Escola Secundária de Penafiel

Ficha Formativa nº 7
Proposta de resolução

12.º Ano de Escolaridade | Turma G-K

1. $|w| = \sqrt{1^2 + \left(-\frac{\sqrt{3}}{3}\right)^2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

   Seja $\theta = Arg(w)$

   $\tan(\theta) = \frac{-\frac{\sqrt{3}}{3}}{1}$ e $\theta \in 4°$ Q $\therefore \tan(\theta) = -\frac{\sqrt{3}}{3}$ e $\theta \in 4°$ Q Logo, $\theta = -\frac{\pi}{6}$

   Então, $w = \frac{2\sqrt{3}}{3}e^{i\left(-\frac{\pi}{6}\right)}$

   Assim,

   $w^3 = \left[\frac{2\sqrt{3}}{3}e^{i\left(-\frac{\pi}{6}\right)}\right]^3 = \frac{8\sqrt{3}}{9}e^{i\left(-\frac{3\pi}{6}\right)} = \frac{8\sqrt{3}}{9}e^{i\left(-\frac{\pi}{2}\right)} = -\frac{8\sqrt{3}}{9}i$

   $w^6 = \left[\frac{2\sqrt{3}}{3}e^{i\left(-\frac{\pi}{6}\right)}\right]^6 = \frac{64}{27}e^{i\left(-\frac{6\pi}{6}\right)} = \frac{64}{27}e^{i(-\pi)} = -\frac{64}{27}$

   $99 = 4 \times 24 + 3$, logo, $i^{99} = i^3$

   Portanto,

   $$x \times z^6 + y \times z^3 = 1 - i^{99} \Leftrightarrow -\frac{64}{27}x - \frac{8\sqrt{3}}{9}yi = 1 - i^3 \Leftrightarrow -\frac{64}{27}x - \frac{8\sqrt{3}}{9}yi = 1 - (-i) \Leftrightarrow$$
   $$\Leftrightarrow -\frac{64}{27}x - \frac{8\sqrt{3}}{9}yi = 1 + i \Leftrightarrow -\frac{64}{27}x = 1 \wedge -\frac{8\sqrt{3}}{9}y = 1 \Leftrightarrow x = -\frac{27}{64} \wedge y = -\frac{9}{8\sqrt{3}} \Leftrightarrow$$
   $$\Leftrightarrow x = -\frac{27}{64} \wedge y = -\frac{9\sqrt{3}}{8 \times 3} \Leftrightarrow x = -\frac{27}{64} \wedge y = -\frac{3\sqrt{3}}{8}$$

2. .

   2.1. $\frac{1}{4}\overline{w_2} = \frac{1}{4} \times 2e^{i\left(\frac{\pi}{3}\right)} = \frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right) + i\cos\left(\frac{\pi}{3}\right)\right] = \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{1}{4} + \frac{\sqrt{3}}{4}i$

   $\frac{1}{8}\overline{w_1} = \frac{1}{8} \times (2 - 2\sqrt{3}i) = \frac{1}{4} - \frac{\sqrt{3}}{4}i$

   Assim,

   $\frac{1}{4}\overline{w_2} + \frac{1}{8}\overline{w_1} = \frac{1}{4} + \frac{\sqrt{3}}{4}i + \frac{1}{4} - \frac{\sqrt{3}}{4}i = \frac{2}{4} = \frac{1}{2}$

Representação do afixo deste número complexo no plano complexo

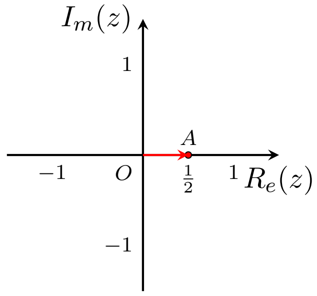

Figura 1

2.2. $|w_1| = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = \sqrt{4+12} = 4$

Seja $\theta = Arg(w_1)$
$\tan(\theta) = \dfrac{2\sqrt{3}}{2}$ e $\theta \in 1^\circ$ Q
$\therefore \tan(\theta) = \sqrt{3}$ e $\theta \in 1^\circ$ Q
Logo, $\theta = \dfrac{\pi}{3}$
Então, $w_1 = 4e^{i\left(\frac{\pi}{3}\right)}$

Assim,
$w_1^3 = \left[4e^{i\left(\frac{\pi}{3}\right)}\right]^3 = 64e^{i\left(\frac{3\pi}{3}\right)} = 64e^{i\pi}$
e
$-w_1^3 = 64e^{i(\pi+\pi)} = 64e^{i(2\pi)} = 64e^{i(0)}$

logo,

$$w = \frac{-w_1^3}{\overline{w_2}} = \frac{64e^{i(0)}}{2e^{i\left(\frac{\pi}{3}\right)}} = 32e^{i\left(0-\frac{\pi}{3}\right)} = 32e^{i\left(-\frac{\pi}{3}\right)}$$

As raízes cúbicas de $w = \dfrac{-w_1^3}{\overline{w_2}}$, são as soluções da equação $z^3 = w$

Então, tem-se,

$z^3 = w \Leftrightarrow z = \sqrt[3]{w} \Leftrightarrow z = \sqrt[3]{32e^{i\left(-\frac{\pi}{3}\right)}} \Leftrightarrow z = \sqrt[3]{32}e^{i\left(\frac{-\frac{\pi}{3}+2k\pi}{3}\right)}, k \in \{0;1;2\} \Leftrightarrow$
$\Leftrightarrow z = 2\sqrt[3]{4}e^{i\left(-\frac{\pi}{9}+\frac{2k\pi}{3}\right)}, k \in \{0;1;2\}$

Atribuindo valores a k, vem,

$k = 0 \to z_0 = 2\sqrt[3]{4}e^{i\left(-\frac{\pi}{9}\right)}$
$k = 1 \to z_1 = 2\sqrt[3]{4}e^{i\left(-\frac{\pi}{9}+\frac{2\pi}{3}\right)} = 2\sqrt[3]{4}e^{i\frac{5\pi}{9}}$
$k = 2 \to z_2 = 2\sqrt[3]{4}e^{i\left(-\frac{\pi}{9}+\frac{4\pi}{3}\right)} = 2\sqrt[3]{4}e^{i\frac{11\pi}{9}} = 2\sqrt[3]{4}e^{i\left(-\frac{7\pi}{9}\right)}$

Concluindo, as raízes cúbicas de $w = \dfrac{-w_1^3}{\overline{w_2}}$ são: $2\sqrt[3]{4}e^{i\left(-\frac{\pi}{9}\right)}; 2\sqrt[3]{4}e^{i\frac{5\pi}{9}}; 2\sqrt[3]{4}e^{i\left(-\frac{7\pi}{9}\right)}$

3. .

3.1. $z_1=\sum\limits_{j=1}^{7} i^j+2+\sqrt{3}i=i+i^2+i^3+i^4+i^5+i^6+i^7+2+\sqrt{3}i=i-1-i+1+i-1-i+2+\sqrt{3}i=$
$=1+\sqrt{3}i$

$|z_1|=\sqrt{1^2+\left(\sqrt{3}\right)^2}=\sqrt{4}=2$

Seja $\theta=Arg(z_1)$
$\tan(\theta)=\dfrac{\sqrt{3}}{1}$ e $\theta\in 1^\circ$ Q
$\therefore \tan(\theta)=\sqrt{3}$ e $\theta\in 1^\circ$ Q
Logo, $\theta=\dfrac{\pi}{3}$
Então, $z_1=2e^{i\left(\dfrac{\pi}{3}\right)}$
logo, $\overline{z_1}\times z_2^2$
$\overline{z_1}=2e^{i\left(-\frac{\pi}{3}\right)}$
e,
$z_2^2=\left[2e^{i\left(-\frac{\pi}{3}\right)}\right]^2=4e^{i\left(-\frac{2\pi}{3}\right)}$

Portanto,
$\overline{z_1}\times z_2^2=2e^{i\left(-\frac{\pi}{3}\right)}\times 4e^{i\left(-\frac{2\pi}{3}\right)}=8e^{i\left(-\dfrac{\pi}{3}-\frac{2\pi}{3}\right)}=8e^{i(-\pi)}=8e^{i\pi}$

3.2. Se z_1 e z_2 são duas raízes consecutivas de índice n de um complexo w, então os seus argumentos estão em progressão aritmética de razão $\dfrac{2\pi}{n}$, ou seja,

$\dfrac{\pi}{3}-\left(-\dfrac{\pi}{3}\right)=\dfrac{2\pi}{n}\Leftrightarrow\dfrac{\pi}{3}+\dfrac{\pi}{3}=\dfrac{2\pi}{n}\Leftrightarrow\dfrac{2\pi}{3}=\dfrac{2\pi}{n}\Leftrightarrow n=3$

e $w=z_1^3=\left[2e^{i\left(\frac{\pi}{3}\right)}\right]^3=2^3e^{i\left(\frac{3\pi}{3}\right)}=8e^{i\pi}$

3.3. $z^4-\overline{z_2}z=0\Leftrightarrow z(z^3-\overline{z_2})=0\Leftrightarrow z=0\vee z^3-\overline{z_2}=0\Leftrightarrow z=0\vee z^3=\overline{z_2}\Leftrightarrow$
$\Leftrightarrow z=0\vee z=\sqrt[3]{\overline{z_2}}\Leftrightarrow z=0\vee z=\sqrt[3]{2e^{i\left(\frac{\pi}{3}\right)}}$
Determinemos as soluções da equação $z=\sqrt[3]{2e^{i\left(\frac{\pi}{3}\right)}}$

$z=\sqrt[3]{2e^{i\left(\frac{\pi}{3}\right)}}\Leftrightarrow z=\sqrt[3]{2}e^{i\left(\frac{\frac{\pi}{3}+2k\pi}{3}\right)}, k\in\{0,1;2\}\Leftrightarrow z=\sqrt[3]{2}e^{i\left(\frac{\pi}{9}+\frac{2k\pi}{3}\right)}, k\in\{0,1;2\}$

Atribuindo valores a k, vem,

$k=0\rightarrow w_0=\sqrt[3]{2}e^{i\frac{\pi}{9}}$
$k=1\rightarrow w_1=\sqrt[3]{2}e^{i\frac{7\pi}{9}}$
$k=2\rightarrow w_3=\sqrt[3]{2}e^{i\frac{13\pi}{9}}=\sqrt[3]{2}e^{i\left(-\frac{5\pi}{9}\right)}$

Concluindo,
$C.S.=\left\{0, \sqrt[3]{2}e^{i\frac{\pi}{9}}; \sqrt[3]{2}e^{i\frac{7\pi}{9}}; \sqrt[3]{2}e^{i\left(-\frac{5\pi}{9}\right)}\right\}$

4. $i^{4n+3}+\sum_{j=1}^{4} i^j = 1-\dfrac{2x+yi}{1-i} \Leftrightarrow i^{4n}\times i^3+i+i^2+i^3+i^4 = 1-\dfrac{(2x+yi)(1+i)}{(1-i)(1+i)} \Leftrightarrow$
$\Leftrightarrow 1\times(-i)+i-1-i+1 = 1-\dfrac{2x+yi+2xi+yi^2}{1^2+1^2} \Leftrightarrow -i = 1-\dfrac{2x-y+(2x+y)i}{2} \Leftrightarrow$
$\Leftrightarrow -i = 1-\dfrac{2x-y}{2}-\dfrac{2x+y}{2}i \Leftrightarrow 1-\dfrac{2x-y}{2}=0 \wedge \dfrac{2x+y}{2}=1 \Leftrightarrow \dfrac{2x-y}{2}=1 \wedge 2x+y=2 \Leftrightarrow$
$\Leftrightarrow 2x-y=2 \wedge 2x+y=2 \Leftrightarrow y=2x-2 \wedge 2x+y=2 \Leftrightarrow y=2x-2 \wedge 2x+2x-2=2 \Leftrightarrow$
$\Leftrightarrow y=2x-2 \wedge 4x=4 \Leftrightarrow y=2x-2 \wedge x=1 \Leftrightarrow y=0 \wedge x=1$

5. $w=\cos\left(\dfrac{\pi}{6}\right)+i\sin\left(\dfrac{\pi}{6}\right)=e^{i\frac{\pi}{6}}$
$-w=e^{i\left(\frac{\pi}{6}+\pi\right)}=e^{i\frac{7\pi}{6}}=e^{i\left(-\frac{5\pi}{6}\right)}$
$\overline{w}=e^{i\left(-\frac{\pi}{6}\right)}$
$(\overline{w})^4=\left[e^{i\left(-\frac{\pi}{6}\right)}\right]^4=e^{i\left(-\frac{4\pi}{6}\right)}=e^{i\left(-\frac{2\pi}{3}\right)}$

Então,

$$-w\times(\overline{w})^4=e^{i\left(-\frac{5\pi}{6}\right)}\times e^{i\left(-\frac{2\pi}{3}\right)}=e^{i\left(-\frac{5\pi}{6}-\frac{2\pi}{3}\right)}=e^{i\left(-\frac{5\pi}{6}-\frac{4\pi}{6}\right)}=e^{i\left(-\frac{9\pi}{6}\right)}=e^{i\frac{3\pi}{6}}=e^{i\frac{\pi}{2}}$$

Logo,

$$\frac{-w\times(\overline{w})^4}{\sqrt{2}e^{i\frac{\pi}{4}}}=\frac{e^{i\frac{\pi}{2}}}{\sqrt{2}e^{i\frac{\pi}{4}}}=\frac{1}{\sqrt{2}}e^{i\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}=\frac{\sqrt{2}}{2}e^{i\frac{\pi}{4}}$$

Portanto, o argumento deste complexo é $\dfrac{\pi}{4}$

6. As n raízes de índice n de 1, são as soluções da equação $z^n=1$

$$z^n=1 \Leftrightarrow z=\sqrt[n]{1} \Leftrightarrow z=\sqrt[n]{e^{i(0)}} \Leftrightarrow z=e^{i\left(\frac{0+2k\pi}{n}\right)}, k\in\{0;1;2;\cdots n-1\}$$

Atribuindo valores k, vem,

$k=0\to z_0=e^{i(0)}$
$k=1\to z_1=e^{i\frac{2\pi}{n}}$
$k=2\to z_2=e^{i\frac{4\pi}{n}}$
$k=3\to z_3=e^{i\frac{6\pi}{n}}$
$\vdots\; k=n-1\to z_{n-1}=e^{i\left(\frac{2(n-1)\pi}{n}\right)}$

Assim, o produto das n raízes é igual a

$$z_0\times z_1\times z_2\times\cdots z_{n-1}=e^{i(0)}\times e^{i\frac{2\pi}{n}}\times e^{i\frac{4\pi}{n}}\times\cdots\times e^{i\left(\frac{2(n-1)\pi}{n}\right)}=e^{i\left(0+\frac{2\pi}{n}+\frac{4\pi}{n}+\frac{6\pi}{n}+\cdots+\frac{2(n-1)\pi}{n}\right)}=$$
$$=e^{i\left[\frac{\pi}{n}(0+2+4+6+\cdots+2n-2)\right]}=e^{i\left[\frac{\pi}{n}\left(\frac{0+2n-2}{2}\times n\right)\right]}=e^{i[(n-1)\pi]}$$

7. .

7.1. $(2-2i)z^4-4i=0 \Leftrightarrow (2-2i)z^4=4i \Leftrightarrow z^4=\dfrac{4i}{2-2i} \Leftrightarrow z^4=\dfrac{4i(2+2i)}{(2-2i)(2+2i)} \Leftrightarrow$
$\Leftrightarrow z^4=\dfrac{4i(2+2i)}{2^2+2^2} \Leftrightarrow z^4=\dfrac{8i+8i^2}{8} \Leftrightarrow z^4=\dfrac{-8+8i}{8} \Leftrightarrow z^4=-1+i \Leftrightarrow z=\sqrt[4]{-1+i} \Leftrightarrow$
$\Leftrightarrow z=\sqrt[4]{\sqrt{2}e^{i\frac{3\pi}{4}}} \Leftrightarrow z=\sqrt[4]{\sqrt{2}}e^{i\left(\frac{\frac{3\pi}{4}+2k\pi}{4}\right)}, k\in\{0;1;2;3\} \Leftrightarrow$
$\Leftrightarrow z=\sqrt[8]{2}e^{i\left(\frac{3\pi}{16}+\frac{k\pi}{2}\right)}, k\in\{0;1;2;3\}$

Atribuindo valores a k

$k=0 \rightarrow z_0 = \sqrt[8]{2}e^{i\frac{3\pi}{16}}$
$k=1 \rightarrow z_1 = \sqrt[8]{2}e^{i\left(\frac{3\pi}{16}+\frac{\pi}{2}\right)} = \sqrt[8]{2}e^{i\frac{11\pi}{16}}$
$k=2 \rightarrow z_2 = \sqrt[8]{2}e^{i\left(\frac{3\pi}{16}+\frac{2\pi}{2}\right)} = \sqrt[8]{2}e^{i\frac{19\pi}{16}} = \sqrt[8]{2}e^{i\left(-\frac{13\pi}{16}\right)}$
$k=3 \rightarrow z_3 = \sqrt[8]{2}e^{i\left(\frac{3\pi}{16}+\frac{3\pi}{2}\right)} = \sqrt[8]{2}e^{i\frac{27\pi}{16}} = \sqrt[8]{2}e^{i\left(-\frac{5\pi}{16}\right)}$
Assim, $C.S. = \left\{\sqrt[8]{2}e^{i\frac{3\pi}{16}}; \sqrt[8]{2}e^{i\frac{11\pi}{16}}; \sqrt[8]{2}e^{i\left(-\frac{13\pi}{16}\right)}; \sqrt[8]{2}e^{i\left(-\frac{5\pi}{16}\right)}\right\}$

Cálculos auxiliares:

Seja $w_1 = -1+i$
$|w_1| = \sqrt{(-1)^2+1^2} = \sqrt{2}$
Seja $\theta = Arg(w_1)$
$\tan(\theta) = \dfrac{1}{-1}$ e $\theta \in 2^\circ$ Q
$\therefore \tan(\theta) = -1$ e $\theta \in 2^\circ$ Q
Logo, $\theta = \dfrac{3\pi}{4}$
Então, $w_1 = \sqrt{2}e^{i\frac{3\pi}{4}}$

7.2. $z^2+z+i=-z-1 \Leftrightarrow z^2+2z+1+i=0 \Leftrightarrow z = \dfrac{-2\pm\sqrt{2^2-4\times 1\times(1+i)}}{2} \Leftrightarrow$
$\Leftrightarrow z = \dfrac{-2\pm\sqrt{4-4-4i}}{2} \Leftrightarrow z = \dfrac{-2\pm\sqrt{-4i}}{2} \Leftrightarrow z = \dfrac{-2\pm 2\sqrt{-i}}{2} \Leftrightarrow z = -1\pm\sqrt{-i}$

Determinemos as raízes quadradas de $-i$
$-i = e^{i\left(-\frac{\pi}{2}\right)}$
$w = \sqrt{-i} \Leftrightarrow w = \sqrt{e^{i\left(-\frac{\pi}{2}\right)}} \Leftrightarrow w = e^{i\left(\frac{-\frac{\pi}{2}+2k\pi}{2}\right)}, k \in \{0;1\} \Leftrightarrow w = e^{i\left(-\frac{\pi}{4}+k\pi\right)}, k \in \{0;1\}$
Atribuindo valores a k, vem,
$k=0 \rightarrow w_0 = e^{i\left(-\frac{\pi}{4}\right)} = \cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i$
$k=1 \rightarrow w_1 = e^{i\left(-\frac{\pi}{4}+\pi\right)} = e^{i\frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i$

Voltando à equação inicial, resulta,
$z=-1\pm\sqrt{-i} \Leftrightarrow z=-1-\sqrt{-i} \vee z=-1+\sqrt{-i}$

Assim,
Se $\sqrt{-i} = \dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i$, tem-se,
$z=-1-\sqrt{-i} \vee z=-1+\sqrt{-i} \Leftrightarrow z=-1-\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i \vee z=-1+\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i$

Se $\sqrt{-i} = -\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i$, tem-se,
$z=-1-\sqrt{-i} \vee z=-1+\sqrt{-i} \Leftrightarrow z=-1+\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i \vee z=-1-\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i$

Portanto, $C.S. = \left\{-1-\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i; -1+\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i\right\}$

7.3. $z^2 \times |z| + 1 - i = 0 \Leftrightarrow z^2 \times |z| = -1 + i$
Seja $w_1 = -1 + i$
$|w_1| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$
Seja $\theta = Arg(w_1)$
$\tan(\theta) = \dfrac{1}{-1}$ e $\theta \in 2^\circ$ Q
$\therefore \tan(\theta) = -1$ e $\theta \in 2^\circ$ Q
Logo, $\theta = \dfrac{3\pi}{4}$
Então, $w_1 = \sqrt{2}e^{i\frac{3\pi}{4}}$

E seja, $z = |z|e^{i\theta}$, com $\theta \in \mathbb{R}$
Então,
$z^2 \times |z| + 1 - i = 0 \Leftrightarrow z^2 \times |z| = -1 + i \Leftrightarrow \left(|z|e^{i\theta}\right)^2 \times |z| = \sqrt{2}e^{i\frac{3\pi}{4}} \Leftrightarrow |z|^2|z|e^{i(2\theta)} = \sqrt{2}e^{i\frac{3\pi}{4}} \Leftrightarrow$
$\Leftrightarrow |z|^3 e^{i(2\theta)} = \sqrt{2}e^{i\frac{3\pi}{4}} \Leftrightarrow |z|^3 = \sqrt{2} \wedge 2\theta = \dfrac{3\pi}{4} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow$
$\Leftrightarrow |z| = \sqrt[3]{\sqrt{2}} \wedge \theta = \dfrac{3\pi}{8} + k\pi, k \in \mathbb{Z} \Leftrightarrow |z| = \sqrt[6]{2} \wedge \theta = \dfrac{3\pi}{8} + k\pi, k \in \mathbb{Z}$
Atribuindo valores k, vem,
$k = 0 \rightarrow z_0 = \sqrt[6]{2}e^{i\frac{3\pi}{8}}$
$k = 1 \rightarrow z_1 = \sqrt[6]{2}e^{i\left(\frac{3\pi}{8} + \pi\right)} = \sqrt[6]{2}e^{i\frac{11\pi}{8}} = \sqrt[6]{2}e^{i\left(-\frac{5\pi}{8}\right)}$
$k = 2 \rightarrow z_2 = \sqrt[6]{2}e^{i\left(\frac{3\pi}{8} + 2\pi\right)} = \sqrt[6]{2}e^{i\frac{3\pi}{8}} = z_0$
A partir deste valor de $k(k = 2)$,começam a repetir as soluções z_0 e z_1
Logo, $C.S. = \left\{\sqrt[6]{2}e^{i\frac{3\pi}{8}}; \sqrt[6]{2}e^{i\left(-\frac{5\pi}{8}\right)}\right\}$

7.4. Seja $w_1 = 2 - 2\sqrt{3}i$
$|w_1| = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{16} = 4$
Seja $\theta = Arg(w_1)$
$\tan(\theta) = \dfrac{-2\sqrt{3}}{2}$ e $\theta \in 4^\circ$ Q
$\therefore \tan(\theta) = -\sqrt{3}$ e $\theta \in 4^\circ$ Q
Logo, $\theta = -\dfrac{\pi}{3}$
Então, $w_1 = 4e^{i\left(-\frac{\pi}{3}\right)}$

E seja, $z = |z|e^{i\theta}$, com $\theta \in \mathbb{R}$
Então,
$\overline{z} = |z|e^{i(-\theta)}$, com $\theta \in \mathbb{R}$

$z^2 \times e^{i\pi} = \overline{z} \times (2 - 2\sqrt{3}i) \Leftrightarrow \left(|z|e^{i\theta}\right)^2 \times e^{i\pi} = |z|e^{i(-\theta)} \times 4e^{i\left(-\frac{\pi}{3}\right)} \Leftrightarrow$
$\Leftrightarrow |z|^2 e^{i(2\theta)} \times e^{i\pi} = 4|z|e^{i\left(-\theta - \frac{\pi}{3}\right)} \Leftrightarrow |z|^2 e^{i(2\theta + \pi)} = 4|z|e^{i\left(-\theta - \frac{\pi}{3}\right)} \Leftrightarrow$
$\Leftrightarrow |z|^2 = 4|z| \wedge 2\theta + \pi = -\theta - \dfrac{\pi}{3} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow$
$\Leftrightarrow |z|^2 - 4|z| = 0 \wedge 3\theta = -\pi - \dfrac{\pi}{3} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow |z|(|z| - 4) = 0 \wedge 3\theta = -\dfrac{4\pi}{3} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow$
$\Leftrightarrow (|z| = 0 \vee |z| - 4 = 0) \wedge \theta = -\dfrac{4\pi}{9} + \dfrac{2k\pi}{3}, k \in \mathbb{Z} \Leftrightarrow (|z| = 0 \vee |z| = 4) \wedge \theta = -\dfrac{4\pi}{9} + \dfrac{2k\pi}{3}, k \in \mathbb{Z}$
Se $|z| = 0$, então, $z = 0$
Se $|z| = 4$, então, $z = 4e^{i\left(-\frac{4\pi}{9} + \frac{2k\pi}{3}\right)}, k \in \mathbb{Z}$
Atribuindo valores a k, vem,
$k = 0 \rightarrow z_0 = 4e^{i\left(-\frac{4\pi}{9}\right)}$
$k = 1 \rightarrow z_1 = 4e^{i\left(-\frac{4\pi}{9} + \frac{2\pi}{3}\right)} = 4e^{i\frac{2\pi}{9}}$
$k = 2 \rightarrow z_2 = 4e^{i\left(-\frac{4\pi}{9} + \frac{4\pi}{3}\right)} = 4e^{i\frac{8\pi}{9}}$
$k = 3 \rightarrow z_3 = 4e^{i\left(-\frac{4\pi}{9} + \frac{6\pi}{3}\right)} = 4e^{i\left(-\frac{4\pi}{9} + 2\pi\right)} = 4e^{i\left(-\frac{4\pi}{9}\right)} = z_0$
A partir deste valor de $k(k = 3)$,começam a repetir as soluções z_0, z_1 e z_2
Logo, $C.S. = \left\{0; 4e^{i\left(-\frac{4\pi}{9}\right)}; 4e^{i\frac{2\pi}{9}}; 4e^{i\frac{8\pi}{9}}\right\}$

8. .

8.1. $\frac{z_2}{z_1}=\frac{-1+\sqrt{3}i}{1+i}=\frac{(-1+\sqrt{3}i)(1-i)}{(1+i)(1-i)}=\frac{-1+\sqrt{3}i+i-\sqrt{3}i^2}{1^2+(-1)^2}=\frac{-1+\sqrt{3}i+i+\sqrt{3}}{2}=$

$=\frac{-1+\sqrt{3}}{2}+\frac{1+\sqrt{3}}{2}i$

Por outro lado,

Seja $z_2=-1+\sqrt{3}i$

$|z_2|=\sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{4}=2$

Seja $\theta=Arg(z_2)$

$\tan(\theta)=\frac{\sqrt{3}}{-1}$ e $\theta\in 2^\circ$ Q

$\therefore \tan(\theta)=-\sqrt{3}$ e $\theta\in 2^\circ$ Q

Logo, $\theta=\frac{2\pi}{3}$

Então, $w_1=2e^{i\frac{2\pi}{3}}$

Seja $z_1=1+i$

$|z_1|=\sqrt{1^2+1^2}=\sqrt{2}$

Seja $\alpha=Arg(z_1)$

$\tan(\alpha)=\frac{1}{1}$ e $\theta\in 1^\circ$ Q

$\therefore \tan(\alpha)=1$ e $\theta\in 1^\circ$ Q

Logo, $\alpha=\frac{\pi}{4}$

Então, $z_1=\sqrt{2}e^{i\frac{\pi}{4}}$

Assim, $\frac{z_2}{z_1}=\frac{2e^{i\frac{2\pi}{3}}}{\sqrt{2}e^{i\frac{\pi}{4}}}=\frac{2}{\sqrt{2}}e^{i\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)}=\frac{2\sqrt{2}}{2}e^{i\frac{5\pi}{12}}=\sqrt{2}e^{i\frac{5\pi}{12}}=\sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right)+i\sin\left(\frac{5\pi}{12}\right)\right)=$

$=\sqrt{2}\cos\left(\frac{5\pi}{12}\right)+i\sqrt{2}\sin\left(\frac{5\pi}{12}\right)$

Logo, das duas representações, vem,

$\sqrt{2}\cos\left(\frac{5\pi}{12}\right)=\frac{-1+\sqrt{3}}{2}\wedge\sqrt{2}\sin\left(\frac{5\pi}{12}\right)=\frac{1+\sqrt{3}}{2}\Leftrightarrow$

$\Leftrightarrow\cos\left(\frac{5\pi}{12}\right)=\frac{-1+\sqrt{3}}{2\sqrt{2}}\wedge\sin\left(\frac{5\pi}{12}\right)=\frac{1+\sqrt{3}}{2\sqrt{2}}\Leftrightarrow$

$\Leftrightarrow\cos\left(\frac{5\pi}{12}\right)=\frac{-\sqrt{2}+\sqrt{6}}{4}\wedge\sin\left(\frac{5\pi}{12}\right)=\frac{\sqrt{2}+\sqrt{6}}{4}\Leftrightarrow$

$\Leftrightarrow\cos\left(\frac{5\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}\wedge\sin\left(\frac{5\pi}{12}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$

8.2. $\left(\frac{z_2}{z_1}\right)^n=\left(\frac{2\sqrt{2}}{2}e^{i\frac{5\pi}{12}}\right)^n=\left(\frac{2\sqrt{2}}{2}\right)^n e^{i\frac{5n\pi}{12}}$

Para que $\left(\frac{z_2}{z_1}\right)^n$ seja imaginário puro, deverá ter-se $\frac{5n\pi}{12}=\frac{\pi}{2}+k\pi, k\in\mathbb{Z}\Leftrightarrow$

$\Leftrightarrow 5n\pi=\frac{12\pi}{2}+12k\pi, k\in\mathbb{Z}\Leftrightarrow n=\frac{6}{5}+\frac{12k}{5}, k\in\mathbb{Z}\Leftrightarrow n=\frac{6+12k}{5}, k\in\mathbb{Z}$

O menor valor de n , com $n\in\mathbb{N}$, que transforma $\left(\frac{z_2}{z_1}\right)^n$ num imaginário puro é obtido quando $k=2$, ou seja, é, $n=6$

Nota: para que n seja natural, $6+12k$ tem de ser múltiplo de 5