elliptic function

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Let $\Lambda \in \mathbb{C}$ be a lattice in the sense of number theory, i.e. a 2-dimensional free group over $\mathbb{Z}$ which generates $\mathbb{C}$ over $\mathbb{R}$.

An *elliptic function* ϕ, with respect to the lattice Λ, is a meromorphic funtion $\phi : \mathbb{C} \to \mathbb{C}$ which is Λ-periodic:

$$\phi(z + \lambda) = \phi(z), \quad \forall z \in \mathbb{C}, \quad \forall \lambda \in \Lambda$$

Remark: An elliptic function which is holomorphic is constant. Indeed such a function would induce a holomorphic function on $\mathbb{C}/\Lambda$, which is compact (and it is a standard result from Complex Analysis that any holomorphic function with compact domain is constant, this follows from Liouville's Theorem).

Example: The Weierstrass $\wp$-function (see elliptic curve) is an elliptic function, probably the most important. In fact:

Theorem 1. *The field of elliptic functions with respect to a lattice Λ is generated by $\wp$ and $\wp'$ (the derivative of $\wp$).*

Proof. See [**?**], chapter 1, theorem 4. □

References

[1] James Milne, *Modular Functions and Modular Forms*, online course notes. `http://www.jmilne.org/math/CourseNotes/math678.html`http://www.jmilne.org/math/Co

[2] Serge Lang, *Elliptic Functions*. Springer-Verlag, New York.

[3] Joseph H. Silverman, *The Arithmetic of Elliptic Curves*. Springer-Verlag, New York, 1986.