linear differential equation of first order

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An ordinary linear differential equation of first order has the form

$$\frac{dy}{dx} + P(x)y \;=\; Q(x), \tag{1}$$

where y means the unknown function, P and Q are two known continuous functions.

For finding the solution of (1), we may seek a function y which is product of two functions:

$$y(x) \;=\; u(x)v(x) \tag{2}$$

One of these two can be chosen freely; the other is determined according to (1).

We substitute (2) and the derivative $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ in (1), getting $u\frac{dv}{dx} + v\frac{du}{dx} + Puv = Q$, or

$$u\left(\frac{dv}{dx} + Pv\right) + v\frac{du}{dx} \;=\; Q. \tag{3}$$

If we chose the function v such that

$$\frac{dv}{dx} + Pv \;=\; 0,$$

this condition may be written

$$\frac{dv}{v} \;=\; -P\,dx.$$

Integrating here both sides gives $\ln v = -\int P\,dx$ or

$$v \;=\; e^{-\int P dx},$$

where the exponent means an arbitrary antiderivative of $-P$. Naturally, $v(x) \neq 0$.

Considering the chosen property of v in (3), this equation can be written

$$v\frac{du}{dx} \;=\; Q,$$

i.e.
$$\frac{du}{dx} = \frac{Q(x)}{v(x)},$$
whence
$$u = \int \frac{Q(x)}{v(x)}\,dx + C = C + \int Qe^{\int P dx} dx.$$

So we have obtained the solution
$$y = e^{-\int P(x)dx}\left[C + \int Q(x)e^{\int P(x)dx} dx\right] \tag{4}$$
of the given differential equation (1).

The result (4) presents the general solution of (1), since the arbitrary C may be always chosen so that any given initial condition
$$y = y_0 \quad \text{when} \quad x = x_0$$
is fulfilled.