Find the derivative matrices in Exercises 5–8 and evaluate at the given points.

5. $\partial(x, y)/\partial(u, v)$; $x = u \sin v$, $y = e^{uv}$; at $(0, 1)$.

$$x = u \sin(v) \qquad y = e^{uv}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \sin(v) & u\cos(v) \\ ve^{uv} & ue^{uv} \end{pmatrix} = \begin{pmatrix} \sin 1 & 0 \\ 1 & 0 \end{pmatrix}$$

6. $\partial(x, y, z)/\partial(r, \theta, \phi)$; where $x = r \sin\phi \cos\theta$, $y = r \sin\phi \sin\theta$, $z = r\cos\phi$; at $(2, \pi/3, \pi/4)$.

$$x = r \sin\phi \cos\theta \qquad y = r \sin\phi \sin\theta \qquad x = r\cos\phi$$

$$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \begin{pmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\sqrt{2} \end{pmatrix}$$

Compute $\partial z/\partial x$ and $\partial z/\partial y$ in Exercises 21–24 using matrix multiplication and by direct substitution.

21. $z = u^2 + v^2$; $u = 2x + 7$, $v = 3x + y + 7$.
22. $z = u^2 + 3uv - v^2$; $u = \sin x$, $v = -\cos x + \cos y$.
23. $z = \sin u \cos v$; $u = 3x^2 - 2y$, $v = x - 3y$.
24. $z = u/v^2$; $u = x + y$, $v = xy$.

21.

$$z = u^2 + v^2 \qquad u = 2x + 7 \qquad v = 3x + y + 7$$

$$\begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2u & 2v \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4u + 6v & 0u + 2v \end{pmatrix} = \begin{pmatrix} 26x + 6y + 70 & 6x + 2y + 14 \end{pmatrix}$$

22.

$$z = u^2 + 3uv - v^2 \qquad u = \sin x \qquad v = -cosx + \cos y$$

$$\begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2u + 3v & 3u - 2v \end{pmatrix} \begin{pmatrix} \cos x & 0 \\ \cos x & -\sin y \end{pmatrix}$$
$$= \begin{pmatrix} (2u + 3v)\cos x + \cos x(3u - 2v) & 0 - \sin y(3u - 2v) \end{pmatrix}$$
$$= \begin{pmatrix} 5\sin x \cos x + 5\cos^2 x + \cos y \cos x & -3\sin^2 y \end{pmatrix}$$

25. (a) Compute derivative matrices $\partial(x, y)/\partial(t, s)$ and $\partial(u, v)/\partial(x, y)$ if

$$x = t + s, \qquad y = t - s,$$
$$u = x^2 + y^2, \qquad v = x^2 - y^2.$$

(b) Express (u, v) in terms of (t, s) and calculate $\partial(u, v)/\partial(t, s)$.

(c) Verify that the chain rule holds.

25.

(a)

$$x = t + s \qquad y = t - s \qquad u = x^2 + y^2 \qquad v = x^2 - y^2$$

$$\frac{\partial(x, y)}{\partial(t, s)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix}$$

(b)

$$u = (t + s)^2 + (t - s)^2 \qquad v = (t + s)^2 - (t - s)^2$$
$$u = 2t^2 + 2s^2 \qquad v = 4ts$$
$$\begin{pmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial s} \end{pmatrix} = \begin{pmatrix} 4t & 4s \\ 4s & 4t \end{pmatrix}$$

(c)

$$\frac{\partial(u,v)}{\partial(t,s)} = \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(t,s)} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}2t+2s & 2t-2s\\ 2t+2s & -2t+2s\end{pmatrix}$$
$$= \begin{pmatrix}4t & 4s\\ 4s & 4t\end{pmatrix}$$

26.

(a)

$$x = t^2 + s^2 \qquad y = ts \qquad u = \sin(x+y) \qquad v = \cos(x-y)$$

$$\frac{\partial(x,y)}{\partial(t,s)} = \begin{pmatrix}2t & -2s\\ s & t\end{pmatrix}$$
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix}\cos(x+y) & \cos(x+y)\\ -\sin(x-y) & \sin(x-y)\end{pmatrix}$$

(b)

$$u = \sin(t^2 - s^2 + ts) \qquad v = \cos(t^2 - s^2 - ts)$$

$$\begin{pmatrix}\frac{\partial u}{\partial t} & \frac{\partial u}{\partial s}\\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial s}\end{pmatrix} = \begin{pmatrix}(2t+s)\cos(t^2-s^2+ts) & -(2s+t)\cos(t^2-s^2+ts)\\ -(2t-s)\sin(t^2-s^2-ts) & -(2s-t)\sin(t^2-s^2-ts)\end{pmatrix}$$

29. Suppose that a function is given in terms of rectangular coordinates by $u = f(x, y, z)$. If

$$x = r\cos\theta\sin\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\phi,$$

express $\partial u/\partial r$, $\partial u/\partial\theta$, and $\partial u/\partial\phi$ in terms of $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$.

29.

$$x = r\cos\theta\sin\phi \qquad y = r\sin\theta\sin\phi \qquad u = r\cos\phi$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \phi}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta\sin\phi + \frac{\partial u}{\partial y}\sin\theta\sin\phi + \frac{\partial u}{\partial z}\cos\phi$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta\sin\phi) + \frac{\partial u}{\partial y}(r\cos\theta\sin\phi) + \frac{\partial u}{\partial z}\cdot 0$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}(r\cos\theta\cos\phi) + \frac{\partial u}{\partial y}(r\sin\theta\cos\phi) + \frac{\partial u}{\partial z}(-r\sin\phi)$$

30.

$$x = r\cos\theta\sin\phi \qquad y = r\sin\theta\sin\phi \qquad u = r\cos\phi$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \phi}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta\sin\phi + \frac{\partial u}{\partial y}\sin\theta\sin\phi + \frac{\partial u}{\partial z}\cos\phi$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta\sin\phi) + \frac{\partial u}{\partial y}(r\cos\theta\sin\phi) + \frac{\partial u}{\partial z}\cdot 0$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}(r\cos\theta\cos\phi) + \frac{\partial u}{\partial y}(r\sin\theta\cos\phi) + \frac{\partial u}{\partial z}(-r\sin\phi)$$