elliptic integrals and Jacobi elliptic functions

Canonical name	EllipticIntegralsAndJacobiEllipticFunctions
Date of creation	2013-03-22 13:58:28
Last modified on	2013-03-22 13:58:28
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	7
Author	mathcam (2727)
Entry type	Definition
Classification	msc 33E05
Related topic	ArithmeticGeometricMean
Related topic	PerimeterOfEllipse
Defines	elliptic integral
Defines	Jacobi elliptic function
Defines	Jacobian elliptic function
Defines	complementary modulus
Defines	complete elliptic integral

Elliptic integrals

For a modulus $0 < k < 1$ (while here, we define the *complementary modulus* to k to be the positive number k' with $k^2 + k'^2 = 1$) , write

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \tag{1}$$

$$E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta}\, d\theta \tag{2}$$

$$\Pi(n, \phi, k) = \int_0^\phi \frac{d\theta}{(1 + n \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} \tag{3}$$

The change of variable $x = \sin \phi$ turns these into

$$F_1(x, k) = \int_0^x \frac{dv}{\sqrt{(1 - v^2)(1 - k^2 v^2)}} \tag{4}$$

$$E_1(x, k) = \int_0^x \sqrt{\frac{1 - k^2 v^2}{1 - v^2}}\, dv \tag{5}$$

$$\Pi_1(n, x, k) = \int_0^x \frac{dv}{(1 + nv^2)\sqrt{(1 - v^2)(1 - k^2 v^2)}} \tag{6}$$

The first three functions are known as Legendre's form of the incomplete elliptic integrals of the first, second, and third kinds respectively. Notice that (2) is the special case $n = 0$ of (3). The latter three are known as Jacobi's form of those integrals. If $\phi = \pi/2$, or $x = 1$, they are called complete rather than incomplete integrals, and we refer to the auxiliary elliptic integrals $K(k) = F(\pi/2, k)$, $E(k) = E(\pi/2, k)$, etc.

One use for elliptic integrals is to systematize the evaluation of certain other integrals. In particular, let p be a third- or fourth-degree polynomial in one variable, and let $y = \sqrt{p(x)}$. If q and r are any two polynomials in two variables, then the indefinite integral

$$\int \frac{q(x, y)}{r(x, y)}\, dx$$

has a "closed form" in terms of the above incomplete elliptic integrals, together with elementary functions and their inverses.

Jacobi's elliptic functions

In (1) we may regard ϕ as a function of F, or vice versa. The notation used is

$$\phi = \operatorname{am} u \qquad u = \arg \phi$$

and ϕ and u are known as the amplitude and argument respectively. But $x = \sin \phi = \sin \operatorname{am} u$. The function $u \mapsto \sin \operatorname{am} u = x$ is denoted by sn and is one of four *Jacobi (or Jacobian) elliptic functions*. The four are:

$$\begin{aligned}
\operatorname{sn} u &= x \\
\operatorname{cn} u &= \sqrt{1 - x^2} \\
\operatorname{tn} u &= \frac{\operatorname{sn} u}{\operatorname{cn} u} \\
\operatorname{dn} u &= \sqrt{1 - k^2 x^2}
\end{aligned}$$

When the Jacobian elliptic functions are extended to complex arguments, they are doubly periodic and have two poles in any parallelogram of periods; both poles are simple.