orthogonality of Chebyshev polynomials from recursion

Canonical name	OrthogonalityOfChebyshevPolynomialsFromRecursion
Date of creation	2013-03-22 18:54:46
Last modified on	2013-03-22 18:54:46
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	6
Author	rspuzio (6075)
Entry type	Proof
Classification	msc 33C45
Classification	msc 33D45
Classification	msc 42C05

In this entry, we shall demonstrate the orthogonality relation of the Chebyshev polynomials from their recursion relation. Recall that this relation reads as

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1} = 0$$

with initial conditions $T_0(x) = 1$ and $T_1(x) = x$. The relation we seek to demonstrate is

$$\int_{-1}^{+1} dx \, \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} = 0$$

when $m \neq n$.

We start with the observation that T_n is an even function when n is even and an odd function when n is odd. That this is true for T_0 and T_1 follows immediately from their definitions. When $n > 1$, we may induce this from the recursion. Suppose that $T_m(-x) = (-1)^m T_m(x)$ when $m < n$. Then we have

$$\begin{aligned}
T_{n+1}(-x) &= 2(-x)T_n(-x) - T_{n-1}(-x) \\
&= -(-1)^n 2xT_n(x) - (-1)^{n-1}T_{n-1}(x) \\
&= (-1)^{n+1}(2xT_n(x) - T_{n-1}(x)) \\
&= (-1)^{n+1}T_{n+1}(x).
\end{aligned}$$

From this observation, we may immediately conclude half of orthogonality. Suppose that m and n are nonnegative integers whose difference is odd. Then $T_m(-x)T_n(-x) = -T_m(x)T_n(x)$, so we have

$$\int_{-1}^{+1} dx \, \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} = 0$$

because the integrand is an odd function of x.

To cover the remaining cases, we shall proceed by induction. Assume that T_k is orthogonal to T_m whenever $m \leq n$ and $k \leq n$ and $m \neq k$. By the conclusions of last paragraph, we know that T_{n+1} is orthogonal to T_n. Assume then that $m \leq n - 1$. Using the recursion, we have

$$\begin{aligned}
\int_{-1}^{+1} dx \, \frac{T_m(x)T_{n+1}(x)}{\sqrt{1-x^2}} &= 2\int_{-1}^{+1} dx \, \frac{xT_m(x)T_n(x)}{\sqrt{1-x^2}} - \int_{-1}^{+1} dx \, \frac{T_m(x)T_{n-1}(x)}{\sqrt{1-x^2}} \\
&= \int_{-1}^{+1} dx \, \frac{T_{m+1}(x)T_n(x)}{\sqrt{1-x^2}} + \int_{-1}^{+1} dx \, \frac{T_{m-1}(x)T_n(x)}{\sqrt{1-x^2}} - \int_{-1}^{+1} dx \, \frac{T_m(x)T_{n-1}(x)}{\sqrt{1-x^2}}
\end{aligned}$$

By our assumption, each of the three integrals is zero, hence T_{n+1} is orthogonal to T_m, so we conclude that T_k is orthogonal to T_m when $m \leq n+1$ and $k \leq n+1$ and $m \neq k$.