orthogonality of Legendre polynomials

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We start from the first order differential equation

$$(1-x^2)\frac{du}{dx}+2nxu \;=\; 0, \tag{1}$$

where one can http://planetmath.org/SeparationOfVariablesseparate the variables and then get the general solution

$$u \;=\; C(1-x^2)^n. \tag{2}$$

Differentiating $n+1$ times the equation (1) it takes the form

$$(1-x^2)\frac{d^{n+2}u}{dx^{n+2}}-2x\frac{d^{n+1}u}{dx^{n+1}}+n(n+1)\frac{d^nu}{dx^n} \;=\; 0$$

or

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y \;=\; 0 \tag{3}$$

where

$$y \;=\; \frac{d^nu}{dx^n} \;=\; C\frac{d^n}{dx^n}(1-x^2)^n.$$

Especially, the particular solution

$$y \;=\; P_n(x) \;:=\; \frac{1}{2^nn!}\frac{d^n}{dx^n}(1-x^2)^n, \tag{4}$$

which which is the Legendre polynomial of degree n, has been seen to satisfy the Legendre's differential equation (3).

The equality (4) is http://planetmath.org/RodriguesFormulaRodrigues formula. We use it to find the leading coefficient of $P_n(x)$ and to show the http://planetmath.org/OrthogonalPolynomialsorthogonality of the Legendre polynomials $P_0(x),\ P_1(x),\ P_2(x)$, ...

0.1 The coefficient of x^n

By the binomial theorem,

$$\begin{aligned} P_n(x) \;&=\; \frac{1}{2^nn!}\frac{d^n}{dx^n}\sum_{j=0}^{n}\binom{n}{j}x^{2(n-j)}(-1)^j \\ &=\; \frac{1}{2^nn!}\sum_{j=0}^{n}\binom{n}{j}(2n-2j)(2n-2j-1)\cdots(2n-2j-n+1)x^{n-2j}(-1)^j. \end{aligned}$$

From the term with $j = 0$ we get as the coefficient of x^n the following:

$$\frac{1}{2^n n!}\binom{n}{0}(2n)(2n-1)(2n-2)\cdots(2n-n+1)(-1)^0 \;=\; \frac{1}{2^n n!}\cdot\frac{(2n)!}{(2n-n)!} \;=\; \frac{(2n)!}{2^n(n!)^2} \tag{5}$$

0.2 Orthogonality

Let $f_m(x) := a_0 + a_1x + \ldots + a_mx^m$ be any polynomial of degree $m < n$. http://planetmath.org/IntegrationByPartsIntegrating by parts m times we obtain

$$\begin{aligned}
\int_{-1}^{1} f_m(x)P_n(x)\,dx &= \frac{1}{2^n n!}\int_{-1}^{1} f_m(x)\frac{d^n}{dx^n}(x^2-1)^n\,dx \\
&= \frac{1}{2^n n!}\Big/_{-1}^{1} f_m(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n - \frac{1}{2^n n!}\int_{-1}^{1} f'_m(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx \\
\ldots \quad & \quad \ldots \\
&= (-1)^m\frac{a_m m!}{2^n n!}\int_{-1}^{1}\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\,dx \\
&= (-1)^m\frac{a_m m!}{2^n n!}\Big/_{-1}^{1} f_m(x)\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n \;=\; 0,
\end{aligned}$$

since $x = \pm 1$ are zeros of the derivatives $\frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n$.

If, on the other hand, $m = n$, the calculation gives firstly

$$\int_{-1}^{1} f_n(x)P_n(x)\,dx \;=\; 2(-1)^n\frac{a_n}{2^n}\int_0^1(x^2-1)^n\,dx \;=\; 2(-1)^n\frac{a_n}{2^n}\cdot I_n, \tag{6}$$

where the integral I_n is gotten from

$$I_n \;=\; \Big/_{0}^{1} x(x^2-1)^n - 2n\int_0^1 x^2(x^2-1)^{n-1}dx \;=\; -2n\int_0^1\left[(x^2-1)^n + (x^2-1)^{n-1}\right]dx \;=\; -2nI_n - 2nI_{n-}$$

Thus we infer the recurrence relation

$$I_n \;=\; -\frac{2n}{2n+1}I_{n-1}.$$

Using this and $I_0 = 1$ one easily arrives at

$$I_n \;=\; (-1)^n\frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)} \;=\; (-1)^n\frac{[2\cdot 4\cdot 6\cdots(2n)]^2}{(2n+1)!} \;=\; (-1)^n\frac{2^{2n}(n!)^2}{(2n+1)!}. \tag{7}$$

If $f_n(x)$ also is a Legendre polynomial $P_n(x)$, we can in (6) by (5) put

$$a_n \;=\; \frac{(2n)!}{2^n(n!)^2}$$

and taking into account (7), too, (6) reads

$$\int_{-1}^{1}[P_n(x)]^2\,dx \;=\; \frac{(-1)^n}{2^{n-1}}\cdot\frac{(2n)!}{2^n(n!)^2}\cdot(-1)^n\frac{2^{2n}(n!)^2}{(2n+1)!} \;=\; \frac{2}{2n+1}.$$

Our results imply the `http://planetmath.org/Orthonormal`orthonormality condition

$$\int_{-1}^{1}P_m(x)P_n(x)\,dx \;=\; \frac{2}{2n+1}\delta_{mn}, \tag{8}$$

where δ_{mn} is the Kronecker delta.

References

[1] K. Kurki-Suonio: *Matemaattiset apuneuvot*. Limes r.y., Helsinki (1966).