construction of Dirac delta function

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The Dirac delta function is notorious in mathematical circles for having no actual as a function. However, a little known secret is that in the domain of nonstandard analysis, the Dirac delta function admits a completely legitimate construction as an actual function. We give this construction here.

Choose any positive infinitesimal ε and define the hyperreal valued function $\delta : {}^*\mathbb{R} \longrightarrow {}^*\mathbb{R}$ by

$$\delta(x) := \begin{cases} 1/\varepsilon & -\varepsilon/2 < x < \varepsilon/2, \\ 0 & \text{otherwise.} \end{cases}$$

We verify that the above function satisfies the required properties of the Dirac delta function. By definition, $\delta(x) = 0$ for all nonzero real numbers x. Moreover,

$$\int_{-\infty}^{\infty} \delta(x)\ dx = \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{\varepsilon}\ dx = 1,$$

so the integral property is satisfied. Finally, for any *continuous* real function $f : \mathbb{R} \longrightarrow \mathbb{R}$, choose an infinitesimal $z > 0$ such that $|f(x) - f(0)| < z$ for all $|x| < \varepsilon/2$; then

$$\varepsilon \cdot \frac{f(0) - z}{\varepsilon} < \int_{-\infty}^{\infty} \delta(x) f(x)\ dx < \varepsilon \cdot \frac{f(0) + z}{\varepsilon}$$

which implies that $\int_{-\infty}^{\infty} \delta(x) f(x)\ dx$ is within an infinitesimal of $f(0)$, and thus has real part equal to $f(0)$.