planetmath.org

Math for the people, by the people.

convergence of arithmetic-geometric mean

Canonical name	ConvergenceOfArithmeticgeometricMean
Date of creation	2013-03-22 17:09:46
Last modified on	2013-03-22 17:09:46
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	13
Author	rspuzio (6075)
Entry type	Theorem
Classification	msc 33E05
Classification	msc 26E60

In this entry, we show that the arithmetic-geometric mean converges. By the arithmetic-geometric means inequality, we know that the sequences of arithmetic and geometric means are both monotonic and bounded, so they converge individually. What still needs to be shown is that they converge to the same limit.

Define $x_n = a_n/g_n$. By the arithmetic-geometric inequality, we have $x_n \geq 1$. By the defining recursions, we have

$$x_{n+1} = \frac{a_{n+1}}{g_{n+1}} = \frac{a_n + g_n}{2\sqrt{a_n g_n}} = \frac{1}{2}\left(\sqrt{\frac{a_n}{g_n}} + \sqrt{\frac{g_n}{a_n}}\right) = \frac{1}{2}\left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}}\right)$$

Since $x_n \geq 1$, we have $1/\sqrt{x_n} \leq 1$, and $\sqrt{x_n} \leq x_n$, hence

$$x_{n+1} - 1 = \frac{1}{2}\left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}} - 2\right) \leq \frac{1}{2}(x_n + 1 - 2) \leq \frac{1}{2}(x_n - 1).$$

From this inequality

$$0 \leq x_{n+1} - 1 \leq \frac{1}{2}(x_n - 1),$$

we may conclude that $x_n \to 1$ as $n \to \infty$, which , by the definition of x_n, is equivalent to

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} a_n.$$

Not only have we proven that the arithmetic-geometric mean converges, but we can infer a rate of convergence from our proof. Namely, we have that $0 \leq x_n - 1 \leq (x_0 - 1)/2^n$. Hence, we see that the rate of convergence of a_n and g_n to the answer goes as $O(2^{-n})$.

By more carefully bounding the recursion for x_n above, we may obtain better estimates of the rate of convergence. We will now derive an inequality. Suppose that $y \geq 0$.

$$\begin{aligned} 0 &\leq y^5 + y^4 + 4y^3 + 3y^2 \\ y^2 + 4y + 4 &\leq y^5 + y^4 + 4y^3 + 4y^2 + 4y + 4 \\ (y+2)^2 &\leq (y+1)(y^2+2)^2 \end{aligned}$$

Set $x = y + 1$ (so we have $x \geq 1$).

$$\begin{aligned}
(x+1)^2 &\leq x((x-1)^2+2)^2 \\
x &\leq \frac{x^2((x-1)^2+2)^2}{(x+1)^2} \\
\sqrt{x} &\leq \frac{x((x-1)^2+2)}{x+1} \\
\frac{x+1}{x}\sqrt{x} &\leq (x-1)^2+2 \\
\frac{1}{2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right) &\leq 1+\frac{1}{2}(x-1)^2
\end{aligned}$$

Thus, because $x_{n+1} = (\sqrt{x_n} + 1/\sqrt{x_n})/2$, we have

$$x_{n+1} - 1 \leq \frac{1}{2}(x_n - 1)^2.$$

From this equation, we may derive the bound

$$x_n - 1 \leq \frac{1}{2^{2^n-1}}(x_0 - 1)^{2^n}.$$

This is a much better bound! It approaches zero far more rapidly than any exponential function, so we have superlinear convergence.