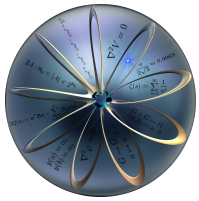

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all solution of the Lorenz equation enter an ellipsoid

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If $\sigma, \tau, \beta > 0$ then all solutions of the Lorenz equation

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\tau - z) - y \\ \dot{z} &= xy - \beta z \end{aligned}$$

will enter an ellipsoid centered at $(0, 0, 2\tau)$ in finite time. In addition the solution will remain inside the ellipsoid once it has entered. To observe this we define a Lyapunov function

$$V(x, y, z) = \tau x^2 + \sigma y^2 + \sigma(z - 2\tau)^2.$$

It then follows that

$$\begin{aligned} \dot{V} &= 2\tau x\dot{x} + 2\sigma y\dot{y} + 2\sigma(z - 2\tau)\dot{z} \\ &= 2\tau x\sigma(y - x) + 2\sigma y(x(\tau - z) - y) + 2\sigma(z - 2\tau)(xy - \beta z) \\ &= -2\sigma(\tau x^2 + y^2 + \beta(z - r)^2 - b\tau^2). \end{aligned}$$

We then choose an ellipsoid which all the solutions will enter and remain inside. This is done by choosing a constant $C > 0$ such that the ellipsoid

$$\tau x^2 + y^2 + \beta(z - r)^2 = b\tau^2$$

is strictly contained in the ellipsoid

$$\tau x^2 + \sigma y^2 + \sigma(z - 2\tau)^2 = C.$$

Therefore all solution will eventually enter and remain inside the above ellipsoid since $\dot{V} < 0$ when a solution is located at the exterior of the ellipsoid.