

Find the derivative matrices in Exercises 5–8 and evaluate at the given points.

5. $\partial(x, y)/\partial(u, v)$; $x = u \sin v$, $y = e^{uv}$; at $(0, 1)$.

$$x = u \sin(v) \quad y = e^{uv}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{pmatrix} \sin(v) & u \cos(v) \\ ve^{uv} & ue^{uv} \end{pmatrix} = \begin{pmatrix} \sin 1 & 0 \\ 1 & 0 \end{pmatrix}$$

6. $\partial(x, y, z)/\partial(r, \theta, \phi)$; where $x = r \sin \phi \cos \theta$,
 $y = r \sin \phi \sin \theta$, $z = r \cos \phi$; at $(2, \pi/3, \pi/4)$.

$$x = r \sin \phi \cos \theta \quad y = r \sin \phi \sin \theta \quad z = r \cos \phi$$

$$\frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} = \begin{pmatrix} \sin \phi \cos \theta & -r \sin \phi \sin \theta & r \cos \phi \cos \theta \\ \sin \phi \sin \theta & r \sin \phi \cos \theta & r \cos \phi \sin \theta \\ \cos \phi & 0 & -r \sin \phi \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\sqrt{2} \end{pmatrix}$$

Compute $\partial z/\partial x$ and $\partial z/\partial y$ in Exercises 21–24 using matrix multiplication and by direct substitution.

21. $z = u^2 + v^2$; $u = 2x + 7$, $v = 3x + y + 7$.

22. $z = u^2 + 3uv - v^2$; $u = \sin x$,
 $v = -\cos x + \cos y$.

23. $z = \sin u \cos v$; $u = 3x^2 - 2y$, $v = x - 3y$.

24. $z = u/v^2$; $u = x + y$, $v = xy$.

21.

$$z = u^2 + v^2 \quad u = 2x + 7 \quad v = 3x + y + 7$$

$$\begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = (2u \quad 2v) \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$

$$= (4u + 6v \quad 0u + 2v) = (26x + 6y + 70 \quad 6x + 2y + 14)$$

22.

$$z = u^2 + 3uv - v^2 \quad u = \sin x \quad v = -\cos x + \cos y$$

$$\begin{aligned} \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} &= \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = (2u + 3v \quad 3u - 2v) \begin{pmatrix} \cos x & 0 \\ \cos x & -\sin y \end{pmatrix} \\ &= ((2u + 3v) \cos x + \cos x(3u - 2v) \quad 0 - \sin y(3u - 2v)) \\ &= (5 \sin x \cos x + 5 \cos^2 x + \cos y \cos x \quad -3 \sin^2 y) \end{aligned}$$

25. (a) Compute derivative matrices $\partial(x, y)/\partial(t, s)$ and $\partial(u, v)/\partial(x, y)$ if

$$\begin{aligned} x &= t + s, & y &= t - s, \\ u &= x^2 + y^2, & v &= x^2 - y^2. \end{aligned}$$

(b) Express (u, v) in terms of (t, s) and calculate $\partial(u, v)/\partial(t, s)$.

(c) Verify that the chain rule holds.

25.

(a)

$$x = t + s \quad y = t - s \quad u = x^2 + y^2 \quad v = x^2 - y^2$$

$$\frac{\partial(x, y)}{\partial(t, s)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix}$$

(b)

$$u = (t + s)^2 + (t - s)^2 \quad v = (t + s)^2 - (t - s)^2$$

$$u = 2t^2 + 2s^2 \quad v = 4ts$$

$$\begin{pmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial s} \end{pmatrix} = \begin{pmatrix} 4t & 4s \\ 4s & 4t \end{pmatrix}$$

(c)

$$\begin{aligned}\frac{\partial(u, v)}{\partial(t, s)} &= \frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(t, s)} = \begin{pmatrix} 2t + 2s & 2t - 2s \\ 2t + 2s & -2t + 2s \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4t & 4s \\ 4s & 4t \end{pmatrix}\end{aligned}$$

26.

(a)

$$x = t^2 + s^2 \quad y = ts \quad u = \sin(x + y) \quad v = \cos(x - y)$$

$$\begin{aligned}\frac{\partial(x, y)}{\partial(t, s)} &= \begin{pmatrix} 2t & -2s \\ s & t \end{pmatrix} \\ \frac{\partial(u, v)}{\partial(x, y)} &= \begin{pmatrix} \cos(x + y) & \cos(x + y) \\ -\sin(x - y) & \sin(x - y) \end{pmatrix}\end{aligned}$$

(b)

$$u = \sin(t^2 - s^2 + ts) \quad v = \cos(t^2 - s^2 - ts)$$

$$\begin{pmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial s} \end{pmatrix} = \begin{pmatrix} (2t + s) \cos(t^2 - s^2 + ts) & -(2s + t) \cos(t^2 - s^2 + ts) \\ -(2t - s) \sin(t^2 - s^2 - ts) & -(2s - t) \sin(t^2 - s^2 - ts) \end{pmatrix}$$

29. Suppose that a function is given in terms of rectangular coordinates by $u = f(x, y, z)$. If

$$x = r \cos \theta \sin \phi,$$

$$y = r \sin \theta \sin \phi,$$

$$z = r \cos \phi,$$

express $\partial u / \partial r$, $\partial u / \partial \theta$, and $\partial u / \partial \phi$ in terms of $\partial u / \partial x$, $\partial u / \partial y$, and $\partial u / \partial z$.

29.

$$x = r \cos \theta \sin \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \phi$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial \theta}$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial \phi}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cos \theta \sin \phi + \frac{\partial u}{\partial y} \sin \theta \sin \phi + \frac{\partial u}{\partial z} \cos \phi$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} (-r \sin \theta \sin \phi) + \frac{\partial u}{\partial y} (r \cos \theta \sin \phi) + \frac{\partial u}{\partial z} \cdot 0$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x} (r \cos \theta \cos \phi) + \frac{\partial u}{\partial y} (r \sin \theta \cos \phi) + \frac{\partial u}{\partial z} (-r \sin \phi)$$

30. Suppose that x, y, z are as in Exercise 29 and $u = x^2 + y^2 + z^2$. Find $\partial u / \partial r$, $\partial u / \partial \theta$, and $\partial u / \partial \phi$.

30.

$$u = x^2 + y^2 + z^2 \quad x = r \cos \theta \sin \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \phi$$

$$u = (r \cos \theta \sin \phi)^2 + (r \sin \theta \sin \phi)^2 + (r \cos \phi)^2 = r^2 (\cos^2 \theta \sin^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \phi)$$

$$= r^2 (\sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + \cos^2 \phi)$$

$$= r^2$$

$$\frac{\partial u}{\partial r} = 2r$$

$$\frac{\partial u}{\partial \theta} = 0$$

$$\frac{\partial u}{\partial \phi} = 0$$

31. Express the polar coordinates r and θ in terms of the cartesian coordinates x and y , and find the derivative matrix $\partial(r, \theta)/\partial(x, y)$.

31.

$$x = r \cos \theta \quad y = r \sin \theta \quad r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$

$$\frac{\partial(r, \theta)}{\partial(x, y)} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

$$\begin{pmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{pmatrix}$$

32. Let A be the derivative matrix $\partial(x, y)/\partial(r, \theta)$ for $x = r \cos \theta$, $y = r \sin \theta$. Let B be the derivative matrix $\partial(r, \theta)/\partial(x, y)$ of Exercise 31, with its entries expressed in terms of r and θ . Find AB and BA .

32.

$$x = r \cos \theta \quad y = r \sin \theta \quad r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$

$$A = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} \end{pmatrix}$$

$$AB = \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \cos \theta \sin \theta \\ \sin \theta \cos \theta - \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$