.3. Seja τ o período positivo mínimo de g

$g(x+\tau)=g(x) \Leftrightarrow$

$\Leftrightarrow -4\cos\left(x+\tau+\frac{\pi}{6}\right)=-4\cos\left(x+\frac{\pi}{6}\right)$

$\Leftrightarrow \cos\left(x+\tau+\frac{\pi}{6}\right)=\cos\left(x+\frac{\pi}{6}\right)$

Como 2π rad é o período positivo mínimo da função cosseno, vem,

$\tau = 2\pi$

Portanto, 2π rad é o período positivo mínimo da função g

3. .

3.1. Domínio da função h: $D_h = \left\{x \in \mathbb{R} : 3x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\} = \left\{x \in \mathbb{R} : x \neq \frac{\pi}{6} + k\frac{\pi}{3}, k \in \mathbb{Z}\right\} =$

$= \mathbb{R} \setminus \left\{\frac{\pi}{6} + k\frac{\pi}{3}, k \in \mathbb{Z}\right\}$

Cálculo auxiliar

$3x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \Leftrightarrow x = \frac{\pi}{6} + k\frac{\pi}{3}, k \in \mathbb{Z}$

3.2. Pretende-se determinar a expressão geral das soluções da equação $h(x) = 0$

$h(x) = 0 \Leftrightarrow 3 - \tan^2(3x) = 0 \Leftrightarrow \tan^2(3x) = 3 \Leftrightarrow \tan(3x) = \pm\sqrt{3} \Leftrightarrow$

$\Leftrightarrow \tan(3x) = \sqrt{3} \vee \tan(3x) = -\sqrt{3} \Leftrightarrow \tan(3x) = \tan\left(\frac{\pi}{3}\right) \vee \tan(3x) = \tan\left(-\frac{\pi}{3}\right) \Leftrightarrow$

$\Leftrightarrow 3x = \frac{\pi}{3} + k\pi \vee 3x = -\frac{\pi}{3} + k\pi, k \in \mathbb{Z} \Leftrightarrow x = \frac{\pi}{9} + k\frac{\pi}{3} \vee x = -\frac{\pi}{9} + k\frac{\pi}{3}, k \in \mathbb{Z}$

As expressões algébricas dos zeros da função h, são $x = \frac{\pi}{9} + k\frac{\pi}{3} \vee x = -\frac{\pi}{9} + k\frac{\pi}{3}, k \in \mathbb{Z}$

3.3. Seja τ o período positivo mínimo de h

$h(x+\tau) = h(x) \Leftrightarrow$

$\Leftrightarrow 3 - \tan^2(3(x+\tau)) = 3 - \tan^2(3x)$

$\Leftrightarrow \tan^2(3x + 3\tau) = \tan^2(3x)$

Como π rad é o período positivo mínimo da função tangente, vem,

$3\tau = \pi \Leftrightarrow \tau = \frac{\pi}{3}$

Portanto, $\frac{\pi}{3}$ rad é o período positivo mínimo da função h

3.4. Ora,

$h(-x) = 3 - \tan^2[3(-x)] = 3 - \tan^2(-3x) = 3 - [\tan(-3x)]^2 = 3 - [-\tan(3x)]^2 =$

$= 3 - \tan^2(3x) = h(x), \forall x, -x \in D_h$

4. .

- Função i

Domínio da função i: $D_i = \left\{x \in \mathbb{R} : 2\sin(2x) - \sqrt{2} \neq 0\right\} =$

$= \left\{x \in \mathbb{R} : x \neq \frac{\pi}{8} + k\pi \wedge x \neq \frac{3\pi}{8} + k\pi, k \in \mathbb{Z}\right\} = \mathbb{R} \setminus \left\{\frac{\pi}{8} + k\pi; \frac{3\pi}{8} + k\pi, k \in \mathbb{Z}\right\}$

Cálculo auxiliar

$2\sin(2x)-\sqrt{2}=0 \Leftrightarrow 2\sin(2x)=\sqrt{2} \Leftrightarrow \sin(2x)=\frac{\sqrt{2}}{2} \Leftrightarrow \sin(2x)=\sin\left(\frac{\pi}{4}\right) \Leftrightarrow$

$\Leftrightarrow 2x=\frac{\pi}{4}+k2\pi \vee 2x=\pi-\frac{\pi}{4}+k2\pi, k\in\mathbb{Z} \Leftrightarrow x=\frac{\pi}{8}+k\pi \vee 2x=\frac{3\pi}{4}+k2\pi, k\in\mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x=\frac{\pi}{8}+k\pi \vee x=\frac{3\pi}{8}+k\pi, k\in\mathbb{Z}$

- Função j

Domínio da função j: $D_j=\left\{x\in\mathbb{R}: \sqrt{3}+2\cos\left(2x+\frac{\pi}{4}\right)\neq 0\right\}=$

$=\left\{x\in\mathbb{R}: x\neq\frac{7\pi}{24}+k\pi \wedge x\neq -\frac{13\pi}{24}+k\pi, k\in\mathbb{Z}\right\}=\mathbb{R}\setminus\left\{\frac{7\pi}{24}+k\pi; -\frac{13\pi}{24}+k\pi, k\in\mathbb{Z}\right\}$

Cálculo auxiliar

$\sqrt{3}+2\cos\left(2x+\frac{\pi}{4}\right)=0 \Leftrightarrow 2\cos\left(2x+\frac{\pi}{4}\right)=-\sqrt{3} \Leftrightarrow \cos\left(2x+\frac{\pi}{4}\right)=-\frac{\sqrt{3}}{2} \Leftrightarrow$

$\Leftrightarrow \cos\left(2x+\frac{\pi}{4}\right)=\cos\left(\frac{5\pi}{6}\right) \Leftrightarrow 2x+\frac{\pi}{4}=\frac{5\pi}{6}+k2\pi \vee 2x+\frac{\pi}{4}=-\frac{5\pi}{6}+k2\pi, k\in\mathbb{Z} \Leftrightarrow$

$\Leftrightarrow 2x=\frac{5\pi}{6}-\frac{\pi}{4}+k2\pi \vee 2x=-\frac{5\pi}{6}-\frac{\pi}{4}+k2\pi, k\in\mathbb{Z} \Leftrightarrow$

$\Leftrightarrow 2x=\frac{10\pi}{12}-\frac{3\pi}{12}+k2\pi \vee 2x=-\frac{10\pi}{12}-\frac{3\pi}{12}+k2\pi, k\in\mathbb{Z} \Leftrightarrow$

$\Leftrightarrow 2x=\frac{7\pi}{12}+k2\pi \vee 2x=-\frac{13\pi}{12}+k2\pi, k\in\mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x=\frac{7\pi}{24}+k\pi \vee x=-\frac{13\pi}{24}+k\pi, k\in\mathbb{Z}$

5. .

5.1. $\sin(a-b)=\sin[a+(-b)]=\sin a\cos(-b)+\cos a\sin(-b)=\sin a\cos b-\cos a\sin b$

5.2. $\sin(4x)=\sin(2x+2x)=\sin(2x)\cos(2x)+\cos(2x)\sin(2x)=2\sin(2x)\cos(2x)$

6. .

6.1. $\cos(a-b)=\cos[a+(-b)]=\cos a\cos(-b)-\sin a\sin(-b)=\cos a\cos b+\sin a\sin b$

6.2. $\cos(6x)=\cos(3x+3x)=\cos(3x)\cos(3x)-\sin(3x)\sin(3x)=\cos^2(3x)-\sin^2(3x)$

7. .

7.1. Sabe-se que $\sin(\pi-\alpha)=\frac{2}{3} \Leftrightarrow \sin(\alpha)=\frac{2}{3}$

De $\sin^2\alpha+\cos^2\alpha=1$, vem,

$\left(\frac{2}{3}\right)^2+\cos^2\alpha=1 \Leftrightarrow \frac{4}{9}+\cos^2\alpha=1 \Leftrightarrow \cos^2\alpha=1-\frac{4}{9} \Leftrightarrow \cos^2\alpha=\frac{5}{9} \Leftrightarrow \cos\alpha=\pm\sqrt{\frac{5}{9}} \Leftrightarrow$

$\Leftrightarrow \cos\alpha=\pm\frac{\sqrt{5}}{3}$

Como $\alpha \in \left]\frac{\pi}{2};\pi\right[$, vem, $\cos\alpha = -\frac{\sqrt{5}}{3}$

Assim, de $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$, resulta,

$$\tan\alpha = \frac{\frac{2}{3}}{-\frac{\sqrt{5}}{3}} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

7.2. Ora,

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\times\frac{2}{3}\times\left(-\frac{\sqrt{5}}{3}\right) = -\frac{4\sqrt{5}}{9}$$

7.3. Ora,

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = \frac{5}{9} - \left(\frac{2}{3}\right)^2 = \frac{5}{9} - \frac{4}{9} = \frac{1}{9}$$