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Stable Subspaces of Positive Maps of Matrix Algebras

Marek Miller[1] and Robert Olkiewicz[2]

Institute of Theoretical Physics, Uniwersytet Wrocławski,
pl. Maksa Borna 9, 50–204 Wrocław, Poland

[1] *e-mail: marek.miller@ift.uni.wroc.pl*

[2] *e-mail: robert.olkiewicz@ift.uni.wroc.pl*

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Abstract. We study stable subspaces of positive extremal maps of finite dimensional matrix algebras that preserve trace and matrix identity (so-called bistochastic maps). We have established the existence of the isometric-sweeping decomposition for such maps. As the main result of the paper, we have shown that all extremal bistochastic maps acting on the algebra of matrices of size 3×3 fall into one of the three possible categories, depending on the form of the stable subspace of the isometric-sweeping decomposition. Our example of an extremal atomic positive map seems to be the first one that handles the case of that subspace being non-trivial. Lastly, we compute the entanglement witness associated with the extremal map and specify a large family of entangled states detected by it.

1. Introduction

Positive maps of operator algebras is an increasingly popular subject of research, both as a mathematical theory interesting in itself, as well as a prominent domain of applications to quantum theory [25]. Started in the pioneering work of Kadison [14], and Stinespring [22], the theory reached its first major breakthrough, when Størmer and Woronowicz presented a structure theorem for positive maps of low dimensional matrix algebras [23, 28]. It is also necessary to mention the work of M.-D. Choi here, as the major contribution to the theory at every stage [4 − 6]. The second turning point came in 1990s, when Peres and P. M. R. Horodeckis pointed out at the intrinsic relation between separable states of composite quantum systems and positive maps of algebras of observables [12, 21]. It turns out that there is a one-to-one correspondence between positive maps and entanglement witnesses [8], and the Peres–Horodecki criterion, whether a quantum state is separable, is computationally feasible as long as a structure theorem similar to that proven by Størmer and Woronowicz holds true. Unfortunately, whereas that correspondence does exist even for the most general infinite-dimensional quantum systems [18, 24], higher dimensional situation lacks the complete description of positive maps and one needs a deeper understanding of their highly non-trivial structure. Because the set of positive maps forms a convex cone, its elements can be characterised as convex combinations of extremal ones. This is a consequence of the celebrated Krein-Milman theorem [15], as for every positive number $r > 0$, the set of positive maps such that their operator norm: $||S|| \leq r$, is compact. This is certainly true for maps on finite-dimensional matrix algebras, but applies also to the general setting of von Neumann algebras [18]. Despite considerable effort, examples of extremal positive maps, even in the low-dimensional case, are scarce [6, 7, 10, 20]. In this paper, we deal exclusively with extremal maps as extreme points in the cone of all positive maps on matrix algebras (see below).

In order to facilitate the further study of extremal positive maps, we propose to take a closer look at their stable subspaces. In general, by a 'stable subspace' of a positive map $S : M_n \to M_n$, where M_n stands for the algebra of square complex matrices of size n, we mean a particular subspace $K_S \subset M_n$ (in fact, a JB*-algebra, see Corollary 3 below), such that the map is a Jordan automorphism on K_S and $S^k \to 0$ strongly on the orthogonal complement of K_S, as $k \to \infty$ (see the more precise definition below). We say that the map S is strongly ergodic, if K_S is isomorphic to the set of complex numbers. Adapting the previous results in this matter from [19], we establish the existence of the isometric-sweeping decomposition for a positive map that preserves trace and the matrix identity (so-called bistochastic maps). Such decomposition says precisely that S must be a Jordan automorphism of K_S, and its powers tend strongly to 0 for any matrix orthogonal to K_S. What is interesting in this respect, is that if we narrow our interest to the maps acting on M_3, it turns out that there are only three possible forms the algebra K_S can have. Moreover, it seems that all previously known examples of extremal positive maps of M_3 fall into only two of the three possible categories: in particular, it is easy to see that for a Jordan automorphism $S(A) = U^* A\, U$, or $S(A) = U^* A^t\, U$, where U is unitary, $K_S = M_3$; whereas the Choi map [6, 20] is strongly ergodic. The remaining possibility among those three, namely the two-dimensional commutative subalgebra, requires from us to expand on an example of a positive map that cannot be classified as belonging to one of the types previously encountered in the literature. One of the main results of this paper is the proof that the map provided by us as the example is indeed extremal and atomic [9]. Moreover, we show that the map is also exposed. This fact gives immediately a useful entanglement witness that the authors hope could serve as a prototypic case in further study of the entanglement of three-level quantum systems [1, 3].

The structure of the paper is as follows. In Sect. 2, we introduce some basic notions and notation. In Sect. 3, following [19], we establish the existence of the isometric-sweeping decomposition of bistochastic maps of finite dimensional matrix algebra M_n. Next, in Sect. 4, we provide the main result of this paper and prove that for bistochastic map of M_3 the stable subspace of the decomposition takes only one of three possible forms. We end the paper with an original example of an extremal positive map of the algebra M_3; we compute the entanglement witness associated with the extremal map and specify a large family of entangled states detected by it.

2. Preliminaries

Let $n \in \mathbb{N}$. The complex linear space $\mathbb{C}^n$ consists of column vectors: η, ξ, etc.; the respective row vectors are denoted η^t, ξ^t, etc. The bar over a number, vector or a matrix always denotes the element-wise complex conjugation: e.g. $\overline{z}$, $\overline{\eta}$, $\overline{A}$, etc. The space $\mathbb{C}^n$ is equipped with the inner product $\langle \eta, \xi \rangle = \eta^* \xi$, where $\eta^* = \overline{\eta}^t$, and $||\eta||^2 = \eta^* \eta$. To avoid confusion when two different spaces, say $\mathbb{C}^m$ and $\mathbb{C}^n$, are involved, we distinguish vectors belonging to one of them with an arrow: e.g. $\vec{\eta}$, $\vec{\xi}$. Let $M_n = M_n(\mathbb{C})$ be the algebra of square complex matrices of size n. We will use letters A, B, C, etc. to specify a matrix of M_n. The norm of A, denoted by $||A||$, is understood as the operator norm of A as a linear map acting on $\mathbb{C}^n$. For a vector $\eta \in \mathbb{C}^n$, $\eta \neq 0$, by P_η or $P(\eta) \in M_n$, we denote the rank-one operator $P_\eta = \eta\eta^*$. Of course, when $||\eta|| = 1$, P_η is the orthogonal projection onto one-dimensional space spanned by η. For $A \in M_n$, we denote its trace by $\mathrm{Tr}A$; and by A^t and $A^* = \overline{A}^t$ its transpose and conjugate transpose, respectively. We say that a matrix A is positive-semidefinite, or simply *positive*, if $\eta^* A \eta \geq 0$ for any $\eta \in \mathbb{C}^n$ (i.e. $A = A^*$ and A has a non-negative spectrum).

A linear map $S \colon M_n \to M_n$ is said to be positive, indicated as $S \geq 0$, if for any $A \in M_n$ such that $A \geq 0$, we have $S(A) \geq 0$. For a positive map S, its operator norm is given by $||S|| = S(\mathbf{1})$, where $\mathbf{1}_n$, or simply $\mathbf{1}$, is the identity matrix of M_n. Any positive map is Hermitian, i.e. $S(A^*) = S(A)^*$, for all $A \in M_n$. The identity map of M_n is labelled I_n, or simply I. The convex cone of all positive maps of M_n is denoted by $\mathcal{P}(M_n)$. For $k \in \mathbb{N}$, a map $S \in \mathcal{P}(M_n)$, such that the map $I_k \otimes S : M_k \otimes M_n \to M_k \otimes M_n$ is positive, is called k-positive. If a map is a k-positive map for every k, it is called completely positive. Similarly, a map is k-*co*positive, or completely copositive, if $I_k \otimes (S \circ t)$ is positive for some k, or for every k, respectively, where $t \colon A \mapsto A^t$, $A \in M_n$, is the transposition map. A positive map S is called decomposable, if it can be written in the form $S(A) = \Lambda_1(A) + \Lambda_2(A^t)$, where both maps Λ_1, Λ_2 are completely positive, possibly zero (see [4] for more details on completely positive maps). It is called atomic [9], if it cannot be written as a sum of 2-positive and 2-copositive maps. A positive map S is extremal, if for any positive map $T : M_n \to M_n$ such that $S - T \in \mathcal{P}(M_n)$, i.e. $0 \leq T \leq S$, we have $T = \alpha S$ for some number $0 \leq \alpha \leq 1$. We denote the set of extremal maps of $\mathcal{P}(M_n)$ by $\mathrm{Ext}(M_n)$. It is true that every positive map can be written as a convex combination of extremal ones. Nevertheless, it could be of use to specify a dense subset of $\mathrm{Ext}(M_n)$, such that it is easier to handle its elements instead of general extremal maps. Hence, we say that a positive map S of the matrix algebra M_n is exposed [17], if for all $T \in \mathcal{P}(M_n)$ such that $\mathrm{Tr}\, P_\xi\, T(P_\eta) = 0$, where $\mathbb{C}^n \ni \eta, \xi \neq 0$ is any pair for which $\mathrm{Tr}\, P_\xi\, S(P_\eta) = 0$, we have that $T = \alpha S$, $\alpha \geq 0$. Due to the Straszewicz theorem [26], the set of exposed maps of M_n is indeed dense in $\mathrm{Ext}(M_n)$.

By a JB*-algebra K we understand a complex Banach space which is also a complex Jordan algebra. We always assume that $\mathbf{1} \in K$. See [11] for the overview of the theory of JB*-algebras.

3. Decomposition of Bistochastic Maps

Let $S\colon M_n \to M_n$ be a positive map such that $S(\mathbf{1}) = \mathbf{1}$, $\mathrm{Tr}S(A) = \mathrm{Tr}A$, for any $A \in M_n$. We call such a map *bistochastic*. It is easy to see that S is a contraction in the Hilbert–Schmidt norm (*HS-norm*) on M_n, defined as $||A||_{HS} = (\mathrm{Tr}\, A^* A)^{1/2}$. Indeed, since S fulfils the Kadison–Schwarz inequality,

$$S(A^*)\, S(A) \;\leq\; S(A^* A)\,, \tag{3.1}$$

for any normal element $A \in M_n$ (see Prop. 3.6 of [5]), assuming at first that $A = A^*$, we have

$$||S(A)||_{HS}^2 \;=\; \mathrm{Tr}S(A)^2 \;\leq\; \mathrm{Tr}S(A^2) \;=\; \mathrm{Tr}A^2 \;=\; ||A||_{HS}^2\,. \tag{3.2}$$

By representing a general element $A \in M_n$ as a sum $A = A_1 + iA_2$, where both A_1, A_2 are self-adjoint, and repeating essentially the same calculation as in (3.2), we obtain the assertion.

Next, we are going to define an isometric splitting of S, drawing from the ideas presented in [19]. Because S is a contraction in the HS-norm on the Hilbert space M_n, equipped with the Hilbert–Schmidt inner product, that space can be decomposed into a direct sum of a space K_S, defined as

$$K_S \ = \ \Big\{ A \in M_n \ : \ ||S^k A||_{HS} = ||S^{*k} A||_{HS} = ||A||_{HS}, \ \forall \ k \in \mathbb{N} \Big\} , \qquad (3.3)$$

and its orthogonal complement $K_S^{\perp}$. By S^*, we denote the bistochastic map of M_n, which is the adjoint of S as a linear operator on the Hilbert space M_n, i.e. $\mathrm{Tr}\, S^*(A)\, B \ = \ \mathrm{Tr} A\, S(B)$, for every $A, B \in M_n$. It is true that $A \in K_S$, if and only if $S^{*k} S^k A = S^k S^{*k} A = A$, for any $k \in \mathbb{N}$. The following proposition, taken with slight modification from [19], puts together some of the characteristics of K_S.

PROPOSITION 1 *Suppose S is a bistochastic map of M_n, and the subspace K_S is defined as in (3.3). Then:*

a) $\mathbf{1} \in K_S$;

b) $A \in K_S$ *implies that* $A^* \in K_S$;

c) $A \in K_S$ *implies that* $|A| = (A^* A)^{1/2} \in K_S$;

d) *if* $A, B \in K_S$, *then* $AB + BA \in K_S$;

e) *if* $A = A^* \in K_S$ *and* $A = \sum_{i=1}^k \lambda_i P_i$, *where each* $\lambda_i \neq 0$ *is different, and each* P_i *is an orthogonal projection in* M_n, *then* $P_i \in K_S$ *for any* $i = 1, 2, \dots, k$;

f) *if* $P \in K_S$ *is an orthogonal projection, then* $S(P)$ *and* $S^*(P)$ *are orthogonal projections as well, and* $\dim S(P) = \dim S^*(P) = \dim P$;

g) *if* $P, Q \in K_S$ *are orthogonal projections such that* $PQ = 0$, *then*

$$S(P) S(Q) \ = \ S^*(P) S^*(Q) \ = \ 0 \, .$$

Proof. a) obvious, since S is bistochastic; b), c) as in Prop. 5 a) and 5 b) of [19].

d) Suppose at first that $A = A^* \in K_S$. Now, $S^{*k} S^k (A) = S^k S^{*k} (A) = A$, and, as a result of the Kadison–Schwarz inequality applied to the map $S^{*k} S^k$,

$$A^2 \ = \ \Big(S^{*k} S^k (A) \Big) \Big(S^{*k} S^k (A) \Big) \ \leq \ S^{*k} S^k (A^2) \, . \qquad (3.4)$$

Hence

$$||A^2||_{HS} \ \leq \ ||S^{*k} S^k (A^2)||_{HS} \ \leq \ ||S^k (A^2)||_{HS} \ \leq \ ||A^2||_{HS} \, , \qquad (3.5)$$

i.e. $||S^k(A^2)||_{HS} = ||A^2||_{HS}$. By a similar argument $||S^{*k}(A^2)||_{HS} = ||A^2||_{HS}$, and thus $A^2 \in K_S$. Now, for any $A, B \in K_S$, such that $A = A^*$, $B = B^*$, we have $AB + BA = (A+B)^2 - A^2 - B^2 \in K_S$. For a general $A \in K_S$, we write $A = A_1 + iA_2$, where A_1, A_2 are both Hermitian. Then $A^2 = A_1^2 - A_2^2 + i(A_1A_2 + A_2A_1) \in K_S$. It is now evident that $AB + BA \in K_S$ for general $A, B \in K_S$.

e) as in Prop. 5 d); f), g) as in Prop. 6 a) and 6 c) of [19]. □

From what has been said above, follows immediately the next observation.

COROLLARY 3 *The space K_S, with the matrix norm $||\cdot||$, and the multiplication $A \circ B = \frac{1}{2}(AB + BA)$, is a JB*-algebra. The map S is a Jordan automorphism on K_S and*

$$\lim_{k\to\infty} S^k(A) \;=\; \lim_{k\to\infty} S^{*k}(A) \;=\; 0\,, \tag{3.6}$$

for any $A \in K_S^{\perp}$.

Proof. It is obvious that K_S is a complex Banach space. From Proposition 1 b), we have that K_S is equipped with involution, and from point d) that K_S is a Jordan algebra. Since the map S is invariant with respect to the subspaces K_S and $K_S^{\perp}$, and of course $M_n = K_S \oplus K_S^{\perp}$, we see that S splits into direct sum $S = S_1 \oplus S_2$, where $S_1 = S|_{K_S}$, $S_2 = S|_{K_S^{\perp}}$. From Prop. 7a) of [19], we have that $S(A^*A) = S(A)^*S(A)$, i.e. S is a Jordan homomorphism on K_S. Moreover, because $S^*S = SS^* = I$ on K_S, the map S_1 is invertible, and thus it is a Jordan automorphism (see Definition 3.2.1(6) in [2]). From definition of the space K_S (3.3), follows easily that (3.6) holds. □

THEOREM 1 *Let $K \subset M_n$ be a JB*-subalgebra of M_n. There is a bistochastic map $S\colon M_n \to M_n$ such that $K = K_S$.*

Proof. We consider the space M_n to be a Hilbert space equipped with the Hilbert–Schmidt inner product. Let $S\colon M_n \to K \subset M_n$ be the orthogonal projection onto K. It is evident that $S(\mathbf{1}) = \mathbf{1}$, and because $S = S^*$, the map S also preserves trace and the space K is a stable subspace for S, provided S is a positive map. Let then $A \in M_n$ be a positive matrix. Since K is JB*-algebra, $(SA)^* \in K$. Because $||A - SA||_{HS} = ||A - (SA)^*||_{HS}$, and SA is the best approximation of A in the space K, we have that $(SA)^* = SA$. We write then $SA = B_+ - B_-$, where both B_+, B_- are positive and $B_+B_- = 0$.

We have assumed that $\mathbf{1} \in K$, and because $(SA)^k \in K$ for any $k \in \mathbb{N}$, we have that also the modulus $|SA| \in K$ (compare the explicit formula for the square-root of a positive matrix in [2], p. 34). Hence both $B_+, B_- \in K$. We compute

$$\begin{aligned} ||A - SA||_{HS}^2 &= \mathrm{Tr}\,(A - SA)^2 \;=\; \mathrm{Tr}\,(A - B_+ + B_-)^2 \\ &= \mathrm{Tr}\,(A - B_+)^2 + 2\,\mathrm{Tr}\,(A - B_+)B_- + \mathrm{Tr}\,B_-^2 \\ &= ||A - B_+||_{HS}^2 + 2\,\mathrm{Tr}\,A\,B_- + \mathrm{Tr}\,B_-^2 \;\geq\; ||A - B_+||_{HS}^2\,. \end{aligned} \tag{3.7}$$

Again, since SA is the best approximation of A in K, we have $SA = B_+$, i.e. $SA \geq 0$, which ends the proof. □

From the above, we know that K_S has additional structure of a JB*-algebra. We will see in the following that, with additional assumption imposed on S, this algebra must necessarily have a specific structure, at least for low dimensional matrices.

4. Extremal Positive Maps

In this section, we focus on maps of the algebra M_3. For convenience, let us label four orthogonal projections:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.1}$$

and $P_{12} = P_1 + P_2$.

It is of importance to us that Theorems 5.3.8 and 6.2.3 of [11] imply that every JB*-algebra contained in M_3 is isomorphic to one of the following: $\mathbb{C}\mathbf{1}$, $\mathbb{C}P_{12} \oplus \mathbb{C}P_3$, $\mathbb{C}P_1 \oplus \mathbb{C}P_2 \oplus \mathbb{C}P_3$, $M_2^s \oplus \mathbb{C}P_3$, $M_2 \oplus \mathbb{C}P_3$, M_3^s, and M_3 itself, where M_n^s is the Jordan algebra of symmetric matrices of size n: $M_2^s = \{A \in M_n : A = A^t\}$.

THEOREM 2 *Let $S\colon M_3 \to M_3$ be an extremal bistochastic map. Then the JB*-algebra K_S is isomorphic to one of the following: $\mathbb{C}\mathbf{1}$, $\mathbb{C}P_{12} \oplus \mathbb{C}P_3$, or M_3.*

Proof. 1. We prove the assertion by excluding other possible forms of K_S in the first place. Let K_S be one of the following JB*-algebras: $\mathbb{C}P_1 \oplus \mathbb{C}P_2 \oplus \mathbb{C}P_3$,

$M_2^s \oplus P_3$, $M_2 \oplus \mathbb{C}P_3$, or M_3^s. Then K_S contains the projections P_1, P_2, P_3, and because S is a Jordan automorphism on K_S:

$$\begin{aligned} \mathrm{Tr}\, S(P_i)S(P_j) &= \frac{1}{2}\,\mathrm{Tr}\,(S(P_i)S(P_j) + S(P_j)S(P_i)) \\ \frac{1}{2}\,\mathrm{Tr}\, S(P_iP_j + P_jP_i) &= \mathrm{Tr}\, P_iP_j = \delta_{ij}\,, \end{aligned} \tag{4.2}$$

where $i, j = 1, 2, 3$ and δ_{ij} is the Kronecker delta. Hence, $\{S(P_i)\}_{i=1}^3$ is a triple of rank-one, mutually orthogonal projections. There is a unitary matrix $U \in M_3$, such that $U^*S(P_i)U = P_i$, for $i = 1, 2, 3$. Let us define $\tilde{S}(A) = U^*S(A)U$. Then $\tilde{S}$ is an extremal [25, Lemma 3.1.2b, p. 27] bistochastic map such that $\tilde{S}(P_i) = P_i$, $i = 1, 2, 3$. By [16, Theorem 4.1], $\tilde{S}$ is decomposable, which contradicts either the fact that S is extremal or $K_S \neq M_3$.

2. While it is easy to see that there are extremal maps for which $K_S = M_3$ (take e.g. $S(A) = UAU^*$ for a unitary matrix U), or $K_S = \mathbb{C}\mathbf{1}$ (take the celebrated Choi map [6]); it is not that straightforward to provide an example of an extremal bistochastic map which has $K_S = \mathbb{C}P_{12} \oplus \mathbb{C}P_3$.

Let then $S\colon M_3 \to M_3$ be a linear map defined as

$$S(A) = \begin{bmatrix} \frac{1}{2}(a_{11} + a_{22}) & 0 & \frac{1}{\sqrt{2}}a_{13} \\ 0 & \frac{1}{2}(a_{11} + a_{22}) & \frac{1}{\sqrt{2}}a_{32} \\ \frac{1}{\sqrt{2}}a_{31} & \frac{1}{\sqrt{2}}a_{23} & a_{33} \end{bmatrix}, \tag{4.3}$$

for $A = [a_{ij}]_{i,j=1}^3 \in M_3$. If we use the notation $A = \begin{bmatrix} B & \vec{u} \\ \vec{w}^t & z \end{bmatrix}$, for $B \in M_2$, $\vec{u}, \vec{w} \in \mathbb{C}^2$ are column vectors, and $z \in \mathbb{C}$, the map S acts by

$$S(A) = S\begin{bmatrix} B & \vec{u} \\ \vec{w}^t & z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\mathrm{Tr}\, B)\,\mathbf{1}_2 & \frac{1}{\sqrt{2}}(\hat{P}_1\vec{u} + \hat{P}_2\vec{w}) \\ \frac{1}{\sqrt{2}}(\hat{P}_1\vec{w} + \hat{P}_2\vec{u})^t & z \end{bmatrix}, \tag{4.4}$$

where $\hat{P}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{P}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $\mathbf{1}_2$ is the identity matrix of M_2. Provided the next Lemma 1 is true, this example ends the proof. □

LEMMA 1 *S is a bistochastic, extremal and atomic map.*

Proof. In order to prove that S is a positive map, it is enough to show that for any $\eta \in \mathbb{C}^3$, $SP_\eta \geq 0$, where P_η is a rank-one operator, $P_\eta = \eta\eta^*$. Let us then take $\eta = (\eta_1, \eta_2, \eta_3) = (\vec{\eta}, \eta_3)$, $\vec{\eta} = (\eta_1, \eta_2) \in \mathbb{C}^2$, $\eta_3 \in \mathbb{C}$. We have then

$$\begin{aligned} SP_\eta &= S\begin{bmatrix} \vec{\eta}\vec{\eta}^* & \overline{\eta}_3\vec{\eta} \\ \eta_3\vec{\eta}^* & |\eta_3|^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{||\vec{\eta}||^2}{2}\mathbf{1}_2 & \frac{1}{\sqrt{2}}\left(\overline{\eta}_3\hat{P}_1\vec{\eta} + \eta_3\hat{P}_2\overline{\vec{\eta}}\right) \\ \frac{1}{\sqrt{2}}\left(\overline{\eta}_3\hat{P}_1\vec{\eta} + \eta_3\hat{P}_2\overline{\vec{\eta}}\right)^* & |\eta_3|^2 \end{bmatrix}. \end{aligned} \tag{4.5}$$

If $\eta_3 = 0$, then of course $SP_\eta \geq 0$. In the case when $\eta_3 \neq 0$, taking the Schur complement (see [29, Theorem 1.12, p.34]), we have that $SP_\eta \geq 0$, if and only if

$$\left(\overline{\eta}_3\hat{P}_1\vec{\eta} + \eta_3\hat{P}_2\overline{\vec{\eta}}\right)\left(\overline{\eta}_3\hat{P}_1\vec{\eta} + \eta_3\hat{P}_2\overline{\vec{\eta}}\right)^* \leq |\eta_3|^2\,||\vec{\eta}||^2\,\mathbf{1}_2\,, \tag{4.6}$$

but it is easy to see that this inequality is fulfilled:

$$\begin{aligned} \left(\overline{\eta}_3\hat{P}_1\vec{\eta} + \eta_3\hat{P}_2\overline{\vec{\eta}}\right)\left(\overline{\eta}_3\hat{P}_1\vec{\eta} + \eta_3\hat{P}_2\overline{\vec{\eta}}\right)^* &\leq ||\overline{\eta}_3\hat{P}_1\vec{\eta} + \eta_3\hat{P}_2\overline{\vec{\eta}}||^2\,\mathbf{1}_2 \\ = \left(||\overline{\eta}_3\hat{P}_1\vec{\eta}||^2 + ||\eta_3\hat{P}_2\overline{\vec{\eta}}||^2\right)\mathbf{1}_2 &= |\eta_3|^2\,||\vec{\eta}||^2\,\mathbf{1}_2\,. \end{aligned} \tag{4.7}$$

Thus, S is a bistochastic map. We also have $K_S = \mathbb{C}P_{12} \oplus \mathbb{C}P_3$. Indeed, one computes immediately that for $A \in M_3$:

$$\lim_{k\to\infty} S^k(A) = \frac{1}{2}(\mathrm{Tr}\,P_{12}A)\,P_{12} + (\mathrm{Tr}\,P_3A)\,P_3 \in K_S\,. \tag{4.8}$$

From that and from the definition (3.3), follows the particular form of the JB*-algebra K_S.

We are going to show now that S is extremal. Let $S_0 : M_3 \to M_3$ be a positive map such that $0 \leq S_0 \leq S$. We have $S_0(P_3) \leq S(P_3) = P_3$, and hence $S_0(P_3) = \alpha P_3$ for some $0 \leq \alpha \leq 1$. Let us consider the space M_2 embedded into M_3 as follows: $A \in M_2 \subset M_3$ whenever

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a_{ij} \in \mathbb{C}\,, \quad i,j = 1,2\,. \tag{4.9}$$

We want to show that $S_0(A) \in M_2 \subset M_3$ for any $A \in M_2 \subset M_3$. Let $B \in M_2$ and suppose first that $B \geq 0$. Then

$$0 \leq S_0\begin{bmatrix} B & \vec{0} \\ \vec{0}^t & 0 \end{bmatrix} = \begin{bmatrix} \hat{S}_0(B) & \vec{u} \\ \vec{w}^t & r \end{bmatrix}, \tag{4.10}$$

for some vectors $\vec{u}, \vec{w} \in \mathbb{C}^2$ and $r \geq 0$. The map $\hat{S}_0 : B \mapsto \hat{S}_0(B) \in M_2$ must be a positive map of M_2 such that $\hat{S}_0(B) \leq \frac{1}{2}(\mathrm{Tr}B)\mathbf{1}_2$. On the other hand,

$$0 \;\leq\; \begin{bmatrix} \hat{S}_0(B) & \vec{u} \\ \vec{w}^t & r \end{bmatrix} \;\leq\; S \begin{bmatrix} B & \vec{0} \\ \vec{0}^t & 0 \end{bmatrix} \;=\; \begin{bmatrix} \frac{1}{2}(\mathrm{Tr}B)\,\mathbf{1}_2 & \vec{0} \\ \vec{0}^t & 0 \end{bmatrix}, \tag{4.11}$$

and hence $r = 0$, $\vec{u} = \vec{w} = \vec{0}$. Because any matrix $B \in M_2$ is a complex combination of four positive matrices, we have that indeed $S_0(A) \in M_2 \subset M_3$ for any $A \in M_2 \subset M_3$.

Let $\{e_i\}_{i=1}^3$ be the standard orthonormal basis of $\mathbb{C}^3$ and $\{E_{jk}\}_{j,k=1}^3$ be the set of matrix units in M_3, $E_{jk} = e_j e_k^*$. We show that for $i = 1, 2, 3$ and $j = 1, 2$; $\langle e_i, S_0(E_{j3})e_i\rangle = 0$ and $\langle e_1, S_0(E_{j3})e_2\rangle = \langle e_2, S_0(E_{j3})e_1\rangle = 0$. Fix i, j, and take $X = |z_1|^2 P_j + z_1\overline{z_2}E_{j3} + \overline{z_1}z_2E_{3j} + |z_2|^2 P_3$, for some $z_1, z_2 \in \mathbb{C}$. It is evident that $X \geq 0$. Hence, since $S_0(E_{j3}) = S_0(E_{3j})^*$,

$$0 \;\leq\; \langle e_i, S_0(X)e_i\rangle \;=\; |z_1|^2\,\delta_{ij} + 2\,\mathrm{Re}\, z_1\overline{z_2}\,\langle e_i, S_0(E_{j3})e_i\rangle + |z_2|^2\,\delta_{i3}\,, \tag{4.12}$$

where δ_{ij} is the Kronecker delta. Because the above equation is true for every $z_1, z_2 \in \mathbb{C}$, and $j \neq 3$, it follows that necessarily $\langle e_i, S_0(E_{j3})e_i\rangle = 0$. Now, since $X \geq 0$, then so is $Y = P_{12}\,S_0(X)P_{12}$. From what has been just shown, $Y_{jj} = |z_1|^2$, but the second diagonal element of Y equals 0, and hence $Y_{12} = Y_{21} = 0$, which means that $\langle e_1, S_0(E_{j3})e_2\rangle = \langle e_2, S_0(E_{j3})e_1\rangle = 0$.

Therefore, we can write that for any $A = \begin{bmatrix} B & \vec{u} \\ \vec{w}^t & z \end{bmatrix} \in M_3$,

$$S_0(A) \;=\; S_0 \begin{bmatrix} B & \vec{u} \\ \vec{w}^t & z \end{bmatrix} \;=\; \begin{bmatrix} \hat{S}_0(B) & S_1\vec{u} + S_2\vec{w} \\ (\overline{S}_1\vec{w} + \overline{S}_2\vec{u})^t & \alpha z \end{bmatrix}, \tag{4.13}$$

and $S_1, S_2 \in M_2$, with matrix elements given by

$$(S_1)_{ij} \;=\; \langle e_i, S(E_{j3})e_3\rangle\,, \qquad (S_2)_{ij} \;=\; \langle e_i, S(E_{3j})e_3\rangle\,. \tag{4.14}$$

Suppose now that $\alpha = 0$. Then, since $S_0P_\eta \geq 0$ for every $\eta \in \mathbb{C}^3\backslash\{0\}$, we have that $S_1 = S_2 = 0$. Because $0 \leq (S - S_0)P(1, 1, 1)$,

$$0 \;\leq\; \hat{S}_0 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \;\leq\; \mathbf{1}_2 - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \;=\; \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{4.15}$$

i.e. for some $\beta \geq 0$,

$$\hat{S}_0 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \beta \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Repeating the same calculation, but this time for $P(i, i, 1)$, we obtain that for some $\beta' \geq 0$,

$$\hat{S}_0 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \beta' \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and so

$$\hat{S}_0 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0.$$

Similarly,

$$\hat{S}_0 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0,$$

and hence $\hat{S}_0(\mathbf{1}_2) = 0$, i.e. $\hat{S}_0 = 0$. Therefore, we can assume in the following that $\alpha > 0$.

Next, by direct computation, one proves that for any $\vec{\eta} \in \mathbb{C}^2$:

$$\mathrm{Tr}\, P(-2\vec{v}, ||\vec{\eta}||^2)\, SP(\vec{\eta}, 1) = 0, \tag{4.16}$$

where $\vec{v} = \frac{1}{\sqrt{2}}\left(\hat{P}_1\vec{\eta} + \hat{P}_2\overline{\vec{\eta}}\right)$, $||\vec{v}||^2 = ||\vec{\eta}||^2/2$. Because $0 \leq S_0 \leq S$, (4.16) holds also for S_0. Writing out the equation, and assuming from now on that $||\vec{\eta}|| = 1$, we get

$$4\,\vec{v}^*\hat{S}_0(\vec{\eta}\vec{\eta}^*)\vec{v} - 4\,\mathrm{Re}\,\vec{v}^*\vec{v}_0 + \alpha = 0, \tag{4.17}$$

where $\vec{v}_0 = S_1\vec{\eta} + S_2\overline{\vec{\eta}}$. To keep the notation simple, we remember that $\vec{v}$ and $\vec{v}_0$ depend on $\vec{\eta}$, without signifying it explicitly.

Since $S_0P(\vec{\eta}, 1)$ is a positive matrix, using again the Schur complement, $\vec{v}_0\vec{v}_0^* \leq \alpha\hat{S}_0(\vec{\eta}\vec{\eta}^*)$, and thus $|\vec{v}^*\vec{v}_0|^2 \leq \alpha\vec{v}^*\hat{S}_0(\vec{\eta}\vec{\eta}^*)\vec{v}$. Therefore

$$4\,(\mathrm{Im}\vec{v}^*\vec{v}_0)^2 + (2\mathrm{Re}\vec{v}^*\vec{v}_0 - \alpha)^2 \leq 4\alpha\vec{v}^*\hat{S}_0(\vec{\eta}\vec{\eta}^*)\vec{v} - 4\alpha\mathrm{Re}\vec{v}^*\vec{v}_0 + \alpha^2 = 0, \tag{4.18}$$

i.e. $\vec{v}^*\vec{v}_0 = \alpha/2$. Substituting both $\vec{v}$ and $\vec{v}_0$, we obtain:

$$\vec{\eta}^*\left(\hat{P}_1S_1 + S_2^t\hat{P}_2\right)\vec{\eta} + \vec{\eta}^*\hat{P}_1S_2\overline{\vec{\eta}} + \vec{\eta}^t\hat{P}_2S_1\vec{\eta} = \frac{\alpha}{\sqrt{2}}. \tag{4.19}$$

Because this holds true for any $\vec{\eta} \in \mathbb{C}^2$, $||\vec{\eta}|| = 1$, we can repeat the whole argument, but this time changing $\vec{\eta} \mapsto i\vec{\eta}$, to obtain:

$$\vec{\eta}^* \left(\hat{P}_1 S_1 + S_2^t \hat{P}_2\right) \vec{\eta} - \vec{\eta}^* \hat{P}_1 S_2 \overline{\vec{\eta}} - \vec{\eta}^t \hat{P}_2 S_1 \vec{\eta} \;=\; \frac{\alpha}{\sqrt{2}} . \tag{4.20}$$

Adding together (4.19) and (4.20), we have $\vec{\eta}^*(\hat{P}_1 S_1 + S_2^t \hat{P}_2)\vec{\eta} = \alpha/\sqrt{2}$, for any $\vec{\eta}$, $||\vec{\eta}|| = 1$. Hence

$$\hat{P}_1 S_1 + S_2^t \hat{P}_2 \;=\; \frac{\alpha}{\sqrt{2}} \mathbf{1}_2 . \tag{4.21}$$

Subtracting (4.20) from (4.19), we get

$$\vec{\eta}^* \hat{P}_1 S_2 \overline{\vec{\eta}} + \vec{\eta}^t \hat{P}_2 S_1 \vec{\eta} \;=\; 0 . \tag{4.22}$$

Let $\vec{\eta}$ be each of the following vectors: $(1, 0)$, $(0, 1)$, $(i, 0)$, $(0, i)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}})$, then we can see that $\hat{P}_1 S_2 = \hat{P}_2 S_1 = 0$. Combining this with (4.21), we obtain

$$S_1 = \begin{bmatrix} \frac{\alpha}{\sqrt{2}} & z_0 \\ 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 \\ -z_0 & \frac{\alpha}{\sqrt{2}} \end{bmatrix}, \quad z_0 \in \mathbb{C} . \tag{4.23}$$

Putting $\vec{v}^* \vec{v}_0 = \alpha/2$ into (4.17), we get that $\vec{v}^* \hat{S}_0(\vec{\eta}\vec{\eta}^*)\vec{v} = \alpha/4$. Hence

$$\vec{v}^* \left(\alpha \, \hat{S}_0(\vec{\eta}\vec{\eta}^*) - \vec{v}_0 \vec{v}_0^*\right) \vec{v} \;=\; 0 , \tag{4.24}$$

and since $\vec{v}_0 \vec{v}_0^* \leq \alpha \hat{S}_0(\vec{\eta}\vec{\eta}^*)$, it must be that $\alpha \, \hat{S}_0(\vec{\eta}\vec{\eta}^*)\vec{v} = (\vec{v}_0^* \vec{v}) \, \vec{v}_0$, or simply $\hat{S}_0(\vec{\eta}\vec{\eta}^*)\vec{v} = \frac{1}{2}\vec{v}_0$. Again, we can repeat the whole argument, changing $\vec{\eta} \mapsto i\vec{\eta}$, adding the obtained result to and subtracting from the previous one, and we arrive at:

$$\eta_1 \, \hat{S}_0(\vec{\eta}\vec{\eta}^*)\vec{e}_1 \;=\; \left(\frac{\alpha}{2}\eta_1 + \frac{\sqrt{2}}{2} z_0 \eta_2\right) \vec{e}_1 , \tag{4.25a}$$

$$\overline{\eta}_2 \, \hat{S}_0(\vec{\eta}\vec{\eta}^*)\vec{e}_2 \;=\; \left(-\frac{\sqrt{2}}{2} z_0 \overline{\eta}_1 + \frac{\alpha}{2}\overline{\eta}_2\right) \vec{e}_2 , \tag{4.25b}$$

where $\vec{\eta} = (\eta_1, \eta_2)$, and $\vec{e}_1 = (1, 0)$, $\vec{e}_2 = (0, 1) \in \mathbb{C}^2$. Taking e.g. (4.25a) and multiplying it by $\overline{\eta}_1\vec{e}_1^{\,*}$, we have

$$0 \;\le\; |\eta_1|^2\,\vec{e}_1^{\,*}\,\hat{S}_0(\vec{\eta}\vec{\eta}^{\,*})\vec{e}_1 \;=\; \frac{\alpha}{2}|\eta_1|^2 + \frac{\sqrt{2}}{2}z_0\overline{\eta}_1\eta_2\,, \tag{4.26}$$

for every $\vec{\eta} \in \mathbb{C}^2$, $||\vec{\eta}|| = 1$. Therefore $z_0 = 0$, and $S_1 = \frac{\alpha}{2}\hat{P}_1$, $S_2 = \frac{\alpha}{2}\hat{P}_2$. As a consequence of (4.25a) and (4.25b), $\hat{S}_0(\vec{\eta}\vec{\eta}^{\,*}) = \frac{\alpha}{2}\mathbf{1}_2$ for every $\vec{\eta}$ such that $\eta_1 \ne 0$ and $\eta_2 \ne 0$. For $0 < \epsilon < 1$, take $\vec{\eta}_\epsilon = (\sqrt{1-\epsilon^2}, \epsilon)$. Then $\hat{S}_0(\hat{P}_1) = \lim_{\epsilon\to 0}\hat{S}_0(\vec{\eta}_\epsilon\vec{\eta}_\epsilon^{\,*}) = \frac{\alpha}{2}\mathbf{1}_2$. Similarly, $\hat{S}_0(\hat{P}_2) = \frac{\alpha}{2}\mathbf{1}_2$, which results in $\hat{S}_0(\vec{\eta}\vec{\eta}^{\,*}) = \frac{\alpha}{2}\mathbf{1}_2$ for every $\vec{\eta} \in \mathbb{C}^2$, $||\vec{\eta}|| = 1$. This is sufficient to say that $\hat{S}_0(B) = \frac{\alpha}{2}(\mathrm{Tr}B)\mathbf{1}_2$ for any $B \in M_2$. We have shown that for an arbitrary positive map such that $0 \le S_0 \le S$, $S_0 = \alpha S$ for $0 \le \alpha \le 1$, which means that S is an extremal positive map of M_3.

It is of interest to note that S is not 2-positive. Indeed, we can prove even more and say that the map S does not fulfil the Kadison–Schwarz inequality for any matrix $B \in M_3$ (see Prop. 4.1 of [5]). Take the matrix $B = P_{12} + E_{32}$. Then it is easy to verify that $S(B^*B) - S(B)^*S(B)$ is not positive. Obviously, the map S is not 2-*co*positive either: to see this, take $B = P_{12} + E_{31}$. That, together with the fact that S is extremal, makes it atomic, and ends the proof. $\square$

Remark. In order to prove extremality of S, we showed that $S_0 = \alpha S$, for every $0 \le S_0 \le S$. But in fact, only a weaker assumption is needed. Indeed, let us assume that $S_0 \in \mathcal{P}(M_3)$ and $\mathrm{Tr}P_\xi\,S_0(P_\eta) = 0$, for every $\mathbb{C}^3 \ni \xi, \eta \ne 0$ such that $\mathrm{Tr}P_\xi\,S(P_\eta) = 0$. Then $\mathrm{Tr}P_{12}\,S_0(P_3) = 0$ and $\mathrm{Tr}P_3\,S_0(P_{12}) = 0$. Because S_0 is positive, so $S_0(P_3) = \alpha P_3$ and $S_0(P_{12}) \in M_2 \subset M_3$. Then the rest of the proof goes exactly the same as in Lemma 1, beginning with (4.10) onward. This proves that the map S is not only extremal, but also *exposed*.

For the map S specified in (4.3), we have $K_S = \mathbb{C}P_{12} \oplus \mathbb{C}P_3$. To the best of the authors' knowledge, all the examples of the extremal maps of M_3 that are not completely or co-completely positive, which have been so far specified in the literature, have $K_S = \mathbb{C}\,\mathbf{1}$. The map S would be the first one with a two-dimensional commutative stable algebra.

Given a positive map $S : M_n \to M_n$, the entanglement witness associated with S is a matrix W_S of the tensor matrix algebra $M_{n^2} = M_n \otimes M_n$, defined by

$$W_S \;=\; \sum_{i,j=1}^{n} E_{ij} \otimes S(E_{ij})\,, \tag{4.27}$$

where $\{E_{ij}\}_{i,j=1}^{n}$ are standard matrix units. The theorem by Choi and

Jamiołkowski [4, 13] states that the matrix W_S is a positive element of M_{n^2}, if and only if S is completely positive. Therefore, for a positive, non-completely positive map S there is at least one density matrix $\rho \in M_{n^2}$ such that $\mathrm{Tr}\, W_S\rho < 0$. This density matrix cannot be separable [27], and hence we say that the entanglement witness *detects* the entangled state ρ.

It would be of interest to provide the explicit form of the entanglement witness associated with the map S of (4.3), and the family of states on the composite quantum system space $M_9 = M_3 \otimes M_3$, detected by S. It is easy to see that in this case:

$$W_S = \left[\begin{array}{ccc|ccc|ccc} \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{2}} & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot \\ \frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right], \tag{4.28}$$

where dots mean matrix elements equal to zero. W_S is not a positive matrix of $M_9 = M_3 \otimes M_3$. Let $U \in \mathrm{U}(9)$ be a unitary matrix such that $U^* W_S U = W_S^{(2)} \oplus W_S^{(7)}$, where

$$W_S^{(2)} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \in M_2\,, \quad W_S^{(7)} = \begin{bmatrix} \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \in M_7\,. \tag{4.29}$$

It is evident that $W_S^{(7)} \geq 0$. Let also $\vec{v}$ be one of the eigenvectors of $W_S^{(2)}$: $\vec{v} = \frac{1}{\sqrt{2}}(1,-1)^t \in \mathbb{C}^2$, $W_S^{(2)}\vec{v} = -\frac{1}{\sqrt{2}}\vec{v}$, and $P_{\vec{v}} = \vec{v}\vec{v}^*$ be the orthogonal projection onto the space spanned by $\vec{v}$. We define $\rho = \frac{1}{2}U(P_{\vec{v}} \oplus \rho_0)U^*$, for a density matrix $\rho_0 \in M_7$, $\rho_0 \geq 0$, $\mathrm{Tr}\rho_0 = 1$, such that $\mathrm{Tr}\, W_S^{(7)}\rho_0 < \frac{1}{\sqrt{2}}$. Then $\rho \geq 0$ and $\mathrm{Tr}\rho = 1$, i.e. ρ is a density matrix. Moreover,

$$\mathrm{Tr}\, W_S\rho = \frac{1}{2}\mathrm{Tr}\, W_S U(P_{\vec{v}} \oplus \rho_0)U^* = \frac{1}{2}\mathrm{Tr}\,(W_S^{(2)} \oplus W_S^{(7)})(P_{\vec{v}} \oplus \rho_0)$$

$$\frac{1}{2}\mathrm{Tr}\, W_S^{(2)} P_{\vec{v}} + \frac{1}{2}\mathrm{Tr}\, W_S^{(7)} \rho_0 \quad < \quad 0\,, \tag{4.30}$$

which means that W_S detects the entangled state ρ. In particular, the state ρ given by

$$\rho \;=\; \frac{1}{7}\left[\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right] \tag{4.31}$$

is a PPT state, meaning that $(I \otimes t)\rho$ is still a density matrix, but $\mathrm{Tr}\, W_S \rho = \frac{2}{7}(1 - \sqrt{2}) < 0$.

Bibliography

[1] R. Bertlmann *et al.*, Phys. Rev. A **72**, 052331 (2005).

[2] O. Bratteli, and D. W Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, Springer, Berlin Heidelberg New York, 2003.

[3] C. M. Caves and G. J. Milburn, Opt. Commun. **179**, 439 (2000).

[4] M.-D. Choi, Linear Algebra Appl. **10**, 285 (1975).

[5] M.-D.Choi, J. Operat. Theor. **4**, 271 (1980).

[6] M.-D. Choi and T.-Y. Lam, Math. Ann. **231**, 1 (1977).

[7] D. Chruściński and G. Sarbicki, arXiv:1111.2012.

[8] D. Chruściński and G. Sarbicki, arXiv:1402.2413.

[9] K.-C. Ha, Publ. Res. I. Math. Sci. **34**, 591 (1998).

[10] K.-C. Ha and S.-H. Kye, Open Sys. Information Dyn. **18**, 323 (2011).

[11] H. Hanche-Olsen and E. Størmer, *Jordan Operator Algebras*, Pitman Advanced Pub. Program, Boston London Melbourne, 1984.

[12] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[13] A. Jamiołkowski, Rep. Math. Phys. **5**, 415 (1974).

[14] R. V. Kadison, Ann. Math. **56**, 494 (1952).

[15] M. Krein and D. Milman, Stud. Math. **9**, 133 (1940).

[16] S.-H. Kye, Lin. Alg. Appl. **216**, 239 (1995).

[17] M. Marciniak, Lin. Multilinear A. **61**, 970 (2013).

[18] M. Miller and R. Olkiewicz, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **17**, 1450028 (2014).

[19] R. Olkiewicz, Commun. Math. Phys. **208**, 245 (1999).

[20] H. Osaka, Publ. Res. I. Math. Sci. **28**, 747 (1992).

[21] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[22] W. F. Stinespring, P. Am. Math. Soc. **6**, 211 (1955).

[23] E. Størmer, Acta Math. **110**, 233 (1963).

[24] E. Størmer, Math. Scand. **105**, 188 (2009).

[25] E. Størmer, *Positive Linear Maps of Operator Algebras*, Springer, Berlin Heidelberg New York, 2013.

[26] S. Straszewicz, Fund. Math. **24**, 139 (1935).

[27] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).

[28] S. L. Woronowicz, Rep. Math. Phys. **10**, 165 (1976).

[29] F. Zhang, *The Schur Complement and Its Applications*, Springer, 2006.