9. *(1 valor)* Considere, no espaço euclidiano complexo $\mathbb{C}^3$ munido do produto escalar usual, o operador $S(x,y,z) = (x+iy-z, 2iy+3z, iz)$. Determine o operador S^* e a composição S^*S.

$$S = \begin{pmatrix} 1 & i & -1 \\ 0 & 2i & 3 \\ 0 & 0 & i \end{pmatrix}$$

$$\bar{S} = \begin{pmatrix} 1 & -i & -1 \\ 0 & -2i & 3 \\ 0 & 0 & -i \end{pmatrix}$$

$$S^* = \begin{pmatrix} 1 & 0 & 0 \\ -i & -2i & 0 \\ -1 & 3 & -i \end{pmatrix}$$

$$S^*S = \begin{pmatrix} 1 & i & -1 \\ -i & 5 & -5i \\ -1 & 5i & 11 \end{pmatrix}$$

10. *(1 valor)* Considere, no espaço euclidiano complexo $\mathbb{C}^2$ munido do produto escalar usual, o operador $T : \mathbb{C}^2 \to \mathbb{C}^2$ definido por $T(x,y) = (2x - iy, ix + y)$. Determine uns operadores auto-adjuntos X e Y tais que $T = X + iY$.

$$T = \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix}$$

$$\bar{T} = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix}$$

$$T^* = \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix}$$

$$X = \frac{T + T^*}{2}$$

$$Y = \frac{T - T^*}{2i}$$

$$X = \frac{1}{2}(T+T^*) = \frac{1}{2}(\begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix} + \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix}) = \frac{1}{2}(\begin{pmatrix} 4 & -2i \\ 2i & 2 \end{pmatrix} = \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix}$$

$$Y = \frac{1}{2i}(T-T^*) = \frac{1}{2i}(\begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix} - \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix}) = \frac{1}{2i}(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

12. *(1 valor)* Calcule valores e vetores próprios da matriz hermítica

$$C = \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix}$$

Valores Próprios:

$$\lambda = 2 \qquad \lambda = -2$$

Vetores Próprios:

$$\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

13. *(1 valor)* Considere a matriz C definida no exercício 12. Determine uma matriz unitária U e uma matriz diagonal Λ tais que $C = U\Lambda U^{-1}$.

Valores Próprios:

$$\lambda = 2 \qquad \lambda = -2$$

$$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

Vetores Próprios:

$$\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$U^{-1} = \begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

10. *(1 valor)* Determine a matriz que define, relativamente à base canónica, um operador ortogonal $R : \mathbb{R}^2 \to \mathbb{R}^2$ tal que $R(1, 0) = (0, -1)$.

Base canónica $\mathbb{R}^2$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & b \\ -1 & d \end{pmatrix}$$

As colunas devem formar uma base ortogonal:

$$R(1,0) = (0,-1) \implies \sqrt{0^2 + (-1)^2} = 1$$

Precisamos de encontrar $R(0,1)$:

$$(0,-1) \cdot (b,d) = 0 \Leftrightarrow d = 0$$

(b,d) tem de ter norma 1:

$$\sqrt{b^2 + 0} = 1 \Leftrightarrow b = \pm 1$$

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Diagonalização de matriz
$A = PDP^{-1}$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \cdot (\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix})^{-1}$$

5. *(2 valores)* Seja $P : \mathbb{R}^2 \to \mathbb{R}^2$ a projeção ortogonal sobre a reta $y = -2x$ do plano euclidiano. Determine a matriz que representa P na base canónica.

$$u = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$proj_u v = \frac{v \cdot u}{u \cdot u} u$$

$$\frac{u}{||u||} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$v \cdot u = x \cdot \frac{1}{\sqrt{5}} + y \cdot -\frac{2}{\sqrt{5}} = \frac{x - 2y}{\sqrt{5}}$$

$$proj_u v = \frac{x-2y}{\sqrt{5}} \cdot (\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}) = (\frac{x-2y}{5}, \frac{-2x+4y}{5})$$

$$P = \begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

6. *(2 valores)* Seja $R : \mathbb{R}^2 \to \mathbb{R}^2$ a reflexão na reta $y = \sqrt{3}\,x$. Determine a matriz que representa R na base canónica.

$$u = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$

$$proj_u v = \frac{v \cdot u}{u \cdot u} u$$

$$R(v) = 2 \cdot proj_u v - v$$

$$\frac{u}{||u||} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$v \cdot u = x \cdot \frac{1}{2} + y \cdot \frac{\sqrt{3}}{2} = \frac{x + \sqrt{3}y}{2}$$

$$proj_u v = \frac{x + \sqrt{3}y}{2} \cdot (\frac{1}{2}, \frac{\sqrt{3}}{2}) = (\frac{x + \sqrt{3}y}{4}, \frac{\sqrt{3}x + 3y}{4})$$

$$R(v) = 2 \cdot (\frac{x + \sqrt{3}y}{4}, \frac{\sqrt{3}x + 3y}{4}) - (x, y) \Leftrightarrow (\frac{x + \sqrt{3}y}{2}, \frac{\sqrt{3}x + 3y}{2}) - (x, y)$$

$$R(v) = \begin{pmatrix} \frac{-x + \sqrt{3}y}{2} \\ \frac{\sqrt{3}x + y}{2} \end{pmatrix}$$

$$R(v) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

5. *(2 valores)* Seja $R : \mathbb{R}^2 \to \mathbb{R}^2$ a reflexão na reta $y = \sqrt{3}\,x$. Determine valores e vetores próprios de R.

$$R = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$det(R - \lambda I)$

$$\lambda = -1 \qquad \lambda = 1$$

$$\begin{vmatrix} -\frac{1}{2}-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}-\lambda \end{vmatrix}$$

$(R - \lambda I) = 0$
Se $\lambda = -1$:

$$\begin{pmatrix} -\frac{1}{2}+1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}+1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{pmatrix}$$

Se $\lambda = 1$:

$$\begin{pmatrix} -\frac{1}{2}-1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}-1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{\sqrt{3}}{3} \\ 0 & 0 \end{pmatrix}$$

Vetores próprios:

$$span = \{\left(-\frac{\sqrt{3}}{3}, -1\right), \left(\sqrt{3}, -1\right)\}$$

7. *(1 valor)* Determine o ponto do plano $P = \{x - 2y + z = 0\} \subset \mathbb{R}^3$ mais próximo do ponto $\mathbf{v} = (1, 0, 2)$.

$$\vec{n} = (1, -2, 1)$$

$$r(t) = (1, 0, 2) + t(1, -2, 1)$$

$$x = 1 + t \qquad y = -2t \qquad z = 2 + t$$

Substituir na equação:

$$1 + t + 4t + 2 + t = 0 \Leftrightarrow t = -\frac{1}{2}$$

$$x = \frac{1}{2} \qquad y = 1 \qquad z = \frac{3}{2}$$

6. *(1 valor)* Determine uma base ortonormada do plano $P = \{2x - y - z = 0\} \subset \mathbb{R}^3$.

Encontrar dois vetores perpendiculares:
Se $x = 1$ e $y = 2$

$$2 - 2 - z = 0 \Leftrightarrow z = 0$$

$$u_1 = v_1 = (1, 2, 0)$$

Se $x = 0$ e $y = 1$

$$0 - 1 - z = 0 \Leftrightarrow z = -1$$

$$v_2 = (0, 1, -1)$$

$$u_2 = v_2 - proj_{u_1} u_2 = \frac{u_1 \cdot v_2}{||u_1||^2} u_1 = (0, 1, -1) - (\frac{2}{5}, \frac{4}{5}, 0) = (-\frac{2}{5}, \frac{1}{5}, -1)$$

$$||u_2|| = \sqrt{\frac{4}{25}, \frac{1}{25}, \frac{25}{25}} = \frac{\sqrt{30}}{5}$$

$$||u_1|| = \sqrt{5}$$

$$\frac{u_1}{||u_1||} = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0)$$

$$\frac{u_2}{||u_2||} = (-\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{5}{\sqrt{30}})$$

$$\{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right), \left(-\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{5}{\sqrt{30}}\right)\}$$

7. *(1 valor)* Calcule a projeção ortogonal do vetor $\mathbf{v} = (3, 2, 1)$ sobre o plano P definido no exercício 6.

Encontrar dois vetores perpendiculares:

$$\vec{n} = (2, -1, -1)$$
$$\vec{v} = (3, 2, 1)$$

$$v_p = v - proj_n v = \frac{n \cdot v}{||n||^2} n = (3, 2, 1) - \frac{1}{2}(2, -1, -1) = (2, \frac{3}{2}, \frac{1}{2})$$

Determine a forma quadrática $13x^2 + 16xy + 5y^2$ nas coordenadas $x' = 2x + y$ e $y' = 3x + 2y$.

$$Q(x, y) = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$$

Método 1:

$$z = 2x + y \qquad w = 3x + 2y$$

$$2w - 3z = (6x + 4y) - (6x + 3y) = y \Leftrightarrow y = 2w - 3z$$
$$2z - w = (4x + 2y) - (3x + 2y) = x \Leftrightarrow x = 2z - w$$

$$Q(2z - w, 2w - 3z)$$
$$P(z, w) = 13(2z - w)^2 + 16(2z - w)(2w - 3z) + 5(2w - 3z)^2$$
$$P(z, w) = z^2 + w^2$$
$$P(x', y') = (x')^2 + (y')^2$$

Método 2:

$$T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$$

$$U = T^{-1} = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow \frac{1}{2} R_1} \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow -3R_1} \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow \frac{1}{2} R_2} \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -3 & 2 \end{array}\right)$$
$$\xrightarrow{R_1 \leftrightarrow -\frac{1}{2} R_1} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right)$$
$$U^* = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$
$$U^* \cdot A \cdot U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$