proof of Gelfand spectral radius theorem

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For any $\epsilon > 0$, consider the matrix

$$\tilde{A} = (\rho(A) + \epsilon)^{-1} A$$

Then, obviously,

$$\rho(\tilde{A}) = \frac{\rho(A)}{\rho(A) + \epsilon} < 1$$

and, by a well-known result on convergence of matrix powers,

$$\lim_{k \to \infty} \tilde{A}^k = 0.$$

That means, by sequence limit definition, a natural number $N_1 \in \mathbf{N}$ exists such that

$$\forall k \geq N_1 \Rightarrow \|\tilde{A}^k\| < 1$$

which in turn means:

$$\forall k \geq N_1 \Rightarrow \|A^k\| < (\rho(A) + \epsilon)^k$$

or

$$\forall k \geq N_1 \Rightarrow \|A^k\|^{1/k} < (\rho(A) + \epsilon).$$

Let's now consider the matrix

$$\breve{A} = (\rho(A) - \epsilon)^{-1} A$$

Then, obviously,

$$\rho(\breve{A}) = \frac{\rho(A)}{\rho(A) - \epsilon} > 1$$

and so, by the same convergence theorem,$\|\breve{A}^k\|$ is not bounded. This means a natural number $N_2 \in \mathbf{N}$ exists such that

$$\forall k \geq N_2 \Rightarrow \|\breve{A}^k\| > 1$$

which in turn means:

$$\forall k \geq N_2 \Rightarrow \|A^k\| > (\rho(A) - \epsilon)^k$$

or

$$\forall k \geq N_2 \Rightarrow \|A^k\|^{1/k} > (\rho(A) - \epsilon).$$

Taking $N := max(N_1, N_2)$ and putting it all together, we obtain:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall k \geq N \Rightarrow \rho(A) - \epsilon < \|A^k\|^{1/k} < \rho(A) + \epsilon$$

which, by definition, is

$$\lim_{k \to \infty} \|A^k\|^{1/k} = \rho(A). \square$$

Actually, in case the norm is http://planetmath.org/SelfConsistentMatrixNormself-consistent, the proof shows more than the thesis; in fact, using the fact that $|\lambda| \leq \rho(A)$, we can replace in the limit definition the left lower bound with the spectral radius itself and write more precisely:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall k \geq N \Rightarrow \rho(A) \leq \|A^k\|^{1/k} < \rho(A) + \epsilon$$

which, by definition, is

$$\lim_{k \to \infty} \|A^k\|^{1/k} = \rho(A)^+.$$