Bessel's equation

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The linear differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y \;=\; 0, \tag{1}$$

in which p is a constant (non-negative if it is real), is called the *Bessel's equation*. We derive its general solution by trying the series form

$$y \;=\; x^r\sum_{k=0}^{\infty} a_k x^k \;=\; \sum_{k=0}^{\infty} a_k x^{r+k}, \tag{2}$$

due to Frobenius. Since the parameter r is indefinite, we may regard a_0 as distinct from 0.

We substitute (2) and the derivatives of the series in (1):

$$x^2\sum_{k=0}^{\infty}(r+k)(r+k-1)a_k x^{r+k-2} + x\sum_{k=0}^{\infty}(r+k)a_k x^{r+k-1} + (x^2-p^2)\sum_{k=0}^{\infty} a_k x^{r+k} \;=\; 0.$$

Thus the coefficients of the powers x^r, x^{r+1}, x^{r+2} and so on must vanish, and we get the system of equations

$$\begin{cases} [r^2 - p^2]a_0 \;=\; 0, \\ [(r+1)^2 - p^2]a_1 \;=\; 0, \\ [(r+2)^2 - p^2]a_2 + a_0 \;=\; 0, \\ \qquad\ldots \\ [(r+k)^2 - p^2]a_k + a_{k-2} \;=\; 0. \end{cases} \tag{3}$$

The last of those can be written

$$(r+k-p)(r+k+p)a_k + a_{k-2} \;=\; 0.$$

Because $a_0 \neq 0$, the first of those (the indicial equation) gives $r^2 - p^2 = 0$, i.e. we have the roots

$$r_1 \;=\; p, \quad r_2 \;=\; -p.$$

Let's first look the the solution of (1) with $r = p$; then $k(2p+k)a_k + a_{k-2} = 0$, and thus

$$a_k \;=\; -\frac{a_{k-2}}{k(2p+k).}$$

From the system (3) we can solve one by one each of the coefficients a_1, a_2, $\ldots$ and express them with a_0 which remains arbitrary. Setting for k the integer values we get

$$\begin{cases} a_1 \;=\; 0, \quad a_3 \;=\; 0, \;\ldots,\; a_{2m-1} \;=\; 0; \\ a_2 \;=\; -\frac{a_0}{2(2p+2)}, \quad a_4 \;=\; \frac{a_0}{2\cdot 4(2p+2)(2p+4)}, \;\ldots,\; a_{2m} \;=\; \frac{(-1)^m a_0}{2\cdot 4\cdot 6\cdots(2m)(2p+2)(2p+4)\ldots(2p+2m)} \end{cases} \tag{4}$$

(where $m = 1, 2, \ldots$). Putting the obtained coefficients to (2) we get the particular solution

$$y_1 \;:=\; a_0 x^p \left[\frac{x^2}{2(2p+2)} + \frac{x^4}{2\cdot 4(2p+2)(2p+4)} - \frac{x^6}{2\cdot 4\cdot 6(2p+2)(2p+4)(2p+6)} + - \ldots\right] \tag{5}$$

In order to get the coefficients a_k for the second root $r_2 = -p$ we have to look after that

$$(r_2 + k)^2 - p^2 \;\neq\; 0,$$

or $r_2 + k \neq p = r_1$. Therefore

$$r_1 - r_2 \;=\; 2p \;\neq\; k$$

where k is a positive integer. Thus, when p is not an integer and not an integer added by $\frac{1}{2}$, we get the second particular solution, gotten of (5) by replacing p by $-p$:

$$y_2 \;:=\; a_0 x^{-p} \left[1 - \frac{x^2}{2(-2p+2)} + \frac{x^4}{2\cdot 4(-2p+2)(-2p+4)} - \frac{x^6}{2\cdot 4\cdot 6(-2p+2)(-2p+4)(-2p+6)} + - \ldots\right] \tag{6}$$

The power series of (5) and (6) converge for all values of x and are linearly independent (the ratio y_1/y_2 tends to 0 as $x \to \infty$). With the appointed value

$$a_0 \;=\; \frac{1}{2^p\,\Gamma(p+1)},$$

the solution y_1 is called the *Bessel function of the first kind and of order* p and denoted by J_p. The similar definition is set for the first kind Bessel function of an arbitrary order $p \in \mathbb{R}$ (and $\mathbb{C}$). For $p \notin \mathbb{Z}$ the general solution of the Bessel's differential equation is thus

$$y \;:=\; C_1 J_p(x) + C_2 J_{-p}(x),$$

where $J_{-p}(x) = y_2$ with $a_0 = \frac{1}{2^{-p}\Gamma(-p+1)}$.

The explicit expressions for $J_{\pm p}$ are

$$J_{\pm p}(x) \;=\; \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m \pm p + 1)} \left(\frac{x}{2}\right)^{2m \pm p}, \qquad (7)$$

which are obtained from (5) and (6) by using the last for gamma function.

E.g. when $p = \frac{1}{2}$ the series in (5) gets the form

$$y_1 \;=\; \frac{x^{\frac{1}{2}}}{\sqrt{2}\,\Gamma(\frac{3}{2})} \left[1 - \frac{x^2}{2{\cdot}3} + \frac{x^4}{2{\cdot}4{\cdot}3{\cdot}5} - \frac{x^6}{2{\cdot}4\,{\cdot}6{\cdot}3{\cdot}5{\cdot}7} + - \ldots\right] \;=\; \sqrt{\frac{2}{\pi x}} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \ldots\right).$$

Thus we get

$$J_{\frac{1}{2}}(x) \;=\; \sqrt{\frac{2}{\pi x}}\,\sin x;$$

analogically (6) yields

$$J_{-\frac{1}{2}}(x) \;=\; \sqrt{\frac{2}{\pi x}}\,\cos x,$$

and the general solution of the equation (1) for $p = \frac{1}{2}$ is

$$y \;:=\; C_1 J_{\frac{1}{2}}(x) + C_2 J_{-\frac{1}{2}}(x).$$

In the case that p is a non-negative integer n, the "+" case of (7) gives the solution

$$J_n(x) \;=\; \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n},$$

but for $p = -n$ the expression of $J_{-n}(x)$ is $(-1)^n J_n(x)$, i.e. linearly dependent on $J_n(x)$. It can be shown that the other solution of (1) ought to be searched in the form $y = K_n(x) = J_n(x)\ln x + x^{-n}\sum_{k=0}^{\infty} b_k x^k$. Then the general solution is $y := C_1 J_n(x) + C_2 K_n(x)$.

Other formulae

The first kind Bessel functions of integer order have the generating function F:

$$F(z,\,t) \;=\; e^{\frac{z}{2}(t - \frac{1}{t})} \;=\; \sum_{n=-\infty}^{\infty} J_n(z) t^n \qquad (8)$$

This function has an essential singularity at $t = 0$ but is analytic elsewhere in $\mathbb{C}$; thus F has the Laurent expansion in that point. Let us prove (8) by using the general expression

$$c_n \;=\; \frac{1}{2\pi i}\oint_\gamma \frac{f(t)}{(t-a)^{n+1}}\,dt$$

of the coefficients of Laurent series. Setting to this $a := 0$, $f(t) := e^{\frac{z}{2}(t-\frac{1}{t})}$, $\zeta := \frac{zt}{2}$ gives

$$c_n \;=\; \frac{1}{2\pi i}\oint_\gamma \frac{e^{\frac{zt}{2}}e^{-\frac{z}{2t}}}{t^{n+1}}\,dt \;=\; \frac{1}{2\pi i}\left(\frac{z}{2}\right)^n\oint_\delta \frac{e^\zeta e^{-\frac{z^2}{4\zeta}}}{\zeta^{n+1}}\,d\zeta \;=\; \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{z}{2}\right)^{2m+n}\frac{1}{2\pi i}\oint_\delta \zeta^{-m-n-1}e^\zeta\,d\zeta.$$

The paths γ and δ go once round the origin anticlockwise in the t-plane and ζ-plane, respectively. Since the residue of $\zeta^{-m-n-1}e^\zeta$ in the origin is $\frac{1}{(m+n)!} = \frac{1}{\Gamma(m+n+1)}$, the `http://planetmath.org/CauchyResidueTheorem`residue theorem gives

$$c_n \;=\; \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+n+1)}\left(\frac{z}{2}\right)^{2m+n} \;=\; J_n(z).$$

This that F has the Laurent expansion (8).

By using the generating function, one can easily derive other formulae, e.g. the of the Bessel functions of integer order:

$$J_n(z) \;=\; \frac{1}{\pi}\int_0^\pi \cos(n\varphi - z\sin\varphi)\,d\varphi$$

Also one can obtain the addition formula

$$J_n(x+y) \;=\; \sum_{\nu=-\infty}^{\infty} J_\nu(x)J_{n-\nu}(y)$$

and the series of cosine and sine:

$$\cos z \;=\; J_0(z) - 2J_2(z) + 2J_4(z) - + \ldots$$

$$\sin z \;=\; 2J_1(z) - 2J_3(z) + 2J_5(z) - + \ldots$$

References

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[2] K. Kurki-Suonio: *Matemaattiset apuneuvot*. Limes r.y., Helsinki (1966).