Notes on Numerical Analysis
数值分析

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1 Solving Nonlinear Equations 求解非线性方程组

1.1 The bisection method 二分法

二分法通过重复地将间隔减小到根存在的半间隔来求连续函数 f: R → R.

Input: $f:[a,b]\to R, a\in R, b\in R, M\in N^+, \delta\in R^+, \epsilon\in R^+$
Preconditions(前置条件): $f\in C[a,b], sgn(f(a))\neq sgn(f(b))$
Output: c, h, k
Postconditions(终止条件): |f(c)| < ϵ or |h| < δ or k = M

```
u <- f(a)
v <- f(b)
for k = 1:M do
    h <- b-a
    c <- a+h/2
    w <- f(c)
    if |h| < delta or |w| < epsilon Then
       break
    else if sgn(w) =! sgn(u) then
         b <- c  v <- w
         else
         a <- c  u <- w
    end
end
```

1.2 The signature of an algorithm 算法的签名

Definition 1.2. 算法是一个逐步的过程，它将一组值作为输入，并生成一组值，作为输出。

(An algorithm is a step-by-step procedure that takes some set of values as its input and produces some set of values as its output.)

定义 1.3. 先决条件是在执行算法之前保持输入的条件。

(A precondition is a condition that holds for the input prior to the execution of an algorithm.)

定义 1.4. 后置条件是在执行算法后对输出有效的条件

(A postcondition is a condition that holds for the output after the execution of an algorithm.)

定义 1.5. 算法的签名包括其输入、输出、先决条件、后置条件以及如何处理违反先决条件的输入参数。

(The signature of an algorithm consists of its input, output, preconditions, postconditions, and how input parameters violating preconditions are handled.)

1.3 Proof of correctness and simplification of algorithms 算法的正确性证明和简化

定义 1.6. **不变量是在算法执行期间保持不变的条件。**

(An invariant is a condition that holds during the execution of an algorithm.)

定义 1.7. **如果变量是在循环中初始化的，则该变量是临时变量或从循环中派生的变量。如果变量在循环之前初始化，并且其值在不同迭代期间发生变化，则变量对于循环是持久的或主要的。**

(A variable is temporary or derived for a loop if it is initialized inside the loop. A variable is persistent or primary for a loop if it is initialized before the loop and its value changes across different iterations.)

定义 1.8. **算法 1.1 中的不变量是什么？a、b、c、h、u、v、w 代表哪些量？其中哪些是主要的？以下哪些变量是临时变量？绘制图片以说明这些变量的寿命。**

(What are the invariants in Algorithm 1.1? Which quantities do a, b, c, h, u, v, w represent? Which of them are primary? Which of these variables are temporary? Draw pictures to illustrate the life spans of these variables.)

定义 1.9. **一种简化的二等分算法。**

(A simplified bisection algorithm.)

Input: $f : [a, b] \to R, a \in R, b \in R, M \in N^+, \delta \in R^+, \epsilon \in R^+$
Preconditions(前置条件): $f \in C[a, b], sgn(f(a)) \neq sgn(f(b))$
Output: c, h, k
Postconditions(终止条件): $|f(c)| < \epsilon$ or $|h| < \delta$ or $k = M$

```
h <- b-a
u <- f(a)
for k = 1:M do
    h <- h/2
    c <- a+h
    w <- f(c)
    if |h| < delta or |w| < epsilon Then
       break
    else if sgn(w) = sgn(u) then
         a <- c
    end
end
```

1.4 Q-order convergence Q 阶收敛

定义 1.10. **(Q-order convergence) 收敛序列 $\{x_n\}$ 收敛于 L, 若:**

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|^p} = c > 0; (1.1)$$

常数 c 称为渐近因子。特别地，当 p=1 时，$\{x_n\}$ 具有 Q-线性收敛性，而当 p=2 时，具有 Q-二次收敛性。

(A convergent sequence x_n is said to converge to L with Q-order p (p $\geq$ 1),the constant c is called the asymptotic factor. In particular, x_n has Q-linear convergence if p = 1 and Q-quadratic convergence if p = 2.)

定义 1.11. 迭代序列 $\{x_n\}$ 被称为线性收敛到 L, 如果:

$$\exists c \in (0,1), \exists d > 0, s.t. \forall n \in N, |x_n - L| \leq c^n d.(1.2)$$

对于一个收敛到 L 的序列 $\{x_n\}$，其收敛阶数是所有满足 $p \in R^+$ 中最大的那个 p。

$$\exists c > 0, \exists N \in N s.t. \forall n > N, |x_{n+1} - L| \leq c|x_n - L|^p.(1.3)$$

特别的, 如果 p=2, 则 $\{x_n\}$ 二次收敛

(A sequence of iterates $\{x_n\}$ is said to converge linearly to L if: For a sequence $\{x_n\}$ that converges to L, its order of convergence is the maximum $p \in R^+$ satisfying.In particular, $\{x_n\}$ converges quadratically if p = 2.)

定义 1.12. (单调序列定理 Monotonic sequence theorem) 每个有界单调序列都是收敛的。

(Every bounded monotonic sequence is convergent.)

定义 1.13. (二分法的收敛性 Convergence of the bisection method). 对于满足 $sgn(f(a_1)) \neq sgn(f(b_2))$ 的连续函数 $f : [a_0, b_0] \to R$, 二分法中的迭代序列线性收敛于渐近因子 $\frac{1}{2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \alpha, (1.4)$$

$$f(\alpha) = 0, (1.5)$$

$$|c_n - \alpha| \leq 2^{-(n+1)}(b_0 - a_0), (1.6)$$

其中 $[a_n, b_n]$ 是二分法第 n 次迭代的间隔, 并且 $c_n = \frac{1}{2}(a_n + b_n)$。

(For a continuous function $f : [a_0, b_0] \to R$ satisfying $sgn(f(a_0)) \neq sgn(f(b_0))$, the sequence of iterates in the bisection method converges linearly with asymptotic factor $\frac{1}{2}$,where $[a_n, b_n]$ is the interval in the nth iteration of the bisection method and $c_n = \frac{1}{2}(a_n + b_n)$.)

证明. 由二分法得出:

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq b_0,$$

$$b_0 \geq b_1 \geq b_2 \geq \cdots \geq a_0,$$

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n).$$

另外，“lim” 是 “$\lim_{n\to\infty}$.” 的缩写。根据定理 1.12, $\{a_n\}$ 和 $\{b_n\}$ 都收敛。此外 $\lim(b_n - a_n) = \lim \frac{1}{2^n}(b_0 - a_0) = 0$, 因此 $\lim b_n = \lim a_n = \alpha$。通过给定的条件和算法, 不变量 $f(a_n)f(b_n) \leq 0$ 始终成立。由于 f 是连续的, 所以 $limf(a_n)f(b_n) = f(\lim a_n)f(limb_n)$, 则 $f^2(\alpha)$ 表示 $f(\alpha) = 0$. (1.6) 是另一个重要的不变量, 可以通过归纳证明。将 (1.6) 与 (1.2) 进行比较, 得出二分法的收敛性。此外收敛性与渐近因子 $c = \frac{1}{2}$ 成线性关系。

(In the rest of this proof, “lim” is a shorthand for “$\lim_{n\to\infty}$.” By Theorem 1.12, both $\{a_n\}$ and $\{b_n\}$ converge. Also, $\lim(b_n - a_n) = \lim \frac{1}{2^n}(b_0 - a_0) = 0$, hence $\lim b_n = \lim a_n = \alpha$. By the given condition and the algorithm, the invariant $f(a_n)f(b_n) \leq 0$ always holds. Since f is continuous, $\lim f(a_n)f(b_n) = f(\lim a_n)f(limb_n)$, then $f_2(\alpha) \leq 0$ implies $f(\alpha) = 0$. (1.6) is another important invariant that can be proven by induction. Comparing (1.6) to (1.2) yields convergence of the bisection method. Also, the convergence is linear with asymptotic factor as $c = \frac{1}{2}$.)

1.5 Newton’s method 牛顿法

算法 1.14. 牛顿法通过迭代公式在初始值 x_0 附近找到 $f : R \to R$ 的根

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n \in N(1.7)$$

Input: $f : R \to R, f', x_0 \in R, M \in N^+, \epsilon \in R^+$
Preconditions(前置条件): $f \in C^2$ and x_0 is sufficiently close to a root of f
Output: x, k
Postconditions(终止条件): |f(c)| < ϵ or k = M

```
x <- x_0
for k = 0 : M do
    u <- f(x)
    if |u| < delta Then
       break
    end
    x <- x-u/f'(x)
end
```

(Algorithm 1.14. Newton’s method finds the root of $f : R \to R$ near an initial guess x_0 by the iteration formula)

定理 1.15 (牛顿法的收敛性)。考虑 $B = [\alpha - \delta, \alpha - \delta]$ 上的函数 $f : B \to R$ ($f \in C^2$) 满足 $f(\alpha) = 0$ 和 $f'(\alpha) \neq 0$。如果选取的 x_0 充分接近 α，则牛顿方法中的迭代序列 $\{x_n\}$ 二次收敛到根 α，即:

(Theorem 1.15 (Convergence of Newton’s method). Consider a C^2 function $f : B \to R$ on $B = [\alpha - \delta, \alpha + \delta]$ satisfying $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. If x_0 is chosen sufficiently close to α, then the sequence of iterates x_n in the Newton’s method converges quadratically to the root α, i.e.)

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^2} = -\frac{f''(\alpha)}{2f'(\alpha)}.(1.8)$$

证明. 根据泰勒定理（定理 C.60）和所假设的函数 $f(f \in C^2)$，我们有

(Proof. By Taylor’s theorem (Theorem C.60) and the assumption $f \in C^2$, we have)

$$f(\alpha) = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\xi)$$

当 ξ 在 α 和 x_n 之间，$f(\alpha) = 0$ 得:

(When ξ is between α and x_n. $f(\alpha)) = 0$ yields)

$$-\alpha = -x_n + \frac{f(x_n)}{f'(x_n)} + \frac{(\alpha - x_n)^2}{2} \frac{f''(\xi)}{f'(x_n)}.$$

通过式子 (1.7)，我们有:

(By (1.7), we have)

$$(*) : x_{n+1} - \alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \alpha = (x_n - \alpha)^2 \frac{f''(\xi)}{2f'(x_n)}.$$

通过 f' 的连续性以及 $f'(\alpha) \neq 0$ 的假设:

(The continuity of f' and the assumption $f'(\alpha) \neq 0$ yield)

$$\exists \delta_1 \in (0, \delta) s.t. \forall x \in B_1, f'(x) \neq 0$$

其中 $B_1 = [\alpha - \delta_1, \alpha + \delta_1]$. 定义:

(Where $B_1 = [\alpha - \delta_1, \alpha + \delta_1]$. Define)

$$M = \frac{max_{x \in B_1} |f''(x)|}{2min_{x \in B_1} |f'(x)|}$$

选取 x_0，使它接近 α，使得:

(And pick x_0 sufficiently close to α such that)

(i) $|x_0 - \alpha| = \delta_0 \leq \delta_1$;

(ii) $M\delta_0 \leq 1$.

由 M 和 (*) 的定义有:

(The definition of M and (*) imply)

$$|x_{n+1} - \alpha| \leq M|x_n - \alpha|^2.$$

将上述与不等式与 (1.3) 进行比较可得，如果 $\{x_n\}$ 收敛，则收敛阶数为 2。我们仍然必须证明 (a) 它收敛，并且 (b) 它收敛到 α。

通过 (i) 和 (ii)，我们有 $M|x_0 - \alpha| < 1$。然后通过归纳可以很容易获得以下结果:

$$|x_n - \alpha| \leq \frac{1}{M}(M|x_0 - \alpha|)^{2^n},$$

这说明了 (a) 和 (b)，且完成了证明.

(Comparing the above to (1.3) implies that if $\{x_n\}$ converges, then the order of convergence is 2. We must still show that (a) it converges and (b) it converges to α.)

(By (i) and (ii), we have $M|x_0 - \alpha| < 1$. Then it is easy to obtain the following via induction, which shows both (a) and (b) and completes the proof.)

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