Logistic Regression

Linear Regression Review

$y = \beta_0 + \beta_1 X_1 + \ldots + \beta_n X_n$

```
import pandas as pd
import numpy as np
from sklearn.datasets import make_regression

X, y = make_regression(n_samples=1000, n_features=5)
df = pd.DataFrame(data=X)
df.columns = ['x1', 'x2', 'x3', 'x4', 'x5']
df['y'] = y
print(df.head())
df.describe()
```

```
         x1        x2        x3        x4        x5           y
0  1.796004 -0.244652  0.657822 -0.103979  0.214390  197.208861
1 -1.086812 -0.444955 -1.450538  0.562638 -0.102427 -150.713633
2  0.959988  1.849409  1.598702 -0.945212  0.227278  133.849201
3  0.458813 -1.910318 -0.715464  1.722429  0.033708  133.298814
4  1.757517 -0.383854 -0.293895  1.727209 -0.853108  191.398191
```

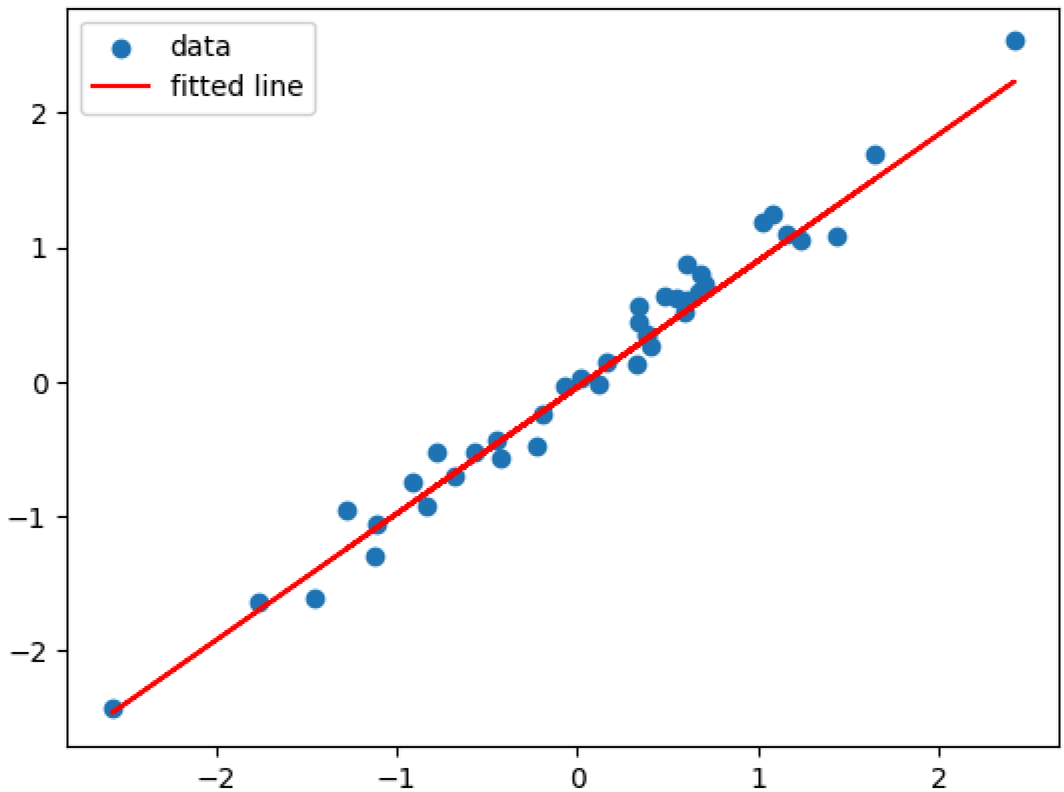
	x1	x2	x3	x4	x5	y
count	1000.000000	1000.000000	1000.000000	1000.000000	1000.000000	1000.000000
mean	0.013542	-0.006065	-0.020402	-0.017844	0.018806	-0.328668
std	0.968533	0.982815	0.983748	0.968562	1.005675	172.147916
min	-3.328911	-3.532877	-2.660831	-3.052230	-2.729183	-492.402207
25%	-0.622896	-0.684911	-0.661850	-0.682803	-0.662060	-119.221042
50%	0.012580	-0.000915	-0.045584	0.000186	0.042087	-1.733974
75%	0.662652	0.625220	0.634073	0.648692	0.674581	121.261788
max	3.182635	3.440037	3.087405	2.994076	3.633502	587.702845

Simple Linear Regression

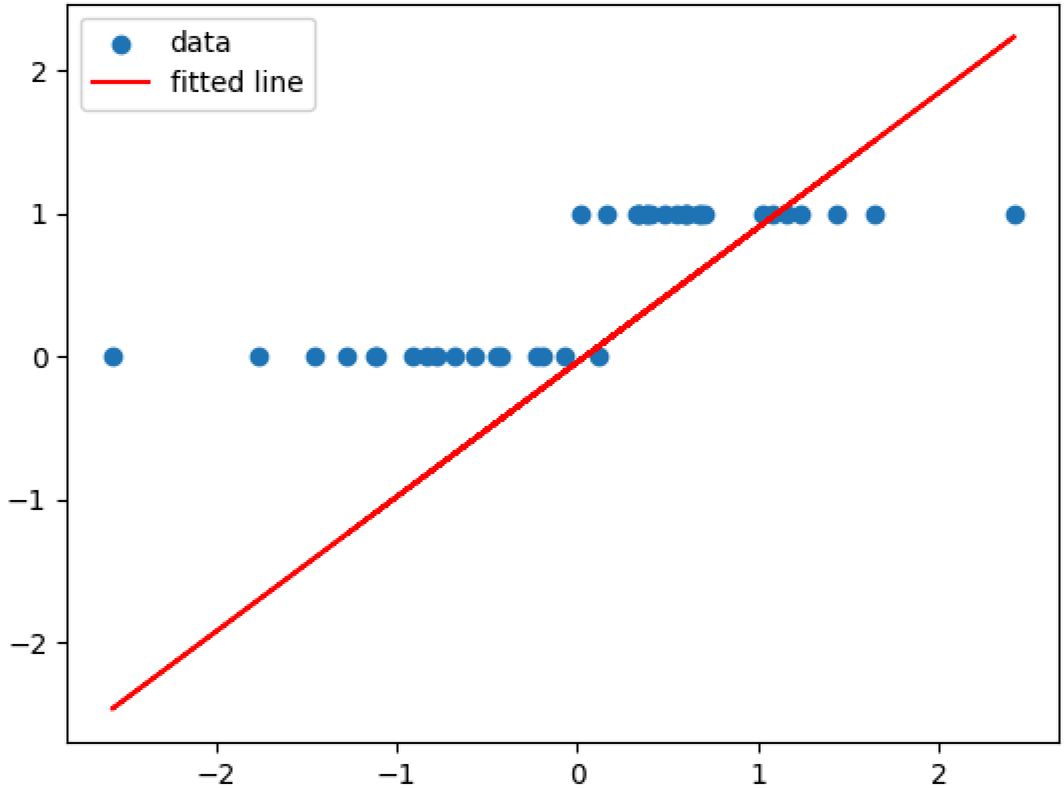
```
# create and train the model
import pandas as pd
import matplotlib.pyplot as plt
from sklearn.linear_model import LinearRegression
from sklearn.datasets import make_regression
from sklearn.model_selection import train_test_split
```

```
from sklearn.preprocessing import StandardScaler

X, y = make_regression(n_samples=50, n_features=1, noise=10)
df = pd.DataFrame(data=X, columns=['x'])
df['y'] = y
print(df.head())

scaler = StandardScaler()
df[['x', 'y']] = scaler.fit_transform(df[['x', 'y']])
print('\n Dataframe after scaling')
print(df.head())

# decision threshold = .5
df['class'] = df['y'].apply(lambda x: 0 if x <= 0 else 1)

X_train, X_test, y_train, y_test = train_test_split(df.drop(['y'], axis=1), df['y'], test

model = LinearRegression()
model.fit(X_train, y_train)

# test set prediction results
predictions = model.predict(X_test)
print('\nx, y scatter plot with line of best fit as per our line equation')
print(f'y = {model.intercept_} + {model.coef_[0]}X')

plt.scatter(X_train['x'], y_train, label='data')
plt.plot(X_train['x'], model.intercept_ + model.coef_[0] * X_train['x'], color='red', lab
# plt.axhline(0.5, color='green', linestyle='dashed', label='threshold')
plt.legend();
```

```
          x          y
0  0.074011   2.803995
1 -1.011085 -48.805477
2  0.683727  54.348082
3  0.382338  26.663093
4 -1.129127 -62.257658

 Dataframe after scaling
          x         y
0  0.377758  0.331632
1 -0.781449 -0.527449
2  1.029118  1.189624
3  0.707144  0.728785
4 -0.907552 -0.751371

x, y scatter plot with line of best fit as per our line equation
y = -0.04327650267858249 + 0.940839972851991X
```




```
# does a line of best fit over a binary dependent variable make sense?
plt.scatter(X_train['x'], X_train['class'], label='data')
plt.plot(X_train['x'], model.intercept_ + model.coef_[0] * X_train['x'], color='red', lal
plt.legend();
```



Logistic Regression

The Sigmoid

Sigmoid curves are common in statistics as cumulative distribution functions (which go from 0 to 1), such as the integrals of the logistic density, the normal density, and Student's t probability density functions. The logistic sigmoid function is invertible, and its inverse is the logit function.

https://en.wikipedia.org/wiki/Sigmoid_function

Sigmoidal growth is a modification of exponential growth in which the percent change gets smaller as the population approaches the carrying capacity.

http://matcmath.org/textbooks/quantitativereasoning/sigmoidal-growth/

What is this Thing Called Threshold?

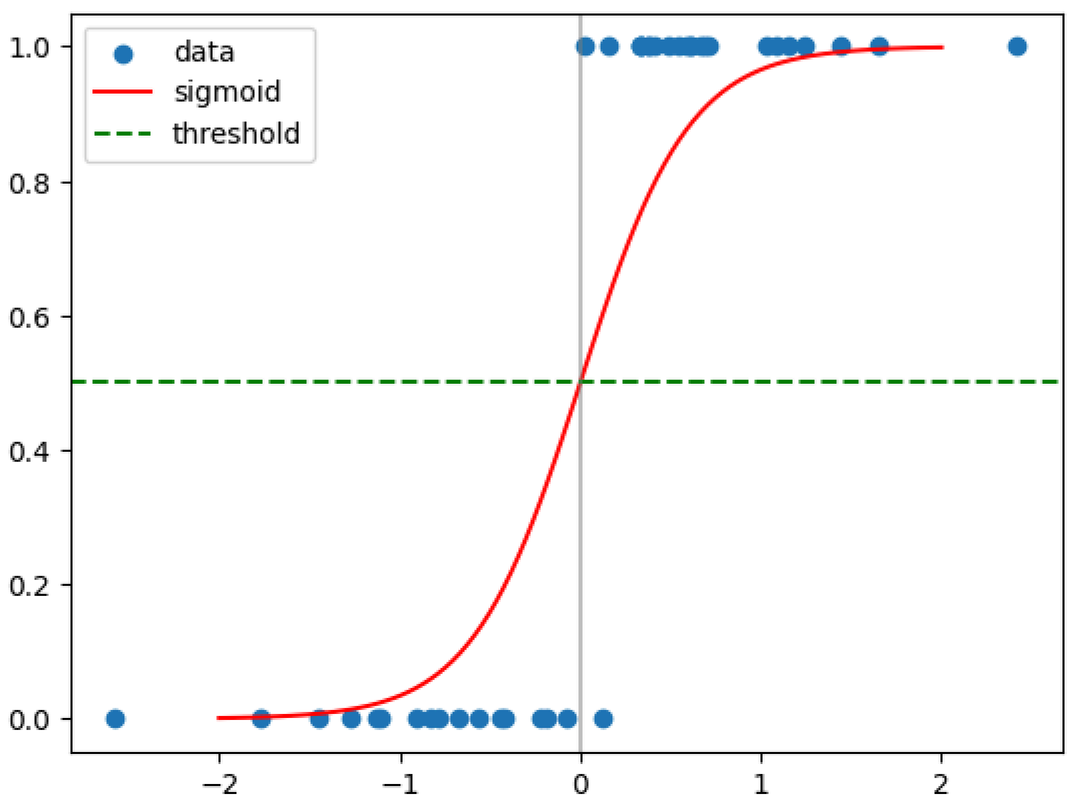
Logistic regression returns a probability. You can use the returned probability "as is" (for example, the probability that the user will click on this ad is 0.00023) or convert the returned probability to a binary value (for example, this email is spam).

A logistic regression model that returns 0.9995 for a particular email message is predicting that it is very likely to be spam. Conversely, another email message with a prediction score of 0.0003 on that same logistic regression model is very likely not spam. However, what about an email message with a prediction score of 0.6? In order to map a logistic regression value to a binary category, you must define a classification threshold (also called the decision threshold). A value above that threshold indicates "spam"; a value below indicates "not spam." It is tempting to assume that the classification threshold should always be 0.5, but thresholds are problem-dependent, and are therefore values that you must tune.

https://developers.google.com/machine-learning/crash-course/classification/thresholding

```
# what we'd really like
import numpy as np
import matplotlib.pyplot as plt
import scipy.stats as stats

# df with x and class
print(df.drop('y', axis=1).head())

x = np.linspace(-2, 2, 100)
s = .3 # scale parameter
print('\nAn S curve that is bound by 1 and 0')
plt.scatter(X_train['x'], X_train['class'], label='data')
plt.plot(x, 1 / (1 + np.exp(-(np.mean(x)/s + (1/s)*x))), color='red', label='sigmoid')
plt.axhline(.5, color='green', linestyle='dashed', label='threshold')
plt.axvline(0, color='gray', alpha=0.5)
plt.legend()
plt.show()
```

```
          x  class
0  0.377758      1
1 -0.781449      0
2  1.029118      1
3  0.707144      1
4 -0.907552      0

An S curve that is bound by 1 and 0
```



Probability

Probability = observed outcomes / possible outcomes

A naive idea (not what we actually use) would be:

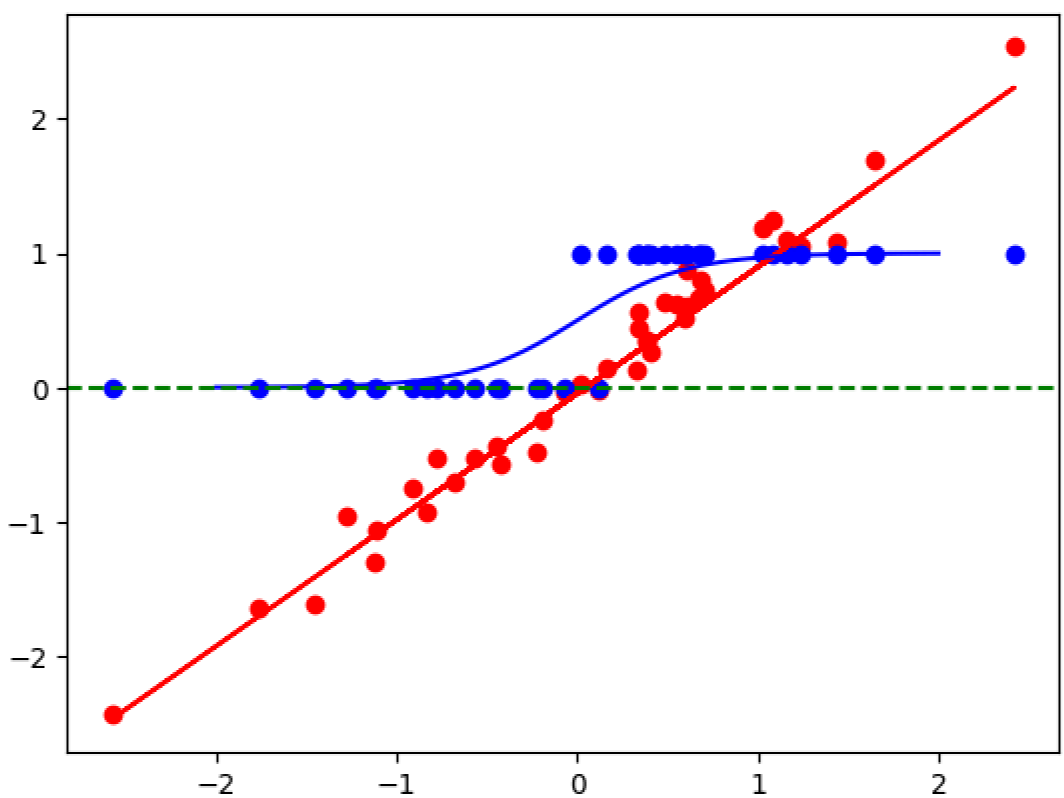
$$P(y = 1) \approx \beta_0 + \beta_1 X$$

But this can produce values below 0 or above 1, which is not valid for probabilities. With binary classification we want to show a better relationship with x. This motivates transforming the problem using odds and then log-odds (logit).

Review our line of best fit given x and y

```
print('\nx, y scatter plot with line of best fit as per our line equation')
print(f'y = {model.intercept_} + {model.coef_[0]}X')

plt.scatter(X_train['x'], y_train, label='data', color='red')
plt.plot(X_train['x'], model.intercept_ + model.coef_[0] * X_train['x'], color='red', lal

plt.scatter(X_train['x'], X_train['class'], label='class', color='blue')
plt.plot(x, 1 / (1 + np.exp(-(np.mean(x)/s + (1/s)*x))), label='sigmoid', color='blue')

plt.axhline(0, color='green', linestyle='dashed')
plt.show()
```

```
x, y scatter plot with line of best fit as per our line equation
y = -0.04327650267858249 + 0.940839972851991X
```



How do we bind this line to fit a binary outcome?

Odds

What happened/ what didn't happen - https://www.graphpad.com/support/faq/probability-vs-odds/

$$\frac{p}{1-p} = \beta_0 + \beta_1 X$$

Odds are positive so we have a range of 0, $+\infty$ so we use log

```
# https://www.statisticshowto.com/log-odds/
import pandas as pd
import math

print('Defining odds based on a probability represented by a decimals')
d = {'Probs': [.1, .2, .3, .4, .5, .6, .7, .8, .9]}
probs = pd.DataFrame(d)
probs['Odds'] = probs['Probs']/(1 - probs['Probs'])
probs.head(9)
```

```
Defining odds based on a probability represented by a decimals
```

	Probs	Odds
0	0.1	0.111111
1	0.2	0.250000
2	0.3	0.428571
3	0.4	0.666667
4	0.5	1.000000
5	0.6	1.500000
6	0.7	2.333333
7	0.8	4.000000
8	0.9	9.000000

Log Odds

We use the log of odds so that we can get to $(-\infty, +\infty)$

$$log(\frac{p}{1-p}) = \beta_0 + \beta_1 X$$

Now our range is unrestricted $-\infty, +\infty$ where the odds for and odds against meet at 0 with potential symmetrical range 0 to infinity. See images at https://towardsdatascience.com/https-towardsdatascience-com-what-and-why-of-log-odds-64ba988bf704

In binomial logistic regression, the dependent variable takes only two values, which can be coded 0 and 1. So, it can't be that $Y = b0 + b1X$ because that could result in any value. The logit transformation solves this problem.

We take the odds to make the value continuous. We take the odds ratio to get a parameter estimate and we take the log of that ratio to make the variable range from negative infinity to infinity and be symmetric around 0 instead of 1.

Peter Flom (https://stats.stackexchange.com/users/686/peter-flom), Why $\log(\frac{p}{1-p}) = b_0 + b_1 x$ in Logistic regression, URL (version: 2017-09-25): https://stats.stackexchange.com/q/304844

```
# log odds example
import math

p = .2
odds = p/(1-p)
print('.2/.8 =', odds)
print('log(.25) =', math.log(odds))
```

```
.2/.8 = 0.25
log(.25) = -1.3862943611198906
```

```
# add log odds to our probs dataframe
probs['Log Odds'] = probs['Odds'].apply(lambda x: math.log(x))
probs.head(9)
```

	Probs	Odds	Log Odds
0	0.1	0.111111	-2.197225
1	0.2	0.250000	-1.386294
2	0.3	0.428571	-0.847298
3	0.4	0.666667	-0.405465
4	0.5	1.000000	0.000000
5	0.6	1.500000	0.405465
6	0.7	2.333333	0.847298
7	0.8	4.000000	1.386294
8	0.9	9.000000	2.197225

Logistic Function Proof

- $p(y=1) = \beta_0 + \beta_1 x$
- $\frac{p}{1-p} = \beta_0 + \beta_1 x$ # bounded by 0
- $log(\frac{p}{1-p}) = \beta_0 + \beta_1 x$ # what we want because it is unbounded
- $e^{ln[\frac{p}{1-p}]} = e^{\beta_0 + \beta_1 x}$

- $\frac{p}{1-p} = e^{\beta_0+\beta_1 x}$
- $p = e^{\beta_0+\beta_1 x} - pe^{\beta_0+\beta_1 x}$
- $= p[\frac{e^{\beta_0+\beta_1 x}}{p} - e^{\beta_0+\beta_1 x}]$
- $p[1 + e^{\beta_0+\beta_1 x}] = e^{\beta_0+\beta_1 x}$
- $= \frac{e^{\beta_0+\beta_1 x}}{1+e^{\beta_0+\beta_1 x}}$
- $= \frac{1}{1+e^{-(\beta_0+\beta_1 x+\ldots)}}$
- $= \frac{1}{1+e^{-(x)}}$

Why the Exponential

Exponentials are also used to express growth which is instrumental with the sigmoid function. The function $y = e^x$ gives a curve at which the slope at any value x is also the value of y, which means the derivative is equal to the function itself, or $f'(x) = e^x$ and $f(x) = e^x$.

https://www.ml-science.com/eulers-number

Summary

Logistic functions are used in logistic regression to model how the probability p of an event may be affected by one or more explanatory variables: an example would be to have the model $p = f(a + bx)$, where x is the explanatory variable, a and b are model parameters to be fitted, and f is the standard logistic function.

https://en.wikipedia.org/wiki/Logistic_function#In_statistics_and_machine_learning

- $f(x) = \frac{1}{1+e^{-(x)}}$

https://www.analyticsvidhya.com/blog/2021/08/conceptual-understanding-of-logistic-regression-for-data-science-beginners/

Example

```
# Create dataset
import pandas as pd
from sklearn.datasets import make_classification

n = 1000 # number of observations
f = 2 # number of features
inform = 2 # number of meaningful features

# unpack X and y
features, binary_class = make_classification(n_samples=n, n_features=f,
                                             n_informative=inform, n_redundant=0,
                                             n_clusters_per_class=1, random_state=42)

# Create a dataframe of the features and add the binary class (label, output)
```

```
df = pd.DataFrame(features)
df.columns = ['Feature_1', 'Feature_2']
df['Class'] = binary_class
df.head()
```

	Feature_1	Feature_2	Class
0	0.601034	1.535353	1
1	0.755945	-1.172352	0
2	1.354479	-0.948528	0
3	3.103090	0.233485	0
4	0.753178	0.787514	1

$p(y=1) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$

or

$\pi(x) = \sigma(\beta_0 + \beta_1 x_1 + \beta_2 x_2)$

Logistic Model

In statistics, the logistic model (or logit model) is a statistical model that models the probability of an event taking place by having the log-odds for the event be a linear combination of one or more independent variables. In regression analysis, logistic regression (or logit regression) is estimating the parameters of a logistic model (the coefficients in the linear combination).

- $p = \frac{1}{1+e^{-(\beta_0+\beta_1 x)}}$
- https://en.wikipedia.org/wiki/Logistic_regression
- https://en.m.wikipedia.org/wiki/Logistic_function

```
# get our parameters
from statsmodels.formula.api import logit

model = logit(formula='Class ~ Feature_1 + Feature_2', data=df).fit()
model.summary()
```

```
Optimization terminated successfully.
         Current function value: 0.242088
         Iterations 8
```

Logit Regression Results

Dep. Variable:	Class	**No. Observations:**	1000
Model:	Logit	**Df Residuals:**	997
Method:	MLE	**Df Model:**	2
Date:	Thu, 10 Oct 2024	**Pseudo R-squ.:**	0.6507
Time:	21:53:44	**Log-Likelihood:**	-242.09
converged:	True	**LL-Null:**	-693.15
Covariance Type:	nonrobust	**LLR p-value:**	1.283e-196

	coef	std err	z	P>|z|	[0.025	0.975]
Intercept	1.1197	0.365	3.070	0.002	0.405	1.834
Feature_1	-0.4561	0.171	-2.673	0.008	-0.790	-0.122
Feature_2	3.4780	0.237	14.649	0.000	3.013	3.943

```
import numpy as np

print(model.params)

def linear_function(row):
    b0 = model.params.Intercept
    b1 = model.params.Feature_1
    b2 = model.params.Feature_2

    x1 = row['Feature_1']
    x2 = row['Feature_2']

    return b0 + (b1 * x1) + (b2 * x2)

def logistic_function(row):
    b0 = model.params.Intercept
    b1 = model.params.Feature_1
    b2 = model.params.Feature_2

    x1 = row['Feature_1']
    x2 = row['Feature_2']

    return 1 / (1 + np.exp(-(b0+(b1*x1)+(b2*x2))))

df['y'] = df.apply(linear_function, axis=1)
df['p(y=1)'] = df.apply(logistic_function, axis=1)
df['Predicted Class'] =  df['p(y=1)'].apply(lambda x: 0 if x < .5 else 1)
df['Odds'] = df['p(y=1)']/(1 - df['p(y=1)'])
df['Log Odds'] = df['Odds'].apply(lambda x: math.log(x))
df.head()
```

```
Intercept     1.119669
Feature_1    -0.456101
Feature_2     3.478029
dtype: float64
```

	Feature_1	Feature_2	Class	y	p(y=1)	Predicted Class	Odds	Log Odds
0	0.601034	1.535353	1	6.185537	0.997945	1	485.673902	6.185537
1	0.755945	-1.172352	0	-3.302593	0.035482	0	0.036788	-3.302593
2	1.354479	-0.948528	0	-2.797120	0.057480	0	0.060985	-2.797120
3	3.103090	0.233485	0	0.516414	0.626309	1	1.676007	0.516414

Threshold

Logistic regression returns a probability. You can use the returned probability "as is" (for example, the probability that the user will click on this ad is 0.00023) or convert the returned probability to a binary value (for example, this email is spam).

A logistic regression model that returns 0.9995 for a particular email message is predicting that it is very likely to be spam. Conversely, another email message with a prediction score of 0.0003 on that same logistic regression model is very likely not spam. However, what about an email message with a prediction score of 0.6? In order to map a logistic regression value to a binary category, you must define a classification threshold (also called the decision threshold). A value above that threshold indicates "spam"; a value below indicates "not spam." It is tempting to assume that the classification threshold should always be 0.5, but thresholds are problem-dependent, and are therefore values that you must tune.

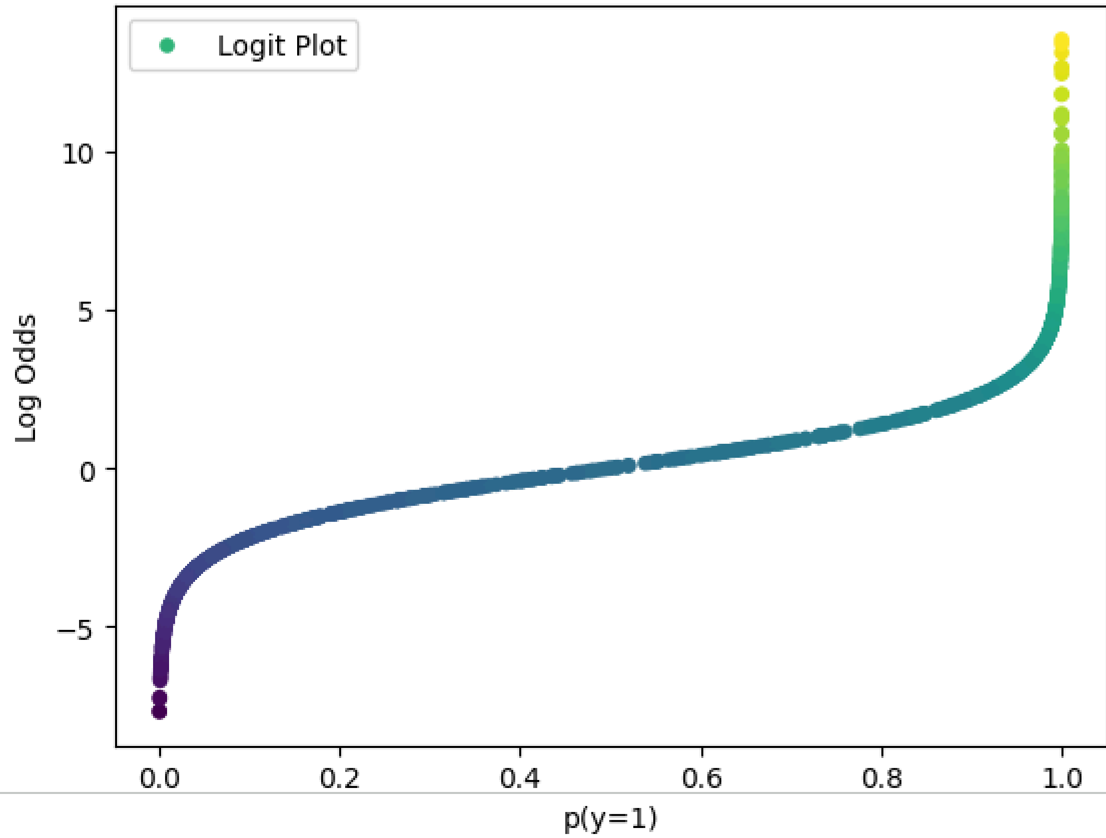
https://developers.google.com/machine-learning/crash-course/classification/thresholding

Logit

A Logit function, the inverse of the logistic sigmoid, also known as the log-odds function, is a function that represents probability values from 0 to 1, and negative infinity to infinity.

- https://deepai.org/machine-learning-glossary-and-terms/logit
- https://en.m.wikipedia.org/wiki/Logit

```
# logit curve
import matplotlib.pyplot as plt

df.plot.scatter(x='p(y=1)', y='Log Odds', label='Logit Plot', c=df['Log Odds'])
plt.show()
```



Logit vs Sigmoid

The inverse of the logit curve is the inverse-logit or sigmoid function. The sigmoid function transforms the numbers ($-\infty$ to $+\infty$) back to values between 0 and 1.

- https://ajaytech.co/python-logistic-regression/

```
import matplotlib.pyplot as plt
```