$\int_\gamma \frac{f(z)}{z-a}\ \mathrm{d}z = 2\pi i \cdot f(a)$

$\int_\gamma \frac{f(z)}{z-a}^{n+1}\ \mathrm{d}z = \frac{2\pi i}{n!} f^n(a)$

$f^n(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{z-a}^{n+1}\ \mathrm{d}z$

$Res_{z=a} f(z) = \lim_{z\to a}(z-a)f(z)$

$Res_{z=a} f(z) = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right]$

$Res_{z=a}\left(\frac{g(z)}{h(z)}\right) = \frac{g(a)}{h'(a)}$

$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$

$Res_{z=a} f(z) = c_{-1}$

$\gamma \to |z| = 3$

$\int_\gamma \frac{1}{z^2-3z-4}$

Method 1:

$$\begin{aligned}\int_\gamma \frac{1}{z^2-3z-4} &= \int_\gamma \frac{1}{(z-4)\,(z+1)}\ \mathrm{d}z\\ &= \int_\gamma \frac{\frac{1}{(z-4)}}{z+1}\ \mathrm{d}z\end{aligned}$$

$\frac{1}{z-4}$ is holomorphic.
$\frac{1}{z+1}$:

$$f\,(-1)\cdot 2\pi i = -\frac{2}{5}\pi i$$

Method 2:

$$\begin{aligned}\int_\gamma \frac{1}{z^2-3z-4} &= \int_\gamma \frac{1}{(z-4)\,(z+1)}\ \mathrm{d}z\\ &= \frac{1}{5}\int_\gamma \left[\frac{1}{z-4} - \frac{1}{z+1}\right]\ \mathrm{d}z\end{aligned}$$

$$f(-1) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z+1}\ \mathrm{d}z \Leftrightarrow 1 = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z+1}\ \mathrm{d}z \Leftrightarrow 1 = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z+1}\ \mathrm{d}z$$

$$f(-1) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z+1}\ \mathrm{d}z \Leftrightarrow 1 = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z+1}\ \mathrm{d}z$$
$$\int_\gamma \frac{f(z)}{z+1}\ \mathrm{d}z = 2\pi i$$
$$= \frac{1}{5}\int_\gamma \left[\frac{1}{z-4} - \frac{1}{z+1}\right]\ \mathrm{d}z$$
$$= \frac{1}{5}\left(0 - 2\pi i\right) = -\frac{2}{5}\pi i$$

$\gamma \to |z| = 2$

$\int_\gamma \frac{\sin{(3z)}}{2z-\pi}\ \mathrm{d}z$

Method 1:

$$\int_\gamma \frac{\sin{(3z)}}{2z-\pi} = \frac{1}{2}\int_\gamma \frac{1}{z-\frac{\pi}{2}}\ \mathrm{d}z$$
$$f(\frac{\pi}{2}) = \frac{1}{2\pi i}\int_\gamma \frac{\sin{(3z)}}{2z-\pi}\ \mathrm{d}z \Leftrightarrow 2\pi i f(\frac{\pi}{2}) = \int_\gamma \frac{\sin{(3z)}}{2z-\pi}\ \mathrm{d}z \Leftrightarrow 2\pi i \sin{(\frac{3\pi}{2})}$$
$$\Leftrightarrow \int_\gamma \frac{\sin{(3z)}}{2z-\pi}\ \mathrm{d}z = \frac{1}{2}(-2\pi i) = -\pi i$$

Method 2 (Residue Theorem):

$$\frac{1}{2}\lim_{z\to\frac{\pi}{2}}(z-\frac{\pi}{2})\frac{\sin{(\frac{3\pi}{2})}}{z-\frac{\pi}{2}} = -\pi i$$

$\int_\gamma \frac{\cos{(z)}}{z^3+16z}\ \mathrm{d}z$

Method 1:

$$\int_\gamma \frac{\cos{(z)}}{z^3+16z}\ \mathrm{d}z = \int_\gamma \frac{\frac{\cos{(z)}}{z^2+16}}{z}\ \mathrm{d}z$$
$$f(0) = \frac{1}{2\pi i}\int_\gamma \frac{\frac{\cos{(z)}}{z^2+16}}{z}\ \mathrm{d}z \Leftrightarrow 2\pi i f(0) = \int_\gamma \frac{\frac{\cos{(z)}}{z^2+16}}{z}\ \mathrm{d}z \Leftrightarrow 2\pi i\frac{1}{16}$$
$$\Leftrightarrow \int_\gamma \frac{\cos{(z)}}{z^3+16z}\ \mathrm{d}z = \frac{\pi}{8}i$$

Method 2 (Residue Theorem):

$$\lim_{z \to 0} z \frac{\cos(z)}{z(z^2+16)} = \frac{\pi}{8} i$$

$\int_\gamma \frac{ze^z}{2z-3} \, \mathrm{d}z$

$$\frac{1}{2} \int_\gamma \frac{ze^z}{z - \frac{3}{2}} \, \mathrm{d}z$$

$$f(\frac{3}{2}) = \frac{1}{4\pi i} \int_\gamma \frac{f(z)}{z - \frac{3}{2}} \, \mathrm{d}z$$

$$\frac{3}{2} e^{\frac{3}{2}} \pi i = \int_\gamma \frac{f(z)}{2z-3} \, \mathrm{d}z$$

$\int_\gamma \frac{4z^3+3z^2-1}{(z+i)^3} \, \mathrm{d}z$

$$f'(z) = 12z^2 + 6z \qquad f''(z) = 24z + 6 \qquad f''(-i) = -24i + 6$$

$$f''(-i) = \frac{2!}{2\pi i} \int_\gamma \frac{f(z)}{(z+i)^3} \, \mathrm{d}z$$

$$-24i + 6 = \frac{2!}{2\pi i} \int_\gamma \frac{f(z)}{(z+i)^3} \, \mathrm{d}z$$

$$(-24i + 6)\pi i = \int_\gamma \frac{f(z)}{(z+i)^3} \, \mathrm{d}z$$

$$\int_\gamma \frac{4z^3 + 3z^2 - 1}{(z+i)^3} \, \mathrm{d}z = 24\pi + 6\pi i$$

$\int_\gamma \frac{e^{-z}}{z^2(z-3)} \, \mathrm{d}z$

$$\int_\gamma \frac{\frac{1}{e^z(z-3)}}{z^2} \, \mathrm{d}z$$

$$f'(z) = -\frac{e^z(z-3) + e^z}{e^{2z}(z-3)^2} = -\frac{z-2}{e^z(z-3)^2}$$

$$f'(0) = \frac{2}{9}$$

$$f'(0) = \frac{1}{2\pi i} \int_\gamma \frac{\frac{1}{e^z(z-3)}}{z^2} \ \mathrm{d}z$$

$$\frac{2}{9} 2\pi i = \int_\gamma \frac{\frac{1}{e^z(z-3)}}{z^2} \ \mathrm{d}z$$

$$\int_\gamma \frac{e^{-z}}{z^2(z-3)} \ \mathrm{d}z = \frac{4}{9}\pi i$$

$\int_\gamma \frac{\sin(z)}{(z+\frac{\pi}{2})^2} \ \mathrm{d}z$

$$f'(z) = \cos z \qquad f'(-\frac{\pi}{2}) = 0$$

$$f'(-\frac{\pi}{2}) = \frac{1}{2\pi i} \int_\gamma \frac{\sin(z)}{(z+\frac{\pi}{2})^2} \ \mathrm{d}z$$

$$0 = \int_\gamma \frac{\sin(z)}{(z+\frac{\pi}{2})^2} \ \mathrm{d}z$$

$$\int_\gamma \frac{\sin(z)}{(z+\frac{\pi}{2})^2} \ \mathrm{d}z = 0$$

$\int_\gamma \frac{e^{-z}}{(2z+1)(z-1)} \ \mathrm{d}z$

$\int_\gamma \frac{e^{iz}}{z^2+1} \ \mathrm{d}z$

$\int_\gamma \frac{e^{iz}}{(z^2+1)^2} \ \mathrm{d}z$

$$\frac{1}{(z^2+1)^2} = \frac{A}{z-i} + \frac{B}{(z-i)^2} + \frac{C}{z+i} + \frac{D}{(z+i)^2}$$

$$1 = A(z-i)(z+i)^2 + B(z+i)^2 + C(z+i)(z-1)^2 + D(z-i)^2$$

$z = i$

$$B = -\frac{1}{4}$$

$z = -i$

$$D = -\frac{1}{4}$$

$$1 = A(z-i)(z+i)^2 - \frac{1}{4}(z+i)^2 + C(z+i)(z-1)^2 + -\frac{1}{4}(z-i)^2$$

$$0 = A + C \qquad C = -A$$

$$1 = A(-i)i^2 - \frac{1}{4}i^2 + Ci(-i)^2 + -\frac{1}{4}(-i)^2 \qquad (a-c)i = -\frac{1}{2s}$$

$$\begin{cases} C = -A \\ (A-C)i = -\frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} C = -A \\ 2Ai = -\frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} C = -\frac{1}{4}i \\ A = \frac{1}{4}i \end{cases}$$

$$\frac{1}{(z^2+1)^2} = \frac{\frac{1}{4}i}{z-i} + \frac{-\frac{1}{4}}{(z-i)^2} + \frac{-\frac{1}{4}i}{z+i} + \frac{-\frac{1}{4}}{(z+i)^2}$$

$$\int_\gamma \frac{e^{iz}}{(z^2+1)^2}\ \mathrm{d}z = \frac{1}{4}i\int_\gamma \frac{e^{iz}}{z-i} - \frac{1}{4}\int_\gamma \frac{e^{iz}}{(z-i)^2} - \frac{1}{4}i\int_\gamma \frac{e^{iz}}{z+i} - \frac{1}{4}\int_\gamma \frac{e^{iz}}{(z+i)^2}$$

$$\frac{1}{4}i(2\pi i f(i)) - \frac{1}{4}(2\pi i f'(i)) - \frac{1}{4}i(2\pi i f(-i)) - \frac{1}{4}(2\pi i f'(-i))$$

$$= \frac{1}{4}i(2\pi e^{-1}i) - \frac{1}{4}(2\pi i(ie^{-1})) - \frac{1}{4}i(2\pi i(2\pi e i)) - \frac{1}{4}(2\pi i(ie))$$

$$= -\frac{\pi}{2}e^{-1} + \frac{\pi}{2}e^{-1} + \frac{\pi}{2}e + \frac{\pi}{2}e$$
$$= \pi e$$

$\int_\gamma \frac{1}{z^2(z^2+16)}\ \mathrm{d}z$

$$f'(z) = -\frac{2z}{(z^2+16)^2} \qquad f'(0) = 0$$

$$f'(0) = \frac{1!}{2\pi i}\frac{f(z)}{z^2}\ \mathrm{d}z$$

$$0 = \frac{1}{2\pi i}\frac{\frac{1}{z^2+16}}{z^2}\ \mathrm{d}z$$

$\int_\gamma \frac{1}{z^3+2z^2-3z}\ \mathrm{d}z$

$\int_\gamma \frac{\sin(z)}{(z^4}\ \mathrm{d}z$