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variation of parameters

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The method of *variation of parameters* is a way of finding a particular solution to a nonhomogeneous linear differential equation.

Suppose that we have an nth order linear differential operator

$$L[y] := y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y,$$

and a corresponding nonhomogeneous differential equation

$$L[y] = g(t). \tag{1}$$

Suppose that we know a fundamental set of solutions $y_1, y_2, \ldots, y_n$ of the corresponding homogeneous differential equation $L[y_c] = 0$. The general solution of the homogeneous equation is

$$y_c(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t),$$

where $c_1, c_2, \ldots, c_n$ are constants. The general solution to the nonhomogeneous equation $L[y] = g(t)$ is then

$$y(t) = y_c(t) + Y(t),$$

where $Y(t)$ is a particular solution which satisfies $L[Y] = g(t)$, and the constants $c_1, c_2, \ldots, c_n$ are chosen to satisfy the appropriate boundary conditions or initial conditions.

The key step in using variation of parameters is to suppose that the particular solution is given by

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t), \tag{2}$$

where $u_1(t), u_2(t), \ldots, u_n(t)$ are as yet to be determined functions (hence the name *variation of parameters*). To find these n functions we need a set of n independent equations. One obvious condition is that the proposed ansatz satisfies Eq. (**??**). Many possible additional conditions are possible, we choose the ones that make further calculations easier. Consider the following set of $n-1$ conditions

$$\begin{aligned}
y_1u_1' + y_2u_2' + \cdots + y_nu_n' &= 0 \\
y_1'u_1' + y_2'u_2' + \cdots + y_n'u_n' &= 0 \\
&\vdots \\
y_1^{(n-2)}u_1' + y_2^{(n-2)}u_2' + \cdots + y_n^{(n-2)}u_n' &= 0.
\end{aligned}$$

Now, substituting Eq. (**??**) into $L[Y] = g(t)$ and using the above conditions, we can get another equation

$$y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n' = g.$$

So we have a system of n equations for $u_1', u_2', \ldots, u_n'$ which we can solve using Cramer's rule:

$$u_m'(t) = \frac{g(t)W_m(t)}{W(t)}, \quad m = 1, 2, \ldots, n.$$

Such a solution always exists since the Wronskian $W = W(y_1, y_2, \ldots, y_n)$ of the system is nowhere zero, because the $y_1, y_2, \ldots, y_n$ form a fundamental set of solutions. Lastly the term W_m is the Wronskian determinant with the mth column replaced by the column $(0, 0, \ldots, 0, 1)$.

Finally the particular solution can be written explicitly as

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int \frac{g(t)W_m(t)}{W(t)} dt.$$

References

[1] W. E. Boyce and R. C. DiPrima. *Elementary Differential Equations and Boundary Value Problems* John Wiley & Sons, 6th edition, 1997.