Problem 1.11. Find the gradients of the following functions:

(a) $f(x, y, z) = x^2 + y^3 + z^4$.

(b) $f(x, y, z) = x^2 y^3 z^4$.

(c) $f(x, y, z) = e^x \sin(y) \ln(z)$.

Problem 1.12. The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

where y is the distance (in miles) north, x the distance east, of South Hadley.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point 1 mile north and 1 mile east of South Hadley? In what direction is the slope steepest, at that point?

Problem 1.13. Let $\boldsymbol{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

(a) $\nabla(\mathscr{r}^2) = 2\boldsymbol{\mathscr{r}}$;

(b) $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.

(c) What is the *general* formula for $\nabla(\mathscr{r}^n)$?

Problem 1.11

(a)

$$\nabla f = \left(2x, 3y^2, 4z^3\right)$$

(b)

$$\nabla f = \left(2xy^3z^4, 3y^2x^2z^4, 4z^3x^2y^3\right)$$

(c)

$$\nabla f = \left(e^x \sin{(y)} \ln{(z)}, \cos{(y)} e^x \ln{(z)}, \frac{1}{z} e^x \sin{(y)}\right)$$

Problem 1.12

(a)

$$\nabla h = (10(2y - 6x - 18), 10(2x - 8y + 28))$$

$$\nabla h = \vec{0}$$

$$h(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

$$H(-2,3) \begin{pmatrix} -60 & 20 \\ 20 & -80 \end{pmatrix}$$

local maximum

$$D > 0, h_{xx} < 0$$

local minimum

$$D > 0, h_{xx} > 0$$

saddle point

$$D < 0$$

inconclusive

$$D = 0$$

This point is a maximum

$$x = -2 \qquad y = 3$$

(b)

$$h(-2,3) = 720ft$$

(c)

$$\nabla h(1,1) = (-220, 220)$$

$$\vec{v} = \frac{1}{||\nabla h||} \cdot \nabla h = \frac{1}{\sqrt{(-220)^2 + (220)^2}} \cdot (-220, 220) = \frac{1}{220\sqrt{2}} \cdot (-220, 220) = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

Problem 1.13

(a)

$$\vec{r} = (x, y, z) - (x', y', z') = (x - x', y - y', z - z')$$

$$= (x - x')\mathbf{\hat{x}} + (y - y')\mathbf{\hat{y}} + (z - z')\mathbf{\hat{z}}$$

$$|\vec{r}| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\nabla(r^2) = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\left[(x-x') + (y-y') + (z-z')\right]$$

$$= \hat{\mathbf{x}}\left[2(x-x')\right] + \hat{\mathbf{y}}\left[2(y-y')\right] + \hat{\mathbf{z}}\left[2(z-z')\right]$$

$$= 2\left[(x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}\right]$$

$$= 2\vec{r}$$

(b)

$$\nabla(\frac{1}{r}) = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\left[\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\right]$$