Find the derivative matrices in Exercises 5–8 and evaluate at the given points.

5. $\partial(x, y)/\partial(u, v)$; $x = u \sin v$, $y = e^{uv}$; at $(0, 1)$.

$$x = u \sin(v) \qquad y = e^{uv}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{pmatrix} \sin(v) & u\cos(v) \\ ve^{uv} & ue^{uv} \end{pmatrix} = \begin{pmatrix} \sin 1 & 0 \\ 1 & 0 \end{pmatrix}$$

6. $\partial(x, y, z)/\partial(r, \theta, \phi)$; where $x = r \sin \phi \cos \theta$, $y = r \sin \phi \sin \theta$, $z = r \cos \phi$; at $(2, \pi/3, \pi/4)$.

$$x = r \sin \phi \cos \theta \qquad y = r \sin \phi \sin \theta \qquad x = r \cos \phi$$

$$\frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} = \begin{pmatrix} \sin \phi \cos \theta & -r \sin \phi \sin \theta & r \cos \phi \cos \theta \\ \sin \phi \sin \theta & r \sin \phi \cos \theta & r \cos \phi \sin \theta \\ \cos \phi & 0 & -r \sin \phi \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\sqrt{2} \end{pmatrix}$$

Compute $\partial z/\partial x$ and $\partial z/\partial y$ in Exercises 21–24 using matrix multiplication and by direct substitution.

21. $z = u^2 + v^2$; $u = 2x + 7$, $v = 3x + y + 7$.
22. $z = u^2 + 3uv - v^2$; $u = \sin x$,
    $v = -\cos x + \cos y$.
23. $z = \sin u \cos v$; $u = 3x^2 - 2y$, $v = x - 3y$.
24. $z = u/v^2$; $u = x + y$, $v = xy$.

21.

$$z = u^2 + v^2 \qquad u = 2x + 7 \qquad v = 3x + y + 7$$

$$\begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2u & 2v \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4u + 6v & 0u + 2v \end{pmatrix} = \begin{pmatrix} 26x + 6y + 70 & 6x + 2y + 14 \end{pmatrix}$$

22.

$$z = u^2 + 3uv - v^2 \qquad u = \sin x \qquad v = -\cos x + \cos y$$

$$\left(\tfrac{\partial z}{\partial x} \quad \tfrac{\partial z}{\partial y}\right) = \left(\tfrac{\partial z}{\partial u} \quad \tfrac{\partial z}{\partial v}\right) \begin{pmatrix} \tfrac{\partial u}{\partial x} & \tfrac{\partial u}{\partial y} \\ \tfrac{\partial v}{\partial x} & \tfrac{\partial v}{\partial y} \end{pmatrix} = \left(2u + 3v \quad 3u - 2v\right) \begin{pmatrix} \cos x & 0 \\ \cos x & -\sin y \end{pmatrix}$$

$$= \left((2u + 3v)\cos x + \cos x(3u - 2v) \quad 0 - \sin y(3u - 2v)\right)$$

$$= \left(5 \sin x \cos x + 5 \cos^2 x + \cos y \cos x \quad -3 \sin^2 y\right)$$

25. (a) Compute derivative matrices $\partial(x, y)/\partial(t, s)$
and $\partial(u, v)/\partial(x, y)$ if

$$x = t + s, \qquad y = t - s,$$
$$u = x^2 + y^2, \qquad v = x^2 - y^2.$$

(b) Express (u, v) in terms of (t, s) and calculate $\partial(u, v)/\partial(t, s)$.

(c) Verify that the chain rule holds.

26.

(a)

$$x = t + s \qquad y = t - s \qquad u = x^2 + y^2 \qquad v = x^2 - y^2$$

$$\frac{\partial(x, y)}{\partial(t, s)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix}$$

(b)

$$u = (t + s)^2 + (t - s)^2 \qquad v = (t + s)^2 - (t - s)^2$$
$$u = 2t^2 + 2s^2 \qquad v = 4ts$$
$$\begin{pmatrix} \tfrac{\partial u}{\partial t} & \tfrac{\partial u}{\partial s} \\ \tfrac{\partial v}{\partial t} & \tfrac{\partial v}{\partial s} \end{pmatrix} = \begin{pmatrix} 4t & 4s \\ 4s & 4t \end{pmatrix}$$

2

(c)

$$\frac{\partial(u, v)}{\partial(t, s)} = \frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(t, s)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2t + 2s & 2t - 2s \\ 2t + 2s & -2t + 2s \end{pmatrix}$$

$$= \begin{pmatrix} 4t & 4s \\ 4s & 4t \end{pmatrix}$$

27.

(a)

$$x = t^2 + s^2 \qquad y = ts \qquad u = \sin(x + y) \qquad v = \cos(x - y)$$

$$\frac{\partial(x, y)}{\partial(t, s)} = \begin{pmatrix} 2t & -2s \\ s & t \end{pmatrix}$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} \cos(x + y) & \cos(x + y) \\ -\sin(x - y) & \sin(x - y) \end{pmatrix}$$

(b)

$$u = \sin(t^2 - s^2 + ts) \qquad v = \cos(t^2 - s^2 - ts)$$

$$\begin{pmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial s} \end{pmatrix} = \begin{pmatrix} (2t + s)\cos(t^2 - s^2 + ts) & -(2s + t)\cos(t^2 - s^2 + ts) \\ -(2t - s)\sin(t^2 - s^2 - ts) & -(2s - t)\sin(t^2 - s^2 - ts) \end{pmatrix}$$