L = $\int_R e^{-\frac{x^2}{2}}\ \mathrm{d}x$
Coordenadas Polares

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

$$L^2 = \int_R e^{-\frac{x^2}{2}}\ \mathrm{d}x \cdot \int_R e^{-\frac{y^2}{2}}\ \mathrm{d}y = \iint_{R^2} e^{-\frac{1}{2}(x^2+y^2)}\ \mathrm{d}x$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$L^2 = \iint_{R^2} e^{-\frac{r^2}{2}} r\ \mathrm{d}r\,\mathrm{d}\theta = \int_0^\infty e^{-\frac{r^2}{2}} r\ \mathrm{d}r \cdot \int_0^{2\pi}\ \mathrm{d}\theta$$

$$u = -\frac{r^2}{2} \qquad -du = r\ \mathrm{d}r$$

$$u(\infty) = -\infty \qquad u(0) = 0$$

$$-\int_0^{-t} e^u\ \mathrm{d}u = \lim_{t\to\infty} [e^u]_0^{-t} = -(0-1) = 1$$

$$-\int_0^\infty e^u\ \mathrm{d}u \cdot 2\pi = 1 \cdot 2\pi$$

Então a integral é $\sqrt{2\pi}$
L = $\int_R e^{-x^2}\ \mathrm{d}x$

$$L^2 = \int_R e^{-x^2}\ \mathrm{d}x \cdot \int_R e^{-y^2}\ \mathrm{d}y = \iint_{R^2} e^{-(x^2+y^2)}\ \mathrm{d}x$$

$$\int_0^\infty e^{-r^2} r\ \mathrm{d}r = -\frac{1}{2}\lim_{t\to\infty}\left[e^{-r^2}\right]_0^t = -\frac{1}{2}(0-1) = \frac{1}{2}$$

$$L^2 = \iint_{R^2} e^{-r^2} r\ \mathrm{d}r\,\mathrm{d}\theta = \int_0^\infty e^{-r^2} r\ \mathrm{d}r \cdot \int_0^{2\pi}\ \mathrm{d}\theta = \frac{1}{2}\cdot 2\pi = \pi$$

Então a integral é $\sqrt{\pi}$
Comprimento de uma arco:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ \mathrm{d}x$$

$$L = \int\ \mathrm{d}s$$

$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x$ se $y = f(x)$, $a \le x \le b$
$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y$ se $x = h(y)$, $c \le x \le d$

$$L = \int\ \mathrm{d}s$$

1.

Determine the length of $y = \ln(\sec x)$ between $0 \leq x \leq \frac{\pi}{4}$.

$$f'(\ln(\sec x)) = \tan(x)$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)} = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2(x)}\ \mathrm{d}x = \int_0^{\frac{\pi}{4}} \sec(x)\ \mathrm{d}x$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec x + \tan x}{\sec x + \tan x} \sec(x)\ \mathrm{d}x$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\ \mathrm{d}x$$

$$u = \sec x + \tan x \qquad du = \sec^2 x + \sec x \tan x dx$$

$$u(0) = 1 \qquad u(\frac{\pi}{4}) = \sqrt{2} + 1$$

$$= \int_1^{\sqrt{2}+1} \frac{1}{u}\ \mathrm{d}u$$

$$= [\ln|u|]_1^{\sqrt{2}+1} = \ln(\sqrt{2} + 1)$$

2.

Determine the length of x=$\frac{2}{3}(y-1)^{\frac{3}{2}}$ between $1 \leq y \leq 4$.

$$f'(\frac{2}{3}(y-1)^{\frac{3}{2}}) = \sqrt{(y-1)}$$

$$L = \int_1^4 \sqrt{1 + (\sqrt{(y-1)})^2}\ \mathrm{d}y = \int_1^4 \sqrt{y}\ \mathrm{d}y$$

$$= \left[\frac{2}{3}y^{\frac{3}{2}}\right]_1^4 = \frac{2}{3}4^{\frac{3}{2}} - \frac{2}{3} = \frac{14}{3}$$

3.

Redo the previous example using the function in the form $y = f(x)$ instead.

4.

Determine the length of $x = \frac{1}{2}y^2$ for $0 \leq x \leq \frac{1}{2}$ .Assume that y is positive.

Comprimento de uma curva:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ \mathrm{d}t = \int_a^b ||\vec{r}'(t)||\ \mathrm{d}t$$

1.

Determine the length of the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t)\rangle$ on the interval $0 \leq t \leq 2\pi$.

$$\vec{r}'(t) = \langle 2, 6\cos(2t), -6\sin(2t)\rangle$$

$$||\vec{r}'(t)|| = \sqrt{2^2 + (6\cos(2t))^2 + (-6\sin(2t))^2}$$

$$= 2\sqrt{10}$$

$$\int_0^{2\pi} ||\vec{r}'(t)||\ \mathrm{d}t = \int_0^{2\pi} 2\sqrt{10}\ \mathrm{d}t = 4\pi\sqrt{10}$$

2.

Determine the arc length function for $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t)\rangle$.

$$||\vec{r}'(t)|| = 2\sqrt{10}$$

$$s(t) = \int_0^t ||\vec{r}'(u)||\ \mathrm{d}u$$

$$s(t) = \int_0^t 2\sqrt{10}\ \mathrm{d}u = \left[2\sqrt{10}u\right]_0^t 2\sqrt{10}t$$

$$t = \frac{s}{2\sqrt{10}}$$

Parametrizar:

$$\vec{r}(t(s)) = \langle \frac{s}{\sqrt{10}}, 3\sin(\frac{s}{\sqrt{10}}), 3\cos(\frac{s}{\sqrt{10}})\rangle$$

3.

Where on the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t)\rangle$ are we after traveling for a distance of $\frac{\pi\sqrt{10}}{3}$?

$$\vec{r}(t(\frac{\pi\sqrt{10}}{3})) = \langle \frac{\pi}{3}, 3\sin(\frac{\pi}{3}), 3\cos(\frac{\pi}{3})\rangle$$
$$= \langle \frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2}\rangle$$

Practice Problems

For problems 1&2 determine the length of the vector function on the given interval.

1.

$\vec{r}(t) = (3-4t)\hat{\mathbf{i}} + 6t\hat{\mathbf{j}} - (9+2t)\hat{\mathbf{k}}$ from $-6 \le t \le 8$.

$$||\vec{r}'(t)|| = \sqrt{16+36+4} = 2\sqrt{14}$$

$$= \int_{-6}^{8} ||\vec{r}'(t)||\ \mathrm{d}t = \int_{-6}^{8} 2\sqrt{14}\ \mathrm{d}t$$

$$2\sqrt{14}\,[t]_{-6}^{8} = 28\sqrt{14}$$

2.

$\vec{r}(t) = \langle \frac{1}{3}t^3, 4t, \sqrt{2}t^2\rangle$ from $0 \le t \le 2$.

$$||\vec{r}'(t)|| = \sqrt{t^4+16+8t^2} = \sqrt{(t^2+4)^2} = t^2+4$$

$$= \int_{0}^{2} ||\vec{r}'(t)||\ \mathrm{d}t = \int_{0}^{2} t^2+4\ \mathrm{d}t$$

$$= \left[\frac{1}{3}t^3 + 4t\right]_0^2 = \frac{32}{3}$$

or problems 3&4 find the arc length function for the given vector function.

3.

$\vec{r}(t) = \langle t^2, 2t^3, 1-t^3\rangle$

$$||\vec{r}'(t)|| = \sqrt{4t^2+45t^4} = t\sqrt{4+45t^2}$$

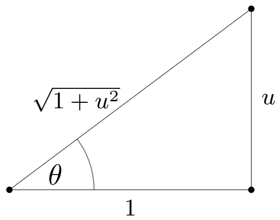
$$s(t) = \int_0^t u\sqrt{4+45u^2}\ \mathrm{d}u = \frac{1}{135}\left[\left(4+45u^2\right)^{\frac{3}{2}}\right]_0^t$$
$$= \frac{1}{135}\left[\left(4+45t^2\right)^{\frac{3}{2}} - 8\right]$$

4.

$\vec{r}(t) = \langle 4t, -2t, \sqrt{5}t^2 \rangle$

$$||\vec{r}'(t)|| = \sqrt{20+20t^2} = 2\sqrt{5}\sqrt{1+t^2}$$

$$s(t) = 2\sqrt{5}\int_0^t \sqrt{1+u^2}\ \mathrm{d}u$$



$$u = \tan(\theta) \qquad \mathrm{d}u = \sec^2(\theta)\ \mathrm{d}\theta$$

$$\sqrt{1+(\tan\theta)^2} = |\sec^2\theta| = \sec\theta$$

$$0 = \tan\theta \qquad \tan^{-1}(t) = \theta$$

$$s(t) = 2\sqrt{5}\int_0^{\tan^{-1}(t)} \sec^3(\theta)\ \mathrm{d}\theta$$

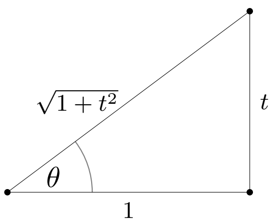
$$f = \sec(\theta) \qquad f' = \sec(\theta)\tan(\theta)$$

$$g' = \sec^2(\theta)\ \mathrm{d}\theta \qquad g = \tan(\theta)$$

$$2\sqrt{5}\left(\left[\sec(\theta)\tan(\theta)\right]_0^{tan^{-1}(t)} - \int_0^{tan^{-1}(t)} \sec(\theta)\tan^2(\theta)\right)$$

$$2\sqrt{5}\left(\sec(tan^{-1}(t))\tan(tan^{-1}(t)) - \int_0^{tan^{-1}(t)} \sec^3(\theta)\ \mathrm{d}\theta + \int_0^{tan^{-1}(t)} \sec(\theta)\ \mathrm{d}\theta\right)$$

$$\sqrt{5}\left(\sec(tan^{-1}(t))\tan(tan^{-1}(t))+ln|\sec(tan^{-1}(t))+\tan(tan^{-1}(t))|\right)$$



$$\theta=\tan^{-1}(t) \qquad \sec(\theta)=\sqrt{1+t^2} \qquad \sec(tan^{-1}(t))=\sec\theta=\sqrt{1+t^2}$$

$$s(t)=\sqrt{5}\left(t\sqrt{1+t^2}+ln|\sqrt{1+t^2}+t|\right)$$

5.

$$t=\sqrt{\frac{(135s+8)^{\frac{2}{3}}-4}{45}}$$

$$t(20)=\sqrt{\frac{(2708)^{\frac{2}{3}}-4}{45}}=2.05633$$

$$r(t(\vec{20}))=\langle 4.22849, 17.39035, -7.69518\rangle$$

For problems 1&2 determine the length of the vector function on the given interval.

1.

$\vec{r}(t)=4\cos(2t)\mathbf{\hat{i}}+3t\mathbf{\hat{j}}-4\sin(2t)\mathbf{\hat{k}}$ from $0\le t\le 3\pi$.

$$||\vec{r}'(t)||=\sqrt{(-8\sin 2t)^2+9+(-8\cos 2t)^2}=\sqrt{73}$$

$$=\int_0^{3\pi}||\vec{r}'(t)||\ \mathrm{d}t=\int_0^{3\pi}\sqrt{73}\ \mathrm{d}t$$

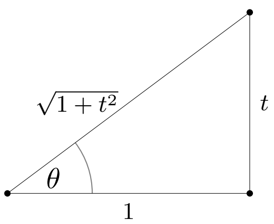
$$\sqrt{73}\,[t]_0^{3\pi}=3\pi\sqrt{73}$$

2.

$\vec{r}(t) = \langle 9-2t, 4+2t, \sqrt{2}t^2 \rangle$ from $0 \le t \le 1$.

$$||\vec{r}'(t)|| = \sqrt{(-2)^2 + 2^2 + (2\sqrt{2}t)^2} = 2\sqrt{2}\sqrt{t^2+1}$$

$$= \int_0^1 ||\vec{r}'(t)||\ \mathrm{d}t = \int_0^1 2\sqrt{2}\sqrt{t^2+1}\ \mathrm{d}t$$



$$t = \tan(\theta) \qquad \mathrm{d}t = \sec^2(\theta)\ \mathrm{d}\theta$$

$$\sqrt{1+(\tan\theta)^2} = |\sec^2\theta| = \sec\theta$$

$$0 = \tan\theta \qquad 1 = \tan\theta = \frac{\pi}{4}$$

$$= 2\sqrt{2}\int_0^{\frac{\pi}{4}} \sec^3\theta\ \mathrm{d}\theta$$

$$= 2\sqrt{2}\left([\sec(\theta)\tan(\theta)]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^3(\theta)\ \mathrm{d}\theta + \int_0^{\frac{\pi}{4}} \sec(\theta)\ \mathrm{d}\theta \right) = 3.25$$

3.

$\vec{r}(t) = 2t\vec{i} + \frac{1}{2}t^2\vec{j} + \ln(t^2)\vec{k}, \sqrt{2}t^2$ from $1 \le t \le 3$.

$$||\vec{r}'(t)|| = \sqrt{t^4 + 16 + 8t^2} = \sqrt{(t^2+4)^2} = t^2 + 4$$

$$= \int_0^2 ||\vec{r}'(t)||\ \mathrm{d}t = \int_0^2 t^2 + 4\ \mathrm{d}t$$

$$= \left[\frac{1}{3}t^3 + 4t\right]_0^2 = \frac{32}{3}$$

The curvature measures how fast a curve is changing direction at a given point.

$$k = \left\| \frac{\mathrm{d}\vec{T}}{\mathrm{d}s} \right\|$$

Alternative:

$$k = \frac{\left\|\vec{T}'(t)\right\|}{\|\vec{r}'(t)\|} \qquad k = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

If $y = f(x)$:

$$\vec{r}(x) = x\vec{i} + f(x)\vec{j}$$

Curvature formula:

$$k = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{\frac{3}{2}}}$$

Determine the curvature for $\vec{r}(t) = \langle t, 3\sin(t), 3\cos(t)\rangle$.

$$\vec{r}'(t) = \langle 1, 3\cos(t), -3\sin(t)\rangle$$

$$||\vec{r}'(t)|| = \sqrt{10}$$

$$\vec{T}(t) = \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos(t), -\frac{3}{\sqrt{10}}\sin(t)\rangle$$

$$\vec{T}'(t) = \langle 0, -\frac{3}{\sqrt{10}}\sin(t), -\frac{3}{\sqrt{10}}\cos(t)\rangle$$

$$||\vec{T}'(t)|| = \frac{3}{\sqrt{10}}$$

$$k = \frac{\left\|\vec{T}'(t)\right\|}{\|\vec{r}'(t)\|} = \frac{\frac{3}{\sqrt{10}}}{\sqrt{10}} = \frac{3}{10}$$

Determine the curvature of $\vec{r}(t) = t^2\vec{i} + t\vec{k}$

$$\vec{r}'(t) = 2t\vec{i} + \vec{k}$$

$$||\vec{r}'(t)|| = \sqrt{4t^2 + 1}$$

$$\vec{r}''(t) = 2\vec{i}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 2\vec{j}$$

$$\left\|\vec{r}'(t) \times \vec{r}''(t)\right\| = 2$$

$$k = \frac{2}{(4t^2+1)^{\frac{3}{2}}}$$

Find the curvature for each the following vector functions.

1.

$\vec{r}(t) = \langle \cos(2t), -\sin(2t), 4t\rangle$

$$\vec{r}'(t) = \langle -2\sin(2t), -2\cos(2t), 4\rangle$$

$$||\vec{r}'(t)|| = \sqrt{20} = 2\sqrt{5}$$

$$\vec{T}(t) = \langle -\frac{1}{\sqrt{5}}\sin(2t), -\frac{1}{\sqrt{5}}\cos(2t), \frac{2}{\sqrt{5}}\rangle$$

$$\vec{T}'(t) = \langle -\frac{2}{\sqrt{5}}\cos(2t), \frac{2}{\sqrt{5}}\sin(2t), 0\rangle$$

$$||\vec{T}'(t)|| = \sqrt{\frac{4}{\sqrt{5}}\cos^2(2t) + \frac{4}{\sqrt{5}}\sin^2(2t)} = \frac{2}{\sqrt{5}}$$

$$k = \frac{\left\|\vec{T}'(t)\right\|}{\|\vec{r}'(t)\|} = \frac{\frac{2}{\sqrt{5}}}{2\sqrt{5}} = \frac{1}{5}$$

2.

$\vec{r}(t) = \langle 4t, -t^2, 2t^3\rangle$

$$\vec{r}'(t) = \langle 4, -2t, 6t^2\rangle$$

$$||\vec{r}'(t)|| = \sqrt{16 + 4t^2 + 36t^4}$$

$$\vec{r}''(t) = \langle 0, -2, 12t\rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2t & 6t^2 \\ 0 & -2 & 12t \end{vmatrix} = -12t^2\vec{i} - 48t\vec{j} - 8\vec{k}$$

$$\left\|\vec{r}'(t) \times \vec{r}''(t)\right\| = \sqrt{144t^4 + 2304t^2 + 64}$$

$$k = \frac{\sqrt{144t^4 + 2304t^2 + 64}}{(16 + 4t^2 + 36t^4)^{\frac{3}{2}}}$$

1.

$\vec{r}(t) = \langle 5t, 1 - 2t, 4t^{\frac{3}{2}} \rangle$

$$\vec{r}'(t) = \langle 5, -2, 6t^{\frac{1}{2}} \rangle$$

$$||\vec{r}'(t)|| = \sqrt{29 + 36t}$$

$$\vec{r}''(t) = \langle 0, 0, 3t^{-\frac{1}{2}} \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -2 & 6t^{\frac{1}{2}} \\ 0 & 0 & 3t^{-\frac{1}{2}} \end{vmatrix} = -6t^{-\frac{1}{2}}\vec{i} - 15t^{-\frac{1}{2}}\vec{j}$$

$$\left\|\vec{r}'(t) \times \vec{r}''(t)\right\| = \sqrt{261}t^{-\frac{1}{2}}$$

$$k = \frac{\sqrt{261}t^{-\frac{1}{2}}}{(29 + 36t)^{\frac{3}{2}}}$$

2.

$\vec{r}(t) = \langle 6, e^{-5t}, 3te^{-5t} \rangle$

$$\vec{r}'(t) = \langle 5, -2, 6t^{\frac{1}{2}} \rangle$$

$$||\vec{r}'(t)|| = \sqrt{29 + 36t}$$

$$\vec{r}''(t) = \langle 0, 0, 3t^{-\frac{1}{2}} \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -2 & 6t^{\frac{1}{2}} \\ 0 & 0 & 3t^{-\frac{1}{2}} \end{vmatrix} = -6t^{-\frac{1}{2}}\vec{i} - 15t^{-\frac{1}{2}}\vec{j}$$

$$\left\|\vec{r}'(t) \times \vec{r}''(t)\right\| = \sqrt{261}t^{-\frac{1}{2}}$$

$$k = \frac{\sqrt{261}t^{-\frac{1}{2}}}{(29+36t)^{\frac{3}{2}}}$$

3.

$\vec{r}(t) = \langle \cos(\omega t), t, \sin(\omega t) \rangle$

$$\vec{r}'(t) = \langle 5, -2, 6t^{\frac{1}{2}} \rangle$$

$$||\vec{r}'(t)|| = \sqrt{29+36t}$$

$$\vec{r}''(t) = \langle 0, 0, 3t^{-\frac{1}{2}} \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -2 & 6t^{\frac{1}{2}} \\ 0 & 0 & 3t^{-\frac{1}{2}} \end{vmatrix} = -6t^{-\frac{1}{2}}\vec{i} - 15t^{-\frac{1}{2}}\vec{j}$$

$$\left\|\vec{r}'(t) \times \vec{r}''(t)\right\| = \sqrt{261}t^{-\frac{1}{2}}$$

$$k = \frac{\sqrt{261}t^{-\frac{1}{2}}}{(29+36t)^{\frac{3}{2}}}$$

Área de superfície:

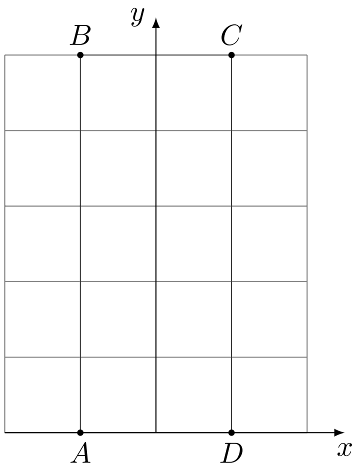
$$A(S) = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ \mathrm{d}x\,\mathrm{d}y$$

Valores Médios:

$$A = \iint_D \mathrm{d}x\,\mathrm{d}y$$

$$\bar{f} = \frac{\iint_D (x,y)\ \mathrm{d}x\,\mathrm{d}y}{A}$$

$$\bar{f} = \frac{1}{A(R)} \iint_R f(x,y)\ \mathrm{d}A$$



$$A(R) = 2 \cdot 5 = 10$$

$$\bar{f} = \frac{1}{10} \int_{-1}^{1} \int_{0}^{5} x^2 y \; \mathrm{d}y \, \mathrm{d}x$$

$$= \frac{1}{10} \int_{-1}^{1} \left[x^2 \frac{y^2}{2} \right]_0^5 \mathrm{d}x = \frac{1}{10} \int_{-1}^{1} \frac{25}{2} x^2 \; \mathrm{d}x = \frac{25}{20} \left[\frac{x^3}{3} \right]_{-1}^{1} = \frac{5}{6}$$

Centro de Massa:

$$M = \iint_R \rho(x,y) \; \mathrm{d}A \qquad M_x = \iint_R y\rho(x,y) \; \mathrm{d}A \qquad M_y = \iint_R x\rho(x,y) \; \mathrm{d}A$$

$$R = (\bar{x}.\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right)$$

$$\bar{x} = \frac{1}{A} \iint_D x f(x,y) \; \mathrm{d}x \, \mathrm{d}y$$

$$\bar{y} = \frac{1}{A} \iint_D y f(x,y) \; \mathrm{d}x \, \mathrm{d}y$$

Find the volume of the solid that is bounded about by $f(x,y) = y \sin(xy)$ and below $R = [1,2] \times [0,\pi]$

$$V = \int_0^{\pi} \int_1^2 y \sin(xy) \; \mathrm{d}x \, \mathrm{d}y$$

$$= \int_0^{\pi} \left[-\frac{y}{y} \cos(xy) \right]_1^2 \mathrm{d}y = \int_0^{\pi} -\cos(2y) + \cos(y) \; \mathrm{d}y$$

$$= \left[-\frac{1}{2} \sin(2y) + \sin(y) \right]_0^{\pi} = 0$$

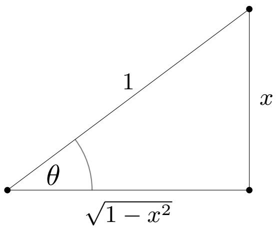
If $R = \{(x,y) | -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate $\iint_R \sqrt{1-x^2}\ \mathrm{d}A$
Método 1:
Como $z = \sqrt{1-x^2}$ é metade de um cilindro então:

$$V_{cilindro} = \pi r^2 h$$

$$V = \frac{1}{2}\pi r^2 h = \frac{1}{2}\pi 1 \cdot 4 = 2\pi$$

Método 2:

$$\int_{-2}^{2}\int_{-1}^{1} \sqrt{1-x^2}\ \mathrm{d}x\,\mathrm{d}y$$



Substituição trigonométrica:

$$x = \sin\theta \qquad \mathrm{d}x = \cos\theta\ \mathrm{d}\theta$$

$$\cos\theta = \sqrt{1-x^2}$$

Se $x = -1$ então $\theta = -\frac{\pi}{2}$
Se $x = 1$ então $\theta = \frac{\pi}{2}$

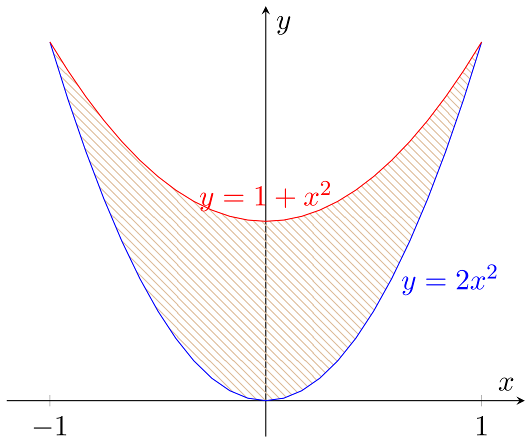
$$\int_{-2}^{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\ \mathrm{d}\theta\,\mathrm{d}y$$

$$\cos 2\theta = \sin^2\theta - \cos^2\theta \Leftrightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\int_{-2}^{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left(1+\cos 2\theta\right)\ \mathrm{d}\theta\,\mathrm{d}y = \int_{-2}^{2}\frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\ \mathrm{d}y$$

$$= \left[\frac{\pi}{2}y\right]_{-2}^{2} = 2\pi$$

Evaluate $\iint_D (x+2y)\ \mathrm{d}A$ where D is the region bounded by $y = 2x^2$ and $y = 1 + x^2$



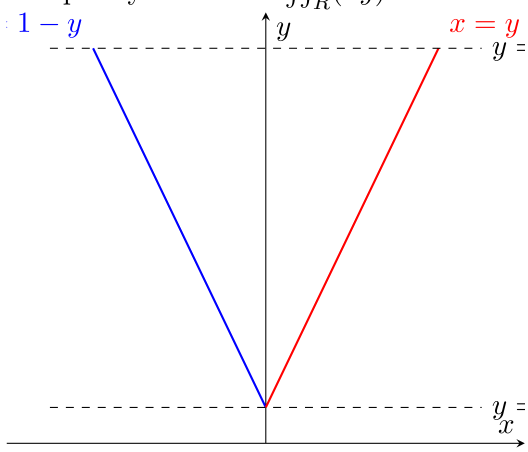
$$\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)\ \mathrm{d}y\,\mathrm{d}x = \int_{-1}^{1}\left[xy+y^2\right]_{2x^2}^{1+x^2}\ \mathrm{d}x$$

$$= \int_{-1}^{1}\left[(x(1+x^2)+(1+x^2)^2)-(x(2x^2)+(2x^2)^2)\right]\ \mathrm{d}x$$

$$= \int_{-1}^{1}(-x^3-3x^4+x+2x^2+1)\ \mathrm{d}x = \left[-\frac{1}{4}x^4-\frac{3}{5}x^5+\frac{1}{2}x^2+\frac{2}{3}x^3+x\right]_{-1}^{1}$$

$$= \left[(-\frac{1}{4}-\frac{3}{5}+\frac{1}{2}+\frac{2}{3}+1)-(-\frac{1}{4}+\frac{3}{5}+\frac{1}{2}-\frac{2}{3}-1)\right] = \frac{32}{15}$$

Setup only! Evaluate $\iint_R (xy)\ \mathrm{d}A$ where R is the region bounded by $y=-x+1, y=x+1$ and $y=3$



Horizontal fixamos o x

$$\int_{1}^{3}\int_{1-y}^{y-1}(xy)\ \mathrm{d}x\,\mathrm{d}y$$

Find the volume of the solid that lies under $z=xy$ and and about D where D is the region bounded by

$$\int_{-2}^{4}\int_{\frac{1}{2}y^3-3}^{y+1}(xy)\ \mathrm{d}x\,\mathrm{d}y$$

Calculate $\iiint_R = (x+y+2z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ R: x^2+z^2=4, y=2, y=3.$
Coordenadas cilíndricas:

$$x = r\cos(\theta) \qquad z = r\sin\theta$$

$$\int_2^3\int_0^{2\pi}\int_0^2 (r\cos(\theta)+y+2r\sin(\theta))\,r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}y$$

$$= \int_2^3\int_0^{2\pi}\int_0^2 (r^2\cos(\theta)+yr+2r^2\sin(\theta))\ \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}y$$

$$= \int_2^3\int_0^{2\pi}\left[(\frac{1}{3}r^3\cos(\theta)+\frac{1}{2}yr^2+\frac{2}{3}r^3\sin(\theta))\right]_0^2\ \mathrm{d}\theta\,\mathrm{d}y$$

$$= \int_2^3\int_0^{2\pi}(\frac{8}{3}\cos(\theta)+2y+\frac{16}{3}\sin(\theta))\ \mathrm{d}\theta\,\mathrm{d}y$$

$$= \int_2^3\left[(\frac{8}{3}\sin(\theta)+2y\theta-\frac{16}{3}\cos(\theta))\right]_0^{2\pi}\ \mathrm{d}y$$

$$= \left[2\pi y^2\right]_2^3 = 18\pi - 8\pi = 10\pi$$

Calculate the volume of the region $\iiint_R = (x+y+2z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\ R: x^2+z^2=4, y=2, y=3.$

$$\int_2^3\int_0^{2\pi}\int_0^2 r\ \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}y$$

$$= \int_2^3\int_0^{2\pi}\left[\frac{1}{2}r^2\right]_0^2\ \mathrm{d}\theta\,\mathrm{d}y$$

$$= \int_2^3 [2\theta]_0^{2\pi}\ \mathrm{d}y$$

$$= [4\pi y]_2^3 = 12\pi - 8\pi = 4\pi$$

Calculate $\iiint_R = y\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ R: x^2+y^2=3, z=-1, z=2.$
Coordenadas cilíndricas:

$$y = r\sin\theta$$

$$\int_{-1}^{2}\int_0^{2\pi}\int_0^{\sqrt{3}} r\sin(\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

$$= \int_{-1}^{2} \int_{0}^{2\pi} \int_{0}^{\sqrt{3}} r^2 \sin(\theta) \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}z$$

$$= \int_{-1}^{2} \int_{0}^{2\pi} \left[\frac{1}{3} r^3 \sin(\theta)\right]_0^{\sqrt{3}} \mathrm{d}\theta \,\mathrm{d}z$$

$$= \int_{-1}^{2} \int_{0}^{2\pi} \sqrt{3} \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}z$$

$$= [0]_{-1}^{2} = 0$$

Calculate the volume of the region $\iiint_R = y \;\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z,\; R : x^2 + y^2 = 3, z = -1, z = 2$.

$$\int_{-1}^{2} \int_{0}^{2\pi} \int_{0}^{\sqrt{3}} 1 \, r \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}z$$

$$= \int_{-1}^{2} \int_{0}^{2\pi} \int_{0}^{\sqrt{3}} r \;\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}z$$

$$= \int_{-1}^{2} \int_{0}^{2\pi} \left[\frac{1}{2} r^2\right]_0^{\sqrt{3}} \mathrm{d}\theta \,\mathrm{d}z$$

$$= \int_{-1}^{2} \left[\frac{3}{2}\theta\right]_0^{2\pi} \mathrm{d}z$$

$$= [3\pi z]_{-1}^{2} = 9\pi$$

Line integral with respect to arc length.
Parametric equations:

$$x = h(t) \qquad y = g(t) \qquad a \le t \le b$$

Parametric equations as a vector function:

$$\vec{r}(t) = h(t)\vec{i} + g(t)\vec{j} \qquad a \le t \le b$$

The curve is called smooth if $\vec{r}(t)$ is continuous and $\vec{r}(t) \neq 0$

$$L = \int_a^b \mathrm{d}s \qquad \mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \;\mathrm{d}t$$

$$\int_C f(x, y) \;\mathrm{d}s = \int_a^b f(h(t), g(t)) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \;\mathrm{d}t$$

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = ||\vec{r}(t)||$$

Curve	Parametric Equations	
	Counter-Clockwise	Clockwise
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (Ellipse)	$x=a\cos(t)$ $y=b\sin(t)$ $0\le t\le 2\pi$	$x=a\cos(t)$ $y=-b\sin(t)$ $0\le t\le 2\pi$
	Counter-Clockwise	Clockwise
$x^2+y^2=r^2$ (Circle)	$x=r\cos(t)$ $y=r\sin(t)$ $0\le t\le 2\pi$	$x=r\cos(t)$ $y=-r\sin(t)$ $0\le t\le 2\pi$
$y=f(x)$	$x=t$ $y=f(t)$	
$x=g(y)$	$x=g(t)$ $y=t$	
Line Segment From (x_0,y_0,z_0) to (x_1,y_1,z_1)	$\vec{r}(t)=(1-t)\langle x_0,y_0,z_0\rangle+t\langle x_1,y_1,z_1\rangle\ ,\ 0\le t\le 1$ or $x=(1-t)x_0+t\,x_1$ $y=(1-t)y_0+t\,y_1\quad ,\quad 0\le t\le 1$ $z=(1-t)z_0+t\,z_1$	

Example 1

Evaluate $\int_C xy^4\ \mathrm{d}s$ where C is the right half of the circle, $x^2+y^2=16$ traced out in a counter clockwise direction.

$$x=4\cos(t)\qquad y=4\sin(t)\qquad -\frac{\pi}{2}\le t\le\frac{\pi}{2}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-4\sin(t)\qquad \frac{\mathrm{d}y}{\mathrm{d}t}=4\cos(t)$$

$$\mathrm{d}s=\sqrt{(-4\sin(t))^2+(4\cos(t))^2}\ \mathrm{d}t=4\ \mathrm{d}t$$

$$\int_C xy^4\ \mathrm{d}s=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(4\cos(t))(4\sin(t))^4\,4\ \mathrm{d}t$$

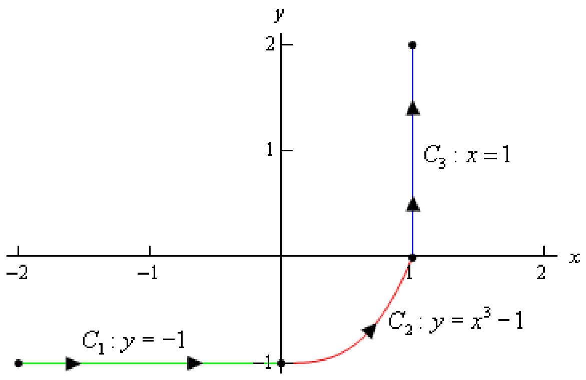
$$=4096\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos(t))(\sin(t))^4\ \mathrm{d}t$$

$$= \frac{4096}{5}\left[(\sin(t))^5\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{4096}{5}[2] = \frac{8192}{5}$$

Example 2

Evaluate $\int_C 4x^3 \, \mathrm{d}s$ where C is the curve shown below.



$$C_1 : x = t \qquad y = -1 \qquad -2 \leq t \leq 0$$

$$C_1 : \ \mathrm{d}x = 1 \qquad \mathrm{d}y = 0 \qquad \mathrm{d}s = \sqrt{1}\ \mathrm{d}t = \ \mathrm{d}t$$

$$C_2 : x = t \qquad y = t^3 - 1 \qquad 0 \leq t \leq 1$$

$$C_2 : \ \mathrm{d}x = 1 \qquad \mathrm{d}y = 3t^2 \qquad \mathrm{d}s = \sqrt{1 + 9t^4}\ \mathrm{d}t$$

$$C_3 : x = 1 \qquad y = t \qquad 0 \leq t \leq 2$$

$$C_3 : \ \mathrm{d}x = 0 \qquad \mathrm{d}y = 1 \qquad \mathrm{d}s = \sqrt{1}\ \mathrm{d}t = \ \mathrm{d}t$$

$$\int_{C_1} 4x^3 \ \mathrm{d}s = \int_{-2}^{0} 4t^3 \ \mathrm{d}t$$

$$\int_{C_2} 4x^3 \ \mathrm{d}s = \int_0^1 4t^3\sqrt{1 + 9t^4} \ \mathrm{d}t = \frac{2}{27}\left[(1 + 9t^4)^{\frac{3}{2}}\right]_0^1 = \frac{2}{27}\left[10^{\frac{3}{2}} - 1\right] = 2.268354$$

$$\int_{C_3} 4x^3 \ \mathrm{d}s = \int_0^2 4(1)^3 \ \mathrm{d}t = [4t]_0^2 = 8$$

Example 3

Evaluate $\int_C 4x^3 \ \mathrm{d}s$ where C is the line segment from $(-2,-1)$ to $(1,2)$.

$$\vec{r}(t) = (1-t)\langle -2,-1\rangle + t\langle 1,2\rangle$$

$$= \langle 3t-2, 3t-1\rangle$$

$$x = 3t-2 \qquad y = 3t-1$$

$$||\vec{r}(t)|| = \sqrt{18} = 3\sqrt{2}\ \mathrm{d}t$$

$$12\sqrt{2}\int_0^1 (3t-2)^3\ \mathrm{d}t = \sqrt{2}\left[(3t-2)^4\right]_0^1 = -15\sqrt{2}$$

Example 4

Evaluate $\int_C 4x^3 \ \mathrm{d}s$ where C is the line segment from $(1,2)$ to $(-2,-1)$.

$$\vec{r}(t) = (1-t)\langle 1,2\rangle + t\langle -2,-1\rangle$$

$$= \langle -3t+1, -3t-+2\rangle$$

$$x = -3t+1 \qquad y = -3t+2$$

$$||\vec{r}(t)|| = \sqrt{18} = 3\sqrt{2}\ \mathrm{d}t$$

$$12\sqrt{2}\int_0^1 (-3t+1)^3\ \mathrm{d}t = -\sqrt{2}\left[(-3t+1)^4\right]_0^1 = -15\sqrt{2}$$

Example 5

Evaluate $\int_C x \ \mathrm{d}s$ for each of the following curves.

a)

$$C_1 : y = x^2, -1 \leq x \leq 1$$

$$C_1 : x = t, y = t^2, -1 \leq t \leq 1$$

$$||\vec{r'}(t)|| = \sqrt{1+4t^2}\ \mathrm{d}t$$

$$\int_{-1}^{1} t\sqrt{1+4t^2}\ \mathrm{d}t = \frac{1}{8}\int_{-1}^{1} 8t(1+4t^2)^{\frac{1}{2}}\ \mathrm{d}t = \frac{1}{12}\left[(1+4t^2)^{\frac{3}{2}}\right]_{-1}^{1} = 0$$

b)

C_2 : The line segment from (-1,1) to (1,1).

$$\begin{aligned}\vec{r}(t) &= (1-t)\langle -1, 1\rangle + t\langle 1, 1\rangle \qquad 0 \leq t \leq 1 \\ &= \langle 2t-1, 1\rangle\end{aligned}$$

$$||\vec{r'}(t)|| = 2$$

$$\int_{0}^{1} (4t-2)\ \mathrm{d}t = \left[2t^2 - 2t\right]_0^1 = 0$$

Método 2:

$$C_2 : x = t, y = 1, -1 \leq t \leq 1$$

$$\vec{r}(t) = 1$$

$$\int_{-1}^{1} t\ \mathrm{d}t = \left[\frac{1}{2}t^2\right]_{-1}^{1} = 0$$

c)

C_3 : The line segment from (1,1) to (-1,1).

$$\begin{aligned}\vec{r}(t) &= (1-t)\langle 1, 1\rangle + t\langle -1, 1\rangle \qquad 0 \leq t \leq 1 \\ &= \langle 1-2t, 1\rangle\end{aligned}$$

$$||\vec{r'}(t)|| = 2$$

$$\int_0^1 (2-4t)\ \mathrm{d}t = \left[2t - t^2\right]_0^1 = 0$$

Example 6

Evaluate $\int_C xyz\ \mathrm{d}s$ where C is the helix given by, $\vec{r}(t) = \langle \cos(t), \sin(t), 3t\rangle$,$0 \leq t \leq 4\pi$.

$$||\vec{r'}(t)|| = \sqrt{10}$$

$$3\int_0^{4\pi} (t\cos(t)\sin(t)\sqrt{10})\ \mathrm{d}t$$

$$\frac{\sin 2t}{2} = \cos(t)\sin(t)$$

$$= \frac{3\sqrt{10}}{2}\left[\frac{1}{4}\sin 2t - \frac{t}{2}\cos 2t\right]_0^{4\pi} = -3\sqrt{10}\pi$$

1.

Evaluate $\int_C 3x^2 - 2y\ \mathrm{d}s$ where C is the line segment from $(3,6)$ to $(-1,1)$.

$$\begin{aligned}\vec{r}(t) &= (1-t)\langle 3,6\rangle + t\langle 1,-1\rangle \qquad 0 \leq t \leq 1\\ &= \langle 3-2t, 6-7t\rangle\end{aligned}$$

$$||\vec{r'}(t)|| = \sqrt{53}\ \mathrm{d}t$$

$$\int_0^1 \left(3(3-2t)^2 - 2(6-7t)\right)\sqrt{53}\ \mathrm{d}t$$

$$= \sqrt{53}\left[-\frac{1}{2}(3-2t)^3 + 7t^2 - 12t\right]_0^1 = 8\sqrt{53}$$

2.

Evaluate $\int_C 2yx^2 - 4x \; \mathrm{d}s$ where C is the lower half of the circle centered at the origin of radius 3 with clo

$$x = 3\cos(t) \qquad y = -3\sin(t)$$

$$\vec{r}(t) = \langle 3\cos(t), -3\sin(t)\rangle \qquad 0 \le t \le \pi$$

$$||\vec{r'}(t)|| = \sqrt{(3\cos(t))^2 + (-3\sin(t))^2} = 3 \; \mathrm{d}t$$

$$\int_0^{\pi} \left(2\left(-3\sin(t)9\cos^2(t)\right) - 12\cos(t)\right) 3 \; \mathrm{d}t$$

$$3\int_0^{\pi} -54\sin(t)\cos^2(t) - 12\cos(t) \; \mathrm{d}t$$

$$= 3\left[18\cos^3(t) - 12\sin(t)\right]_0^{\pi} = -108$$

3.

Evaluate $\int_C 6x \; \mathrm{d}s$ where C is the portion of $y = x^2$ from $x = -1$ to $x = 2$.
The direction of C is in the direction of increasing x.

$$x = t \qquad y = t^2$$

$$\vec{r}(t) = \langle t, t^2\rangle \qquad -1 \le t \le 2$$

$$||\vec{r'}(t)|| = \sqrt{1 + 4t^2} \; \mathrm{d}t$$

$$6\int_{-1}^{2} t\left(1 + 4t^2\right)^{\frac{1}{2}} \; \mathrm{d}t$$

$$\frac{6}{8}\int_{-1}^{2} 8t\left(1 + 4t^2\right)^{\frac{1}{2}} \; \mathrm{d}t$$

$$= \frac{1}{2}\left[\left(1 + 4t^2\right)^{\frac{3}{2}}\right]_{-1}^{2} = \frac{1}{2}\left[17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right]$$

4.

Evaluate $\int_C xy - 4z \text{ d}s$ where C is the line segment from $(1,1,0)$ to $(2,3,-2)$.

$$\begin{aligned}\vec{r}(t) &= (1-t)\langle 1,1,0\rangle + t\langle 2,3,-2\rangle \qquad 0 \le t \le 1 \\ &= \langle 1+t, 1+2t, -2t\rangle\end{aligned}$$

$$||\vec{r'}(t)|| = \sqrt{9} = 3 \text{ d}t$$

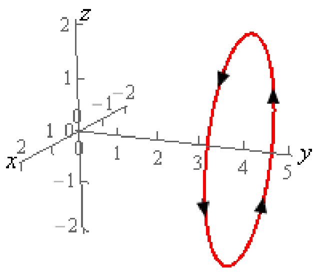
$$\int_0^1 \left((1+t)\left(1+2t\right)+8t\right) 3 \text{ d}t$$

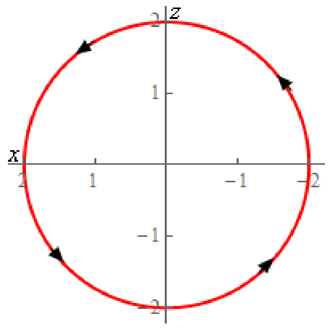
$$3\int_0^1 2t^2 + 11t + 1 \text{ d}t$$

$$= 3\left[\frac{2}{3}t^3 + \frac{11}{2}t^2 + t\right]_0^1 = \frac{43}{2}$$

5.

Evaluate $\int_C x^2y^2 \text{ d}s$ where C is the circle centered at the origin of radius 2 centered on the y-axis at $y = 4$. See the sketches below for orientation. Note the "odd" axis orientation on the 2D circle is intentionally that way to match the 3D axis the direction.





$$x = 2\cos(t) \qquad y = 4 \qquad z = -2\sin(t)$$

$$\vec{r}(t) = \langle 2\cos(t), 4, -2\sin(t)\rangle \qquad 0 \le t \le 2\pi$$

$$||\vec{r'}(t)|| = \sqrt{(2\cos(t))^2 + (-2\sin(t))^2} = 2 \text{ d}t$$

$$128\int_0^{2\pi} \cos^2(t) \text{ d}t$$

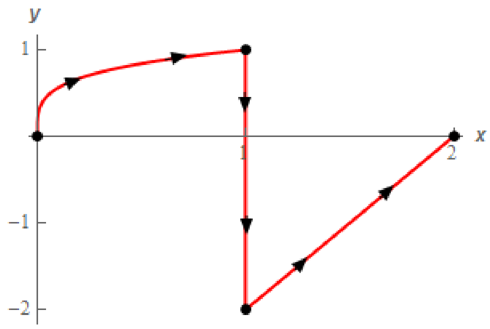
$$128 \int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos(2t)\ \mathrm{d}t$$

$$64\left[t + \frac{1}{2}\sin(2t)\right]_0^{2\pi}$$

$$64\,[2\pi] = 128\pi$$

6.

Evaluate $\int_C 16y^5\ \mathrm{d}s$ where C is the portion of $x = y^4$ from $y = 0$ to $y = 1$ followed by the line segment from $(1,1)$ to $(1,-2)$ which in turn is followed by the line segment from $(1,-2)$ to $(2,0)$. See the sketch below for the direction.



$$C_1 : x = y^4, 0 \le y \le 1$$

$$C_1 : x = t^4, y = t \qquad \vec{r}(t) = \langle t^4, t\rangle, 0 \le t \le 1$$

$$||\vec{r}'(t)|| = \sqrt{1 + 16t^6}\ \mathrm{d}t$$

$$\int_0^1 16t^5\left(1 + 16t^6\right)^{\frac{1}{2}}\ \mathrm{d}t$$

$$= \frac{16}{96}\int_0^1 96t^5\left(1 + 16t^6\right)^{\frac{1}{2}}\ \mathrm{d}t$$

$$= \frac{1}{9}\left[(1 + 16t^6)^{\frac{3}{2}}\right]_0^1 = \frac{1}{9}\left[17^{\frac{3}{2}} - 1\right]$$

$$C_2 : \vec{r}(t) = (1-t)\langle 1, 1\rangle + t\langle 1, -2\rangle \qquad 0 \le t \le 1$$

$$= \langle 1, 1-3t \rangle$$

$$||\vec{r'}(t)|| = 3 \text{ d}t$$

$$48 \int_0^1 (1-3t)^5 \text{ d}t$$

$$-\frac{48}{3} \int_0^1 -3(1-3t)^5 \text{ d}t$$

$$-\frac{8}{3} \left[(1-3t)^6\right]_0^1 = -168$$

$$C_3 : \vec{r}(t) = (1-t)\langle 1, -2 \rangle + t\langle 2, 0 \rangle \qquad 0 \leq t \leq 1$$
$$= \langle 1+t, -2+2t \rangle$$

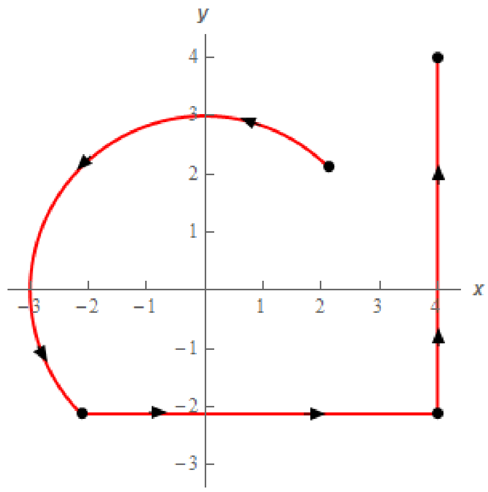
$$||\vec{r'}(t)|| = \sqrt{5} \text{ d}t$$

$$= 16\sqrt{5} \int_0^1 (-2+2t)^5 \text{ d}t$$

$$= \frac{4\sqrt{5}}{3} \left[(-2+2t)^6\right]_0^1 = -\frac{256\sqrt{5}}{3}$$

7.

Evaluate $\int_C 4y - x \text{ d}s$ where C is the upper portion of the circle centered at the origin of radius 3 from $(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$ to $(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$ in the counter clockwise rotation followed by the line segment from $(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$ to $(4, -\frac{3}{\sqrt{2}})$ which in turn is followed by the line segment from $(4, -\frac{3}{\sqrt{2}})$ to $(4, 4)$. See the sketch below for the direction.



$$x = 3\cos(t) \qquad y = 3\sin(t)$$

$$C1 : \vec{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle \qquad \frac{\pi}{4} \leq t \leq \frac{5\pi}{4}$$

$$||\vec{r'}(t)|| = \sqrt{(3\cos(t))^2 + (3\sin(t))^2} = 3 \text{ d}t$$

$$= 9\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 4\sin(t) - \cos(t) \text{ d}t$$

$$= 9\left[-4\cos(t) - \sin(t)\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= 45\sqrt{2}$$

$$C_2 : \vec{r}(t) = \langle t, -\frac{3}{\sqrt{2}}\rangle \qquad -\frac{3}{\sqrt{2}} \leq t \leq 4$$

$$||\vec{r'}(t)|| = \text{ d}t$$

$$= \int_{-\frac{3}{\sqrt{2}}}^{4} -\frac{12}{\sqrt{2}} - t \text{ d}t$$

$$= \left[-\frac{12}{\sqrt{2}}t - \frac{1}{2}t^2\right]_{-\frac{3}{\sqrt{2}}}^{4} = -\frac{48}{\sqrt{2}} - \frac{95}{4}$$

$$C_3 : \vec{r}(t) = \langle 4, t\rangle \qquad -\frac{3}{\sqrt{2}} \leq t \leq 4$$

$$||\vec{r'}(t)|| = \,\mathrm{d}t$$

$$= \int_{-\frac{3}{\sqrt{2}}}^{4} 4t - 4 \,\mathrm{d}t$$

$$= \left[2t^2 - 4t\right]_{-\frac{3}{\sqrt{2}}}^{4} = 7 - \frac{12}{\sqrt{2}}$$

$$\int_C 4y - x \,\mathrm{d}s = 45\sqrt{2} - \frac{48}{\sqrt{2}} - \frac{95}{4} + 7 - \frac{12}{\sqrt{2}} = -\frac{67}{4} + \frac{30}{\sqrt{2}}$$

8.

Evaluate $\int_C y^3 - x^2 \,\mathrm{d}s$ for each of the following curves.

(a)

C is the line segment from $(3, 6)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(3, -6)$.

$$C1 : \vec{r}(t) = (1 - t)\langle 3, 6\rangle + \langle 0, 0\rangle \qquad 0 \leq t \leq 1$$

$$||\vec{r'}(t)|| = \sqrt{45} = 3\sqrt{5} \,\mathrm{d}t$$

$$= 3\sqrt{5} \int_0^1 (6 - 6t)^3 - (3 - 3t)^2 \,\mathrm{d}t$$

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$$= 3\sqrt{5} \left[-\frac{1}{24}(6 - 6t)^4 + \frac{1}{9}(3 - 3t)^3\right]_0^1$$

$$= 153\sqrt{5}$$

(b)

C is the line segment from $(3, 6)$ to $(3, -6)$.

$$C2 : \vec{r}(t) = (1 - t)\langle 0, 0\rangle + \langle 3, -6\rangle \qquad 0 \leq t \leq 1$$

$$||\vec{r'}(t)|| = \sqrt{45} = 3\sqrt{5} \,\mathrm{d}t$$

$$= 3\sqrt{5} \int_0^1 (-6t)^3 - (-3t)^2 \,\mathrm{d}t$$

$$= 3\sqrt{5}\int_0^1 -216t^3 - 9t^2 \text{ d}t$$

$$= 3\sqrt{5}\left[-54t^4 - 3t^3\right]_0^1$$

$$= -171\sqrt{5}$$

9.

Evaluate $\int_C 4x^2 \text{ d}s$ for each of the following curves.

(a)

C is the portion of the circle centered at the origin of radius 2 in the 1st quadrant rotating in the clockwise direction.

$$x = 2\cos(t) \qquad y = -2\sin(t)$$

$$\vec{r}(t) = \langle 2\cos(t), -2\sin(t)\rangle \qquad 0 \le t \le \frac{\pi}{2}$$

$$||\vec{r'}(t)|| = \sqrt{4\sin^2(t) + 4\cos^2(t)} = 2 \text{ d}t$$

$$= 32\int_0^{\frac{\pi}{2}} \cos^2(t) \text{ d}t$$

$$= 16\left[t + \frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}}$$

$$= 8\pi$$

(b)

C is the line segment from $(0, 2)$ to $(2, 0)$.

$$C2 : \vec{r}(t) = (1-t)\langle 0, 2\rangle + \langle 2, 0\rangle \qquad 0 \le t \le 1$$

$$C2 : \vec{r}(t) = \langle 2t, 2-2t\rangle$$

$$||\vec{r'}(t)|| = \sqrt{8} = 2\sqrt{2} \text{ d}t$$

$$= 32\sqrt{2}\int_0^1 t^2 \text{ d}t$$

$$= \frac{32\sqrt{2}}{3}\left[t^3\right]_0^1$$

$$= \frac{32\sqrt{2}}{3}$$

10.

Evaluate $\int_C 2x^3 \text{ d}s$ for each of the following curves.

(a)

C is the portion $y = x^3$ from $x = -1$ to $x = 2$.

$$x = t \qquad y = t^3$$

$$\vec{r}(t) = \langle t, t^3 \rangle \qquad -1 \leq t \leq 2$$

$$||\vec{r'}(t)|| = \sqrt{1 + 9t^4} \text{ d}t$$

$$= 2\int_{-1}^{2} t^3\sqrt{1 + 9t^4} \text{ d}t$$

$$= \frac{1}{18}\int_{-1}^{2} 36t^3\left(1 + 9t^4\right)^{\frac{1}{2}} \text{ d}t$$

$$= \frac{1}{27}\left[\left(1 + 9t^4\right)^{\frac{3}{2}}\right]_{-1}^{2}$$

$$= \frac{1}{27}\left[145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right]$$

(b)

C is the portion $y = x^3$ from $x = 2$ to $x = -1$.
Método 1:

$$x = -t \qquad y = -t^3$$

$$\vec{r}(t) = \langle -t, -t^3 \rangle \qquad -2 \leq t \leq 1$$

$$||\vec{r'}(t)|| = \sqrt{1+9t^4}\ \mathrm{d}t$$

$$= -2\int_{-1}^{2} t^3\sqrt{1+9t^4}\ \mathrm{d}t$$

$$= -\frac{1}{18}\int_{-2}^{1} 36t^3\left(1+9t^4\right)^{\frac{1}{2}}\ \mathrm{d}t$$

$$= -\frac{1}{27}\left[\left(1+9t^4\right)^{\frac{3}{2}}\right]_{-2}^{1}$$

$$= -\frac{1}{27}\left[10^{\frac{3}{2}} - 145^{\frac{3}{2}}\right]$$

Método 2:

$$\int_{C_2} 2x^3\ \mathrm{d}s = \int_{-C_1} 2x^3\ \mathrm{d}s = \int_{C_1} 2x^3\ \mathrm{d}s$$
$$= \frac{1}{27}\left[145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right]$$

Example 1

Evaluate $\int_C 3y\ \mathrm{d}s$ where C is the portion of $x = 9 - y^2$ from $y = -1$ and $y = 2$.
Método 1:

$$\mathrm{d}s = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\ \mathrm{d}y = \sqrt{1+4y^2}\ \mathrm{d}y$$

$$\frac{3}{8}\int_{-1}^{2} 8y\left(1+4y^2\right)^{\frac{1}{2}}\ \mathrm{d}y$$

$$= \frac{1}{4}\left[\left(1+4y^2\right)^{\frac{3}{2}}\right]_{-1}^{2} = \frac{1}{4}\left[17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right]$$

Método 2:

$$x = 9 - t^2 \qquad y = t \qquad -1 \le t \le 2$$

$$||\vec{r'}(t)|| = \sqrt{1+4t^2}\ \mathrm{d}t$$

$$\frac{3}{8}\int_{-1}^{2} 8t\left(1+4t^2\right)^{\frac{1}{2}}\ \mathrm{d}t$$

$$= \frac{1}{4}\left[\left(1+4t^2\right)^{\frac{3}{2}}\right]_{-1}^{2} = \frac{1}{4}\left[17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right]$$

Example 2

Evaluate $\int_C \sqrt{x} + 2xy \; ds$ where C is the line segment from $(7, 3)$ to $(0, 6)$.

Example 3

Evaluate $\int_C y^2 - 10xy \; ds$ where C is the left half
of the circle centered at the origin of radius 6
with counter clockwise rotation.

Example 4

Evaluate $\int_C x^2 - 2y \; ds$ where C is given by
$\vec{r}(t) = \langle 4t^4, t^4 \rangle$ for $-1 \le t \le 0$.

Example 5

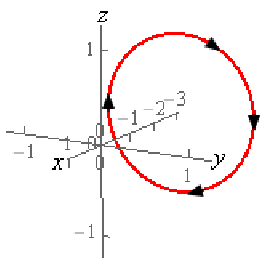
Evaluate $\int_C z^3 - 4x + 2y \; ds$ where C is the line segment
from $(2, 4, -1)$ to $(1, -1, 0)$.

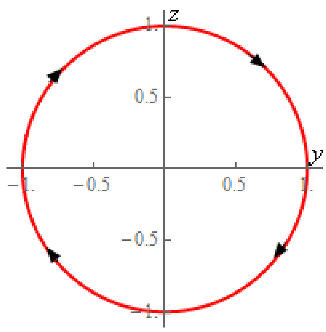
Example 6

Evaluate $\int_C x + 12xz \; ds$ where C is given by
$\vec{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{4}t^4 \rangle$ for $-2 \le t \le 1$.

Example 7

Evaluate $\int_C z^3(x + 7) - 2y \; ds$ where C is a circle centered
at the origin of radius 1 on the x-axis at $x = -3$.
See the sketches below for the direction.





$$x = -3 \qquad y = -\sin(t) \qquad z = \cos(t) \qquad 0 \le t \le 2\pi$$

$$||\vec{r'}(t)|| = \sqrt{\sin^2(t) + \cos^2(t)} = \mathrm{d}t$$

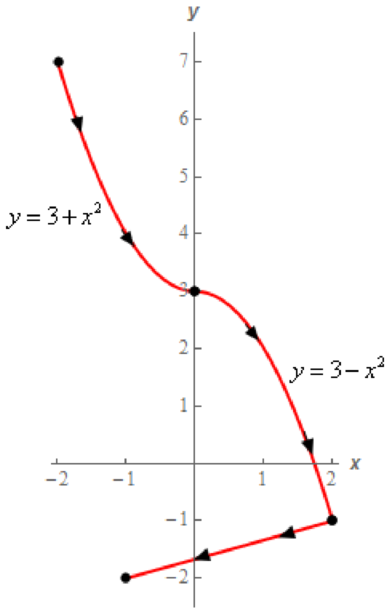
$$\int_0^{2\pi} -4\sin^3(t) - 2\cos(t)\ \mathrm{d}t$$

$$\int_0^{2\pi} -4\sin^3(t)\ \mathrm{d}t = \int_{-\pi}^{\pi} -4\sin^3(t)\ \mathrm{d}t = 0$$

$$\int_0^{2\pi} \cos(t)\ \mathrm{d}t = -2\int_{-\pi}^{\pi} \cos(t)\ \mathrm{d}t = -4\int_0^{\pi} \cos(t)\ \mathrm{d}t = 0$$

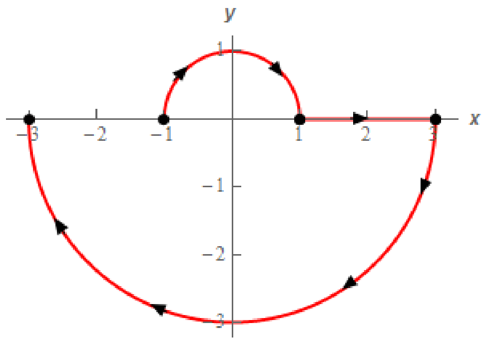
Example 8

Evaluate $\int_C x + 12xz\ \mathrm{d}s$ where C is given by
$\vec{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{4}t^4\rangle$ for $-2 \le t \le 1$.



Example 9

Evaluate $\int_C x + 12xz\ \mathrm{d}s$ where C is given by



$\vec{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{4}t^4 \rangle$ for $-2 \le t \le 1$.
Line Integrals of Vector Fields

1.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x, y) = y^2\vec{i} + (3x - 6y)\vec{j}$ and C is the line segment from $(3, 7)$ to $(0, 12)$.

$$\vec{r}(t) = (1-t)\langle 3, 7\rangle + \langle 0, 12\rangle \qquad 0 \le t \le 1$$

$$\vec{r}(t) = \langle 3 - 3t, 7 + 5t\rangle$$

$$\vec{r}'(t) = (-3, 5)$$

$$\vec{F}(\vec{r}(t)) = (5t+7)^2\vec{i} + (3(3-3t) - 6(5t+7))\vec{j}$$

$$= (5t+7)^2\vec{i} - (39t + 33)\vec{j}$$

$$\int_0^1 -3(5t+7)^2 - 5(39t+33)\ \mathrm{d}t$$

$$= -\frac{3}{5}\int_0^1 5(5t+7)^2\ \mathrm{d}t - 5\int_0^1 39t + 33\ \mathrm{d}t$$

$$= -\frac{1}{5}\left[(5t+7)^3\right]_0^1 - 5\left[\frac{39}{2}t^2 + 33t\right]_0^1$$

$$= -\frac{1079}{2}$$

2.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y) = (x+y)\vec{i} + (1-x)\vec{j}$
and C is the portion of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that
is in the 4th quadrant with the counter clockwise rotation.

$$x = 2\cos(t) \qquad y = 3\sin(t) \qquad \frac{3\pi}{2} \le t \le 2\pi$$

$$\vec{r}(t) = \langle 2\cos(t), 3\sin(t)\rangle$$

$$\vec{r}'(t) = (-2\sin(t), 3\cos(t))$$

$$\vec{F}(\vec{r}(t)) = (2\cos(t) + 3\sin(t))\vec{i} + (1 - 2\cos(t))\vec{j}$$

$$(2\cos(t) + 3\sin(t))(-2\sin(t)) + (1 - 2\cos(t))(3\cos(t))$$

$$= -2\sin(2t) + 3\cos(t) - 6$$

$$\int_{\frac{3\pi}{2}}^{2\pi} -2\sin(2t) + 3\cos(t) - 6 \ \mathrm{d}t$$

$$[\cos(2t) + 3\sin(t) - 6t]_{\frac{3\pi}{2}}^{2\pi}$$

$$= 5 - 3\pi$$

3.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y) = y^2\vec{i} + (x^2-4)\vec{j}$
and C is the portion of $y = (x-1)^2$ from
$x = 0$ to $x = 3$.

$$x = t \qquad y = (t-1)^2 \qquad 0 \le t \le 3$$

$$\vec{r}(t) = \langle t, (t-1)^2\rangle$$

$$\vec{r}'(t) = \langle 1, 2(t-1)\rangle$$

$$\vec{F}(\vec{r}(t)) = (t-1)^4\vec{i} + (t^2-4)\vec{j}$$

$$= (t-1)^4 + 2t^3 - 2t^2 - 8t + 8$$

$$\int_0^3 (t-1)^4 + 2t^3 - 2t^2 - 8t + 8 \ \mathrm{d}t$$

$$\left[\frac{1}{5}(t-1)^5 + \frac{1}{2}t^4 - \frac{2}{3}t^3 - 4t^2 + 8t\right]_0^3$$

$$= \frac{171}{10}$$

13. Find the volume of the region bounded by the surfaces $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 = \frac{1}{4}$.
14. Find the volume of the region enclosed by the cones $z = \sqrt{x^2 + y^2}$ and $z = 1 - 2\sqrt{x^2 + y^2}$.
15. Find the volume inside the ellipsoid $x^2 + y^2 + 4z^2 = 6$.
16. Find the volume of the intersection of the ellipsoid $x^2 + 2(y^2 + z^2) \leqslant 10$ and the cylinder $y^2 + z^2 \leqslant 1$.
17. Find the *normalizing constant* c, depending on σ, such that $\int_{-\infty}^{\infty} ce^{-x^2/\sigma}\,dx = 1$.

13.

$$r^2 = \frac{1}{4} \qquad r = \pm\frac{1}{2}$$

$$z = \pm\sqrt{1-r^2}$$

$$\int_0^{2\pi}\int_0^{\frac{1}{2}}\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r \ \mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{2}} 2r\sqrt{1-r^2} \ \mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} -\frac{2}{3}\left[(1-r^2)^{\frac{3}{2}}\right]_0^{\frac{1}{2}} \ \mathrm{d}\theta$$

$$= \left[-\frac{\sqrt{3}}{4}\theta + \frac{2}{3}\theta\right]_0^{2\pi}$$

$$= -\frac{\sqrt{3}\pi}{2} + \frac{4\pi}{3}$$

14.

$$z = r \qquad z = 1 - 2r \qquad r = \frac{1}{3}$$

$$\int_0^{2\pi} \int_0^{\frac{1}{3}} \int_r^{1-2r} r \;\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{1}{3}} r\,[z]_r^{1-2r}\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{1}{3}} r - 3r^2\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}r^2 - r^3\right]_0^{\frac{1}{3}} \mathrm{d}\theta$$

$$= \left[\frac{1}{54}\theta\right]_0^{2\pi}$$

$$= \frac{\pi}{27}$$

15.

$$V = \frac{4}{3}abc$$

$$x = \sqrt{6} \qquad y = \sqrt{6} \qquad z = \frac{\sqrt{6}}{2}$$

$$V = \frac{4}{3}\sqrt{6}\sqrt{6}\frac{\sqrt{6}}{2} = 4\pi\sqrt{6}$$

16.

$$x \leq \pm\sqrt{10-2r^2} \qquad -1 \leq r \leq 1$$

$$\int_0^{2\pi}\int_0^1\int_{-\sqrt{10-r^2}}^{\sqrt{10-r^2}} r \, \mathrm{d}x \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_0^{2\pi}\int_0^1 r\,[x]_{-\sqrt{10-r^2}}^{\sqrt{10-r^2}} \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_0^{2\pi}\int_0^1 2r\sqrt{10-r^2} \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_0^{2\pi} -\frac{2}{3}\left[(10-r^2)^{\frac{3}{2}}\right]_0^1 \mathrm{d}\theta$$

$$= \left[-18\theta + \frac{20\sqrt{10}}{3}\theta\right]_0^{2\pi}$$

$$= -18\pi + \frac{40\sqrt{10}\pi}{3}$$

1. (a) Find the mass of the box $[0, \frac{1}{2}] \times [0, 1] \times [0, 2]$, assuming the density to be uniform. (b) Same Exercise as part (a), but with a mass density $\rho(x, y, z) = x^2 + 3y^2 + z + 1$.
2. Find the mass of the solid bounded by the cylinder $x^2 + z^2 = 2x$ and the cone $z^2 = x^2 + y^2$ if the density is $\rho = \sqrt{x^2 + y^2}$.

Find the center of mass of the solids in Exercises 3 and 4, assuming them to have constant density.

3. S bounded by $x + y + z = 2$, $x = 0$, $y = 0$, $z = 0$.
4. S bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 6$, $z = 0$.
5. Evaluate the integral in Example 2 by considering the hemisphere as a region of type I.
6. Find the center of mass of the cylinder $x^2 + y^2 \leqslant 1$, $1 \leqslant z \leqslant 2$ if the density is $\rho = (x^2 + y^2)z^2$.
7. Redo Example 3 for the cube

$$W = [-c, c] \times [-c, c] \times [-c, c].$$

[*Hint*: Guess the answer to part (b) first.]

8. Find the average value of $x^2 + y^2$ over the conical region $0 \leqslant z \leqslant 2$, $x^2 + y^2 \leqslant z^2$.
9. Find the average value of $\sin^2 \pi z \cos^2 \pi x$ over the cube $[0, 2] \times [0, 4] \times [0, 6]$.
10. Find the average value of e^{-z} over the ball $x^2 + y^2 + z^2 \leqslant 1$.

1.

a)

$$V = \rho \cdot \frac{1}{2} \cdot 2 \cdot 1 = \rho$$

b)

$$\rho(x, y, z) = x^2 + 3y^2 + z + 1$$

$$M = \int_0^{\frac{1}{2}} \int_0^1 \int_0^2 x^2 + 3y^2 + z + 1 \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$$

a)

Example 2 Give parametric representations for each of the following surfaces. The elliptic paraboloid x = 5 y 2 + 2 z 2 − 10 . The elliptic paraboloid x = 5 y 2 + 2 z 2 − 10 that is in front of the y z -plane. The sphere x 2 + y 2 + z 2 = 30 . The cylinder y 2 + z 2 = 25 . Show All Solutions Hide All Solutions a The elliptic paraboloid x = 5 y 2 + 2 z 2 − 10 . Show Solution b The elliptic paraboloid x = 5 y 2 + 2 z 2 − 10 that is in front of the y z -plane. Show Solution c The sphere x 2 + y 2 + z 2 = 30 . Show Solution d The cylinder y 2 + z 2 = 25 . Show Solution