Escola Secundária de Penafiel

Ficha de Trabalho nº 22
Proposta de resolução

Preparação para exame

12.º Ano de Escolaridade | Turma G-K

Números Complexos

1. $\frac{(-1-i)(2-2i)+i^5-2i^3}{3-4i} = \frac{-2+2i-2i+2i^2+i-2(-i)}{3+4i} = \frac{-2-2+i+2i}{3+4i} = \frac{-4+3i}{3+4i} =$
$= \frac{(-4+3i)(3-4i)}{(3+4i)(3-4i)} = \frac{-12+16i+9i-12i^2}{3^2+4^2} = \frac{-12+16i+9i+12}{25} = \frac{25i}{25} = i$

O afixo deste número complexo é $A(0;1)$

Representação no plano complexo

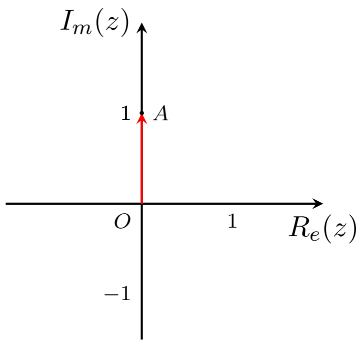

Figura 1

2. $\frac{z_1 - 3i^{41}}{\overline{-z_2}} = \frac{2+3i-3i}{\overline{-1+2i}} = \frac{2}{-1-2i} = \frac{2(-1+2i)}{(-1-2i)(-1+2i)} = \frac{-2+4i}{1^2+2^2} = \frac{-2+4i}{5} = -\frac{2}{5}+\frac{4}{5}i$

3. .

3.1. $\frac{(w-1)^2}{2+i}+1 = \frac{(2-2i-1)^2}{2+i}+1 = \frac{(1-2i)^2}{2+i}+1 = \frac{1-4i+(2i)^2}{2+i}+1 = \frac{1-4i-4}{2+i}+1 =$
$= \frac{-3-4i}{2+i}+1 = \frac{-3-4i+2+i}{2+i} = \frac{-1-3i}{2+i} = \frac{(-1-3i)(2-i)}{(2+i)(2-i)} = \frac{-2+i-6i+3i^2}{2^2+1^2} =$
$= \frac{-2-5i-3}{5} = \frac{-5-5i}{5} = -1-i$
O afixo deste número complexo é $(-1;-1)$, que pertence à bissetriz dos quadrantes ímpares

3.2. .

3.2.1. $zw = i\overline{w} \Leftrightarrow z\times(2-2i) = i\overline{2-2i} \Leftrightarrow z\times(2-2i) = i(2+2i) \Leftrightarrow$
$\Leftrightarrow z\times(2-2i) = 2i+2i^2 \Leftrightarrow z\times(2-2i) = -2+2i \Leftrightarrow z = \frac{-2+2i}{2-2i} \Leftrightarrow z = \frac{-(2-2i)}{2-2i} \Leftrightarrow$
$z = -1$
$C.S. = \{-1\}$

Outro processo

$zw = i\overline{w} \Leftrightarrow z\times(2-2i) = i\overline{2-2i} \Leftrightarrow z\times(2-2i) = i(2+2i) \Leftrightarrow z = \frac{-2+2i}{2-2i} \Leftrightarrow$
$\Leftrightarrow z = \frac{(-2+2i)(2+2i)}{(2-2i)(2+2i)} \Leftrightarrow z = \frac{-4-4i+4i+4i^2}{2^2+2^2} \Leftrightarrow z = \frac{-4-4}{8} \Leftrightarrow z = -1$
$C.S. = \{-1\}$

3.2.2. $z^3 + |\overline{w}|^2 z = 0 \Leftrightarrow z^3 + |\overline{2-2i}|^2 z = 0 \Leftrightarrow z^3 + |2+2i|^2 z = 0 \Leftrightarrow z^3 + (2^2+2^2)z = 0 \Leftrightarrow$
$\Leftrightarrow z^3 + 8z = 0 \Leftrightarrow z(z^2+8) = 0 \Leftrightarrow z = 0 \vee z^2 + 8 = 0 \Leftrightarrow z = 0 \vee z^2 = -8 \Leftrightarrow$
$\Leftrightarrow z = 0 \vee z = \pm\sqrt{-8} \Leftrightarrow z = 0 \vee z = \pm 2\sqrt{2}i$
$C.S. = \{0; -2\sqrt{2}i; 2\sqrt{2}i\}$

3.2.3. $z^3 - 2z^2 + z - |w|^2 + 6 = 0 \Leftrightarrow z^3 - 2z^2 + z - |2-2i|^2 + 6 = 0 \Leftrightarrow$
$\Leftrightarrow z^3 - 2z^2 + z - (2^2 + (-2)^2) + 6 = 0 \Leftrightarrow z^3 - 2z^2 + z - 8 + 6 = 0 \Leftrightarrow z^3 - 2z^2 + z - 2 = 0 \Leftrightarrow$
$\Leftrightarrow (z-2) \times (z^2+1) = 0 \Leftrightarrow z - 2 = 0 \vee z^2 + 1 = 0 \Leftrightarrow z = 2 \vee z^2 = -1 \Leftrightarrow$
$\Leftrightarrow z = 2 \vee z = \pm\sqrt{-1} \Leftrightarrow z = 2 \vee z = \pm i$

$C.S. = \{2; -i; i\}$

Cálculos auxiliares

Como 2 é raiz de $z^3 - 2z^2 + z - 2$, então tem-se que $z^3 - 2z^2 + z - 2 = (z-2) \times Q(z)$

Pela regra de Ruffini,

$$\begin{array}{c|ccc|c} & 1 & -2 & 1 & -2 \\ & & & & \\ 2 & & 2 & 0 & 2 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

logo, $Q(z) = z^2 + 1$

4. $z_1 = \sqrt{2}e^{i\frac{3\pi}{4}} = \sqrt{2} \times \left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = \sqrt{2} \times \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -1 + i$

Logo, $z_1 + z_2^2 = -1 + i + (2+3i)^2 = -1 + i + 4 + 12i + (3i)^2 = 3 + 13i - 9 = -6 + 13i$

5. .

5.1. $w_2 = e^{i\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

Assim, $w_1 + w_2 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = \frac{2\sqrt{2}}{2} = \sqrt{2}$

5.2. $\frac{\overline{x+yi}}{i^{29}} = \sqrt{2}w_1 - \overline{w_3} \Leftrightarrow \frac{x-yi}{i} = \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) - \overline{3i} \Leftrightarrow \frac{(x-yi)(-i)}{1} = 1 - i - (-3i) \Leftrightarrow$
$\Leftrightarrow -xi + yi^2 = 1 - i + 3i \Leftrightarrow -y - xi = 1 + 2i \Leftrightarrow -y = 1 \wedge -x = 2 \Leftrightarrow x = -2 \wedge y = -1$

5.3. .

5.3.1. $|w_1| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$
Seja θ o argumento de w_1
$\tan(\theta) = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$, $\wedge \theta \in 4^{\text{o}}$ quadrante
$\therefore \tan(\theta) = -1$, $\wedge \theta \in 4^{\text{o}}$ quadrante
Logo, $\theta = -\frac{\pi}{4}$
Assim, $w_1 = e^{i\left(-\frac{\pi}{4}\right)}$

5.3.2. $w_1 \times \overline{w_2} - iw_3^2 = e^{i\left(-\frac{\pi}{4}\right)} \times e^{i\left(-\frac{\pi}{4}\right)} - i \times (3i)^2 = e^{i\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)} + 9i = e^{i\left(-\frac{\pi}{2}\right)} + 9i = -i + 9i =$
$= 8i = 8e^{i\frac{\pi}{2}}$

5.3.3. $\frac{-w_2}{w_3} = \frac{e^{i\left(\frac{\pi}{4}+\pi\right)}}{3i} = \frac{e^{i\frac{5\pi}{4}}}{3e^{i\frac{\pi}{2}}} = \frac{1}{3}e^{i\left(\frac{5\pi}{4}-\frac{\pi}{2}\right)} = \frac{1}{3}e^{i\frac{3\pi}{4}}$

6. $z_1 = e^{i(-\alpha)} = \cos(-\alpha) + i\sin(-\alpha) = \cos(\alpha) - i\sin(\alpha)$

   $z_2 = e^{i\left(\frac{\pi}{2}+\alpha\right)} = \cos\left(\frac{\pi}{2}+\alpha\right) + i\sin\left(\frac{\pi}{2}+\alpha\right) = -\sin(\alpha) + i\cos(\alpha)$

   Assim,
   $z_1 - z_2 = \cos(\alpha) - i\sin(\alpha) - (-\sin(\alpha) + i\cos(\alpha)) = \cos(\alpha) - i\sin(\alpha) + \sin(\alpha) - i\cos(\alpha) =$
   $= (\cos(\alpha) + \sin(\alpha)) - i\,(\cos(\alpha) + \sin(\alpha))$
   Logo, o afixo deste número complexo é $(\cos(\alpha) + \sin(\alpha); -\cos(\alpha) - \sin(\alpha))$, que pertence à bissetriz dos quadrantes pares

7. $1 + \cos(2\theta) = 1 + \cos^2(\theta) - \sin^2(\theta) = 1 - \sin^2(\theta) + \cos^2(\theta) = \cos^2(\theta) + \cos^2(\theta) = 2\cos^2(\theta)$

   Assim, vem,

   $|z+1| = |\cos(2\theta) + i\sin(2\theta) + 1| = |\cos(2\theta) + 1 + i\sin(2\theta)| = \sqrt{(\cos(2\theta)+1)^2 + (\sin(2\theta))^2} =$
   $= \sqrt{\cos^2(2\theta) + 2\cos(2\theta) + 1 + \sin^2(2\theta)} = \sqrt{2 + 2\cos(2\theta)} = \sqrt{2(1+\cos(2\theta))} = \sqrt{2 \times 2\cos^2(\theta)} =$
   $= \sqrt{4\cos^2(\theta)} = 2|\cos(\theta)|$