One Variable Calculus with SageMath*

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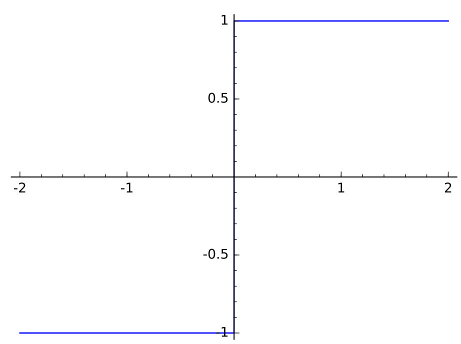
1 Limit

Example 1. Find the $\lim_{x\to 0} \frac{x}{|x|}$.

```
sage: x=var('x')
sage: f(x)=x/abs(x)
```

```
sage: p = plot(f,-2,2,figsize=5)
```




```
sage: f.limit(x=0,dir='+')
x |--> 1
sage: f.limit(x=0,dir='-')
x |--> -1
sage: f.limit(x=0)
x |--> und
```

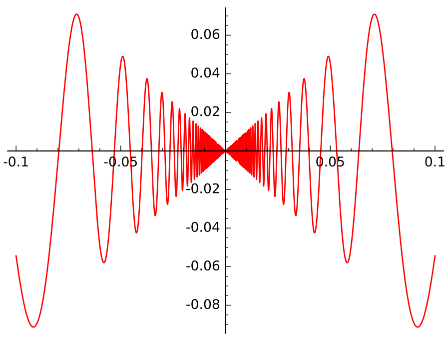
Example 2. Let $a \in \mathbb{R}$. Find $\lim_{x\to\infty} (1 + a/x)^x$.

```
sage: a=var('a')
sage: f = (1+a/x)^x
sage: f.limit(x=infinity)
e^a
```

Example 3. Explore the limit of $g(x) = x\sin(1/x)$ at $x = 0$.

```
sage: g(x)=x*sin(1/x)
sage: p = plot(g,-0.1,0.1,color='red',figsize=5)
```

*Distributed during "Teachers Enrichment Course in Undergraduate Mathematics Curriculum" at IIT Guwahati during July 1-13, 2019



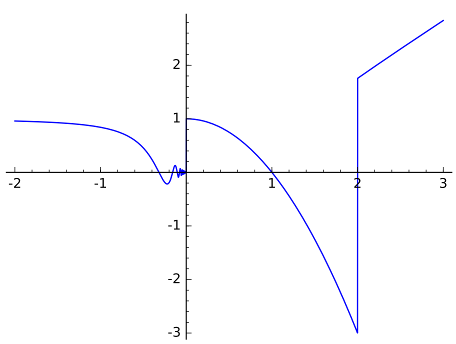

```
sage: g.limit(x=0,dir='-')
x |--> 0
sage: g.limit(x=0,dir='+')
x |--> 0
sage: g.limit(x=0)
x |--> 0
```

Exercise 1. Compute the following limits:

1. $\lim_{x\to 0} \frac{\tan x - x}{x^3}$.
2. $\lim_{x\to 0}(1+\sin x)^{\cot 2x}$.
3. $\lim_{x\to\infty}(1+a/x)^x$.
4. If p dollars is compounded n times per year at an annual interest rate of r, the money will be worth $p\,(1+r/n)^{nt}$ dollars after t years. How much will the money be worth after t years if it is compounded continuously? $(n\to\infty)$

1.1 Piecewise Defined Functions

```
sage: f1(x) = x*sin(1/x)
sage: f2(x) = 1-x^2
sage: f3(x)=x*cos(1/x)
sage: f = piecewise([[(-2,0),f1],[(0,2),f2],[(2,3),f3]])
sage: p=plot(f,(x,-2,3),figsize=5)
```



1.2 Limit of Sequences

Example 4. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

```
sage: n=var('n')
```

```
sage: f(n)=(1+1/n)^n
sage: limit(f(n),n=oo)
e
```

Example 5. Assume $a > 0$. Find $\lim_{n\to\infty} a^{1/n}$.

```
sage: var('a')
a
sage: assume(a>0)
None
sage: limit(a^(1/n),n=oo)
1
```

2 Derivatives

2.1 Derivative Using the First Principle

Example 6. Find the derivative of $h(x) = \log(x) + x^5 + \sin(x)$ using the first principle.

```
sage: a,x=var('a,x')
sage: h(x)=log(x)+x^5+sin(x)
sage: limit((h(a+x)-h(a))/x,x=0)
(5*a^5 + a*cos(a) + 1)/a
```

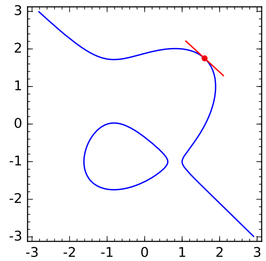
We can verify the this using the SageMath inbuilt function `h.diff()`.

```
sage: h.diff()
x |--> 5*x^4 + 1/x + cos(x)
sage: h.diff().subs(x=a)
x |--> 5*a^4 + 1/a + cos(a)
```

2.2 Implicit Derivative

Example 7. Find the derivative of any function defined implicitly by $yx^2 + e^y = x$.

```
sage: var('x,y')
(x, y)
sage: f(x,y)=x^3+y^3-2*x-3*y-1
sage: c = implicit_plot(f(x,y),(x,-3,3),(y,-3,3),figsize=4)
sage: b=1.75
sage: h(x)=f.subs(y=b)
sage: a = h.find_root(1,2)
sage: pt = point((a,b),color='red',size=20)
sage: c+pt
Graphics object consisting of 2 graphics primitives
sage: dyx=f.implicit_derivative(y,x)
sage: dyx
-1/3*(3*x^2 - 2)/(y^2 - 1)
sage: m = dyx.subs(x=a,y=b)
sage: print(m)
sage: T(x)=b+m*(x-a)
sage: tg=plot(T(x),(x,a-0.5,a+0.5),color='red')
sage: c+pt+tg
Graphics object consisting of 3 graphics primitives
```



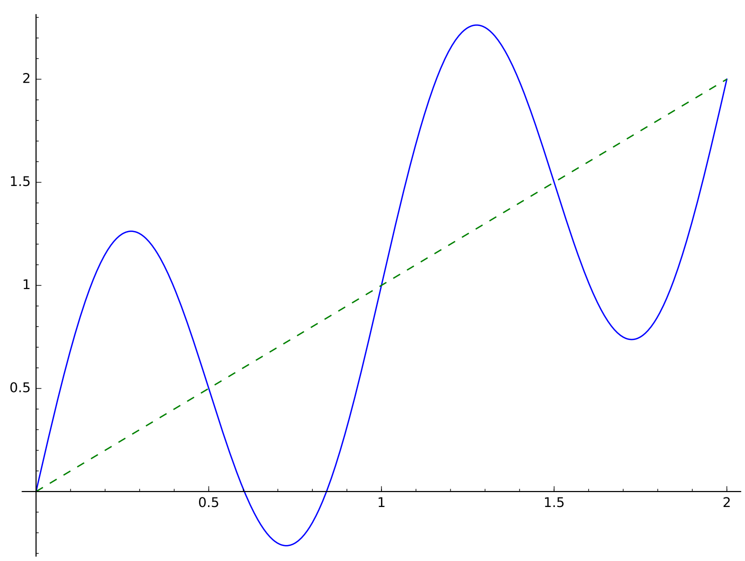
Exercise 2. Consider the implicit function $f(x, y) = x^3 + xy - y^3 + x + y$. Find $\dfrac{dy}{dx}$ at $(0, 1)$ and explain its geometric meaning.

2.3 Mean Value Theorem

Example 8. Find the value(s), c, guaranteed by the Mean Value Theorem for the function $f(x) = x + \sin 2\pi x$ on the interval $[0, 2]$. Also plot the graph that explain the geometric meaning of this result.

First of all let us plot the curve along with the chord oining points $(a, f(a))$ and $(b, f(b))$.

```
sage: f(x)=x+sin(2*pi*x)
sage: a=0
sage: b=2
sage: pf=plot(f,0,2)
sage: cord=line([(0,f(0)),(2,f(2))],color='green',linestyle='--')
sage: pf+cord
Graphics object consisting of 2 graphics primitives
```



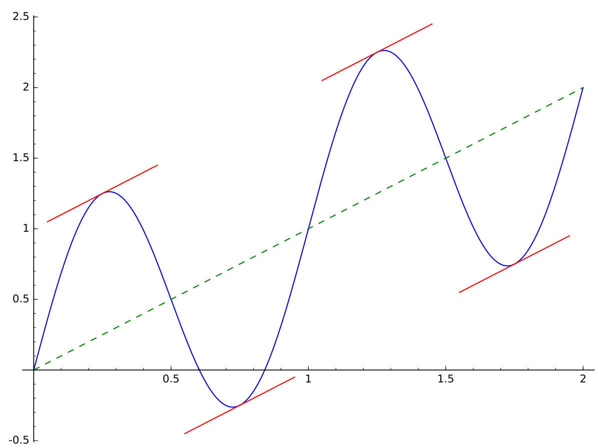
From the above plot, it is clear that there are four points at which the tangent is parallel to the chord joining points $(a, f(a))$ and $(b, f(b))$.

```
sage: m=(f(b)-f(a))/(b-a) # slope of the chord
sage: c1=find_root(f.diff(x)==m,0,0.5)
sage: c2=find_root(f.diff(x)==m,0.5,1)
sage: c3=find_root(f.diff(x)==m,1,1.5)
sage: c4=find_root(f.diff(x)==m,1.5,2)
```

```
sage: c1,c2,c3,c4
(0.25, 0.75, 1.25, 1.75)
```

```
sage: f1=f.diff()
sage: l1(x)=f(c1)+f1(c1)*(x-c1)
```

```
sage: pf=plot(f,0,2)
sage: cord=line([(0,f(0)),(2,f(2))],color='green',linestyle='--')
sage: t1=plot(f(c1)+f1(c1)*(x-c1),c1-0.2,c1+0.2,color='red')
sage: t2=plot(f(c2)+f1(c2)*(x-c2),c2-0.2,c2+0.2,color='red')
sage: t3=plot(f(c3)+f1(c3)*(x-c3),c3-0.2,c3+0.2,color='red')
sage: t4=plot(f(c4)+f1(c4)*(x-c4),c4-0.2,c4+0.2,color='red')
sage: p=pf+cord+t1+t2+t3+t4
```



3 Local Maximum-Minimum

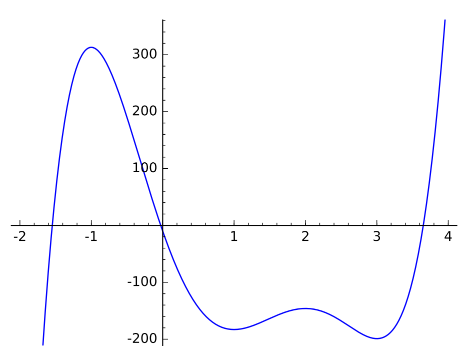
SageMath has inbuilt methods to find local maximum-minimum of a function f in an interval $[a, b]$, using the commands `f.find_local_maximum(a,b)`, `f.find_local_minimum(a,b)`

This computed numerically and it reports both the minimum (maximum) value and the point at which minimum (maximum) occurs in $[a, b]$.

Example 9. Let $f(x) = 12x^5 - 75x^4 + 100x^3 + 150x^2 - 360x - 10$. Find and classify critical point of f. Also find the local maximum and local minimum of f using the inbuilt SageMath methods.

Let us define the function f and plot its graphs in an interval $[-2, 4]$.

```
sage: f(x)=12*x^5 - 75*x^4 + 100*x^3 + 150*x^2 - 360*x-10
sage: pf=plot(f(x),-2,4,ymin=-200,ymax=350,figsize=5,color='blue')
```

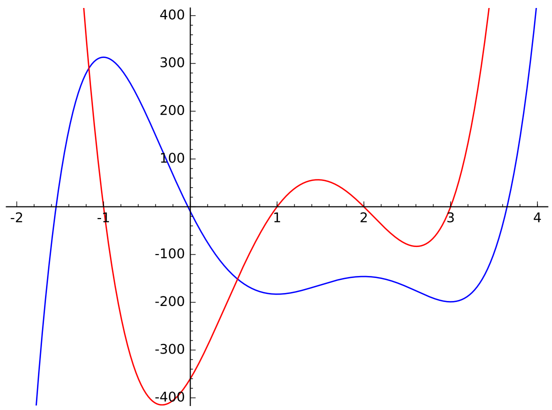


From the graph it looks like that f has four critical points. Let us find them.

```
sage: f1(x)=f.diff()(x)
sage: show(f1(x))
None
```

Let us plot the graph of f and f' together.

```
sage: pf1 = plot(f.diff(),-2,4,ymin=-400,ymax=400,figsize=6,color='
    red')
sage: pf+pf1
Graphics object consisting of 2 graphics primitives
```



We can find all roots of $f'(x) = 0$ using

```
sage: f1.roots()
[(3, 1), (-1, 1), (1, 1), (2, 1)]
```

```
sage: roots =f1.roots()

sage: cpts = []

sage: for i in range(len(roots)):
....:    cpts.append(roots[i][0])

sage: cpts
```

$$[3, -1, 1, 2]$$

```
sage: f2 = f.diff(2)

sage: for a in cpts:
....:    if f2(a)>0:
....:        print("{}  is a point of local minimum".format(a))
....:    if f2(a)<0:
....:         print("{}  is a point of local maximum".format(a))
```

We can find the local maximum and local minimum using the SageMath inbuilt methods.

```
sage: f1,x1=f.find_local_maximum(-2,0)
sage: f1,x1
(313.0, -0.99999999821080732)
```

```
sage: f2,x2=f.find_local_minimum(0,3)
```

```
sage: f2,x2
(-183.00000000000003, 1.0000000068636787)
```

```
sage: f3,x3=f.find_local_maximum(1,3)
sage: f3,x3
(-146.0, 2.0000000347394256)
```

```
sage: f4,x4=f.find_local_minimum(2,4)
sage: f4,x4
(-199.00000000000068, 3.0000000213603717)
```

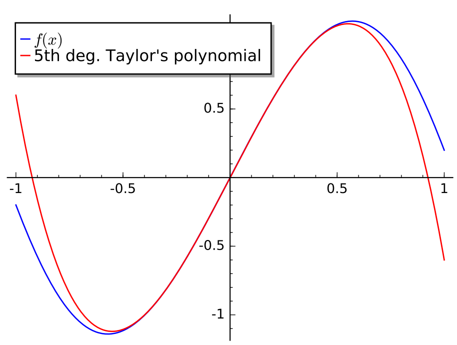
Exercise 3. Find the local maximum and local minimum of $x\sin(x^2) + e^{-x^2}\cos(x)$ in the interval $[1,4]$.

4 Taylor's Approximation

Example 10. Consider the function $f(x) = \sqrt{1+x^2}\sin(3x)$. Find the 5th degree polynomial of f about $x = 0$.

```
sage: f(x) = sqrt(1+x^2)*sin(3*x)
sage: a,n = 0,5
sage: tn(x) = f.taylor(x,a,n)
sage: tn(x)
-3/5*x^5 - 3*x^3 + 3*x
```

```
sage: pf=f.plot(-1,1,legend_label="$f(x)$",figsize=5)
sage: ptn=tn.plot(-1,1,color='red',legend_label="5th␣deg.␣Taylor's␣
   polynomial")
sage: pf+ptn
Graphics object consisting of 2 graphics primitives
```



Exercise 4. Plot the graph of the function $f(x) = \sqrt{1+x^2}\sin(3x)$ and Taylor's polynomial of degrees 1, 2, 3, 4, 5, 6 together. This demonstrate how Taylors polynomial approximate the function.