The line integral of $\vec{F}$ along C is
$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}' \ \mathrm{d}t$

$$\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t), z(t))$$

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \vec{T} \ \mathrm{d}s \qquad \vec{T}(t) = \frac{\vec{r}'}{||\vec{r}'||}$$

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \vec{T} \ \mathrm{d}s$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{||\vec{r}'(t)||} ||\vec{r}'(t)|| \ \mathrm{d}t$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \ \mathrm{d}t$$

Example 1

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x, y, z) = 8x^2yz\vec{i} + 5z\vec{j} - 4xy\vec{k}$ and C is the curve given by $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, $0 \leq t \leq 1$.

$$\vec{F}(\vec{r}(t)) = 8t^2(t^2)(t^3)\vec{i} + 5t^3\vec{j} - 4t(t^2)\vec{k} = 8t^7\vec{i} + 5t^3\vec{j} - 4t^3\vec{k}$$

$$\vec{r}'(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) = (8t^7 + 5t^3 - 4t^3) \cdot (1 + 2t + 3t^2) = 8t^7 + 10t^4 - 12t^5$$

$$= \int_0^1 8t^7 + 10t^4 - 12t^5 \ \mathrm{d}t = \left[t^8 + 2t^5 - 2t^6\right]_0^1 = 1$$

Example 2

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x, y, z) = xz\vec{i} - yz\vec{k}$ and C is the line segment from $(-1, 2, 0)$ to $(3, 0, 1)$.

$$\vec{r}(t) = (1 - t)\langle -1, 2, 0\rangle + t\langle 3, 0, 1\rangle = \langle 4t - 1, -2t + 2, t\rangle$$

$$\vec{r}'(t) = \langle 4, -2, 1\rangle$$

$$\vec{F}(\vec{r}(t)) = (4t - 1)(t)\vec{i} - (-2t + 2)(t)\vec{k} = (4t^2 - t)\vec{i} - (-2t^2 + 2t)\vec{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) = 4(4t^2 - t) - (-2t^2 + 2t) = 18t^2 - 6t$$

$$= \int_0^1 18t^2 - 6t \ \mathrm{d}t = \left[6t^3 - 3t^2\right]_0^1 = 3$$

Given the vector field $\vec{F}(x, y, z) = P\vec{i} + Q\vec{j} + R\vec{k}$ and the curve C parameterized by $\vec{r} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \leq t \leq b$

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_a^b \left(P\vec{i} + Q\vec{j} + R\vec{k}\right) \cdot \left(x'\vec{i} + y'\vec{j} + z'\vec{k}\right) \ \mathrm{d}t$$

$$= \int_a^b Px' + Qy' + Rz' \ \mathrm{d}t$$

$$= \int_a^b Px' \ \mathrm{d}t + \int_a^b Qy' \ \mathrm{d}t + \int_a^b Rz' \ \mathrm{d}t$$

$$= \int_C P \ \mathrm{d}x + \int_C Q \ \mathrm{d}y + \int_C R \ \mathrm{d}z$$

$$= \int_C P \ \mathrm{d}x + Q \ \mathrm{d}y + R \ \mathrm{d}z$$

$$\int_{-C} \vec{F} \cdot \mathrm{d}\vec{r} = -\int_C \vec{F} \cdot \mathrm{d}\vec{r}$$

1.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x, y) = y^2\vec{i} + (3x - 6y)\vec{j}$ and C is the line segment from $(3, 7)$ to $(0, 12)$.

$$\vec{r}(t) = (1 - t)\langle 3, 7\rangle + t\langle 0, 12\rangle = \langle 3 - 3t, 7 + 5t\rangle$$

$$\vec{r}'(t) = \langle -3, 5\rangle$$

$$\vec{F}(\vec{r}(t)) = (7 + 5t)^2\vec{i} + (3(3 - 3t) - 6(7 + 5t))\vec{j} = (7 + 5t)^2\vec{i} + (-33 - 39t)\vec{j}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) = -3(7 + 5t)^2 - 5(33 + 39t)$$

$$= \int_0^1 -3(7 + 5t)^2 - 5(33 + 39t) \ \mathrm{d}t = \left[-\frac{(7 + 5t)^3}{5} - 165t - \frac{195t^2}{2}\right]_0^1$$

$$= -\frac{12}{5} - 165 - \frac{195}{2} - \left(-\frac{1}{5}7^3\right) = -\frac{1079}{2}$$

2.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y) = (x+y)\vec{i} + (1-x)\vec{j}$ and C is the portion of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that is in the 4th quadrant with the counter clockwise rotation.

$$\vec{r}(t) = \langle 2\cos t, 3\sin t \rangle$$

$$\vec{r}'(t) = \langle -2\sin t, 3\cos t \rangle$$

$$\vec{F}(\vec{r}(t)) = (2\cos t + 3\sin t)\vec{i} + (1 - 2\cos t)\vec{j}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) = (2\cos t + 3\sin t)(-2\sin t) + (1 - 2\cos t)(3\cos t) = -2\sin 2t + 3\cos t - 6$$

$$= \int_{\frac{3\pi}{2}}^{2\pi} -2\sin 2t + 3\cos t - 6 \ \mathrm{d}t = [\cos 2t + \sin t - 6t]_{\frac{3\pi}{2}}^{2\pi} = 5 - 3\pi$$

3.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y) = y^2\vec{i} + (x^2 - 4)\vec{j}$ and C is the portion of $y = (x-1)^2$from $x = 0$ to $x = 3$.

$$\vec{r}(t) = \langle t, (t-1)^2 \rangle$$

$$\vec{r}'(t) = \langle 1, 2(t-1) \rangle$$

$$\vec{F}(\vec{r}(t)) = (t-1)^4\vec{i} + (t^2 - 4)\vec{j}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) = (t-1)^4 + 2(t-1)(t^2-4) = (t-1)^4 + 2t^3 - 2t^2 - 8t + 8$$

$$= \int_0^3 (t-1)^4 + 2(t-1)(t^2-4) \ \mathrm{d}t = \left[\frac{1}{5}(t-1)^5 + \frac{1}{2}t^4 - \frac{2}{3}t^3 - 4t^2 + 8t\right]_0^3 = \frac{171}{10}$$

4.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ **where** $\vec{F}(x,y,z) = e^{2x}\vec{i} + z(y+1)\vec{j} + z^3\vec{k}$ **and** C **is given by** $\vec{r}(t) = t^3\vec{i} + (1-3t)\vec{j} + e^t\vec{k}$ **for** $0 \le t \le 2$ **.**

$$\vec{r}'(t) = 3t^2\vec{i} - 3\vec{j} + e^t\vec{k}$$

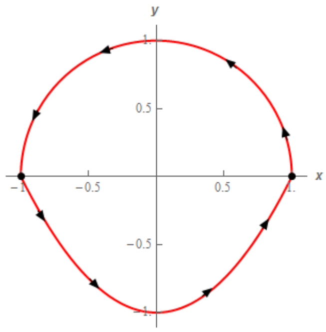
$$\vec{F}(\vec{r}(t)) = e^{2t^3}\vec{i} + e^t(2-3t)\vec{j} + e^{3t}\vec{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) = e^{2t^3}(3t^2) + e^t(2-3t)(-3) + e^{3t}(e^t) = 3t^2e^{2t^3} - 3e^t(2-3t) + e^{4t}$$

$$= \int_0^2 3t^2e^{2t^3} - 3e^t(2-3t) + e^{4t} \,\mathrm{d}t = \left[\frac{1}{2}e^{2t^3} - 3e^t(5-3t) + \frac{1}{4}e^{4t}\right]_0^2 = \frac{57}{4} + 3e^2 + \frac{1}{4}e^8 + \frac{1}{2}e^{16}$$

5.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ **where** $\vec{F}(x,y) = 3y\vec{i} + (x^2-y)\vec{j}$ **and** C **is the upper half of the circle centered at the origin of radius** 1 **with counter clockwise rotation and the portion of** $y = x^2 - 1$ **from** $x = -1$ **to** $x = 1$**. See the example below.**



$$C_1 : \vec{r}(t) = \langle \cos t, \sin t \rangle \qquad 0 \le t \le \pi$$

$$C_1 : \vec{r}'(t) = \langle -\sin t, \cos t \rangle$$

$$C_1 : \vec{F}(\vec{r}(t)) = 3\sin t\vec{i} + (\cos^2 t - \sin t)\vec{j}$$

$$C_1 : \vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) = -3\sin^2 t + \cos^3 t - \cos t \sin t$$

$$= -\frac{3}{2}(1 - \cos 2t) + \cos t(1 - \sin^2 t) - \cos t \sin t$$

$$= -\frac{3}{2}(1-\cos 2t) + \cos t(1-\sin^2 t) - \frac{1}{2}\sin 2t$$

$$= \int_0^{\pi} -\frac{3}{2}(1-\cos 2t) + \cos t - \cos t \sin^2 t - \frac{1}{2}\sin 2t \ \mathrm{d}t$$

$$= \left[-\frac{3}{2}(t - \frac{1}{2}\sin 2t) + \sin t - \frac{1}{3}\sin^3 t + \frac{1}{4}\cos 2t\right]_0^{\pi} = -\frac{3\pi}{2}$$

$$\vec{r}'(t) = 3t^2\vec{i} - 3\vec{j} + e^t\vec{k}$$

$$\vec{F}(\vec{r}(t)) = e^{2t^3}\vec{i} + e^t(2-3t)\vec{j} + e^{3t}\vec{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) = e^{2t^3}(3t^2) + e^t(2-3t)(-3) + e^{3t}(e^t) = 3t^2e^{2t^3} - 3e^t(2-3t) + e^{4t}$$

$$= \int_0^2 3t^2e^{2t^3} - 3e^t(2-3t) + e^{4t} \ \mathrm{d}t = \left[\frac{1}{2}e^{2t^3} - 3e^t(5-3t) + \frac{1}{4}e^{4t}\right]_0^2 = \frac{57}{4} + 3e^2 + \frac{1}{4}e^8 + \frac{1}{2}e^{16}$$

$$C_2 : \vec{r}(t) = \langle t, t^2 - 1\rangle \qquad -1 \le t \le 1$$

$$C_2 : \vec{r}'(t) = \langle 1, 2t\rangle$$

$$C_2 : \vec{F}(\vec{r}(t)) = 3(t^2-1)\vec{i} + \vec{j}$$

$$C_2 : \vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) = 3(t^2-1) + 2t$$

$$= \int_{-1}^1 3(t^2-1) + 2t \ \mathrm{d}t = \left[t^3 - 3t + t^2\right]_{-1}^1 = -4$$

6.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y) = (x+y)\vec{i} + (1-x)\vec{j}$ and C is the portion of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that is in the 4th quadrant with the counter clockwise rotation.

$$\vec{r}(t) = \langle t, (t-1)^2\rangle$$

$$\vec{r}'(t) = \langle 1, 2(t-1)\rangle$$

$$\vec{F}(\vec{r}(t)) = (t-1)^4\vec{i} + (t^2-4)\vec{j}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) = (t-1)^4 + 2(t-1)(t^2-4) = (t-1)^4 + 2t^3 - 2t^2 - 8t + 8$$

$$= \int_0^3 (t-1)^4 + 2(t-1)(t^2-4)\ \mathrm{d}t = \left[\frac{1}{5}(t-1)^5 + \frac{1}{2}t^4 - \frac{2}{3}t^3 - 4t^2 + 8t\right]_0^3 = \frac{171}{10}$$

7.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y) = (6x-2y)\vec{i} + x^2\vec{j}$ for each of the following curves.

a)

C is the line segment from $(6,-3)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(6,3)$.

b)

C is the line segment from $(6,-3)$ to $(6,3)$.

8.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y) = 3\vec{i} + (xy-2x)\vec{j}$ for each of the following curves.

a)

C is the upper half of the circle centered at the origin of radius 4 with counter clockwise rotation.

b)

C is the upper half of the circle centered at the origin of radius 4 with clockwise rotation. Evaluate C
→ F d → r where → F (x , y) = (x + y) → i + (1 − x) → j and C is the portion of x 2 4 + y 2 9 = 1 that is in the 4th quadrant with the counter clockwise rotation. C
→ F d → r = − C
→ F d → r → r (t) = x (t) → i + y (t) → j + z (t) → k , a t b the line integral is,
C

→ F d → r = b a (P → i + Q → j + R → k) (x → i + y → j + z → k) d t = b a P x + Q y + R z d t = b a P x d t + b aGiven the vector field → F (x , y , z) = P → i + Q → j + R → k and the curve C parameterized by → r (t) = x (t) → i + y (t) → j + z (t) → k , a t b the line integral is,
C
→ F d → r = b a (P → i + Q → j + R → k) (x → i + y → j + z → k) d t = b a P x + Q y + R z d t = b a P x d t + b a Q y d t + b a R z d t =
C
P d x + C
Q d y + C
R d z = C
P d x + Q d y +
Q y d t + b a R z d t = C
P d x + C
Q d y + C
R d z = C
P d x + Q d y +

$$f(x) = \sqrt{1+\sqrt{x}}$$

$$f'(x) = (1+(x)^{\frac{1}{2}})^{\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}$$

$$F(x) = \int_0^{\sin x} \frac{t^2}{t^2-1}\, \mathrm{d}t$$

$$F'(x) = \frac{\sin^2 x}{\sin^2 x - 1} \cdot \frac{d}{\mathrm{d}x} \sin x$$

$$= -\frac{\sin^2 x}{1-\sin^2 x} \cdot \cos x$$

$$= -\frac{\sin^2 x}{\cos^{\cancel{2}} x} \cdot \cancel{\cos x}$$

$$= -\frac{\sin^2 x}{\cos x}$$

$$y - y_0 = m(x - x_0)$$

$$m = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{3t^2}{2t} = \frac{3t}{2}$$

$$m(t=5)=\frac{15}{2}$$

$$x(5)=25 \qquad y(5)=125$$

$$y-125=\frac{15}{2}(x-25)=y=125+\frac{15}{2}(x-25)$$

4. *(2 valores)* Calcule uma (apenas uma) das seguintes primitivas

$$\int \frac{x^3}{1+x^8}\,dx \qquad\qquad \int (x+2)\,e^x\,dx$$

$$\int=\frac{x^3}{1+x^8}\ \mathrm{d}x=\frac{1}{4}\int=\frac{4x^3}{1+(x^4)^2}\ \mathrm{d}x=\frac{1}{4}\arctan x^4$$

$$\int=(x+2)e^x\ \mathrm{d}x=\int xe^x+2e^x\ \mathrm{d}x$$

$$=\int xe^x\ \mathrm{d}x+\int 2e^x\ \mathrm{d}x$$

$$\int xe^x\ \mathrm{d}x=xe^x-e^x$$

$$=\int xe^x\ \mathrm{d}x+\int 2e^x\ \mathrm{d}x=(x+1)e^x$$

5. *(2 valores)* Calcule um (apenas um) dos seguintes integrais

$$\int_{-\pi/2}^{\pi/2}\cos^2\theta\,\sin\theta\,d\theta \qquad\qquad \int_1^3 \log(x^3)\,dx$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\sin\theta\ \mathrm{d}\theta=\frac{1}{3}\left[-\cos^3\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int=\frac{4x^3}{1+(x^4)^2}\ \mathrm{d}x=0$$

$$\int_1^3\ln x^3\ \mathrm{d}x=3\int_1^3\ln x\ \mathrm{d}x$$

$$f=\ln x \qquad g'=1$$

$$f'=\frac{1}{x} \qquad g=x$$

$$3\int_1^3 \ln x \; \mathrm{d}x = [x\ln x]_1^3 - \int_1^3 1 = 3x\ln x - 3x = 9\ln 3 - 6$$

6. *(2 valores)* Determine a solução da equação diferencial $\frac{dx}{dt} = 2 - x$ com condição inicial $x(0) = 1$.

$$\dot{y} = 2 - x \qquad x(0) = 1$$

$$Y_c = c_1 e^{-t}$$

$$2 - x = \Leftrightarrow x = 2$$

$$c_1 e^0 + 2 = 3 \Leftrightarrow c_1 = 1$$

$$Y_c = e^{-t} + 2$$

8. *(2 valores)* Calcule o limite
$$\lim_{x\to\pi} \frac{1+\cos x}{x-\pi}$$

Método 1:

$$f(x) = \cos x$$

$$f'(\pi) = \frac{f(x) - f(\pi)}{x - \pi}$$

$$f'(\pi) = -\sin\pi = 0$$

Método 2:

$$\lim_{x\to\pi} = \frac{1+\cos x}{x-\pi} = \frac{\frac{\mathrm{d}}{x}(1+\cos x)}{\frac{\mathrm{d}}{x}(x-\pi)} = 0$$

$\ddot{y} + 9y = \sin(\pi t)$

$$y_c(t) = c_1\cos(3t) + c_2\sin(3t)$$

$$z = b\sin(\pi t)$$

$$z' = \pi b\cos(\pi t)$$

$$z'' = -\pi^2 b \sin(\pi t)$$

$$-\pi^2 b \sin(\pi t) + 9bt \sin(\pi t) = \sin(\pi t)$$

$$b\cancel{\sin(\pi t)}(-\pi^2 + 9) = \cancel{\sin(\pi t)}$$

$$b = \frac{1}{-\pi^2 + 9}$$

$$y_c(t) + y_p(t) = c_1 \cos(3t) + c_2 \sin(3t) + \frac{1}{-\pi^2 + 9} \sin(\pi t)$$

$\ddot{y} + 9y = \sin(3t)$
Método 1:

$$y_c(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

$$z = bt \cos(3t)$$

$$z' = -3bt \sin(3t) + b \cos(3t)$$

$$z'' = -6b \sin(3t) - 9bt \cos(3t)$$

$$-6b \sin(3t) - 9bt \cos(3t) + 9bt \cos(3t) = \sin(3t)$$

$$-6b \sin(3t) = \sin(3t)$$

$$b = -\frac{1}{6}$$

$$y_c(t) + y_p(t) = c_1 \cos(3t) + c_2 \sin(3t) - \frac{1}{6} t \cos(3t)$$

Método 2:

$$y_c(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

$$y_1(t) = \cos(3t) \qquad y_2(t) = \sin(3t)$$

$$W = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3$$

$$Y_p(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)}\ \mathrm{d}t + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)}\ \mathrm{d}t$$

$$Y_p(t) = -\frac{\cos{(3t)}}{3} \int \sin{(3t)} \sin{(3t)}\ \mathrm{d}t + \frac{\sin{(3t)}}{3} \int \cos{(3t)} \sin{(3t)}\ \mathrm{d}t$$

$$sin^2(3t) = \frac{1 - \cos{(6t)}}{2}$$

$$Y_p(t) = -\frac{\cos{(3t)}}{3} \int \sin^2(3t)\ \mathrm{d}t + \frac{\sin{(3t)}}{3} \cdot -\frac{\sin^2{(3t)}}{6}$$

$$Y_p(t) = -\frac{\cos{(3t)}}{3} \int \frac{1}{2} - \frac{1}{2}\cos(6t)\ \mathrm{d}t + \frac{\sin{(3t)}}{3}(-\frac{1}{12} + \frac{1}{12}\cos(6t)))$$