The line integral of $\vec{F}$ along C is
$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,' \ \mathrm{d}t$

$$\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t), z(t))$$

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \vec{T} \ \mathrm{d}s \qquad \vec{T}(t) = \frac{\vec{r}\,'}{||\vec{r}\,'||}$$

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \vec{T} \ \mathrm{d}s$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||} ||\vec{r}\,'(t)|| \ \mathrm{d}t$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \ \mathrm{d}t$$

Example 1

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y,z) = 8x^2yz\vec{i} + 5z\vec{j} - 4xy\vec{k}$ and C is the curve given by $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, $0 \le t \le 1$.

$$\vec{F}(\vec{r}(t)) = 8t^2(t^2)(t^3)\vec{i} + 5t^3\vec{j} - 4t(t^2)\vec{k} = 8t^7\vec{i} + 5t^3\vec{j} - 4t^3\vec{k}$$

$$\vec{r}\,'(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) = (8t^7 + 5t^3 - 4t^3) \cdot (1 + 2t + 3t^2) = 8t^7 + 10t^4 - 12t^5$$

$$= \int_0^1 8t^7 + 10t^4 - 12t^5 \ \mathrm{d}t = \left[t^8 + 2t^5 - 2t^6\right]_0^1 = 1$$

Example 2

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y,z) = xz\vec{i} - yz\vec{k}$ and C is the line segment from $(-1,2,0)$ to $(3,0,1)$.

$$\vec{r}(t) = (1-t)\langle -1,2,0\rangle + t\langle 3,0,1\rangle = \langle 4t-1, -2t+2, t\rangle$$

$$\vec{r}\,'(t) = \langle 4,-2,1\rangle$$

$$\vec{F}(\vec{r}(t)) = (4t-1)(t)\vec{i} - (-2t+2)(t)\vec{k} = (4t^2 - t)\vec{i} - (-2t^2 + 2t)\vec{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r\prime}(t) = 4(4t^2 - t) - (-2t^2 + 2t) = 18t^2 - 6t$$

$$= \int_0^1 18t^2 - 6t \, \mathrm{d}t = \left[6t^3 - 3t^2\right]_0^1 = 3$$

Given the vector field $\vec{F}(x, y, z) = P\vec{i} + Q\vec{j} + R\vec{k}$ and the curve C parameterized by $\vec{r} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \leq t \leq b$

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_a^b \left(P\vec{i} + Q\vec{j} + R\vec{k}\right) \cdot \left(x'\vec{i} + y'\vec{j} + z'\vec{k}\right) \mathrm{d}t$$

$$= \int_a^b Px' + Qy' + Rz' \, \mathrm{d}t$$

$$= \int_a^b Px' \, \mathrm{d}t + \int_a^b Qy' \, \mathrm{d}t + \int_a^b Rz' \, \mathrm{d}t$$

$$= \int_C P \, \mathrm{d}x + \int_C Q \, \mathrm{d}y + \int_C R \, \mathrm{d}z$$

$$= \int_C P \, \mathrm{d}x + Q \, \mathrm{d}y + R \, \mathrm{d}z$$

$$\int_{-C} \vec{F} \cdot \mathrm{d}\vec{r} = -\int_C \vec{F} \cdot \mathrm{d}\vec{r}$$

1.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x, y) = y^2\vec{i} + (3x - 6y)\vec{j}$ and C is the line segment from $(3, 7)$ to $(0, 12)$.

$$\vec{r}(t) = (1 - t)\langle 3, 7\rangle + t\langle 0, 12\rangle = \langle 3 - 3t, 7 + 5t\rangle$$

$$\vec{r}'(t) = \langle -3, 5\rangle$$

$$\vec{F}(\vec{r}(t)) = (7 + 5t)^2\vec{i} + (3(3 - 3t) - 6(7 + 5t))\vec{j} = (7 + 5t)^2\vec{i} + (-33 - 39t)\vec{j}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r\prime}(t) = -3(7 + 5t)^2 - 5(33 + 39t)$$

$$= \int_0^1 -3(7 + 5t)^2 - 5(33 + 39t) \, \mathrm{d}t = \left[-\frac{(7 + 5t)^3}{5} - 165t - \frac{195t^2}{2}\right]_0^1$$

$$= -\frac{12}{5} - 165 - \frac{195}{2} - \left(-\frac{1}{5}7^3\right) = -\frac{1079}{2}$$

2.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ **where** $\vec{F}(x,y) = (x+y)\vec{i} + (1-x)\vec{j}$ **and** C **is the portion of** $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **that is in the 4th quadrant with the counter clockwise rotation.**

3.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ **where** $\vec{F}(x,y) = y^2\vec{i} + (x^2-4)\vec{j}$ **and** C **is the portion of** $y = (x-1)^2$**from** $x = 0$ **to** $x = 3$**.**

4.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ **where** $\vec{F}(x,y,z) = e^{2x}\vec{i} + z(y+1)\vec{j} + z^3\vec{k}$ **and** C **is given by** $\vec{r}(t) = t^3\vec{i} + (1-3t)\vec{j} + e^t\vec{k}$ **for** $0 \leq t \leq 2$ **.**

5.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ **where** $\vec{F}(x,y) = (x+y)\vec{i} + (1-x)\vec{j}$ **and** C **is the portion of** $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **that is in the 4th quadrant with the counter clockwise rotation.**

6.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ **where** $\vec{F}(x,y) = (x+y)\vec{i} + (1-x)\vec{j}$ **and** C **is the portion of** $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **that is in the 4th quadrant with the counter clockwise rotation.**

7.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ **where** $\vec{F}(x,y) = (6x-2y)\vec{i} + x^2\vec{j}$ **for each of the following curves.**

a)

C **is the line segment from** $(6,-3)$ **to** $(0,0)$ **followed by the line segment from** $(0,0)$ **to** $(6,3)$**.**

b)

C **is the line segment from** $(6,-3)$ **to** $(6,3)$**.**

8.

Evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y) = 3\vec{i} + (xy - 2x)\vec{j}$ for each of the following curves.

a)

C is the upper half of the circle centered at the origin of radius 4 with counter clockwise rotation.

b)

C is the upper half of the circle centered at the origin of radius 4 with clockwise rotation. Evaluate C
→ F d → r where → F (x , y) = (x + y) → i + (1 − x) → j and C is the portion of x 2 4 + y 2 9 = 1 that is in the 4th quadrant with the counter clockwise rotation. C
→ F d → r = − C
→ F d → r → r (t) = x (t) → i + y (t) → j + z (t) → k , a t b the line integral is,
C
→ F d → r = b a (P → i + Q → j + R → k) (x → i + y → j + z → k) d t = b a P x + Q y + R z d t = b a P x d t + b aGiven the vector field → F (x , y , z) = P → i + Q → j + R → k and the curve C parameterized by → r (t) = x (t) → i + y (t) → j + z (t) → k , a t b the line integral is,
C
→ F d → r = b a (P → i + Q → j + R → k) (x → i + y → j + z → k) d t = b a P x + Q y + R z d t = b a P x d t + b a Q y d t + b a R z d t =
C
P d x + C
Q d y + C
R d z = C
P d x + Q d y +
Q y d t + b a R z d t = C
P d x + C
Q d y + C
R d z = C
P d x + Q d y +

$$f(x) = \sqrt{1+\sqrt{x}}$$

$$f'(x) = (1+(x)^{\frac{1}{2}})^{\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}$$

$$F(x) = \int_0^{\sin x} \frac{t^2}{t^2 - 1} \, \mathrm{d}t$$

$$F'(x) = \frac{\sin^2 x}{\sin^2 x - 1} \cdot \frac{d}{\mathrm{d}x} \sin x$$

$$= -\frac{\sin^2 x}{1 - \sin^2 x} \cdot \cos x$$

$$= -\frac{\sin^2 x}{\cos^{\cancel{2}} x} \cdot \cancel{\cos x}$$

$$= -\frac{\sin^2 x}{\cos x}$$

$$y - y_0 = m(x - x_0)$$

$$m = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{3t^2}{2t} = \frac{3t}{2}$$

$$m(t = 5) = \frac{15}{2}$$

$$x(5) = 25 \qquad y(5) = 125$$

$$y - 125 = \frac{15}{2}(x - 25) = y = 125 + \frac{15}{2}(x - 25)$$

4. *(2 valores)* Calcule uma (apenas uma) das seguintes primitivas

$$\int \frac{x^3}{1 + x^8} \, dx \qquad\qquad \int (x + 2) \, e^x \, dx$$

$$\int = \frac{x^3}{1 + x^8} \, \mathrm{d}x = \frac{1}{4} \int = \frac{4x^3}{1 + (x^4)^2} \, \mathrm{d}x = \frac{1}{4} \arctan x^4$$

$$\int = (x + 2)e^x \, \mathrm{d}x = \int xe^x + 2e^x \, \mathrm{d}x$$

$$= \int xe^x \, \mathrm{d}x + \int 2e^x \, \mathrm{d}x$$

$$\int xe^x \, \mathrm{d}x = xe^x - e^x$$

$$= \int xe^x \,\mathrm{d}x + \int 2e^x \,\mathrm{d}x = (x+1)e^x$$

5. *(2 valores)* Calcule um (apenas um) dos seguintes integrais

$$\int_{-\pi/2}^{\pi/2} \cos^2\theta \,\sin\theta \,d\theta \qquad\qquad \int_1^3 \log(x^3)\,dx$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \sin\theta \,\mathrm{d}\theta = \frac{1}{3}\left[-\cos^3\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int = \frac{4x^3}{1+(x^4)^2}\,\mathrm{d}x = 0$$

$$\int_1^3 \ln x^3 \,\mathrm{d}x = 3\int_1^3 \ln x \,\mathrm{d}x$$

$$f = \ln x \qquad g' = 1$$

$$f' = \frac{1}{x} \qquad g = x$$

$$3\int_1^3 \ln x \,\mathrm{d}x = [x\ln x]_1^3 - \int_1^3 1 = 3x\ln x - 3x = 9\ln 3 - 6$$

6. *(2 valores)* Determine a solução da equação diferencial $\frac{dx}{dt} = 2 - x$ com condição inicial $x(0) = 1$.

$$\dot{y} = 2 - x \qquad x(0) = 1$$

$$Y_c = c_1 e^{-t}$$

$$2 - x = \Leftrightarrow x = 2$$

$$c_1 e^0 + 2 = 3 \Leftrightarrow c_1 = 1$$

$$Y_c = e^{-t} + 2$$

8. *(2 valores)* Calcule o limite

$$\lim_{x\to\pi} \frac{1+\cos x}{x-\pi}$$

Método 1:

$$f(x) = \cos x$$

$$f'(\pi) = \frac{f(x) - f(\pi)}{x - \pi}$$

$$f'(\pi) = -\sin \pi = 0$$

Método 2:

$$\lim_{x \to \pi} = \frac{1 + \cos x}{x - \pi} = \frac{\frac{\mathrm{d}}{x}(1 + \cos x)}{\frac{\mathrm{d}}{x}(x - \pi)} = 0$$

$\ddot{y} + 9y = \sin(\pi t)$

$$y_c(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

$$z = b \sin(\pi t)$$

$$z' = \pi b \cos(\pi t)$$

$$z'' = -\pi^2 b \sin(\pi t)$$

$$-\pi^2 b \sin(\pi t) + 9bt \sin(\pi t) = \sin(\pi t)$$

$$b\cancel{\sin(\pi t)}(-\pi^2 + 9) = \cancel{\sin(\pi t)}$$

$$b = \frac{1}{-\pi^2 + 9}$$

$$y_c(t) + y_p(t) = c_1 \cos(3t) + c_2 \sin(3t) + \frac{1}{-\pi^2 + 9} \sin(\pi t)$$

$\ddot{y} + 9y = \sin(3t)$
Método 1:

$$y_c(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

$$z = bt \cos(3t)$$

$$z' = -3bt \sin(3t) + b \cos(3t)$$

$$z'' = -6b\sin(3t) - 9bt\cos(3t)$$

$$-6b\sin(3t) - 9bt\cos(3t) + 9bt\cos(3t) = \sin(3t)$$

$$-6b\sin(3t) = \sin(3t)$$

$$b = -\frac{1}{6}$$

$$y_c(t) + y_p(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{1}{6}t\cos(3t)$$

Método 2:

$$y_c(t) = c_1\cos(3t) + c_2\sin(3t)$$

$$y_1(t) = \cos(3t) \qquad y_2(t) = \sin(3t)$$

$$W = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3$$

$$Y_p(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)}\ \mathrm{d}t + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)}\ \mathrm{d}t$$

$$Y_p(t) = -\frac{\cos(3t)}{3} \int \sin(3t)\sin(3t)\ \mathrm{d}t + \frac{\sin(3t)}{3} \int \cos(3t)\sin(3t)\ \mathrm{d}t$$

$$sin^2(3t) = \frac{1-\cos(6t)}{2}$$

$$Y_p(t) = -\frac{\cos(3t)}{3} \int \sin^2(3t)\ \mathrm{d}t + \frac{\sin(3t)}{3} \cdot -\frac{\sin^2(3t)}{6}$$

$$Y_p(t) = -\frac{\cos(3t)}{3} \int \frac{1}{2} - \frac{1}{2}\cos(6t)\ \mathrm{d}t + \frac{\sin(3t)}{3}(-\frac{1}{12} + \frac{1}{12}\cos(6t)))$$