duality of Gudermannian and its inverse function

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There are a lot of formulae concerning the Gudermannian function and its inverse function containing a hyperbolic function or a trigonometric function or both, such that if we change functions of one kind to the corresponding functions of the other kind, then the new formula also is true.

Some exemples:

$$\operatorname{gd} x \;=\; \int_0^x \frac{dt}{\cosh t}, \qquad \operatorname{gd}^{-1} x \;=\; \int_0^x \frac{dt}{\cos t} \tag{1}$$

$$\frac{d}{dx}\operatorname{gd} x \;=\; \frac{1}{\cosh x}, \qquad \frac{d}{dx}\operatorname{gd}^{-1} x \;=\; \frac{1}{\cos x} \tag{2}$$

$$\tan(\operatorname{gd} x) \;=\; \sinh x, \qquad \tanh(\operatorname{gd}^{-1} x) \;=\; \sin x \tag{3}$$

$$\sin(\operatorname{gd} x) \;=\; \tanh x, \qquad \sinh(\operatorname{gd}^{-1} x) \;=\; \tan x \tag{4}$$

$$\tan\frac{\operatorname{gd} x}{2} \;=\; \tanh\frac{x}{2}, \qquad \tanh\frac{\operatorname{gd}^{-1} x}{2} \;=\; \tan\frac{x}{2} \tag{5}$$

For proving (5) we can check that

$$\frac{d}{dx}[2\arctan(\tanh\frac{x}{2})] \;=\; \frac{1}{\cosh x},$$

and since both the expression in the brackets and the `http://planetmath.org/node/11997`Guderm
vanish in the origin, we have

$$\operatorname{gd} x \;\equiv\; 2\arctan(\tanh\frac{x}{2}).$$

This equation implies (5).

The `http://planetmath.org/DualityInMathematics`duality of the formula pairs may be explained by the equality

$$\operatorname{gd} ix \;=\; i\operatorname{gd}^{-1} x. \tag{6}$$