

9. (1 valor) Considere, no espaço euclidiano complexo  $\mathbb{C}^3$  munido do produto escalar usual, o operador  $S(x, y, z) = (x + iy - z, 2iy + 3z, iz)$ . Determine o operador  $S^*$  e a composição  $S^*S$ .

$$S = \begin{bmatrix} 1 & i & -1 \\ 0 & 2i & 3 \\ 0 & 0 & i \end{bmatrix}$$

$$\bar{S} = \begin{bmatrix} 1 & -i & -1 \\ 0 & -2i & 3 \\ 0 & 0 & -i \end{bmatrix}$$

$$S^* = \begin{bmatrix} 1 & 0 & 0 \\ -i & -2i & 0 \\ -1 & 3 & -i \end{bmatrix}$$

$$S^*S = \begin{bmatrix} 1 & i & -1 \\ -i & 5 & -5i \\ -1 & 5i & 11 \end{bmatrix}$$

10. (1 valor) Considere, no espaço euclidiano complexo  $\mathbb{C}^2$  munido do produto escalar usual, o operador  $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  definido por  $T(x, y) = (2x - iy, ix + y)$ . Determine uns operadores auto-adjuntos  $X$  e  $Y$  tais que  $T = X + iY$ .

$$T = \begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix}$$

$$\bar{T} = \begin{bmatrix} 2 & i \\ -i & 1 \end{bmatrix}$$

$$T^* = \begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix}$$

$$X = \frac{T + T^*}{2}$$

$$Y = \frac{T - T^*}{2i}$$

$$X = \frac{1}{2}(T + T^*) = \frac{1}{2}\left(\begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix} + \begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix}\right) = \frac{1}{2}\left(\begin{bmatrix} 4 & -2i \\ 2i & 2 \end{bmatrix}\right) = \begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix}$$

$$Y = \frac{1}{2i}(T - T^*) = \frac{1}{2i}\left(\begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix} - \begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix}\right) = \frac{1}{2i}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

12. (1 valor) Calcule valores e vetores próprios da matriz hermitica

$$C = \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix}$$

Valores Próprios:

$$\lambda = 2 \quad \lambda = -2$$

Vetores Próprios:

$$\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

13. (1 valor) Considere a matriz  $C$  definida no exercício 12. Determine uma matriz unitária  $U$  e uma matriz diagonal  $\Lambda$  tais que  $C = U\Lambda U^{-1}$ .

Valores Próprios:

$$\lambda = 2 \quad \lambda = -2$$

$$\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

Vetores Próprios:

$$\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U^{-1} = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

10. (1 valor) Determine a matriz que define, relativamente à base canónica, um operador ortogonal  $R: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  tal que  $R(1, 0) = (0, -1)$ .

Base canónica  $\mathbb{R}^2$ :

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$R = \begin{bmatrix} 0 & b \\ -1 & d \end{bmatrix}$$

As colunas devem formar uma base ortogonal:

$$R(1, 0) = (0, -1) \implies \sqrt{0^2 + (-1)^2} = 1$$

Precisamos de encontrar  $R(0, 1)$ :

$$(0, -1) \cdot (b, d) = 0 \Leftrightarrow d = 0$$

$(b, d)$  tem de ter norma 1:

$$\sqrt{b^2 + 0} = 1 \Leftrightarrow b = \pm 1$$

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Diagonalização de matriz

$$A = PDP^{-1}$$

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \cdot \left( \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1}$$

5. (2 valores) Seja  $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  a projeção ortogonal sobre a reta  $y = -2x$  do plano euclidiano. Determine a matriz que representa  $P$  na base canônica.

$$u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\text{proj}_u v = \frac{v \cdot u}{u \cdot u} u$$

$$\frac{u}{||u||} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$v \cdot u = x \cdot \frac{1}{\sqrt{5}} + y \cdot -\frac{2}{\sqrt{5}} = \frac{x - 2y}{\sqrt{5}}$$

$$\text{proj}_u v = \frac{x-2y}{\sqrt{5}} \cdot \left( \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right) = \left( \frac{x-2y}{5}, \frac{-2x+4y}{5} \right)$$

$$P = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

6. (2 valores) Seja  $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  a reflexão na reta  $y = \sqrt{3}x$ . Determine a matriz que representa  $R$  na base canônica.

$$u = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

$$\text{proj}_u v = \frac{v \cdot u}{u \cdot u} u$$

$$R(v) = 2 \cdot \text{proj}_u v - v$$

$$\frac{u}{\|u\|} = \frac{1}{2} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$v \cdot u = x \cdot \frac{1}{2} + y \cdot \frac{\sqrt{3}}{2} = \frac{x + \sqrt{3}y}{2}$$

$$\text{proj}_u v = \frac{x + \sqrt{3}y}{2} \cdot \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) = \left( \frac{x + \sqrt{3}y}{4}, \frac{\sqrt{3}x + 3y}{4} \right)$$

$$R(v) = 2 \cdot \left( \frac{x + \sqrt{3}y}{4}, \frac{\sqrt{3}x + 3y}{4} \right) - (x, y) \Leftrightarrow \left( \frac{x + \sqrt{3}y}{2}, \frac{\sqrt{3}x + 3y}{2} \right) - (x, y)$$

$$R(v) = \begin{bmatrix} \frac{-x + \sqrt{3}y}{2} \\ \frac{\sqrt{3}x + y}{2} \end{bmatrix}$$

$$R(v) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

5. (2 valores) Seja  $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  a reflexão na reta  $y = \sqrt{3}x$ . Determine valores e vetores próprios de  $R$ .

$$R = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\det(R - \lambda I)$$

$$\lambda = -1 \quad \lambda = 1$$

$$\begin{vmatrix} -\frac{1}{2} - \lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} - \lambda \end{vmatrix}$$

Se  $\lambda = -1$ :

$$\begin{vmatrix} -\frac{1}{2} + 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} + 1 \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{2} \end{vmatrix} = 0$$

$$= \begin{vmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{vmatrix}$$

Se  $\lambda = 1$ :

$$\begin{vmatrix} -\frac{1}{2} - 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} - 1 \end{vmatrix} = \begin{vmatrix} -\frac{3}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{vmatrix} = 0$$

$$= \begin{vmatrix} 1 & -\frac{\sqrt{3}}{3} \\ 0 & 0 \end{vmatrix}$$

7. (1 valor) Determine o ponto do plano  $P = \{x - 2y + z = 0\} \subset \mathbb{R}^3$  mais próximo do ponto  $\mathbf{v} = (1, 0, 2)$ .

$$\vec{n} = (1, -2, 1)$$

$$r(t) = (1, 0, 2) + t(1, -2, 1)$$

$$x = 1 + t \quad y = -2t \quad z = 2 + t$$

Substituir na equação:

$$1 + t + 4t + 2 + t = 0 \Leftrightarrow t = -\frac{1}{2}$$

$$x = \frac{1}{2} \quad y = 1 \quad z = \frac{3}{2}$$

6. (1 valor) Determine uma base ortonormada do plano  $P = \{2x - y - z = 0\} \subset \mathbb{R}^3$ .

Encontrar dois vetores perpendiculares:

Se  $x = 1$  e  $y = 2$

$$2 - 2 - z = 0 \Leftrightarrow z = 0$$

$$u_1 = v_1 = (1, 2, 0)$$

Se  $x = 0$  e  $y = 1$

$$0 - 1 - z = 0 \Leftrightarrow z = -1$$

$$v_2 = (0, 1, -1)$$

$$u_2 = v_2 - \text{proj}_{u_1} v_2 = \frac{u_1 \cdot v_2}{\|u_1\|^2} u_1 = (0, 1, -1) - \left(\frac{2}{5}, \frac{4}{5}, 0\right) = \left(-\frac{2}{5}, \frac{1}{5}, -1\right)$$

$$\|u_2\| = \sqrt{\frac{4}{25} + \frac{1}{25} + \frac{25}{25}} = \frac{\sqrt{30}}{5}$$

$$\|u_1\| = \sqrt{5}$$

$$\frac{u_1}{\|u_1\|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$\frac{u_2}{\|u_2\|} = \left(-\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{5}{\sqrt{30}}\right)$$

$$\left\{ \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right), \left(-\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{5}{\sqrt{30}}\right) \right\}$$

7. (1 valor) Calcule a projeção ortogonal do vetor  $\mathbf{v} = (3, 2, 1)$  sobre o plano  $P$  definido no exercício 6.

Encontrar dois vetores perpendiculares:

$$\vec{n} = (2, -1, -1)$$

$$\vec{v} = (3, 2, 1)$$

$$v_p = v - \text{proj}_n v = \frac{n \cdot v}{\|n\|^2} n = (3, 2, 1) - \frac{1}{2}(2, -1, -1) = \left(2, \frac{3}{2}, \frac{1}{2}\right)$$