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Horodeckis criterion of separability of mixed states in von Neumann and C*-algebras

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The Horodeckis necessary and sufficient condition of separability of mixed states is generalized to arbitrary composite quantum systems.

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1. Introduction

Entangled states are one of the most astonishing features of quantum mechanics, and a resource to perform various quantum information tasks such as quantum computing, cryptography and teleportation.[13] Generally, to decide whether or not a state of a composite quantum system is separable is a challenging task. One of a few known necessary and sufficient criteria for detecting entanglement is based on positive maps. For quantum systems represented by finite dimensional Hilbert spaces, it was shown that[10]:

Horodeckis' theorem. Let $\mathcal{H}$, $\mathcal{K}$ *be finite dimensional complex Hilbert spaces and let* ρ *be a density matrix acting on* $\mathcal{H} \otimes \mathcal{K}$*. Then* ρ *is separable, if and only if for any positive linear map* $S : B(\mathcal{K}) \to B(\mathcal{H})$*, where* $B(\mathcal{H})$ *stands for the algebra of all linear operators (matrices) on* $\mathcal{H}$*, the operator* $(I \otimes S)\rho$ *is positive on* $\mathcal{H} \otimes \mathcal{H}$*.*

It should be mentioned that the work of Choi[6] played an important role in the formulation of the above theorem. In the case of $\mathcal{H} = \mathbb{C}^2$ and $\mathcal{K} = \mathbb{C}^2$ or $\mathcal{K} = \mathbb{C}^3$, due to the complete characterization of positive linear maps between $B(\mathcal{H})$ and $B(\mathcal{K})$,[16,22] that leads to a useful criterion of separability of density matrices on

$\mathcal{H} \otimes \mathcal{K}$, namely positivity of their partial transpose (PPT).[10,14] It was next generalized to infinite dimensional Hilbert spaces by Størmer,[17] who proved that a state ρ on $\mathcal{H} \otimes \mathcal{K}$ is separable, if and only if for any normal, positive and linear map $S : B(\mathcal{K}) \to B(\mathcal{H})$, the operator $(I \otimes S)\rho$ is positive. The idea was to reduce the problem to finite dimensional subspaces by considering $(P \otimes Q)\rho(P \otimes Q)$, where P and Q are finite dimensional projections in $B(\mathcal{H})$ and $B(\mathcal{K})$, respectively. Recently, Hou[11] improved further the above result and established the elementary operator criterion: if ρ is a state on $\mathcal{H}\otimes\mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces, then ρ is separable, if and only if $(I \otimes S)\rho \geq 0$ for every positive finite rank elementary map $S : B(\mathcal{K}) \to B(\mathcal{H})$. Let us point out that it was conjectured[17] that the positive map criterion should also work for other von Neumann algebras. A growing interest for such a generalization arises from a very active field of entanglement of quantum composite systems with continuous degrees of freedom, the so-called CV-entanglement.[1] In this paper, we prove that conjecture for states on $\mathfrak{M} \bar{\otimes} \mathfrak{N}$, when one of the von Neumann algebras, $\mathfrak{M}$ or $\mathfrak{N}$, is injective. Moreover, we will not assume that $\mathfrak{M}$ or $\mathfrak{N}$ are factors.

It is worth emphasizing that all of the following von Neumann algebras are injective: Abelian, finite dimensional, $B(\mathcal{H})$ with arbitrary Hilbert space $\mathcal{H}$, hyperfinite II_1 factors representing infinite spin systems, the Araki–Woods algebra representing nonrelativistic infinite free Bose gas,[2] factors of type III_1 representing local algebras $\mathcal{M}(O)$ in quantum field theory.[23] As tensor products of injective algebras are again injective, so all quantum systems, even those with central charges, i.e. whose operator algebras have nontrivial centers, are described by injective von Neumann algebras. Therefore, the proposed generalization covers all physically interesting cases.

The paper is organized as follows. In Sec. 2, we introduce notation and present basic mathematical definitions and known results. The main theorem is formulated and proved in Sec. 3. Its reformulation to the C^*-algebra setting, given in Sec. 4, ends the paper.

2. Preliminaries

For Banach spaces X and Y, we denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear maps from X into Y. The pairing between $x \in X$ and $\varphi \in X^*$ is denoted by $\langle \varphi, x\rangle$. We say that a net (T_λ), $T_\lambda \in \mathcal{L}(X, Y^*)$, converges to $T \in \mathcal{L}(X, Y^*)$ in the *weak* operator topology*, if $\langle T_\lambda x, y\rangle \to \langle Tx, y\rangle$ for any $x \in X$ and $y \in Y$. If X and Y are also operator spaces, we denote by $\mathcal{CB}(X, Y)$ the operator space of completely bounded linear maps equipped with the norm $\|\cdot\|_{\mathcal{CB}}$. It is true that the space $\mathcal{F}(X, Y)$ of finite rank maps from X into Y, i.e. the maps with their image being finite dimensional subspaces of Y, is a subspace of $\mathcal{CB}(X, Y)$. By $\mathcal{CB}(X, Y)_+$ we denote the cone of completely bounded positive maps from X and Y, and by $\mathcal{CP}(X, Y)$ the cone of completely positive maps. It is known that $\mathcal{CP}(X, Y) \subset \mathcal{CB}(X, Y)_+$, (see the book by Effros and Ruan[8] for a general outline of the theory of completely bounded and completely positive maps).

Let $\mathfrak{M}$, $\mathfrak{N}$ be arbitrary von Neumann algebras. Without any loss of generality we may assume that $\mathfrak{M} \subset B(\mathcal{H})$ and $\mathfrak{N} \subset B(\mathcal{K})$, where $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, not necessarily separable. Their state spaces, i.e. convex sets of normal, positive and normalized functionals, we denote by $\mathcal{S}(\mathfrak{M})$ and $\mathcal{S}(\mathfrak{N})$, respectively. The algebraic tensor product is denoted by $\mathfrak{M} \otimes \mathfrak{N}$; whereas the spatial tensor product of $\mathfrak{M}$ and $\mathfrak{N}$ by $\mathfrak{M} \bar{\otimes} \mathfrak{N}$, i.e. $\mathfrak{M} \bar{\otimes} \mathfrak{N} = (\mathfrak{M} \otimes \mathfrak{N})''$. Notice that $(\mathfrak{M} \bar{\otimes} \mathfrak{N})_*$ is isometrically isomorphic with $\mathfrak{M}_* \hat{\otimes} \mathfrak{N}_*$, the completion of $\mathfrak{M}_* \otimes \mathfrak{N}_*$ with respect to the projective operator norm $\|\cdot\|_\wedge$, and so $(\mathfrak{M}_* \hat{\otimes} \mathfrak{N}_*)^* = \mathfrak{M} \bar{\otimes} \mathfrak{N}$, with the operator norm on $\mathfrak{M} \bar{\otimes} \mathfrak{N}$ being the dual norm on the dual space $(\mathfrak{M}_* \hat{\otimes} \mathfrak{N}_*)^*$ (see again the book by Effros and Ruan,[8] Theorem 7.2.4; and also Corollary 7.1.5 for the proof of the next theorem). Moreover, one has the following.

Theorem 2.1. *The map* $F : \mathcal{CB}(\mathfrak{M}_*, \mathfrak{N}) \to \mathfrak{M} \bar{\otimes} \mathfrak{N}$, *given by*

$$\langle F(T), \omega \otimes \varphi \rangle = \langle T\omega, \varphi \rangle, \tag{2.1}$$

where $\omega \in \mathfrak{M}_*$, $\varphi \in \mathfrak{N}_*$, *extends to a completely isometric isomorphism between operator spaces.*

Let us recall that a density matrix acting on $\mathcal{H} \otimes \mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional Hilbert spaces, is separable, if it is a convex combination of tensor products of density matrices in $\mathcal{H}$ and $\mathcal{K}$. The definition was next generalized by Werner[21] to infinite dimensional Hilbert spaces by taking the trace norm limits of such convex combinations. In the same spirit, we define a convex set C_1 of separable states in $\mathcal{S}(\mathfrak{M} \bar{\otimes} \mathfrak{N})$ by

$$C_1 = \overline{\mathrm{conv}}^{\|\cdot\|_\wedge} \{\omega \otimes \varphi : \omega \in \mathcal{S}(\mathfrak{M}), \varphi \in \mathcal{S}(\mathfrak{N})\}. \tag{2.2}$$

By C_1^* we denote the dual cone: $C_1^* = \{x \in \mathfrak{M} \bar{\otimes} \mathfrak{N} : \langle x, \varphi \rangle \geq 0 \,\forall\, \varphi \in C_1\}$, and by C_0^* the subcone of C_1^*: $C_0^* = \{x \in \mathfrak{M} \otimes \mathfrak{N} : \langle x, C_1 \rangle \geq 0\}$. It follows easily from Theorem 2.1 that $F(\mathcal{CB}(\mathfrak{M}_*, \mathfrak{N})_+) = C_1^*$ and $F(\mathcal{CP}(\mathfrak{M}_*, \mathfrak{N})) = (\mathfrak{M} \bar{\otimes} \mathfrak{N})_+$, the cone of positive elements.

For any von Neumann algebra $\mathfrak{M}$, there is also a natural completely isometric embedding

$$\theta : \mathfrak{M} \check{\otimes} \mathfrak{M}_* \to \mathcal{CB}(\mathfrak{M}_*, \mathfrak{M}_*), \tag{2.3}$$

where $\check{\otimes}$ stands for injective operator space tensor product (see again Proposition 8.1.2 of Ref. 8). For instance, if $u \in \mathfrak{M} \otimes \mathfrak{M}_*$, $u = \sum_{i=1}^n a_i \otimes \omega_i$, then for $\omega \in \mathfrak{M}_*$,

$$\theta(u)\omega = \sum_{i=1}^n \langle a_i, \omega \rangle \omega_i. \tag{2.4}$$

Using the map F, we show the following.

Proposition 2.1. *Let* $\omega \in \mathcal{S}(\mathfrak{M} \bar{\otimes} \mathfrak{N})$. *Then* $\langle F(A), \omega \rangle \geq 0 \;\forall\, A \in \mathcal{CB}(\mathfrak{M}_*, \mathfrak{N})_+$, *if and only if* ω *is separable.*

Proof. ($\Rightarrow$) Suppose that $\omega \notin C_1$. As C_1 is norm closed and convex, so there exists $x \in \mathfrak{M} \bar{\otimes} \mathfrak{N}$ such that $\langle x, \omega \rangle < 0$ and $\langle x, C_1 \rangle \geq 0$. Therefore, $x \in C_1^*$ and so there exists $A \in \mathcal{CB}(\mathfrak{M}_*, \mathfrak{N})_+$ such that $\langle F(A), \omega \rangle < 0$, a contradiction.
($\Leftarrow$) This part is clear. $\square$

Proposition 2.2. *Let $E \subset \mathcal{CB}(\mathfrak{M}_*, \mathfrak{N})_+$ and let B_1 be a unit ball of $\mathcal{CB}(\mathfrak{M}_*, \mathfrak{N})$ (with respect to the $\mathcal{CB}$-norm). Let also $E \cap B_1$ be dense in $\mathcal{CB}(\mathfrak{M}_*, \mathfrak{N})_+ \cap B_1$ in the weak* operator topology. Suppose that $\omega_0 \in \mathcal{S}(\mathfrak{M} \bar{\otimes} \mathfrak{N})$. If $\forall\, A \in E$ $\langle F(A), \omega_0 \rangle \geq 0$, then $\omega_0 \in C_1$.*

Proof. At first, we show that $F(E)$ is weak* dense in C_1^*. Let $x \in C_1^*$. Then $x = F(A)$, $A \in \mathcal{CB}(\mathfrak{M}_*, \mathfrak{N})_+$. By the assumption, there exists a net (A_λ), $A_\lambda \in E$, $\sup_\lambda \|A_\lambda\|_{\mathcal{CB}} \leq \|A\|_{\mathcal{CB}}$ such that $A_\lambda \to A$ in the weak* operator topology (see p. 2). The map F is continuous with respect to that topology because of Theorem 2.1. Therefore, for any $\omega \in \mathfrak{M}_* \otimes \mathfrak{N}_*$ $\langle F(A_\lambda), \omega \rangle \to \langle x, \omega \rangle$. As $\|F(A_\lambda)\| = \|A_\lambda\|_{\mathcal{CB}}$ is uniformly bounded so $F(A_\lambda)$ converges to x in the weak* topology of von Neumann algebra $\mathfrak{M} \bar{\otimes} \mathfrak{N}$. Hence $\langle x, \omega_0 \rangle \geq 0$ for any $x \in C_1^*$, and so $\omega_0 \in C_1$. $\square$

It follows that a kind of positive uniform approximation in $\mathfrak{M}$ or $\mathfrak{N}$ is necessary to accomplish our task of generalizing the Horodeckis' theorem. This leads to the assumption about injectivity of one (say $\mathfrak{M}$) of the algebras. Let us recall that a von Neumann algebra $\mathfrak{M} \subset B(\mathcal{H})$ is called injective if there is a norm-one Banach space projection (not necessarily normal) $\pi : B(\mathcal{H}) \to \mathfrak{M}$ onto $\mathfrak{M}$. In such an algebra, there is a net (A_λ) of completely positive finite rank contractions on $\mathfrak{M}_*$ converging strongly to the identity map (see the Takesaki's textbook,[20] Vol. III, Theorem XV.3.1):

$$\|A_\lambda \omega - \omega\| \xrightarrow{\lambda} 0, \tag{2.5}$$

for any $\omega \in \mathfrak{M}_*$. This property of injective von Neumann algebras will prove itself of particular use in the following.

3. Horodeckis Criterion in von Neumann Algebras

We embark on an attempt to prove the Horodeckis criterion for separable states. Starting from the simplest case, we shall proceed gradually with the help of the structure theory of von Neumann algebras.

Theorem 3.1. *Suppose that $\mathfrak{M}$ is injective. Then a state $\tilde{\phi} \in \mathcal{S}(\mathfrak{M} \bar{\otimes} \mathfrak{N})$ is separable, if and only if the functional $\tilde{\phi} \circ (\mathbf{1} \otimes S)$ is positive for any normal positive and finite rank map $S : \mathfrak{M} \to \mathfrak{N}$.*

Proof. Suppose that $\tilde{\phi}$ is separable. Let $S : \mathfrak{M} \to \mathfrak{N}$ be a positive and normal map with the predual S_*, and let z be a positive element of $\mathfrak{M} \bar{\otimes} \mathfrak{M}$. Because

$\tilde{\phi} = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i^{(n)} \omega_i \otimes \varphi_i$, where $\alpha_i^{(n)} \geq 0$, $\sum_{i=1}^{n} \alpha_i^{(n)} = 1$, $\omega_i \in \mathcal{S}(\mathfrak{M})$ and $\varphi_i \in \mathcal{S}(\mathfrak{N})$, so

$$\langle (\mathbf{1} \otimes S)(z), \tilde{\phi} \rangle = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i^{(n)} \langle z, \omega_i \otimes S_* \varphi_i \rangle \geq 0. \tag{3.1}$$

The proof of the reverse implication is contained in the following series of steps.

Step 1. Denseness of C_0^*.

Lemma 3.1. *Suppose that the algebra $\mathfrak{M}$ is injective and let (A_λ) be a net of completely positive finite rank contractions on $\mathfrak{M}_*$ such that $A_\lambda \xrightarrow{\lambda} \mathbf{1}$ strongly. If $z \in \mathfrak{M} \,\bar{\otimes}\, \mathfrak{N}$, then $z = \mathrm{w}^* - \lim_\lambda (\mathbf{1} \otimes F^{-1}(z))(\theta^{-1}(A_\lambda))$.*

Proof. As each norm $\|A_\lambda\|_{\mathcal{CB}} = \|A_\lambda\| \leq 1$, it is enough to show that, for any $\omega \in \mathfrak{M}_*$ and $\varphi \in \mathfrak{N}_*$, $\langle z_\lambda, \omega \otimes \varphi \rangle \to \langle z, \omega \otimes \varphi \rangle$, where by (z_λ) we have denoted the net $z_\lambda = (\mathbf{1} \otimes F^{-1}(z))(\theta^{-1}(A_\lambda)) \in \mathfrak{M} \otimes \mathfrak{N}$. Writing $\theta^{-1}(A_\lambda) = \sum_{i=1}^{N_\lambda} a_i^\lambda \otimes \omega_i^\lambda$, we have that

$$\begin{aligned} \langle z_\lambda, \omega \otimes \varphi \rangle &= \sum_{i=1}^{N_\lambda} \langle a_i^\lambda \otimes F^{-1}(z) \omega_i^\lambda, \omega \otimes \varphi \rangle \\ &= \sum_{i=1}^{N_\lambda} \langle a_i^\lambda, \omega \rangle \langle z, \omega_i^\lambda \otimes \varphi \rangle = \langle z, (A_\lambda \omega) \otimes \varphi \rangle. \end{aligned} \tag{3.2}$$

As $A_\lambda \xrightarrow{\lambda} \mathbf{1}$ strongly, we obtain that

$$|\langle z - z_\lambda, \omega \otimes \varphi \rangle| \leq \|A_\lambda \omega - \omega\| \|\varphi\| \|z\| \xrightarrow{\lambda} 0. \tag{3.3}$$

□

Corollary 3.1. *The above lemma shows that C_0^* is weakly* dense in C_1^*. Indeed, for an element $z \in C_1^*$, as (A_λ) is a net of completely positive finite rank maps, each $z_\lambda \in C_0^*$.*

Step 2. Finite injective separable factors.

Suppose now that $\mathcal{R}$ is an injective separable factor of type II_1,[7] and let τ be the normalized trace on $\mathcal{R}$. A von Neumann algebra is by definition *separable*, if it acts on a separable Hilbert space. As $\mathcal{R}$ is finite so there exists an embedding $i : \mathcal{R} \to \mathcal{R}_*$, given by $i(x) = \omega_x$, $\omega_x(y) = \tau(xy)$, for any $x, y \in \mathcal{R}$. It is clear that $i(\mathcal{R})$ is norm dense in $\mathcal{R}_*$. However, it should be noticed that the embedding is positive and completely bounded, although not completely positive. The following construction of the explicit representation of $\mathcal{R}$, taken from the book by Evans,[9] Example 5.16, will prove itself useful in the sequel. Let $\mathcal{A}_0$ be a union of the increasing sequence of matrix algebras

$$M_2(\mathbb{C}) \subset M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \subset M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \subset \cdots. \tag{3.4}$$

On the nth tensor power of $M_2(\mathbb{C})$, denoted by $M_2(\mathbb{C})^{\otimes n}$, we define $\tau(x) = \frac{1}{2^n}\mathrm{Tr}(x)$, and an embedding of $M_2(\mathbb{C})^{\otimes n}$ into $\mathcal{A}_0$, $x \mapsto x \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots$. Then τ is a well-defined tracial state on $\mathcal{A}_0$. Let $\mathcal{H}$ be a separable Hilbert space that arises as a completion of $\mathcal{A}_0$ with respect to the inner product $\langle x, y\rangle = \tau(y^*x)$. We can consider $\mathcal{A}_0$ as a *-subalgebra of $B(\mathcal{H})$ by letting $\mathcal{A}_0$ act on $\mathcal{H}$ by left multiplication. Now, we define the von Neumann algebra $\mathcal{R}$ as a weak closure of $\mathcal{A}_0$ in $B(\mathcal{H})$. It is known that $\mathcal{R}$ is a type II_1 separable factor and the extension of τ, which we denote by the same symbol, is the unique normalized trace on $\mathcal{R}$. Next, we define a linear map β_0 on $\mathcal{A}_0$ such that

$$\beta_0(x^1 \otimes \cdots \otimes x^n \otimes \mathbf{1} \otimes \cdots) = (x^1)^t \otimes \cdots \otimes (x^n)^t \otimes \mathbf{1} \otimes \cdots, \tag{3.5}$$

where $(x^i)^t$ stands for transposition of $x^i \in M_2(\mathbb{C})$, $i = 1, 2, \ldots, n$. It is obvious that β_0 is a *-anti-automorphism on $\mathcal{A}_0$, and so it can be extended to the σ-weakly continuous (normal) *-anti-automorphism β on the algebra $\mathcal{R}$. Although, neither the map i nor β is completely positive, in fact, β is not even completely bounded, we show that the map $i \circ \beta$ is a completely positive contraction. Indeed, let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be arbitrary elements of $\mathcal{R}$. Then

$$\begin{aligned}\sum_{i,j=1}^{n} \langle x_i^* x_j, i\circ\beta(y_i^* y_j)\rangle &= \sum_{i,j=1}^{n} \langle x_i^* x_j, i(\beta(y_j)\beta(y_i^*))\rangle \\ &= \sum_{i,j=1}^{n} \tau(\beta(y_j)\beta(y_i^*)x_i^* x_j) = \sum_{i,j=1}^{n} \tau((x_i\beta(y_i))^* x_j\beta(y_j)) \\ &= \tau\left(\left(\sum_{i=1}^{n} x_i\beta(y_i)\right)^* \left(\sum_{j=1}^{n} x_j\beta(y_j)\right)\right) \geq 0,\end{aligned} \tag{3.6}$$

which proves that $i \circ \beta$ is completely positive. As both i and β are contractive, then so is $i \circ \beta$. From now on, we use for this map the short-hand notation $i\beta$.

Suppose now that $T : \mathcal{R} \to M_2(\mathbb{C})^{\otimes n} \subset \mathcal{R}$ is a normal map of $\mathcal{R}$ onto the nth tensor power of $M_2(\mathbb{C})$, which we treat as a subalgebra of $\mathcal{R}$ in the natural way. Applying a kind of Choi–Jamiolkowski isomorphism,[5,12] we can consider the predual map T_* in terms of tensor product maps on $\mathcal{R} \otimes \mathcal{R}$. Namely, let e_{ij} be the matrix units in $M_2(\mathbb{C})$. We consider the basis elements in $M_2(\mathbb{C})^{\otimes n}$, $e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes \cdots \otimes e_{i_n j_n} \otimes \mathbf{1} \otimes \cdots$, where each of the indices $i_1, j_1, \ldots, i_n, j_n$ takes value 1 or 2. Then, for any $x \in \mathcal{R}$ and $\omega \in \mathcal{R}_*$, there is

$$\begin{aligned}\langle x, T_*\omega\rangle = \langle Tx, \omega\rangle &= 2^n \sum_{i_1, j_1 \ldots i_n, j_n} \tau((Tx)e_{j_1 i_1} \otimes \cdots \otimes e_{j_n i_n} \otimes \mathbf{1} \otimes \cdots) \\ &\quad \cdot \langle e_{i_1 j_1} \otimes \cdots \otimes e_{i_n j_n} \otimes \mathbf{1} \otimes \cdots, \omega\rangle \\ &= 2^n \sum_{i_1, j_1 \ldots i_n, j_n} \tau((Tx)\beta(e_{i_1 j_1} \otimes \cdots \otimes e_{i_n j_n} \otimes \mathbf{1} \otimes \cdots)) \\ &\quad \cdot \langle e_{i_1 j_1} \otimes \cdots \otimes e_{i_n j_n} \otimes \mathbf{1} \otimes \cdots, \omega\rangle\end{aligned}$$

$$= 2^n \sum_{i_1,j_1\ldots i_n,j_n} \langle Tx, i\beta(e_{i_1j_1} \otimes \cdots \otimes e_{i_nj_n} \otimes \mathbf{1} \otimes \cdots)\rangle$$
$$\cdot \langle e_{i_1j_1} \otimes \cdots \otimes e_{i_nj_n} \otimes \mathbf{1} \otimes \cdots, \omega\rangle. \tag{3.7}$$

Hence, if we define a positive element in $\mathcal{R} \otimes \mathcal{R}$ by

$$m_n = 2^n \sum_{i_1,j_1\ldots i_n,j_n} (e_{i_1j_1} \otimes e_{i_2j_2} \otimes \cdots \otimes e_{i_nj_n} \otimes \mathbf{1} \otimes \cdots)$$
$$\otimes (e_{i_1j_1} \otimes e_{i_2j_2} \otimes \cdots \otimes e_{i_nj_n} \otimes \mathbf{1} \otimes \cdots), \tag{3.8}$$

then we can write

$$\theta^{-1}(T_*) = (\mathbf{1} \otimes (T_* \circ i\beta))(m_n). \tag{3.9}$$

Lemma 3.2. *Let $\mathcal{R}$ be a separable injective factor of type II_1 and let $z \in C_1^* \subset \mathcal{R} \bar{\otimes} \mathfrak{N}$. Then, there exists a net S_λ of normal positive and finite rank maps, $S_\lambda : \mathcal{R} \to \mathfrak{N}$, and a sequence $(m_{n(\lambda)})$ of positive elements in $\mathcal{R} \otimes \mathcal{R}$ such that*

$$z = \mathrm{w}^*\!-\lim_\lambda (\mathbf{1} \otimes S_\lambda)(m_{n(\lambda)}). \tag{3.10}$$

Proof. Let (A_λ) be a net of completely positive, finite-rank contractions on $\mathcal{R}_*$, strongly converging to the identity map. We can safely assume that the finite-rank dual maps A_λ^* have their range in $M_2(\mathbb{C})^{\otimes n(\lambda)}$. Thus, by Eq. (3.9),

$$\theta^{-1}(A_\lambda) = (\mathbf{1} \otimes (A_\lambda \circ i\beta))(m_{n(\lambda)}). \tag{3.11}$$

We define a net (S_λ) by $S_\lambda = F^{-1}(z) \circ A_\lambda \circ i\beta$. Since $z \in C_1^*$, each S_λ is a positive normal and finite rank map on $\mathcal{R}$. As

$$(\mathbf{1} \otimes S_\lambda)(m_{n(\lambda)}) = (\mathbf{1} \otimes F^{-1}(z))(\theta^{-1}(A_\lambda)), \tag{3.12}$$

the assertion follows now easily from Lemma 3.1. □

Lemma 3.3. *Let $\tilde{\phi}$ be a state on $\mathfrak{M} \bar{\otimes} \mathfrak{N}$, where $\mathfrak{M}$ is a finite injective separable factor. If $\tilde{\phi} \circ (\mathbf{1} \otimes S)$ is a positive functional on $\mathfrak{M} \bar{\otimes} \mathfrak{M}$ for any positive normal and finite rank map $S : \mathfrak{M} \to \mathfrak{N}$, then $\tilde{\phi}$ is separable.*

Proof. If $\mathfrak{M}$ is a factor of type I, i.e. $\mathfrak{M}$ is in fact a finite dimensional matrix algebra, then the proof occurs already in the work of Horodeckis'.[10] In this case, instead of the sequence $(m_{n(\lambda)})$, it is possible to consider only one, "maximally entangled", element of the tensor product algebra. Hence, suppose that $\mathfrak{M} = \mathcal{R}$. It is enough to prove that $\tilde{\phi}(z) \geq 0$ for any $z \in C_1^* \subset \mathfrak{M} \bar{\otimes} \mathfrak{N}$. By Lemma 3.2, $z = \mathrm{w}^* - \lim_\lambda (\mathbf{1} \otimes S_\lambda)(m_{n(\lambda)})$, where each S_λ is positive normal and finite rank, and each $m_{n(\lambda)}$ is positive in $\mathfrak{M} \otimes \mathfrak{M}$. Then, by the assumption,

$$\langle z, \tilde{\phi}\rangle = \lim_\lambda \langle (\mathbf{1} \otimes S_\lambda)(m_{n(\lambda)}), \tilde{\phi}\rangle \geq 0. \tag{3.13}$$

□

Step 3. Finite injective separable algebras.

In this step, we get rid of the assumption that $\mathfrak{M}$ is a factor. If $\tilde{\phi}$ is a state on $\mathfrak{M}\,\bar{\otimes}\,\mathfrak{N}$, then $\tilde{\phi}$ is separable, if and only if the functional $\tilde{\phi}_p(z) = \langle (p\otimes \mathbf{1})z, \tilde{\phi}\rangle$ is separable on $p\mathfrak{M}\,\bar{\otimes}\,\mathfrak{N}$, for every central projection $p\in\mathfrak{M}$. Thus, we can consider separately von Neumann algebras of type I_n and II_1.

Lemma 3.4. *Lemma* 3.3 *holds for* $\mathfrak{M}$ *being a finite injective and separable von Neumann algebra.*

Proof. If $\mathfrak{M}$ is a type I_n von Neumann algebra, then $\mathfrak{M}$ is isomorphic to an algebra $M_n(\mathbb{C})\,\bar{\otimes}\,\mathcal{A}$, where $\mathcal{A}$ is an Abelian von Neumann algebra (see Blackaddar,[3] III.1.5.12). If $\mathfrak{M}$ is of type II_1, then $\mathfrak{M}\cong\mathcal{R}\,\bar{\otimes}\,\mathcal{C}$, where $\mathcal{C}$ is the centre of $\mathfrak{M}$ (see Takesaki,[20] Theorem XVI.1.5). The rest of the proof relies only on the fact that $\mathfrak{M}$ can be split into a finite injective separable factor and an Abelian von Neumann algebra. Therefore, we provide the proof only for the II_1 case.

We start with representing the Abelian von Neumann algebra $\mathcal{C}$ as $L^\infty(\Gamma,\mu)$, the algebra of essentially bounded functions on a locally compact space Γ with a positive Radon measure μ. Moreover, again since $\mathcal{C}$ is Abelian, $\mathfrak{M}\cong L^\infty_{\mathcal{R}}(\Gamma,\mu)$, the algebra of $\mathcal{R}$-valued bounded functions on Γ, and $\mathfrak{M}_*\cong L^1_{\mathcal{R}_*}(\Gamma,\mu)$, the space of $\mathcal{R}_*$-valued integrable functions on Γ (see Takesaki,[19] Theorem IV.7.17). For any $x\in\mathfrak{M}$ and $\omega\in\mathfrak{M}_*$, there exist functions $x:\Gamma\to\mathcal{R}$ and $\omega:\Gamma\to\mathcal{R}_*$ such that

$$\langle x,\omega\rangle = \int_\Gamma \langle x(\gamma),\omega(\gamma)\rangle d\mu(\gamma), \tag{3.14}$$

together with $\|\omega\| = \int_\Gamma \|\omega(\gamma)\|d\mu(\gamma)$ and $\|x\| = \sup\|x(\gamma)\|$. It should be noted, that by definition of the space $L^1_{\mathcal{R}_*}(\Gamma,\mu)$, for any compact set $K\subset\Gamma$ and $\epsilon>0$, there is a compact subset $K'\subset K$, such that $\mu(K\backslash K')<\epsilon$ and the function $\gamma\mapsto\omega(\gamma)$ is continuous on K'.

For any measurable subset $E\subset\Gamma$, we define a map $i_E:\mathcal{R}\to\mathfrak{M}$ by $i_E(x_0)(\gamma) = \chi_E(\gamma)x_0$, where $x_0\in\mathcal{R}$ and χ_E is the indicator function of the set E. The map is completely positive and normal, in fact it is a *-homomorphism. Let now $\tilde{\phi}_E = \tilde{\phi}\circ(i_E\otimes\mathbf{1})$, for any $E\subset\Gamma$, $0<\mu(E)<\infty$. Then $\tilde{\phi}$ is separable, if and only if every functional $\tilde{\phi}_E$ is (up to a normalization constant) a separable state on $\mathcal{R}\,\bar{\otimes}\,\mathfrak{N}$. In order not to obfuscate the present line of reasoning, we postpone the technical proof of that fact to the next lemma.

Let now $S_0:\mathcal{R}\to\mathfrak{N}$ be a positive normal and finite rank map. We define a map $S_E:\mathfrak{M}\to\mathfrak{N}$ by

$$S_E(x(\cdot)) = \frac{1}{\mu(E)}\int_E S_0x(\gamma)d\mu(\gamma), \tag{3.15}$$

understood as the Bochner integral in $\mathfrak{N}$. The integral exists by the following argument. Since S_0 is finite rank, the function $\gamma\mapsto S_0x(\gamma)$ is weakly measurable. Of course, the range of S_0 is separable, and thus, by the Pettis measurability theorem, the integrated function is Bochner measurable. Its integrability is now obvious. Consequently, S_E is a positive, normal finite rank map. Noticing that for any $x_0\in\mathcal{R}$,

$S_0 x_0 = S_E \circ i_E(x_0)$ and taking a positive element $z_0 \in \mathcal{R} \otimes \mathcal{R}$, we have, by the main assumption,

$$\langle (\mathbf{1} \otimes S_0)(z_0), \tilde{\phi}_E \rangle = \langle (\mathbf{1} \otimes S_E)(i_E \otimes i_E)(z_0), \tilde{\phi} \rangle \geq 0. \tag{3.16}$$

By Lemma 3.3, $\tilde{\phi}_E$ is separable and thus $\tilde{\phi}$ is separable as well, provided the next lemma is true. □

Lemma 3.5. *A state $\tilde{\phi}$ on $\mathfrak{M} \bar{\otimes} \mathfrak{N}$ is separable, if and only if every functional $\tilde{\phi}_E$ is (up to a normalization constant) a separable state on $\mathcal{R} \bar{\otimes} \mathfrak{N}$ for every measurable set $E \subset \Gamma$, such that $\mu(E) < \infty$.*

Proof. From the fact that $\tilde{\phi}$ is separable follows easily that $\tilde{\phi}_E$ is separable as well. Suppose the converse, i.e. that each $\tilde{\phi}_E$ is separable. Let $x(\cdot) \in \mathfrak{M}$. An elementary reasoning, similar to the construction of the Lebesgue integral in real analysis, shows that for $x(\cdot) \in \mathfrak{M}$, because $\mathfrak{M}$ is a separable von Neumann algebra, there is a sequence of simple functions

$$x_n(\cdot) = \sum_{k=1}^{n} \chi_{B_k^{(n)}}(\cdot) x_k^{(n)}, \tag{3.17}$$

where

$$x_k^{(n)} = \frac{1}{\mu(B_k^{(n)})} \int_{B_k^{(n)}} x(\gamma) d\mu(\gamma) \in \mathcal{R}, \tag{3.18}$$

each $B_k^{(n)} \subset \Gamma$ is measurable and $0 < \mu(B_k^{(n)}) < \infty$, such that $\langle x_n, \omega \rangle \xrightarrow{n} \langle x, \omega \rangle$ for any $\omega \in \mathfrak{M}_*$. The integral (3.18) plays the role of an average value of $x(\cdot)$ on the set $B_k^{(n)}$ and is understood in the following sense. Because for any $\omega_0 \in \mathcal{R}_*$, the function $\gamma \mapsto \langle x(\gamma), \omega_0 \rangle$ is measurable, so if $E \subset \Gamma$ is a measurable set, $\mu(E) < \infty$, then the integral $\int_E \langle x(\gamma), \omega \rangle d\mu(\gamma)$ defines a bounded functional on the space $\mathcal{R}_*$, i.e. an element of $\mathcal{R}$.

Now, let $\tilde{\phi} = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i (\omega_i \otimes \varphi_i)$, where the limit is understood in the projective operator space tensor norm $\| \cdot \|_\wedge$ on $\mathfrak{M}_* \hat{\otimes} \mathfrak{N}_*$, and let $z \in C_0^*$, $z = \sum_{j=1}^{M} x_j \otimes y_j$. It is clear that the collection of the sets $B_k^{(n)}$ in Eq. (3.18) can be chosen for all x_j, $j = 1, 2, \ldots, M$; i.e. there are sequences $(x_{j,n})_{n=1}^{\infty}$, given by

$$x_{j,n}(\cdot) = \sum_{k=1}^{n} \chi_{B_k^{(n)}}(\cdot) x_{j,k}^{(n)}, \tag{3.19}$$

where

$$x_{j,k}^{(n)} = \frac{1}{\mu(B_k^{(n)})} \int_{B_k^{(n)}} x_j(\gamma) d\mu(\gamma) \in \mathcal{R}, \tag{3.20}$$

such that $x_{j,n} \xrightarrow{n} x_j$ in the weak* topology on $\mathfrak{M}$.

Next, we define a map $\pi_E : \mathcal{R}_* \to \mathfrak{M}_*$ by $\pi_E(\omega_0)(\gamma) = \frac{1}{\mu(E)} \chi_E(\gamma) \omega_0$. It is easy to check that π_E is a completely positive contraction for any measurable set $E \subset \Gamma$, such that $\mu(E) < \infty$, and so, by the main assumption, each functional

$$\tilde{\phi}_n = \sum_{k=1}^{n} (\pi_{B_k^{(n)}} \otimes \mathbf{1})(\tilde{\phi}_{B_k^{(n)}}) \tag{3.21}$$

is separable on $\mathfrak{M} \bar{\otimes} \mathfrak{N}$. We are now going to show that $\langle z, \tilde{\phi}_n \rangle \xrightarrow{n} \langle z, \tilde{\phi} \rangle$. Let us take an arbitrary $\epsilon > 0$. Next, we choose a natural number L such that $\|\sum_{i=L+1}^{\infty} \alpha_i(\omega_i \otimes \varphi_i)\|_\wedge \le \frac{\epsilon}{4\|z\|}$. Then

$$\begin{aligned}|\langle z, \tilde{\phi} - \tilde{\phi}_n \rangle| &= \left| \left\langle z, \tilde{\phi} - \sum_{k=1}^{n} \tilde{\phi} \circ (i_{B_k^{(n)}} \pi^*_{B_k^{(n)}} \otimes \mathbf{1}) \right\rangle \right| \\ &\le \left| \left\langle z, \sum_{i=1}^{L} \omega_i \otimes \varphi_i - \sum_{i'=1}^{L} \sum_{k=1}^{n} \omega_{i'} \circ (i_{B_k^{(n)}} \pi^*_{B_k^{(n)}}) \otimes \varphi_{i'} \right\rangle \right| + \frac{\epsilon}{2},\end{aligned} \tag{3.22}$$

because the map $\sum_{k=1}^{n} i_{B_k^{(n)}} \pi^*_{B_k^{(n)}}$ is a contraction on $\mathfrak{M}$. The dual map $\pi_E^* : \mathfrak{M} \to \mathcal{R}$ acts on any $x(\cdot) \in \mathfrak{M}$ by

$$\pi_E^*(x(\cdot)) = \frac{1}{\mu(E)} \int_E x(\gamma) d\mu(\gamma). \tag{3.23}$$

We take a natural number N such that for any $i = 1, 2, \ldots, L$, and $j = 1, 2, \ldots, M$, whenever $n \ge N$, we have $|\langle x_j - x_{j,n}, \omega_i \rangle| \le \frac{\epsilon}{2LMc_L}$, where $c_L = \max\{|\langle y_j, \varphi_i \rangle|, i = 1, 2, \ldots, L; j = 1, 2, \ldots, M\}$. Using Eqs. (3.23) and (3.20), we compute

$$\begin{aligned}\sum_{k=1}^{n} i_{B_k^{(n)}} \pi^*_{B_k^{(n)}}(x_j) &= \sum_{k=1}^{n} \chi_{B_k^{(n)}}(\cdot) \pi^*_{B_k^{(n)}}(x_j) \\ &= \sum_{k=1}^{n} \chi_{B_k^{(n)}}(\cdot) \frac{1}{\mu(B_k^{(n)})} \int_{B_k^{(n)}} x_j(\gamma) d\mu(\gamma) \\ &= \sum_{k=1}^{n} \chi_{B_k^{(n)}}(\cdot) x_{j,k}^{(n)} = x_{j,n}.\end{aligned} \tag{3.24}$$

Continuing inequality (3.22), we obtain for $n \ge N$

$$\begin{aligned}|\langle z, \tilde{\phi} - \tilde{\phi}_n \rangle| &\le \left| \left\langle z, \sum_{i=1}^{L} \omega_i \otimes \varphi_i - \sum_{i'=1}^{L} \sum_{k=1}^{n} \omega_{i'} \circ (i_{B_k^{(n)}} \pi^*_{B_k^{(n)}}) \otimes \varphi_{i'} \right\rangle \right| + \frac{\epsilon}{2} \\ &= \left| \sum_{j=1}^{M} \left(\sum_{i=1}^{L} \langle x_j, \omega_i \rangle \langle y_j, \varphi_i \rangle - \sum_{i'=1}^{L} \sum_{k=1}^{n} \langle i_{B_k^{(n)}} \pi^*_{B_k^{(n)}}(x_j), \omega_{i'} \rangle \langle y_j, \varphi_{i'} \rangle \right) \right| + \frac{\epsilon}{2} \\ &= \left| \sum_{j=1}^{M} \left(\sum_{i=1}^{L} \langle x_j, \omega_i \rangle \langle y_j, \varphi_i \rangle - \sum_{i'=1}^{L} \langle x_{j,n}, \omega_{i'} \rangle \langle y_j, \varphi_{i'} \rangle \right) \right| + \frac{\epsilon}{2}\end{aligned}$$

$$\begin{aligned}
&= \left|\sum_{j=1}^{M}\sum_{i=1}^{L}(\langle x_j,\omega_i\rangle\langle y_j,\varphi_i\rangle - \langle x_{j,n},\omega_i\rangle\langle y_j,\varphi_i\rangle)\right| + \frac{\epsilon}{2}\\
&\le \sum_{j=1}^{M}\sum_{i=1}^{L} c_L|\langle x_j,\omega_i\rangle - \langle x_{j,n},\omega_i\rangle| + \frac{\epsilon}{2}\\
&\le \sum_{j=1}^{M}\sum_{i=1}^{L} c_L\frac{\epsilon}{2LMc_L} + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned} \tag{3.25}$$

It follows that $\langle z,\tilde{\phi}\rangle = \lim_n\langle z,\tilde{\phi}_n\rangle \ge 0$, and since z was an arbitrary element of C_0^*, we conclude, by Corollary 3.1, that $\tilde{\phi}$ is separable, which completes the proof. □

Step 4. Finite injective algebras.

In order to no longer depend on the assumption of separability of the algebra $\mathfrak{M}$, we argue as follows. Let $\mathfrak{M}$ be a finite injective von Neumann algebra and $\tilde{\phi}$ be a state on $\mathfrak{M}\,\bar{\otimes}\,\mathfrak{N}$, such that $\tilde{\phi}\circ(\mathbf{1}\otimes S)\ge 0$ for any positive normal and finite rank map $S:\mathfrak{M}\to\mathfrak{N}$. In the light of the discussion in Takesaki's book,[20] p. 178, for any finite set $\{\mathbf{1},x_1,x_2,\dots,x_n\}\subset\mathfrak{M}$, there is a finite separable injective subalgebra $\mathfrak{S}\subset\mathfrak{M}$, such that for each $i=1,2,\dots n$, $x_i\in\mathfrak{S}$. Moreover, $\mathfrak{S}$ is the image of a normal conditional expectation $\mathcal{E}$ of $\mathfrak{M}$. We show that $\tilde{\phi}$ is a weak limit of separable states. Indeed, let all finite subsets of $\mathfrak{M}$, ordered by inclusion, label a net $(\mathcal{E}_\mu)$ of conditional expectations projecting $\mathfrak{M}$ onto separable subalgebras $\mathfrak{S}_\mu$ that contain the respective subsets. Then, the restriction $S|_{\mathfrak{S}_\mu}$ of the map S to $\mathfrak{S}_\mu$ is also positive and normal. By Lemma 3.4, the restriction of $\tilde{\phi}$ to $\mathfrak{S}_\mu\,\bar{\otimes}\,\mathfrak{N}$ is separable, and hence each state $\tilde{\phi}_\mu=\tilde{\phi}|_{\mathfrak{S}_\mu\bar{\otimes}\mathfrak{N}}\circ(\mathcal{E}_\mu\otimes\mathbf{1})$ is separable as well. It is easy to see that the net $(\tilde{\phi}_\mu)$ converges weakly to $\tilde{\phi}$. Since the set of separable states is a convex cone, $\tilde{\phi}$ is also a norm limit of separable states and hence separable as well.

Step 5. Semifinite injective algebras.

In this step, we weaken the assumption that $\mathfrak{M}$ is finite.

Lemma 3.6. *Lemma* 3.3 *is valid for* $\mathfrak{M}$ *being a semifinite injective von Neumann algebra.*

Proof. For any finite nonzero projection $p\in\mathfrak{M}$, by $\mathfrak{M}_p$ we denote a finite von Neumann algebra $p\mathfrak{M}p$. Let $S:\mathfrak{M}_p\to\mathfrak{N}$ be a positive normal map. We define $S_p:\mathfrak{M}\to\mathfrak{N}$ by $S_p(x)=S(pxp)$, $x\in\mathfrak{M}$, which is also positive and normal. Let $T_px=pxp$, $x\in\mathfrak{M}$. As the map T_p is a completely positive normal contraction, it has extension to a positive complete contraction $T_p\otimes\mathbf{1}$ on $\mathfrak{M}\,\bar{\otimes}\,\mathfrak{N}$. Hence, we can define a positive functional $\tilde{\phi}_p$ on $\mathfrak{M}\,\bar{\otimes}\,\mathfrak{N}$, by $\tilde{\phi}_p=\tilde{\phi}\circ(T_p\otimes\mathbf{1})$. Let z be a positive element of $\mathfrak{M}_p\otimes\mathfrak{M}_p$. As there is a positive element $z'\in\mathfrak{M}\otimes\mathfrak{M}$ such that

$z = (T_p \otimes T_p)(z')$, so, by the assumption,

$$\tilde{\phi}_p|_{\mathfrak{M}_p \bar{\otimes} \mathfrak{N}} \circ (\mathbf{1} \otimes S)(z) = \tilde{\phi} \circ (\mathbf{1} \otimes S_p)((T_p \otimes \mathbf{1})(z')) \geq 0. \tag{3.26}$$

Therefore, by Step 4, $\tilde{\phi}_p|_{\mathfrak{M}_p \bar{\otimes} \mathfrak{N}}$ is separable. It follows that $\tilde{\phi}_p$ is also separable. Indeed, for an element $z \in C_0^*$, we have that $(T_p \otimes \mathbf{1})(z) \in \mathfrak{M}_p \bar{\otimes} \mathfrak{N}$, and so

$$\tilde{\phi}_p(z) = \tilde{\phi}_p((T_p \otimes \mathbf{1})z) = \tilde{\phi}_p|_{\mathfrak{M}_p \bar{\otimes} \mathfrak{N}}((T_p \otimes \mathbf{1})z) \geq 0, \tag{3.27}$$

which shows that $\tilde{\phi}_p$ is separable.

Let again $z \in C_0^* \subset \mathfrak{M} \otimes \mathfrak{N}$ be arbitrary. We are now going to show that $\langle z, \tilde{\phi} \rangle \geq 0$. Let p_ν be a net of finite projections of the semifinite algebra $\mathfrak{M}$, such that $\mathrm{w}^* - \lim_\nu p_\nu = \mathbf{1}$. As $\mathfrak{M}$ is an algebra of operators acting on a Hilbert space, we can say that the net p_ν converges to the identity in the σ-weak operator topology. This in turn implies that $p_\nu \xrightarrow{\nu} \mathbf{1}$ in the σ-strong topology, since $p_\nu = p_\nu^2 = p_\nu p_\nu^*$. It follows that for any $x \in \mathfrak{M}$ and $\omega \in \mathfrak{M}_*$, $\langle p_\nu x p_\nu, \omega \rangle \xrightarrow{\nu} \langle x, \omega \rangle$, and because $(T_{p_\nu} \otimes \mathbf{1})(z) = (p_\nu \otimes \mathbf{1}) z (p_\nu \otimes \mathbf{1})$, we have that $\langle z, \tilde{\phi} \rangle = \lim_\nu \langle z, \tilde{\phi}_{p_\nu} \rangle \geq 0$, which completes the proof. □

Step 6. Type III injective algebras.

Lemma 3.7. *Lemma* 3.3 *holds for type* III *injective von Neumann algebras.*

Proof. As $\mathfrak{M}$ is injective and of type III, so there exists a semifinite injective von Neumann algebra $\mathfrak{M}_0$ and a norm-one projection $\mathcal{E} : \mathfrak{M}_0 \to \mathfrak{M}$. By the construction of $\mathfrak{M}_0$, we are allowed to treat $\mathfrak{M}$ as a subalgebra of $\mathfrak{M}_0$. Moreover, $\mathcal{E}$ is a limit in the weak* operator topology of a net $(\mathcal{E}_\lambda)$ of normal unital and completely positive maps $\mathcal{E}_\lambda : \mathfrak{M}_0 \to \mathfrak{M}$, see Brown and Ozawa,[4] Lemma 9.3.6. In fact, $\mathfrak{M}_0 = \mathfrak{M} \rtimes_\alpha \mathbb{R}$, the crossed product with respect of the modular action α on $\mathfrak{M}$. As $\mathfrak{M}$ is of type III so $(\mathfrak{M} \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R} \cong \mathfrak{M} \bar{\otimes} B(L^2(\mathbb{R})) \cong \mathfrak{M}$, where $\hat{\alpha}$ is the action conjugate to α. Hence, there is also a norm-one projection $\tilde{\mathcal{E}} : \mathfrak{M} \to \mathfrak{M}_0$. By analogy, let $(\tilde{\mathcal{E}}_\mu)$ be a net of normal unital and completely positive maps $\tilde{\mathcal{E}}_\mu : \mathfrak{M} \to \mathfrak{M}_0$ that converges to $\tilde{\mathcal{E}}$ in the weak* operator topology.

For any λ, let $\tilde{\phi}_\lambda$ denote a state on $\mathfrak{M}_0 \bar{\otimes} \mathfrak{N}$, given by $\tilde{\phi}_\lambda = \tilde{\phi} \circ (\mathcal{E}_\lambda \otimes \mathbf{1})$. Suppose that each $\tilde{\phi}_\lambda$ is separable. Then, its restriction to $\mathfrak{M} \bar{\otimes} \mathfrak{N}$ is, up to a normalization constant, a separable state and $\tilde{\phi}_\lambda \xrightarrow{\lambda} \tilde{\phi}$ weakly. By the standard convexity argument, $\tilde{\phi}$ is also a strong limit of separable states, which by definition means that $\tilde{\phi}$ is separable. Thus, what remains is to prove that $\tilde{\phi}_\lambda$ is a net of separable states on $\mathfrak{M}_0 \bar{\otimes} \mathfrak{N}$. To this end, let us fix λ and suppose on the contrary that $\tilde{\phi}_\lambda$ is not separable. By what has been mentioned above, there is a positive normal map $S_0 : \mathfrak{M}_0 \to \mathfrak{N}$ and also a positive element $z_0 \in \mathfrak{M}_0 \bar{\otimes} \mathfrak{M}_0$ such that

$$\langle (\mathbf{1} \otimes S_0)(z_0), \tilde{\phi}_\lambda \rangle < 0. \tag{3.28}$$

Moreover, z_0 can be chosen such that it belongs to the algebraic tensor product $\mathfrak{M}_0 \otimes \mathfrak{M}_0$. Thus, since $\tilde{\mathcal{E}}$ is onto, there is a positive $z \in \mathfrak{M} \otimes \mathfrak{M}$, such that

$(\tilde{\mathcal{E}} \otimes \tilde{\mathcal{E}})(z) = z_0$. Let $S_\mu : \mathfrak{M} \to \mathfrak{N}$ be a positive normal map given by $S_\mu = S_0 \circ \tilde{\mathcal{E}}_\mu$. Then, by the assumption, and because both $\mathcal{E}_\lambda$ and $\tilde{\mathcal{E}}$ are completely positive, we have

$$\begin{aligned}\langle(\mathbf{1} \otimes S_0)(z_0), \tilde{\phi}_\lambda\rangle &= \lim_\mu \langle(\mathbf{1} \otimes S_\mu)(\tilde{\mathcal{E}} \otimes \mathbf{1})(z), \tilde{\phi}_\lambda\rangle \\ &= \lim_\mu \langle(\mathbf{1} \otimes S_\mu)(\mathcal{E}_\lambda \circ \tilde{\mathcal{E}} \otimes \mathbf{1})(z), \tilde{\phi}\rangle \geq 0, \end{aligned} \tag{3.29}$$

which is a contradiction to (3.28). Hence, $\tilde{\phi}_\lambda$ is separable and the proof is completed. □

Because for each injective algebra $\mathfrak{M}$ there is a central projection p such that $p\mathfrak{M}$ is semifinite and $(\mathbf{1} - p)\mathfrak{M}$ is of type III, this ends the proof of Theorem 3.1.

4. Horodeckis Criterion in C^*-Algebras

In this section, we are going to present the result analogous to Theorem 3.1, but this time in the C*-algebra setting. It is expected that the previous assumption of injectivity will have to be retained somehow. That leads to the choice of *nuclear* C*-algebras, i.e. algebras whose universal enveloping von Neumann algebras (the second dual spaces) are injective. Not only it allows us to define a tensor product of two C*-algebras uniquely, but also facilitates the usage of the previous result. Nuclear C*-algebras, among others, share also the property of local reflexivity, which will prove itself indispensable for our line of reasoning.

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. By $\mathcal{A} \otimes_{\max} \mathcal{B}$ and $\mathcal{A} \otimes_{\min} \mathcal{B}$, we denote respectively the *maximal* and *minimal* C*-tensor product of $\mathcal{A}$ and $\mathcal{B}$. It is known that if at least one of the algebras, say $\mathcal{A}$, is nuclear, then those two tensor norms coincide and the tensor product C*-algebra is in fact uniquely defined. For a nuclear C*-algebra $\mathcal{A}$, we denote that tensor product by $\mathcal{A} \bar{\otimes} \mathcal{B}$. A C*-algebra $\mathcal{A}$ is by definition *locally reflexive*, if for every finite-dimensional operator system $E \subset \mathcal{A}^{**}$, i.e. a closed self-adjoint subspace containing the identity element, there is a net (T_λ) of completely positive contractions, $T_\lambda : E \to \mathcal{A}$, which converges to the identity map on E in the weak operator topology. Every nuclear C*-algebra is locally reflexive (see Brown and Ozawa,[4] 9.3.1–9.3.3). The algebras $\mathcal{A}$ and $\mathcal{B}$ can be treated as operator spaces in a natural way. Thus, it is possible to introduce the operator space projective tensor norm on the dual operator spaces $\mathcal{A}^*$ and $\mathcal{B}^*$, and the completion of the algebraic tensor product with respect to that norm will be denoted by $\mathcal{A}^* \hat{\otimes} \mathcal{B}^*$. Hence, we can introduce a cone of separable states with respect to $\mathcal{A}$ and $\mathcal{B}$ in the same way as in Sec. 2. Let $\mathcal{S}(\mathcal{A})$ and $\mathcal{S}(\mathcal{B})$ denote the state spaces of $\mathcal{A}$ and $\mathcal{B}$, respectively. We define a positive cone $C_{\mathcal{A},\mathcal{B}}$ of separable states with respect to $\mathcal{A}$ and $\mathcal{B}$ by

$$C_{\mathcal{A},\mathcal{B}} = \overline{\operatorname{conv}}^{\|\cdot\|_\wedge}\{\omega \otimes \varphi : \omega \in \mathcal{S}(\mathcal{A}), \varphi \in \mathcal{S}(\mathcal{B})\}. \tag{4.1}$$

It is clear that a state on $\mathcal{A} \bar{\otimes} \mathcal{B}$ is separable, i.e. $\tilde{\phi} \in C_{\mathcal{A},\mathcal{B}}$, if and only if $\tilde{\phi}$ is a separable state on the von Neumann algebra $\mathcal{A}^{**} \bar{\otimes} \mathcal{B}^{**}$.

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and suppose that $\mathcal{A}$ is nuclear. A state $\tilde{\phi} \in (\mathcal{A} \bar{\otimes} \mathcal{B})^*$ is separable with respect to $\mathcal{A}$ and $\mathcal{B}$, if and only if the functional $\tilde{\phi} \circ (\mathbf{1} \otimes S)$ is positive for any positive and finite rank map $S : \mathcal{A} \to \mathcal{B}$.*

Proof. Let $\mathfrak{M} = \mathcal{A}^{**}$ and $\mathfrak{N} = \mathcal{B}^{**}$. By the assumption, $\mathfrak{M}$ is injective. If $\tilde{\phi}$ is separable with respect to $\mathcal{A}$ and $\mathcal{B}$, then it is also separable on $\mathfrak{M} \bar{\otimes} \mathfrak{N}$. For any positive map $S : \mathcal{A} \to \mathcal{B}$, the second dual map $S^{**} : \mathfrak{M} \to \mathfrak{N}$ is positive and normal. It follows from the first part of Theorem 3.1 that for any positive $z \in \mathfrak{M} \bar{\otimes} \mathfrak{M}$, $\langle (\mathbf{1} \otimes S^{**})(z), \tilde{\phi} \rangle \geq 0$. Therefore, $\tilde{\phi} \circ (\mathbf{1} \otimes S)$ is a positive functional on $\mathcal{A} \bar{\otimes} \mathcal{A}$.

Conversely, suppose that $\tilde{\phi} \circ (\mathbf{1} \otimes S) \geq 0$ for any positive and finite rank map $S : \mathcal{A} \to \mathcal{B}$. We are going to show that the state $\tilde{\phi}$ is separable on $\mathfrak{M} \bar{\otimes} \mathfrak{N}$. Suppose on the contrary that $\tilde{\phi}$ is not separable on $\mathfrak{M} \bar{\otimes} \mathfrak{N}$. By Theorem 3.1, there is a positive normal and finite rank map $\tilde{S} : \mathfrak{M} \to \mathfrak{N}$ and a positive element $z \in \mathfrak{M} \bar{\otimes} \mathfrak{M}$, such that

$$\langle (\mathbf{1} \otimes \tilde{S})(z), \tilde{\phi} \rangle < 0. \tag{4.2}$$

Moreover, we can choose the element z from the algebraic tensor product $\mathfrak{M} \otimes \mathfrak{M}$, i.e. $z = \sum_{j=1}^{N} x_j \otimes y_j$, each x_j and $y_j \in \mathfrak{M}$. Let $E \subset \mathfrak{M}$ be an operator system generated by $\{x_j, y_j\}_{j=1}^{N}$. Since $\mathcal{A}$ is locally reflexive, there exists a net (T_λ) of completely positive contractions $T_\lambda : E \to \mathcal{A}$, such that T_λ converges to the identity map on E in the weak* operator topology, i.e. for any $\omega \in \mathcal{A}^*$ and $x \in E$, $\langle \omega, T_\lambda x \rangle \xrightarrow{\lambda} \langle x, \omega \rangle$. Let a_λ denote the positive element of $\mathcal{A} \otimes \mathcal{A}$, $a_\lambda = (T_\lambda \otimes T_\lambda)(z) = \sum_{j=1}^{N} T_\lambda x_j \otimes T_\lambda y_j$, and let $i : \mathcal{A} \to \mathcal{A}^{**} = \mathfrak{M}$ be the canonical embedding. Then, $i(a_\lambda) \xrightarrow{\lambda} z$ σ-weakly. Since $\mathcal{B}$ is a Banach space, it has a local reflexivity property in the following sense (see Ryan,[15] Sec. 5.5). For any finite-dimensional subspace $F \subset \mathcal{B}^{**}$, there is an operator $P_F : F \to \mathcal{B}$ such that for any $y \in \mathcal{B}^{**} = \mathfrak{N}$ and $\varphi \in \mathcal{B}^*$, one has

$$\langle \varphi, P_F y \rangle = \langle y, \varphi \rangle. \tag{4.3}$$

Let $F = \tilde{S}(\mathfrak{M})$. It is obvious from Eq. (4.3) that P_F is positive. We define a positive map $S : \mathcal{A} \to \mathcal{B}$ by $S = P_F \circ \tilde{S} \circ i$. Then, for any $A \in \mathcal{A}$ and $\varphi \in \mathcal{B}^*$, we have

$$\begin{aligned} \langle \varphi, SA \rangle &= \langle \varphi, P_F(\tilde{S} \circ i(A)) \rangle = \langle \tilde{S} \circ i(A), \varphi \rangle \\ &= \langle i(A), \tilde{S}_* \varphi \rangle = \langle \tilde{S}_* \varphi, A \rangle. \end{aligned} \tag{4.4}$$

Suppose now that $\tilde{\phi} = \lim_n \sum_{i=1}^{n} \omega_i \otimes \varphi_i$, where each $\omega_i \in \mathcal{A}^*$, $\varphi_i \in \mathcal{B}^*$. The limit is understood as the one in the projective operator norm $\|\cdot\|_\wedge$, and hence also in the weak and weak* topology on $(\mathcal{A} \bar{\otimes} \mathcal{B})^*$. Then, by the assumption, $\langle \tilde{\phi}, (\mathbf{1} \otimes S)(a_\lambda) \rangle \geq 0$. Moreover,

$$\begin{aligned} \sum_{i=1}^{n} \langle \omega_i \otimes \varphi_i, (\mathbf{1} \otimes S)(a_\lambda) \rangle &= \sum_{i=1}^{n} \sum_{j=1}^{N} \langle \omega_i, T_\lambda x_j \rangle \langle \tilde{S}_* \varphi_i, T_\lambda y_j \rangle \\ &\xrightarrow{\lambda} \sum_{i=1}^{n} \sum_{j=1}^{N} \langle x_j, \omega_i \rangle \langle y_j, \tilde{S}_* \varphi_i \rangle = \langle (\mathbf{1} \otimes \tilde{S})(z), \sum_{i=1}^{n} \omega_i \otimes \varphi_i \rangle. \end{aligned} \tag{4.5}$$

Taking the limit with respect to n, since (T_λ) is a net of contractions, we get that $\langle(\mathbf{1}\otimes\tilde{S})(z),\tilde{\phi}\rangle \geq 0$, which contradicts (4.2), and so $\tilde{\phi}$ is separable on $\mathfrak{M}\bar{\otimes}\mathfrak{N}$. $\square$

That essentially generalizes the result of Størmer's, who proved a similar theorem under additional conditions. Namely, he used all positive linear maps and assumed that $\mathcal{B}$ was an UHF algebra.[18]

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