Proposta de Resolução do 3º Teste de Avaliação

Matemática A

12.º Ano de Escolaridade | maio de 2021

Turma: 12ºJ

1. $\lim\left(\frac{4n-2}{4n+3}\right)^n = \lim\left[\frac{4n\left(1-\frac{2}{4n}\right)}{4n\left(1+\frac{3}{4n}\right)}\right]^n = \frac{\lim\left(1+\frac{-\frac{2}{4}}{n}\right)^n}{\lim\left(1+\frac{\frac{3}{4}}{n}\right)^n} = \frac{e^{-\frac{2}{4}}}{e^{\frac{3}{4}}} = e^{-\frac{2}{4}-\frac{3}{4}} = e^{-\frac{5}{4}}$

   $\lim\left(\frac{4n-2}{4n+3}\right)^n = e^{3k+2} \Leftrightarrow e^{3k+2} = e^{-\frac{5}{4}} \Leftrightarrow 3k+2 = -\frac{5}{4} \Leftrightarrow 3k = -\frac{5}{4} - 2 \Leftrightarrow 3k = -\frac{13}{4} \Leftrightarrow k = -\frac{13}{12}$

   Resposta:

   Versão 1: (D)

   Versão 2: (C)

2. .

   2.1. $|w_1| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4}+\frac{2}{4}} = \sqrt{\frac{4}{4}} = 1$

   $45 = 11 \times 4 + 1$

   $i^{45} = i^{11\times4+1} = i^1 = i$

   $w_2 = 2e^{i\left(-\frac{\pi}{2}\right)}$

   $\overline{w_2} = 2e^{i\left(\frac{\pi}{2}\right)}$
   $\overline{w_2}^3 = 2^3 e^{i\left(\frac{3\pi}{2}\right)} = 8 \times \left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right] = 8 \times (0-i) = -8i$

   Assim,

   $\overline{-x-yi} \times i^{145} = |w_1| + \overline{w_2}^3 \Leftrightarrow (-x+yi)\times i = 1-8i \Leftrightarrow (-x+yi)\times i = 1-8i \Leftrightarrow$

   $\Leftrightarrow -xi + yi^2 = 1-8i \Leftrightarrow -xi - y = 1-8i \Leftrightarrow -y - xi = 1-8i \Leftrightarrow -y = 1 \land -x = -8 \Leftrightarrow$

   $\Leftrightarrow y = -1 \land x = 8$

   Resposta: $x = 8 \land y = -1$

   2.2. $z^4 - z^3 + 2z - 2 + 2i = w_3 \Leftrightarrow z^4 - z^3 + 2z - 2 + 2i = 2i \Leftrightarrow z^4 - z^3 + 2z - 2 = 0$

   Sabemos que $P(z) = z^4 - z^3 + 2z - 2$ é divisível por $z-1$

   Então, $P(z) = (z-1)\times Q(z)$

Pela regra de Ruffini, vem,

$$\begin{array}{c|cccc|c} & 1 & -1 & 0 & 2 & -2 \\ 1 & & 1 & 0 & 0 & 2 \\ \hline & 1 & 0 & 0 & 2 & 0 \end{array}$$

Logo, $Q(z) = z^3 + 2$

Então, $P(z) = (z-1) \times (z^3 + 2)$

Assim,

$z^4 - z^3 + 2z - 2 = 0 \Leftrightarrow (z-1) \times (z^3 + 2) = 0 \Leftrightarrow z - 1 = 0 \vee z^3 + 2 = 0 \Leftrightarrow$

$\Leftrightarrow z - 1 = 0 \vee z^3 = -2 \Leftrightarrow z = 1 \vee z = \sqrt[3]{-2} \Leftrightarrow z = 1 \vee z = \sqrt[3]{2e^{i\pi}} \Leftrightarrow$

$\Leftrightarrow z = 1 \vee z = \sqrt[3]{2}e^{i\left(\frac{\pi + k2\pi}{3}\right)}, k \in \{0;1;2\}$

Se $k = 0 \to w_0 = \sqrt[3]{2}e^{i\left(\frac{\pi}{3}\right)}$

Se $k = 1 \to w_1 = \sqrt[3]{2}e^{i\left(\frac{3\pi}{3}\right)} = \sqrt[3]{2}e^{i\pi}$

Se $k = 2 \to w_2 = \sqrt[3]{2}e^{i\left(\frac{5\pi}{3}\right)} = \sqrt[3]{2}e^{i\left(-\frac{\pi}{3}\right)}$

$C.S. = \left\{e^{i(0)}; \sqrt[3]{2}e^{i\left(\frac{\pi}{3}\right)}; \sqrt[3]{2}e^{i\pi}; \sqrt[3]{2}e^{i\left(-\frac{\pi}{3}\right)}\right\}$

3. $\ln^2(x) - 5\ln(x) + 4 \geq 0 \wedge x > 0$

Fazendo a mudança de variável $y = \ln(x)$, vem,

$y^2 - 5y + 4 \geq 0 \Leftrightarrow y \leq 1 \vee y \geq 4$

Ou seja,

$(\ln(x) \leq 1 \vee \ln(x) \geq 4) \wedge x > 0 \Leftrightarrow$

$\Leftrightarrow \left(x \leq e \vee x \geq e^4\right) \wedge x > 0$

$C.S. =]0; e] \cup \left[e^4; +\infty\right[$

Cálculos auxiliares

$y^2 - 5y + 4 = 0 \Leftrightarrow y = \dfrac{-(-5) \pm \sqrt{(-5^2) - 4 \times 1 \times 4}}{2 \times 1} \Leftrightarrow$

$\Leftrightarrow y = 4 \vee y = 1$

4. Sabe-se que $\log_b a = \dfrac{1}{4}$, com $b \in \mathbb{R}^+ \setminus \{1\}$ e $a > 0$

Assim,

$$\log_a\left(\sqrt[3]{a^2b^2}\right) = \frac{\log_b\left(\sqrt[3]{a^2b^2}\right)}{\log_b a} = \frac{\frac{1}{3}\log_b\left(a^2b^2\right)}{\log_b a} = \frac{\frac{1}{3} \times \left[\log_b\left(a^2\right) + \log_b\left(b^2\right)\right]}{\log_b a} = \frac{\frac{1}{3} \times \left[2\log_b(a) + 2\right]}{\log_b a} =$$

$$= \frac{\frac{1}{3} \times \left(\frac{2}{4} + 2\right)}{\frac{1}{4}} = \frac{\frac{1}{3} \times \left(\frac{2}{4} + \frac{8}{4}\right)}{\frac{1}{4}} = \frac{\frac{1}{3} \times \frac{10}{4}}{\frac{1}{4}} = \frac{10}{3}$$

5. $\sin(2x) = \sin(x + x) = \sin(x)\cos(x) + \cos(x)\sin(x) = 2\sin(x)\cos(x)$

6. $z_1 = \cos(x) + i\sin(x) = e^{ix}$
$z_2 = \cos(y) + i\sin(y) = e^{iy}$

Então,

$\overline{z_1} = e^{i(-x)}$
$\overline{z_2} = e^{i(-y)}$

Assim,
$\overline{z_1} \times \overline{z_2} = e^{i(-x)} \times e^{i(-y)} = e^{i(-x-y)} = \cos(-x-y) + i\sin(-x-y)$

Outro processo

$\overline{z_1} = \cos(x) - i\sin(x)$

$\overline{z_2} = \cos(y) - i\sin(y)$

$\overline{z_1} \times \overline{z_2} = [\cos(x) - i\sin(x)] \times [\cos(y) - i\sin(y)] =$

$= \cos(x)\cos(y) - i\cos(x)\sin(y) - i\sin(x)\cos(y) + i^2\sin(x)\sin(y) =$

$= \cos(x)\cos(y) - \sin(x)\sin(y) + i\,[-\cos(x)\sin(y) - \sin(x)\cos(y)] =$

$= \cos(-x)\cos(y) + \sin(-x)\sin(y) + i\,[\sin(-x)\cos(-y) + \cos(-x)\sin(-y)] =$

$= \cos(-x-y) + i\sin(-x-y)$

Resposta:

Versão 1: (B)

Versão 2: (A)

7. Determinar as coordenadas do ponto B

$B(0; g(0))$

$g(0) = 5^0 = 1$

Logo, $B(0;1)$

Determinar as coordenadas do ponto A

$A(x;1)$

$f(x) = 1 \Leftrightarrow \log_5(x+3) = 1 \wedge x+3 > 0 \Leftrightarrow x+3 = 5 \wedge x > -3 \Leftrightarrow x = 2 \wedge x > -3 \Leftrightarrow x = 2$

Logo, $A(2;1)$

Assim, $\overline{OB} = 1$

A medida de comprimento da altura do triângulo $[ABO]$ é 2

Portanto, $A_{[ABO]} = \dfrac{\overline{OB} \times 2}{2} = \dfrac{1 \times 2}{2} = 1$

Resposta:
Versão 1: (A)
Versão 2: (D)

8. .

8.1. $f(x) = 2x\ln(4x)$

Determinar a primeira derivada de f

$f'(x) = [2x\ln(4x)]' = (2x)'\times\ln(4x) + 2x\times[\ln(4x)]' = 2\times\ln(4x) + 2x\times\frac{(4x)'}{4x} = 2\ln(4x) + 2x\times\frac{4}{4x} =$

$= 2\ln(4x) + 2x\times\frac{1}{x} = 2\ln(4x) + 2$

O declive da reta tangente é:

$m = f'\left(\frac{e}{4}\right) = 2\ln\left(4\times\frac{e}{4}\right) + 2 = 2\ln(e) + 2 = 2 + 2 = 4$

Resposta:

Versão 1: (B)

Versão 2: (C)

8.2. $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}[2x\ln(4x)] =^{(0\times\infty)}= \lim\limits_{y\to+\infty}\left[2\times\frac{1}{y}\ln\left(\frac{4}{y}\right)\right] = 2\times\lim\limits_{y\to+\infty}\frac{\ln\left(\frac{4}{y}\right)}{y} =$

$= 2\times\lim\limits_{y\to+\infty}\frac{\ln(4)-\ln(y)}{y} = 2\times\left[\lim\limits_{y\to+\infty}\frac{\ln(4)}{y} - \lim\limits_{y\to+\infty}\frac{\ln(y)}{y}\right] = 2\times(0-0) = 0$

Fez-se a mudança de variável

$y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}$

Se $x\to 0^+$ então $y\to+\infty$

Aplicou-se o limite notável $\lim\limits_{x\to+\infty}\frac{\ln(x)}{x} = 0$

9. .

9.1. Ora,

$A(\cos(x);\sin(x))$, com $\cos(x) < 0$ e $\sin(x) < 0$

Seja C, a projeção ortogonal do ponto A sobre o eixo Oy

Assim,

$\overline{AB} = 2\times|\cos(x)| = -2\cos(x)$

$\overline{OC} = |\sin(x)| = -\sin(x)$

Portanto,

$A(x) = \frac{\overline{AB}\times\overline{OC}}{2} = \frac{-2\cos(x)\times(-\sin(x))}{2} = \frac{2\sin(x)\cos(x)}{2} = \frac{1}{2}\sin(2x)$, com $x\in\left]\pi;\frac{3\pi}{2}\right[$

9.2. Calculemos a função primeira derivada de $A(x)$

$A'(x) = \left[\frac{1}{2}\sin(2x)\right]' = \frac{1}{2}\times(2x)'\times\cos(2x) = \frac{1}{2}\times 2\times\cos(2x) = \cos(2x)$

Calculemos os zeros de $A'(x)$, no intervalo $\left]\pi;\frac{3\pi}{2}\right[$

$A'(x) = 0 \Leftrightarrow \cos(2x) = 0 \Leftrightarrow \cos(2x) = \cos\left(\frac{\pi}{2}\right) \Leftrightarrow 2x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{\pi}{4} + k\frac{\pi}{2}, k \in \mathbb{Z}$

Atribuindo valores a k, tem-se,

Se $k = 0 \mapsto x = \frac{\pi}{4} \notin \left]\pi; \frac{3\pi}{2}\right[$

Se $k = 1 \mapsto x = \frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4} + \frac{2\pi}{4} = \frac{3\pi}{4} \notin \left]\pi; \frac{3\pi}{2}\right[$

Se $k = 2 \mapsto x = \frac{\pi}{4} + \frac{2\pi}{2} = \frac{\pi}{4} + \frac{4\pi}{4} = \frac{5\pi}{4} \in \left]\pi; \frac{3\pi}{2}\right[$

Se $k = -1 \mapsto x = \frac{\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4} - \frac{2\pi}{4} = -\frac{\pi}{4} \notin \left]\pi; \frac{3\pi}{2}\right[$

Portanto, $x = \frac{5\pi}{4}$

Elaborando um quadro de sinal de $A'(x)$

x	π		$\frac{5\pi}{4}$		$\frac{3\pi}{2}$
$A'(x)$	$n.d$	$+$	0	$-$	$n.d$
$A(x)$	$n.d$	$\nearrow$	$\frac{1}{2}$	$\searrow$	$n.d$

$A\left(\frac{5\pi}{4}\right) = \frac{1}{2} \times \sin\left(2 \times \frac{5\pi}{4}\right) = \frac{1}{2} \times \sin\left(\frac{5\pi}{2}\right) = \frac{1}{2} \times \sin\left(\frac{\pi}{2}\right) = \frac{1}{2} \times 1 = \frac{1}{2}$

O valor exato de x, para o qual o triângulo $[ABO]$ tem área máxima $\left(igual \quad a \quad \frac{1}{2}\right)$, é $\frac{5\pi}{4}$ rad

10. $0 \in D_f$

A função f é contínua em $x = 0$, se existir $\lim\limits_{x \to 0} f(x)$, ou seja,

se $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0)$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \frac{x^2 + 2x}{e^{x+2} - e^2} =^{\left(\frac{0}{0}\right)} \lim\limits_{x \to 0^+} \frac{x(x+2)}{e^2(e^x - 1)} = \frac{1}{e^2} \times \lim\limits_{x \to 0^+} \frac{x}{e^x - 1} \times \lim\limits_{x \to 0^+} (x + 2)$

$= \frac{1}{e^2} \times \frac{1}{\lim\limits_{x \to 0^+} \frac{e^x - 1}{x}} \times 2 = \frac{2}{e^2} \times \frac{1}{1} = 2e^{-2}$

Aplicou-se o limite notável: $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \frac{k\sin(2x)}{x^2 - 2x} =^{\left(\frac{0}{0}\right)} k \times \lim\limits_{x \to 0^-} \frac{\sin(2x)}{x(x-2)} = k \times \lim\limits_{x \to 0^-} \frac{\sin(2x)}{x} \times \lim\limits_{x \to 0^-} \frac{1}{x-2} =$

$= k \times \lim\limits_{2x \to 0^-} \frac{\sin(2x)}{2x} \times 2 \times \frac{1}{-2} = k \times 1 \times 2 \times \left(-\frac{1}{2}\right) = -k$

Aplicou-se o limite notável: $\lim\limits_{x \to 0} \frac{\sin(x)}{x} = 1$

$f(0) = 2e^{-2}$

Ora, a função f é contínua em $x = 0$, se, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0)$

Então, deverá ter-se,

$2e^{-2} = -k \Leftrightarrow k = -2e^{-2}$

Portanto, a função f é contínua em $x = 0$, se $k = -2e^{-2}$