derivatives of solution of first order ODE

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Suppose that f is a continuously differentiable function defined on an open subset E of $\mathbb{R}^2$, i.e. it has on E the continuous partial derivatives $f'_x(x,\,y)$ and $f'_y(x,\,y)$.

If $y(x)$ is a solution of the ordinary differential equation

$$\frac{dy}{dx} \;=\; f(x,\,y), \tag{1}$$

then we have

$$y'(x) \;=\; f(x,\,y(x)), \tag{2}$$

$$y''(x) \;=\; f'_x(x,\,y(x))+f'_y(x,\,y(x))\,y'(x) \tag{3}$$

(see the http://planetmath.org/node/2798general chain rule). Thus there exists on E the second derivative $y''(x)$ which is also continuous. More generally, we can infer the

Theorem. If $f(x,\,y)$ has in E the continuous partial derivatives up to the order n, then any solution $y(x)$ of the differential equation (1) has on E the continuous derivatives $y^{(i)}(x)$ up to the http://planetmath.org/OrderOfDerivativeorder $n{+}1$.

Note 1. The derivatives $y^{(i)}(x)$ are got from the equation (1) via succesive differentiations. Two first ones are (2) and (3), and the next two ones, with a simpler notation:

$$y''' \;=\; f''_{xx}+2f''_{xy}y'+f''_{yy}y'^2+f'_yy'',$$

$$y^{(4)} \;=\; f'''_{xxx}+3f'''_{xxy}y'+3f'''_{xyy}y'^2+f'''_{yyy}y'^3+3f''_{xy}y''+3f''_{yy}y'y''+f'_yy'''$$

Note 2. It follows from (3) that the curve

$$f'_x(x,\,y)+f'_y(x,\,y)f(x,\,y) \;=\; 0 \tag{4}$$

is the locus of the inflexion points of the integral curves of (1), or more exactly, the locus of the points where the integral curves have with their tangents a http://planetmath.org/OrderOfContactcontact of order more than one. The curve (4) is also the locus of the points of tangency of the integral curves and their isoclines.

References

[1] E. Lindelöf: *Differentiali- ja integralilasku III 1.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1935).