proof of bounds for e

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Multiplying and dividing, we have

$$\left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{m}\right)^m \prod_{k=m+1}^{n} \frac{\left(1+\frac{1}{k}\right)^k}{\left(1+\frac{1}{k-1}\right)^{k-1}}$$

As was shown in the parent entry, the quotients in the product can be simplified to give

$$\left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{m}\right)^m \prod_{k=m+1}^{n} \left(1-\frac{1}{k^2}\right)^k \left(1+\frac{1}{k-1}\right)$$

By the inequality for differences of powers,

$$\left(1-\frac{1}{k^2}\right)^k < 1-\frac{k}{k^2+k-1} = \frac{(k+1)(k-1)}{k^2+k-1}$$

Hence, we have the following upper bound:

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{m}\right)^m \prod_{k=m+1}^{n} \frac{k^2+k}{k^2+k-1}$$

By cross mutliplying, it is easy to see that

$$\frac{k^2+k}{k^2+k-1} \le \frac{k^2}{k^2-1}$$

and, hence,

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{m}\right)^m \prod_{k=m+1}^{n} \frac{k^2}{k^2-1}.$$

Factoring the rational function in the product, terms cancel and we have

$$\prod_{k=m+1}^{n} \frac{k^2}{(k+1)(k-1)} = \frac{n(m+1)}{(n+1)m} = \frac{n}{n+1}\left(1+\frac{1}{m}\right)$$

Combining,

$$\left(1+\frac{1}{n}\right)^n < \frac{n}{n+1}\left(1+\frac{1}{m}\right)^{m+1}$$