EXERCISE: Evaluate the integral

$$L=\iint_R x\cos\bigl(x^2+y\bigr)\ \mathrm{d}x\ \mathrm{d}y,$$

where $R=[-\pi,0]\times[0,\pi]$.

SOLUTION: Letting x be the inner variable of integration, and then setting $z=x^2+y$, we have

$$2L=\int_0^\pi\int_{-\sqrt{\pi}}^0 2x\cos\bigl(x^2+y\bigr)\ \mathrm{d}x\ \mathrm{d}y=\int_0^\pi\int_{y+\pi}^y\cos(z)\ \mathrm{d}z\ \mathrm{d}y=\int_0^\pi \sin(z)\Big|_{y+\pi}^{y}\ \mathrm{d}y=\int_0^\pi 2\sin(y)\ \mathrm{d}y.$$

Dividing the previous equation by 2, we have

$$L=\int_0^\pi\sin(y)\ \mathrm{d}y=-\cos(y)\Big|_0^\pi=2.$$

EXERCISE: Evaluate the integral

$$L=\iint_R xy\sqrt{1+x^2+y^2}\ \mathrm{d}x\ \mathrm{d}y,$$

where $R=[0,1]^2$.

SOLUTION: Let $z=x^2$ and $w=y^2$. Notice that the mapping $(x,y)\mapsto(z,w)$ sends R bijectively to itself. Then,

$$\begin{aligned}4L&=\iint_R 4xy\sqrt{1+x^2+y^2}\ \mathrm{d}x\ \mathrm{d}y\\&=\iint_R\sqrt{1+z+w}\ \mathrm{d}z\ \mathrm{d}w\\&=\int_0^1\int_0^1\sqrt{1+z+w}\ \mathrm{d}w\ \mathrm{d}z\\&=\int_0^1\int_{1+z}^{2+z}w^{1/2}\ \mathrm{d}w\ \mathrm{d}z.\end{aligned}$$

Multiplying the previous equation times $3/2$, we have

$$\begin{aligned}6L&=\int_0^1 w^{3/2}\Big|_{1+z}^{2+z}\ \mathrm{d}z\\&=\int_0^1(2+z)^{3/2}\ \mathrm{d}z-\int_0^1(1+z)^{3/2}\ \mathrm{d}z\\&=\int_2^3 z^{3/2}\ \mathrm{d}z-\int_1^2 z^{3/2}\ \mathrm{d}z.\end{aligned}$$

Multiplying the previous equation times 5/2, we have

$$15L = z^{5/2}\Big|_2^3 - z^{5/2}\Big|_1^2 = 3^{5/2} - 2 \cdot 2^{5/2} + 1 = 9\sqrt{3} - 8\sqrt{2} + 1.$$

Dividing the previous equation by 15, we have the sought result.

EXERCISE: Evaluate the integral

$$L = \iint_R x \sin(xy)\ \mathrm{d}x\ \mathrm{d}y,$$

where $R = [0, 2] \times [\pi, 2\pi]$.

SOLUTION: Letting y be the inner variable of integration, and then setting $z = xy$, we have

$$\begin{aligned} L &= \int_0^2 \int_\pi^{2\pi} x \sin(xy)\ \mathrm{d}y\ \mathrm{d}x = \int_0^2 \int_{\pi x}^{2\pi x} \sin(z)\ \mathrm{d}z\ \mathrm{d}x \\ &= -\int_0^2 \cos(z)\Big|_{\pi x}^{2\pi x}\ \mathrm{d}x = \underbrace{\int_0^2 \cos(\pi x)\ \mathrm{d}x}_{0} - \underbrace{\int_0^2 \cos(2\pi x)\ \mathrm{d}x}_{0} = 0. \end{aligned}$$