

$\mathbf{L} = \int_R e^{-\frac{x^2}{2}} dx$
 Coordenadas Polares

$$\begin{aligned} x &= r \cos \theta & y &= r \sin \theta & r^2 &= x^2 + y^2 \\ L^2 &= \int_R e^{-\frac{x^2}{2}} dx \cdot \int_R e^{-\frac{y^2}{2}} dy = \iint_{R^2} e^{-\frac{1}{2}(x^2+y^2)} dx \\ J &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r \\ L^2 &= \iint_{R^2} e^{-\frac{r^2}{2}} r dr d\theta = \int_0^\infty e^{-\frac{r^2}{2}} r dr \cdot \int_0^{2\pi} d\theta \end{aligned}$$

$$u = -\frac{r^2}{2} \quad -du = r dr$$

$$\begin{aligned} u(\infty) &= -\infty & u(0) &= 0 \\ -\int_0^{-t} e^u du &= \lim_{t \rightarrow \infty} [e^u]_0^{-t} = -(0 - 1) = 1 \\ -\int_0^\infty e^u du \cdot 2\pi &= 1 \cdot 2\pi \end{aligned}$$

Então a integral é $\sqrt{2\pi}$
 $\mathbf{L} = \int_R e^{-x^2} dx$

$$\begin{aligned} L^2 &= \int_R e^{-x^2} dx \cdot \int_R e^{-y^2} dy = \iint_{R^2} e^{-(x^2+y^2)} dx \\ \int_0^\infty e^{-r^2} r dr &= -\frac{1}{2} \lim_{t \rightarrow \infty} [e^{-r^2}]_0^t = -\frac{1}{2}(0 - 1) = \frac{1}{2} \\ L^2 &= \iint_{R^2} e^{-r^2} r dr d\theta = \int_0^\infty e^{-r^2} r dr \cdot \int_0^{2\pi} d\theta = \frac{1}{2} \cdot 2\pi = \pi \end{aligned}$$

Então a integral é $\sqrt{\pi}$

Comprimento de uma arco:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$L = \int ds$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{ se } y = f(x), a \leq x \leq b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \text{ se } x = h(y), c \leq y \leq d$$

$$L = \int ds$$

1.

Determine the length of $y = \ln(\sec x)$ between $0 \leq x \leq \frac{\pi}{4}$.

$$f'(\ln(\sec x)) = \tan(x)$$

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)} \, dx = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2(x)} \, dx = \int_0^{\frac{\pi}{4}} \sec(x) \, dx \\ &= \int_0^{\frac{\pi}{4}} \frac{\sec x + \tan x}{\sec x + \tan x} \sec(x) \, dx \\ &= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \end{aligned}$$

$$u = \sec x + \tan x \quad du = \sec^2 x + \sec x \tan x \, dx$$

$$u(0) = 1 \quad u\left(\frac{\pi}{4}\right) = \sqrt{2} + 1$$

$$\begin{aligned} &= \int_1^{\sqrt{2}+1} \frac{1}{u} \, du \\ &= [\ln |u|]_1^{\sqrt{2}+1} = \ln(\sqrt{2} + 1) \end{aligned}$$

2.

Determine the length of $x = \frac{2}{3}(y-1)^{\frac{3}{2}}$ between $1 \leq y \leq 4$.

$$f'\left(\frac{2}{3}(y-1)^{\frac{3}{2}}\right) = \sqrt{(y-1)}$$

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + (\sqrt{(y-1)})^2} \, dy = \int_1^4 \sqrt{y} \, dy \\ &= \left[\frac{2}{3} y^{\frac{3}{2}} \right]_1^4 = \frac{2}{3} 4^{\frac{3}{2}} - \frac{2}{3} = \frac{14}{3} \end{aligned}$$

3.

Redo the previous example using the function in the form $y = f(x)$ instead.

4.

Determine the length of $x = \frac{1}{2}y^2$ for $0 \leq x \leq \frac{1}{2}$. Assume that y is positive.

Comprimento de uma curva:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b \|\vec{r}'(t)\| dt$$

1.

Determine the length of the curve $\vec{r}(t) = \langle 2t, 3 \sin(2t), 3 \cos(2t) \rangle$ on the interval $0 \leq t \leq 2\pi$.

$$\vec{r}'(t) = \langle 2, 6 \cos(2t), -6 \sin(2t) \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{2^2 + (6 \cos(2t))^2 + (-6 \sin(2t))^2}$$

$$= 2\sqrt{10}$$

$$\int_0^{2\pi} \|\vec{r}'(t)\| dt = \int_0^{2\pi} 2\sqrt{10} dt = 4\pi\sqrt{10}$$

2.

Determine the arc length function for $\vec{r}(t) = \langle 2t, 3 \sin(2t), 3 \cos(2t) \rangle$.

$$\|\vec{r}'(t)\| = 2\sqrt{10}$$

$$s(t) = \int_0^t \|\vec{r}'(u)\| du$$

$$s(t) = \int_0^t 2\sqrt{10} du = \left[2\sqrt{10}u \right]_0^t = 2\sqrt{10}t$$

$$t = \frac{s}{2\sqrt{10}}$$

Parametrizar:

$$\vec{r}(t(s)) = \left\langle \frac{s}{\sqrt{10}}, 3 \sin\left(\frac{s}{\sqrt{10}}\right), 3 \cos\left(\frac{s}{\sqrt{10}}\right) \right\rangle$$

3.

Where on the curve $\vec{r}(t) = \langle 2t, 3 \sin(2t), 3 \cos(2t) \rangle$ are we after traveling for a distance of $\frac{\pi\sqrt{10}}{3}$?

$$\begin{aligned}\vec{r}(t(\frac{\pi\sqrt{10}}{3})) &= \langle \frac{\pi}{3}, 3 \sin(\frac{\pi}{3}), 3 \cos(\frac{\pi}{3}) \rangle \\ &= \langle \frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2} \rangle\end{aligned}$$

Practice Problems

For problems 1&2 determine the length of the vector function on the given interval.

1.

$\vec{r}(t) = (3 - 4t)\hat{i} + 6t\hat{j} - (9 + 2t)\hat{k}$ from $-6 \leq t \leq 8$.

$$\begin{aligned}\|\vec{r}'(t)\| &= \sqrt{16 + 36 + 4} = 2\sqrt{14} \\ &= \int_{-6}^8 \|\vec{r}'(t)\| \, dt = \int_{-6}^8 2\sqrt{14} \, dt \\ &= 2\sqrt{14} [t]_{-6}^8 = 28\sqrt{14}\end{aligned}$$

2.

$\vec{r}(t) = \langle \frac{1}{3}t^3, 4t, \sqrt{2}t^2 \rangle$ from $0 \leq t \leq 2$.

$$\begin{aligned}\|\vec{r}'(t)\| &= \sqrt{t^4 + 16 + 8t^2} = \sqrt{(t^2 + 4)^2} = t^2 + 4 \\ &= \int_0^2 \|\vec{r}'(t)\| \, dt = \int_0^2 t^2 + 4 \, dt \\ &= \left[\frac{1}{3}t^3 + 4t \right]_0^2 = \frac{32}{3}\end{aligned}$$

or problems 3&4 find the arc length function for the given vector function.

3.

$\vec{r}(t) = \langle t^2, 2t^3, 1 - t^3 \rangle$

$$\|\vec{r}'(t)\| = \sqrt{4t^2 + 45t^4} = t\sqrt{4 + 45t^2}$$

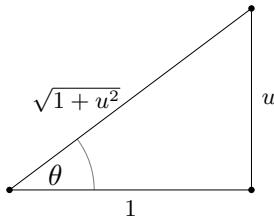
$$\begin{aligned}
 s(t) &= \int_0^t u \sqrt{4 + 45u^2} \, du = \frac{1}{135} \left[(4 + 45u^2)^{\frac{3}{2}} \right]_0^t \\
 &= \frac{1}{135} \left[(4 + 45t^2)^{\frac{3}{2}} - 8 \right]
 \end{aligned}$$

4.

$$\vec{r}(t) = \langle 4t, -2t, \sqrt{5}t^2 \rangle$$

$$||\vec{r}'(t)|| = \sqrt{20 + 20t^2} = 2\sqrt{5}\sqrt{1 + t^2}$$

$$s(t) = 2\sqrt{5} \int_0^t \sqrt{1 + u^2} \, du$$



$$u = \tan(\theta) \quad du = \sec^2(\theta) \, d\theta$$

$$\sqrt{1 + (\tan \theta)^2} = |\sec^2 \theta| = \sec \theta$$

$$0 = \tan \theta \quad \tan^{-1}(t) = \theta$$

$$s(t) = 2\sqrt{5} \int_0^{\tan^{-1}(t)} \sec^3(\theta) \, d\theta$$

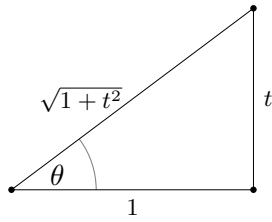
$$f = \sec(\theta) \quad f' = \sec(\theta) \tan(\theta)$$

$$g' = \sec^2(\theta) \, d\theta \quad g = \tan(\theta)$$

$$2\sqrt{5} \left([\sec(\theta) \tan(\theta)]_0^{\tan^{-1}(t)} - \int_0^{\tan^{-1}(t)} \sec(\theta) \tan^2(\theta) \, d\theta \right)$$

$$2\sqrt{5} \left(\sec(\tan^{-1}(t)) \tan(\tan^{-1}(t)) - \int_0^{\tan^{-1}(t)} \sec^3(\theta) \, d\theta + \int_0^{\tan^{-1}(t)} \sec(\theta) \, d\theta \right)$$

$$\sqrt{5} \left(\sec(\tan^{-1}(t)) \tan(\tan^{-1}(t)) + \ln|\sec(\tan^{-1}(t)) + \tan(\tan^{-1}(t))| \right)$$



$$\theta = \tan^{-1}(t) \quad \sec(\theta) = \sqrt{1+t^2} \quad \sec(\tan^{-1}(t)) = \sec \theta = \sqrt{1+t^2}$$

$$s(t) = \sqrt{5} \left(t\sqrt{1+t^2} + \ln|\sqrt{1+t^2} + t| \right)$$

5.

$$t = \sqrt{\frac{(135s+8)^{\frac{2}{3}} - 4}{45}}$$

$$t(20) = \sqrt{\frac{(2708)^{\frac{2}{3}} - 4}{45}} = 2.05633$$

$$r(t(\vec{20})) = \langle 4.22849, 17.39035, -7.69518 \rangle$$

For problems 1&2 determine the length of the vector function on the given interval.

1.

$$\vec{r}(t) = 4 \cos(2t)\hat{\mathbf{i}} + 3t\hat{\mathbf{j}} - 4 \sin(2t)\hat{\mathbf{k}} \text{ from } 0 \leq t \leq 3\pi.$$

$$\|\vec{r}'(t)\| = \sqrt{(-8 \sin 2t)^2 + 9 + (-8 \cos 2t)^2} = \sqrt{73}$$

$$= \int_0^{3\pi} \|\vec{r}'(t)\| \, dt = \int_0^{3\pi} \sqrt{73} \, dt$$

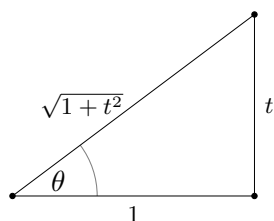
$$\sqrt{73} [t]_0^{3\pi} = 3\pi\sqrt{73}$$

2.

$\vec{r}(t) = \langle 9 - 2t, 4 + 2t, \sqrt{2}t^2 \rangle$ from $0 \leq t \leq 1$.

$$\|\vec{r}'(t)\| = \sqrt{(-2)^2 + 2^2 + (2\sqrt{2}t)^2} = 2\sqrt{2}\sqrt{t^2 + 1}$$

$$= \int_0^1 \|\vec{r}'(t)\| dt = \int_0^1 2\sqrt{2}\sqrt{t^2 + 1} dt$$



$$t = \tan(\theta) \quad dt = \sec^2(\theta) d\theta$$

$$\sqrt{1 + (\tan \theta)^2} = |\sec^2 \theta| = \sec \theta$$

$$0 = \tan \theta \quad 1 = \tan \theta = \frac{\pi}{4}$$

$$= 2\sqrt{2} \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta$$

$$= 2\sqrt{2} \left([\sec(\theta) \tan(\theta)]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^3(\theta) d\theta + \int_0^{\frac{\pi}{4}} \sec(\theta) d\theta \right) = 3.25$$

3.

$\vec{r}(t) = 2t\vec{i} + \frac{1}{2}t^2\vec{j} + \ln(t^2)\vec{k}$, $\sqrt{2}t^2$ from $1 \leq t \leq 3$.

$$\|\vec{r}'(t)\| = \sqrt{t^4 + 16 + 8t^2} = \sqrt{(t^2 + 4)^2} = t^2 + 4$$

$$= \int_0^2 \|\vec{r}'(t)\| dt = \int_0^2 t^2 + 4 dt$$

$$= \left[\frac{1}{3}t^3 + 4t \right]_0^2 = \frac{32}{3}$$

The curvature measures how fast a curve is changing direction at a given point.

$$k = \left\| \frac{d\vec{T}}{ds} \right\|$$

Alternative:

$$k = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} \quad k = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

If $y = f(x)$:

$$\vec{r}(x) = x\vec{i} + f(x)\vec{j}$$

Curvature formula:

$$k = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{\frac{3}{2}}}$$

Determine the curvature for $\vec{r}(t) = \langle t, 3 \sin(t), 3 \cos(t) \rangle$.

$$\vec{r}'(t) = \langle 1, 3 \cos(t), -3 \sin(t) \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{10}$$

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \cos(t), -\frac{3}{\sqrt{10}} \sin(t) \right\rangle$$

$$\vec{T}'(t) = \left\langle 0, -\frac{3}{\sqrt{10}} \sin(t), -\frac{3}{\sqrt{10}} \cos(t) \right\rangle$$

$$\|\vec{T}'(t)\| = \frac{3}{\sqrt{10}}$$

$$k = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\frac{3}{\sqrt{10}}}{\sqrt{10}} = \frac{3}{10}$$

Determine the curvature of $\vec{r}(t) = t^2\vec{i} + t\vec{k}$

$$\vec{r}'(t) = 2t\vec{i} + \vec{k}$$

$$\|\vec{r}'(t)\| = \sqrt{4t^2 + 1}$$

$$\vec{r}''(t) = 2\vec{i}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 2\vec{j}$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = 2$$

$$k = \frac{2}{(4t^2 + 1)^{\frac{3}{2}}}$$

Find the curvature for each the following vector functions.

1.

$$\vec{r}(t) = \langle \cos(2t), -\sin(2t), 4t \rangle$$

$$\vec{r}'(t) = \langle -2\sin(2t), -2\cos(2t), 4 \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{20} = 2\sqrt{5}$$

$$\vec{T}(t) = \langle -\frac{1}{\sqrt{5}}\sin(2t), -\frac{1}{\sqrt{5}}\cos(2t), \frac{2}{\sqrt{5}} \rangle$$

$$\vec{T}'(t) = \langle -\frac{2}{\sqrt{5}}\cos(2t), \frac{2}{\sqrt{5}}\sin(2t), 0 \rangle$$

$$\|\vec{T}'(t)\| = \sqrt{\frac{4}{\sqrt{5}}\cos^2(2t) + \frac{4}{\sqrt{5}}\sin^2(2t)} = \frac{2}{\sqrt{5}}$$

$$k = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\frac{2}{\sqrt{5}}}{2\sqrt{5}} = \frac{1}{5}$$

2.

$$\vec{r}(t) = \langle 4t, -t^2, 2t^3 \rangle$$

$$\vec{r}'(t) = \langle 4, -2t, 6t^2 \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{16 + 4t^2 + 36t^4}$$

$$\vec{r}''(t) = \langle 0, -2, 12t \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2t & 6t^2 \\ 0 & -2 & 12t \end{vmatrix} = -12t^2\vec{i} - 48t\vec{j} - 8\vec{k}$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{144t^4 + 2304t^2 + 64}$$

$$k = \frac{\sqrt{144t^4 + 2304t^2 + 64}}{(16 + 4t^2 + 36t^4)^{\frac{3}{2}}}$$

1.

$$\vec{r}(t) = \langle 5t, 1 - 2t, 4t^{\frac{3}{2}} \rangle$$

$$\vec{r}'(t) = \langle 5, -2, 6t^{\frac{1}{2}} \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{29 + 36t}$$

$$\vec{r}''(t) = \langle 0, 0, 3t^{-\frac{1}{2}} \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -2 & 6t^{\frac{1}{2}} \\ 0 & 0 & 3t^{-\frac{1}{2}} \end{vmatrix} = -6t^{-\frac{1}{2}}\vec{i} - 15t^{-\frac{1}{2}}\vec{j}$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{261}t^{-\frac{1}{2}}$$

$$k = \frac{\sqrt{261}t^{-\frac{1}{2}}}{(29 + 36t)^{\frac{3}{2}}}$$

2.

$$\vec{r}(t) = \langle 6, e^{-5t}, 3te^{-5t} \rangle$$

$$\vec{r}'(t) = \langle 5, -2, 6t^{\frac{1}{2}} \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{29 + 36t}$$

$$\vec{r}''(t) = \langle 0, 0, 3t^{-\frac{1}{2}} \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -2 & 6t^{\frac{1}{2}} \\ 0 & 0 & 3t^{-\frac{1}{2}} \end{vmatrix} = -6t^{-\frac{1}{2}}\vec{i} - 15t^{-\frac{1}{2}}\vec{j}$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{261}t^{-\frac{1}{2}}$$

$$k = \frac{\sqrt{261}t^{-\frac{1}{2}}}{(29 + 36t)^{\frac{3}{2}}}$$

3.

$$\vec{r}(t) = \langle \cos(\omega t), t, \sin(\omega t) \rangle$$

$$\vec{r}'(t) = \langle 5, -2, 6t^{\frac{1}{2}} \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{29 + 36t}$$

$$\vec{r}''(t) = \langle 0, 0, 3t^{-\frac{1}{2}} \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -2 & 6t^{\frac{1}{2}} \\ 0 & 0 & 3t^{-\frac{1}{2}} \end{vmatrix} = -6t^{-\frac{1}{2}}\vec{i} - 15t^{-\frac{1}{2}}\vec{j}$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{261}t^{-\frac{1}{2}}$$

$$k = \frac{\sqrt{261}t^{-\frac{1}{2}}}{(29 + 36t)^{\frac{3}{2}}}$$

Área de superfície:

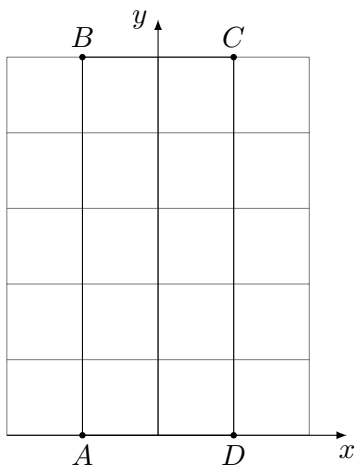
$$A(S) = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dx \, dy$$

Valores Médios:

$$A = \iint_D dx \, dy$$

$$\bar{f} = \frac{\iint_D f(x, y) \, dx \, dy}{A}$$

$$\bar{f} = \frac{1}{A(R)} \iint_R f(x, y) \, dA$$



$$A(R) = 2 \cdot 5 = 10$$

$$\begin{aligned}\bar{f} &= \frac{1}{10} \int_{-1}^1 \int_0^5 x^2 y \, dy \, dx \\ &= \frac{1}{10} \int_{-1}^1 \left[x^2 \frac{y^2}{2} \right]_0^5 dx = \frac{1}{10} \int_{-1}^1 \frac{25}{2} x^2 \, dx = \frac{25}{20} \left[\frac{x^3}{3} \right]_{-1}^1 = \frac{5}{6}\end{aligned}$$

Centro de Massa:

$$M = \iint_R \rho(x, y) \, dA \quad M_x = \iint_R y \rho(x, y) \, dA \quad M_y = \iint_R x \rho(x, y) \, dA$$

$$R = (\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right)$$

$$\bar{x} = \frac{1}{A} \iint_D x f(x, y) \, dx \, dy$$

$$\bar{y} = \frac{1}{A} \iint_D y f(x, y) \, dx \, dy$$

Find the volume of the solid that is bounded about by $f(x, y) = y \sin(xy)$ and below $R = [1, 2] \times [0, \pi]$

$$\begin{aligned}V &= \int_0^\pi \int_1^2 y \sin(xy) \, dx \, dy \\ &= \int_0^\pi \left[-\frac{y}{y} \cos(xy) \right]_1^2 dy = \int_0^\pi -\cos(2y) + \cos(y) \, dy \\ &= \left[-\frac{1}{2} \sin(2y) + \sin(y) \right]_0^\pi = 0\end{aligned}$$

If $R = \{(x, y) | -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate $\iint_R \sqrt{1-x^2} \, dA$

Método 1:

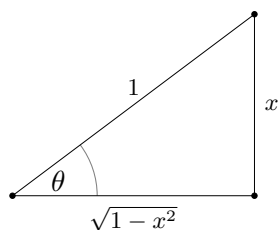
Como $z = \sqrt{1-x^2}$ é metade de um cilindro então:

$$V_{cilindro} = \pi r^2 h$$

$$V = \frac{1}{2} \pi r^2 h = \frac{1}{2} \pi 1 \cdot 4 = 2\pi$$

Método 2:

$$\int_{-2}^2 \int_{-1}^1 \sqrt{1-x^2} \, dx \, dy$$



Substituição trigonométrica:

$$x = \sin \theta \quad dx = \cos \theta \, d\theta$$

$$\cos \theta = \sqrt{1-x^2}$$

Se $x = -1$ então $\theta = -\frac{\pi}{2}$

Se $x = 1$ então $\theta = \frac{\pi}{2}$

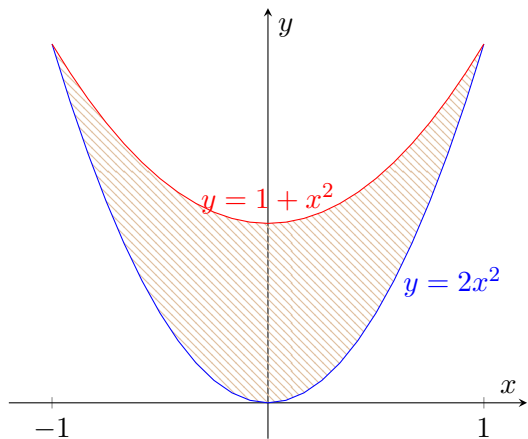
$$\int_{-2}^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta \, dy$$

$$\cos 2\theta = \sin^2 \theta - \cos^2 \theta \Leftrightarrow \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$\int_{-2}^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 2\theta) \, d\theta \, dy = \int_{-2}^2 \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy$$

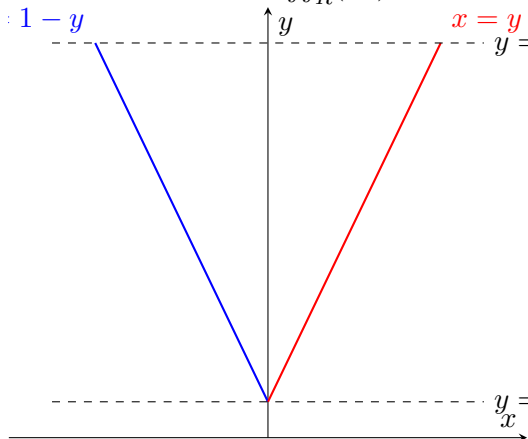
$$= \left[\frac{\pi}{2} y \right]_{-2}^2 = 2\pi$$

Evaluate $\iint_D (x+2y) \, dA$ where D is the region bounded by $y = 2x^2$ and $y = 1+x^2$



$$\begin{aligned}
 \int_{-1}^1 \int_{2x^2}^{1+x^2} (x+2y) \, dy \, dx &= \int_{-1}^1 [xy + y^2]_{2x^2}^{1+x^2} \, dx \\
 &= \int_{-1}^1 [(x(1+x^2) + (1+x^2)^2) - (x(2x^2) + (2x^2)^2)] \, dx \\
 &= \int_{-1}^1 (-x^3 - 3x^4 + x + 2x^2 + 1) \, dx = \left[-\frac{1}{4}x^4 - \frac{3}{5}x^5 + \frac{1}{2}x^2 + \frac{2}{3}x^3 + x \right]_{-1}^1 \\
 &= \left[\left(-\frac{1}{4} - \frac{3}{5} + \frac{1}{2} + \frac{2}{3} + 1 \right) - \left(-\frac{1}{4} + \frac{3}{5} + \frac{1}{2} - \frac{2}{3} - 1 \right) \right] = \frac{32}{15}
 \end{aligned}$$

Setup only! Evaluate $\iint_R (xy) \, dA$ where R is the region bounded by $y = -x + 1$, $y = x + 1$ and $y = 3$



Horizontal fixamos o x

$$\int_1^3 \int_{1-y}^{y-1} (xy) \, dx \, dy$$

Find the volume of the solid that lies under $z = xy$ and above D where D is the region bounded by

$$\int_{-2}^4 \int_{\frac{1}{2}y^3-3}^{y+1} (xy) \, dx \, dy$$

Calculate $\iiint_R (x + y + 2z) \, dx \, dy \, dz$ $R : x^2 + z^2 = 4, y = 2, y = 3$.
 Coordenadas cilíndricas:

$$x = r \cos(\theta) \quad z = r \sin \theta$$

$$\begin{aligned} & \int_2^3 \int_0^{2\pi} \int_0^2 (r \cos(\theta) + y + 2r \sin(\theta)) r \, dr \, d\theta \, dy \\ &= \int_2^3 \int_0^{2\pi} \int_0^2 (r^2 \cos(\theta) + yr + 2r^2 \sin(\theta)) \, dr \, d\theta \, dy \\ &= \int_2^3 \int_0^{2\pi} \left[\left(\frac{1}{3} r^3 \cos(\theta) + \frac{1}{2} y r^2 + \frac{2}{3} r^3 \sin(\theta) \right) \right]_0^2 \, d\theta \, dy \\ &= \int_2^3 \int_0^{2\pi} \left(\frac{8}{3} \cos(\theta) + 2y + \frac{16}{3} \sin(\theta) \right) \, d\theta \, dy \\ &= \int_2^3 \left[\left(\frac{8}{3} \sin(\theta) + 2y\theta - \frac{16}{3} \cos(\theta) \right) \right]_0^{2\pi} \, dy \\ &= [2\pi y^2]_2^3 = 18\pi - 8\pi = 10\pi \end{aligned}$$

Calculate the volume of the region $\iiint_R (x + y + 2z) \, dx \, dy \, dz$, $R : x^2 + z^2 = 4, y = 2, y = 3$.

$$\begin{aligned} & \int_2^3 \int_0^{2\pi} \int_0^2 r \, dr \, d\theta \, dy \\ &= \int_2^3 \int_0^{2\pi} \left[\frac{1}{2} r^2 \right]_0^2 \, d\theta \, dy \\ &= \int_2^3 [2\theta]_0^{2\pi} \, dy \\ &= [4\pi y]_2^3 = 12\pi - 8\pi = 4\pi \end{aligned}$$

Calculate $\iiint_R y \, dx \, dy \, dz$ $R : x^2 + y^2 = 3, z = -1, z = 2$.
 Coordenadas cilíndricas:

$$y = r \sin \theta$$

$$\int_{-1}^2 \int_0^{2\pi} \int_0^{\sqrt{3}} r \sin(\theta) r \, dr \, d\theta \, dz$$

$$\begin{aligned}
&= \int_{-1}^2 \int_0^{2\pi} \int_0^{\sqrt{3}} r^2 \sin(\theta) \, dr \, d\theta \, dz \\
&= \int_{-1}^2 \int_0^{2\pi} \left[\frac{1}{3} r^3 \sin(\theta) \right]_0^{\sqrt{3}} d\theta \, dz \\
&= \int_{-1}^2 \int_0^{2\pi} \sqrt{3} \sin(\theta) \, d\theta \, dz \\
&= [0]_{-1}^2 = 0
\end{aligned}$$

Calculate the volume of the region $\iiint_R y \, dx \, dy \, dz$, $R: x^2 + y^2 = 3, z = -1, z = 2$.

$$\begin{aligned}
&\int_{-1}^2 \int_0^{2\pi} \int_0^{\sqrt{3}} 1 \, r \, dr \, d\theta \, dz \\
&= \int_{-1}^2 \int_0^{2\pi} \int_0^{\sqrt{3}} r \, dr \, d\theta \, dz \\
&= \int_{-1}^2 \int_0^{2\pi} \left[\frac{1}{2} r^2 \right]_0^{\sqrt{3}} d\theta \, dz \\
&= \int_{-1}^2 \left[\frac{3}{2} \theta \right]_0^{2\pi} dz \\
&= [3\pi z]_{-1}^2 = 9\pi
\end{aligned}$$

Line integral with respect to arc length.

Parametric equations:

$$x = h(t) \quad y = g(t) \quad a \leq t \leq b$$

Parametric equations as a vector function:

$$\vec{r}(t) = h(t)\vec{i} + g(t)\vec{j} \quad a \leq t \leq b$$

The curve is called smooth if $\vec{r}(t)$ is continuous and $\vec{r}'(t) \neq 0$

$$\begin{aligned}
L &= \int_a^b ds \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
\int_C f(x, y) \, ds &= \int_a^b f(h(t), g(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \|\vec{r}'(t)\|
\end{aligned}$$

Curve	Parametric Equations	
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Ellipse)	Counter-Clockwise $x = a \cos(t)$ $y = b \sin(t)$ $0 \leq t \leq 2\pi$	Clockwise $x = a \cos(t)$ $y = -b \sin(t)$ $0 \leq t \leq 2\pi$
$x^2 + y^2 = r^2$ (Circle)	Counter-Clockwise $x = r \cos(t)$ $y = r \sin(t)$ $0 \leq t \leq 2\pi$	Clockwise $x = r \cos(t)$ $y = -r \sin(t)$ $0 \leq t \leq 2\pi$
$y = f(x)$	$x = t$ $y = f(t)$	
$x = g(y)$	$x = g(t)$ $y = t$	
Line Segment From (x_0, y_0, z_0) to (x_1, y_1, z_1)	$\vec{r}(t) = (1-t)\langle x_0, y_0, z_0 \rangle + t\langle x_1, y_1, z_1 \rangle, \quad 0 \leq t \leq 1$ or $x = (1-t)x_0 + tx_1$ $y = (1-t)y_0 + ty_1$ $z = (1-t)z_0 + tz_1$, $0 \leq t \leq 1$	

Example 1

Evaluate $\int_C xy^4 \, ds$ where C is the right half of the circle, $x^2 + y^2 = 16$ traced out in a counter clockwise direction.

$$x = 4 \cos(t) \quad y = 4 \sin(t) \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

$$\frac{dx}{dt} = -4 \sin(t) \quad \frac{dy}{dt} = 4 \cos(t)$$

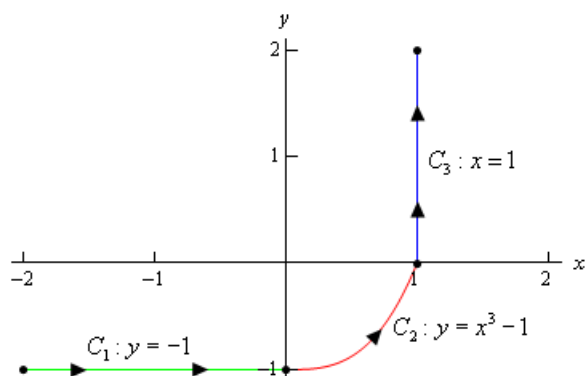
$$ds = \sqrt{(-4 \sin(t))^2 + (4 \cos(t))^2} \, dt = 4 \, dt$$

$$\begin{aligned} \int_C xy^4 \, ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4 \cos(t)) (4 \sin(t))^4 4 \, dt \\ &= 4096 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(t)) (\sin(t))^4 \, dt \end{aligned}$$

$$\begin{aligned}
&= \frac{4096}{5} \left[(\sin(t))^5 \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{4096}{5} [2] = \frac{8192}{5}
\end{aligned}$$

Example 2

Evaluate $\int_C 4x^3 \, ds$ where C is the curve shown below.



$$C_1 : x = t \quad y = -1 \quad -2 \leq t \leq 0$$

$$C_1 : dx = 1 \quad dy = 0 \quad ds = \sqrt{1} \, dt = dt$$

$$C_2 : x = t \quad y = t^3 - 1 \quad 0 \leq t \leq 1$$

$$C_2 : dx = 1 \quad dy = 3t^2 \quad ds = \sqrt{1 + 9t^4} \, dt$$

$$C_3 : x = 1 \quad y = t \quad 0 \leq t \leq 2$$

$$C_3 : dx = 0 \quad dy = 1 \quad ds = \sqrt{1} \, dt = dt$$

$$\int_{C_1} 4x^3 \, ds = \int_{-2}^0 4t^3 \, dt$$

$$\int_{C_2} 4x^3 \, ds = \int_0^1 4t^3 \sqrt{1 + 9t^4} \, dt = \frac{2}{27} \left[(1 + 9t^4)^{\frac{3}{2}} \right]_0^1 = \frac{2}{27} \left[10^{\frac{3}{2}} - 1 \right] = 2.268354$$

$$\int_{C_3} 4x^3 \, ds = \int_0^2 4(1)^3 \, dt = [4t]_0^2 = 8$$

Example 3

Evaluate $\int_C 4x^3 \, ds$ where C is the line segment from $(-2, -1)$ to $(1, 2)$.

$$\vec{r}(t) = (1-t)\langle -2, -1 \rangle + t\langle 1, 2 \rangle$$

$$= \langle 3t - 2, 3t - 1 \rangle$$

$$x = 3t - 2 \quad y = 3t - 1$$

$$\|\vec{r}'(t)\| = \sqrt{18} = 3\sqrt{2} \, dt$$

$$12\sqrt{2} \int_0^1 (3t - 2)^3 \, dt = \sqrt{2} [(3t - 2)^4]_0^1 = -15\sqrt{2}$$

Example 4

Evaluate $\int_C 4x^3 \, ds$ where C is the line segment from $(1, 2)$ to $(-2, -1)$.

$$\vec{r}(t) = (1-t)\langle 1, 2 \rangle + t\langle -2, -1 \rangle$$

$$= \langle -3t + 1, -3t + 2 \rangle$$

$$x = -3t + 1 \quad y = -3t + 2$$

$$\|\vec{r}'(t)\| = \sqrt{18} = 3\sqrt{2} \, dt$$

$$12\sqrt{2} \int_0^1 (-3t + 1)^3 \, dt = -\sqrt{2} [(-3t + 1)^4]_0^1 = -15\sqrt{2}$$

Example 5

Evaluate $\int_C x \, ds$ for each of the following curves.

a)

$$C_1 : y = x^2, -1 \leq x \leq 1$$

$$C_1 : x = t, y = t^2, -1 \leq t \leq 1$$

$$\|\vec{r}'(t)\| = \sqrt{1 + 4t^2} \, dt$$

$$\int_{-1}^1 t \sqrt{1 + 4t^2} \, dt = \frac{1}{8} \int_{-1}^1 8t(1 + 4t^2)^{\frac{1}{2}} \, dt = \frac{1}{12} \left[(1 + 4t^2)^{\frac{3}{2}} \right]_{-1}^1 = 0$$

b)

C_2 : The line segment from (-1,1) to (1,1).

$$\begin{aligned} \vec{r}(t) &= (1-t)\langle -1, 1 \rangle + t\langle 1, 1 \rangle \quad 0 \leq t \leq 1 \\ &= \langle 2t - 1, 1 \rangle \end{aligned}$$

$$\|\vec{r}'(t)\| = 2$$

$$\int_0^1 (4t - 2) \, dt = [2t^2 - 2t]_0^1 = 0$$

Método 2:

$$C_2 : x = t, y = 1, -1 \leq t \leq 1$$

$$\vec{r}'(t) = 1$$

$$\int_{-1}^1 t \, dt = \left[\frac{1}{2} t^2 \right]_{-1}^1 = 0$$

c)

C_3 : The line segment from (1,1) to (-1,1).

$$\begin{aligned} \vec{r}(t) &= (1-t)\langle 1, 1 \rangle + t\langle -1, 1 \rangle \quad 0 \leq t \leq 1 \\ &= \langle 1 - 2t, 1 \rangle \end{aligned}$$

$$\|\vec{r}'(t)\| = 2$$

$$\int_0^1 (2 - 4t) dt = [2t - 2t^2]_0^1 = 0$$

Example 6

Evaluate $\int_C xyz \, ds$ where C is the helix given by, $\vec{r}(t) = \langle \cos(t), \sin(t), 3t \rangle, 0 \leq t \leq 4\pi$.

$$\|\vec{r}'(t)\| = \sqrt{10}$$

$$3 \int_0^{4\pi} (t \cos(t) \sin(t) \sqrt{10}) dt$$

$$\frac{\sin 2t}{2} = \cos(t) \sin(t)$$

$$= \frac{3\sqrt{10}}{2} \left[\frac{1}{4} \sin 2t - \frac{t}{2} \cos 2t \right]_0^{4\pi} = -3\sqrt{10}\pi$$

1.

Evaluate $\int_C 3x^2 - 2y \, ds$ where C is the line segment from $(3, 6)$ to $(-1, 1)$.

Evaluate \int_C

$2y^2 - 4x \, ds$ where C is the lower half of the circle centered at the origin of radius 3 with clockwise rotation. Solution Evaluate \int_C

$6x \, ds$ where C is the portion of $y = x^2$ from $x = -1$ to $x = 2$. The direction of C is in the direction of increasing x . Solution Evaluate \int_C

$xy - 4z \, ds$ where C is the line segment from $(1, 1, 0)$ to $(2, 3, -2)$. Solution Evaluate \int_C

$x^2y^2 \, ds$ where C is the circle centered at the origin of radius 2 centered on the y -axis at $y = 4$. See the sketches below for orientation. Note the “odd” axis orientation on the 2D circle is intentionally that way to match the 3D axis the direction.

Solution Evaluate \int_C

$16y^5 \, ds$ where C is the portion of $x = y^4$ from $y = 0$ to $y = 1$ followed by the line segment from $(1, 1)$ to $(1, -2)$ which in turn is followed by the line segment from $(1, -2)$ to $(2, 0)$. See the sketch below for the direction.

Solution Evaluate \int_C

$4y - x \, ds$ where C is the upper portion of the circle centered at the origin of radius 3 from $(3\sqrt{2}, 3\sqrt{2})$ to $(-3\sqrt{2}, -3\sqrt{2})$ in the counter clockwise rotation followed by the line segment from $(-3\sqrt{2}, -3\sqrt{2})$ to $(4, -3\sqrt{2})$ which in turn is followed by the line segment from $(4, -3\sqrt{2})$ to $(4, 4)$. See the sketch below for the direction.

13. Find the volume of the region bounded by the surfaces $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 = \frac{1}{4}$.
14. Find the volume of the region enclosed by the cones $z = \sqrt{x^2 + y^2}$ and $z = 1 - 2\sqrt{x^2 + y^2}$.
15. Find the volume inside the ellipsoid $x^2 + y^2 + 4z^2 = 6$.
16. Find the volume of the intersection of the ellipsoid $x^2 + 2(y^2 + z^2) \leq 10$ and the cylinder $y^2 + z^2 \leq 1$.
17. Find the *normalizing constant* c , depending on σ , such that $\int_{-\infty}^{\infty} ce^{-x^2/\sigma} dx = 1$.

13.

$$\begin{aligned}
 r^2 &= \frac{1}{4} & r &= \pm \frac{1}{2} \\
 z &= \pm \sqrt{1 - r^2} \\
 \int_0^{2\pi} \int_0^{\frac{1}{2}} \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^{\frac{1}{2}} 2r\sqrt{1-r^2} \, dr \, d\theta \\
 &= \int_0^{2\pi} -\frac{2}{3} \left[(1-r^2)^{\frac{3}{2}} \right]_0^{\frac{1}{2}} d\theta \\
 &= \left[-\frac{\sqrt{3}}{4}\theta + \frac{2}{3}\theta \right]_0^{2\pi} \\
 &= -\frac{\sqrt{3}\pi}{2} + \frac{4\pi}{3}
 \end{aligned}$$

14.

$$\begin{aligned}
 z &= r & z &= 1 - 2r & r &= \frac{1}{3} \\
 \int_0^{2\pi} \int_0^{\frac{1}{3}} \int_r^{1-2r} r \, dz \, dr \, d\theta
 \end{aligned}$$

$$\begin{aligned}
&= \int_0^{2\pi} \int_0^{\frac{1}{3}} r [z]_r^{1-2r} dr d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{1}{3}} r - 3r^2 dr d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{2}r^2 - r^3 \right]_0^{\frac{1}{3}} d\theta \\
&= \left[\frac{1}{54}\theta \right]_0^{2\pi} \\
&= \frac{\pi}{27}
\end{aligned}$$

15.

$$V = \frac{4}{3}abc$$

$$x = \sqrt{6} \quad y = \sqrt{6} \quad z = \frac{\sqrt{6}}{2}$$

$$V = \frac{4}{3}\sqrt{6}\sqrt{6}\frac{\sqrt{6}}{2} = 4\pi\sqrt{6}$$

16.

$$x \leq \pm\sqrt{10-2r^2} \quad -1 \leq r \leq 1$$

$$\begin{aligned}
&\int_0^{2\pi} \int_0^1 \int_{-\sqrt{10-r^2}}^{\sqrt{10-r^2}} r dx dr d\theta \\
&= \int_0^{2\pi} \int_0^1 r [x]_{-\sqrt{10-r^2}}^{\sqrt{10-r^2}} dr d\theta \\
&= \int_0^{2\pi} \int_0^1 2r\sqrt{10-r^2} dr d\theta \\
&= \int_0^{2\pi} -\frac{2}{3} \left[(10-r^2)^{\frac{3}{2}} \right]_0^1 d\theta \\
&= \left[-18\theta + \frac{20\sqrt{10}}{3}\theta \right]_0^{2\pi}
\end{aligned}$$

$$= -18\pi + \frac{40\sqrt{10}\pi}{3}$$

1. (a) Find the mass of the box $[0, \frac{1}{2}] \times [0, 1] \times [0, 2]$, assuming the density to be uniform. (b) Same Exercise as part (a), but with a mass density $\rho(x, y, z) = x^2 + 3y^2 + z + 1$.
2. Find the mass of the solid bounded by the cylinder $x^2 + z^2 = 2x$ and the cone $z^2 = x^2 + y^2$ if the density is $\rho = \sqrt{x^2 + y^2}$.

Find the center of mass of the solids in Exercises 3 and 4, assuming them to have constant density.

3. S bounded by $x + y + z = 2$, $x = 0$, $y = 0$, $z = 0$.
4. S bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 6$, $z = 0$.
5. Evaluate the integral in Example 2 by considering the hemisphere as a region of type I.
6. Find the center of mass of the cylinder $x^2 + y^2 \leq 1$, $1 \leq z \leq 2$ if the density is $\rho = (x^2 + y^2)z^2$.
7. Redo Example 3 for the cube

$$W = [-c, c] \times [-c, c] \times [-c, c].$$

[Hint: Guess the answer to part (b) first.]

8. Find the average value of $x^2 + y^2$ over the conical region $0 \leq z \leq 2$, $x^2 + y^2 \leq z^2$.
9. Find the average value of $\sin^2 \pi z \cos^2 \pi x$ over the cube $[0, 2] \times [0, 4] \times [0, 6]$.
10. Find the average value of e^{-z} over the ball $x^2 + y^2 + z^2 \leq 1$.

1.

a)

$$V = \rho \cdot \frac{1}{2} \cdot 2 \cdot 1 = \rho$$

b)

$$\rho(x, y, z) = x^2 + 3y^2 + z + 1$$

$$M = \int_0^{\frac{1}{2}} \int_0^1 \int_0^2 x^2 + 3y^2 + z + 1 \, dz \, dy \, dx$$

a)