

Find the derivative matrices in Exercises 5–8 and evaluate at the given points.

5.  $\partial(x, y)/\partial(u, v)$ ;  $x = u \sin v$ ,  $y = e^{uv}$ ; at  $(0, 1)$ .

$$x = u \sin(v) \quad y = e^{uv}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{pmatrix} \sin(v) & u \cos(v) \\ ve^{uv} & ue^{uv} \end{pmatrix} = \begin{pmatrix} \sin 1 & 0 \\ 1 & 0 \end{pmatrix}$$

6.  $\partial(x, y, z)/\partial(r, \theta, \phi)$ ; where  $x = r \sin \phi \cos \theta$ ,  
 $y = r \sin \phi \sin \theta$ ,  $z = r \cos \phi$ ; at  $(2, \pi/3, \pi/4)$ .

$$x = r \sin \phi \cos \theta \quad y = r \sin \phi \sin \theta \quad z = r \cos \phi$$

$$\frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} = \begin{pmatrix} \sin \phi \cos \theta & -r \sin \phi \sin \theta & r \cos \phi \cos \theta \\ \sin \phi \sin \theta & r \sin \phi \cos \theta & r \cos \phi \sin \theta \\ \cos \phi & 0 & -r \sin \phi \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\sqrt{2} \end{pmatrix}$$

Compute  $\partial z/\partial x$  and  $\partial z/\partial y$  in Exercises 21–24 using matrix multiplication and by direct substitution.

21.  $z = u^2 + v^2$ ;  $u = 2x + 7$ ,  $v = 3x + y + 7$ .

22.  $z = u^2 + 3uv - v^2$ ;  $u = \sin x$ ,  
 $v = -\cos x + \cos y$ .

23.  $z = \sin u \cos v$ ;  $u = 3x^2 - 2y$ ,  $v = x - 3y$ .

24.  $z = u/v^2$ ;  $u = x + y$ ,  $v = xy$ .

21.

$$z = u^2 + v^2 \quad u = 2x + 7 \quad v = 3x + y + 7$$

$$\begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = (2u \quad 2v) \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$

$$= (4u + 6v \quad 0u + 2v) = (26x + 6y + 70 \quad 6x + 2y + 14)$$

**22.**

$$z = u^2 + 3uv - v^2 \quad u = \sin x \quad v = -\cos x + \cos y$$

$$\begin{aligned} \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} &= \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = (2u + 3v \quad 3u - 2v) \begin{pmatrix} \cos x & 0 \\ \cos x & -\sin y \end{pmatrix} \\ &= ((2u + 3v) \cos x + \cos x(3u - 2v) \quad 0 - \sin y(3u - 2v)) \\ &= (5 \sin x \cos x + 5 \cos^2 x + \cos y \cos x \quad -3 \sin^2 y) \end{aligned}$$

- 25. (a) Compute derivative matrices  $\partial(x, y)/\partial(t, s)$  and  $\partial(u, v)/\partial(x, y)$  if**

$$x = t + s, \quad y = t - s,$$

$$u = x^2 + y^2, \quad v = x^2 - y^2.$$

- (b) Express  $(u, v)$  in terms of  $(t, s)$  and calculate  $\partial(u, v)/\partial(t, s)$ .**  
**(c) Verify that the chain rule holds.**

**(a)**

$$x = t + s \quad y = t - s \quad u = x^2 + y^2 \quad v = x^2 - y^2$$

$$\frac{\partial(x, y)}{\partial(t, s)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix} = \begin{pmatrix} 2t + 2s & 2t - 2s \\ 2t + 2s & -2t + 2s \end{pmatrix}$$