reduction of elliptic integrals to standard form

Canonical name	ReductionOfEllipticIntegralsToStandardForm
Date of creation	2014-02-01 18:13:38
Last modified on	2014-02-01 18:13:38
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	30
Author	rspuzio (6075)
Entry type	Theorem
Classification	msc 33E05
Related topic	ExpressibleInClosedForm

Any integral of the form $\int R(x, \sqrt{P(x)})\, dx$, where R is a rational function and P is a polynomial of degree 3 or 4 can be expressed as a linear combination of elementary functions and elliptic integrals of the first, second, and third kinds.

To begin, we will assume that P has no repeated roots. Were this not the case, we could simply pull the repeated factor out of the radical and be left with a polynomial of degree of 1 or 2 inside the square root and express the integral in terms of inverse trigonometric functions.

Make a change of variables $z = (ax + b)/(cx + d)$. By choosing the coefficients a, b, c, d suitably, one can cast P into either Jacobi's normal form $P(z) = (1-z^2)(1-k^2z^2)$ or Weierstrass' normal form $P(z) = 4z^3 - g_2 z - g_3$.

Note that

$$R(z, \sqrt{P(z)}) = \frac{A(z) + B(z)\sqrt{P(z)}}{C(z) + D(z)\sqrt{P(z)}}$$

for suitable polynomials A, B, C, D. We can rationalize the denominator like so:

$$\frac{A(z) + B(z)\sqrt{P(z)}}{C(z) + D(z)\sqrt{P(z)}} \times \frac{C(z) - D(z)\sqrt{P(z)}}{C(z) - D(z)\sqrt{P(z)}} = F(z) + G(z)\sqrt{P(z)}$$

The rational functions F and G appearing in the foregoing equation are defined like so:

$$\begin{aligned} F(z) &= \frac{A(z)C(z) - B(z)D(z)P(z)}{C^2(z) - D^2(z)P(z)} \\ G(z) &= 2\frac{B(z)C(z) - A(z)D(z)}{C^2(z) - D^2(z)P(z)} \end{aligned}$$

Since $\int F(z)\, dz$ may be expressed in terms of elementary functions, we shall focus our attention on the remaining piece, $\int G(z)\sqrt{P(z)}\, dz$, which we shall write as $\int H(z)/\sqrt{P(z)}\, dz$, where $H = PG$.. Because we may decompose H into partial fractions, it suffices to consider the following cases, which we shall all A_n and B_n:

$$A_n(z) = \int \frac{z^n}{\sqrt{P(z)}}\, dz$$

$$B_n(z, r) = \int \frac{1}{(z-r)^n \sqrt{P(z)}}\, dz$$

Here, n is a non-negative integer and r is a complex number.

We will reduce thes further using integration by parts. Taking antiderivatives, we have:

$$\int \frac{z^{n-1}(zP'(z) + 2nP(z))}{2\sqrt{P(z)}}\, dz = z^n \sqrt{P(z)} + C$$

$$\int \frac{(z-r)P'(z) - 2nP(z)}{2(z-r)^{n+1}\sqrt{P(z)}}\, dz = \frac{\sqrt{P(z)}}{(z-r)^n} + C$$

These identities will allow us to express A_n's and B_n's with large n in terms of ones with smaller n's.

At this point, it is convenient to employ the specific form of the polynominal P. We will first conside the Weierstrass normal form and then the Jacobi normal form.

Substituting into our identities and collecting terms, we find

$$4(2n+3)A_{n+2} = (2n+1)g_2 A_n + 2ng_3 A_{n-1} + z^n \sqrt{4z^3 - g_2 x - g_3} + C$$

$$2n(4r^3 - g_2 r - g_3)B_{n+1} + (2n-1)(12r^2 - g_2)B_n + 24(n-1)rB_{n-1} + 4(2n-3)B_{n-2} + \frac{\sqrt{4z^3 - g_2 x - g_3}}{(z-r)^n} + C =$$

Note that there are some cases which can be integrated in elementary terms. Namely, suppose that the power is odd:

$$\int z^{2m+1} \sqrt{(1-z^2)(1-k^2 z^2)}\, dz$$

Then we may make a change of variables $y = z^2$ to obtain

$$\frac{1}{2} \int y^{2m} \sqrt{(1-y)(1-k^2 y)}\, dy,$$

which may be integrated using elementary functions.

Next, we derive some identities using integration by parts. Since

$$d\left((1-z^2)(1-k^2z^2)\sqrt{(1-z^2)(1-k^2z^2)}\right) = \left(\frac{9}{2}k^2 z^3 - 3(1+k^2)z\right) \sqrt{(1-z^2)(1-k^2z^2)}\, dz,$$

we have

$$
\begin{aligned}
(2m+1) \quad & \int z^{2m}(1-z^2)(1-k^2z^2)\sqrt{(1-z^2)(1-k^2z^2)}\,dz \\
+ \quad & \int z^{2m+1}\left(\frac{9}{2}k^2z^3 - 3(1+k^2)z\right)\sqrt{(1-z^2)(1-k^2z^2)}\,dz \\
= \quad & z^{2m+1}(1-z^2)(1-k^2z^2)\sqrt{(1-z^2)(1-k^2z^2)} + C
\end{aligned}
$$

By colecting terms, this identity may be rewritten as follows:

$$
\begin{aligned}
\left(1+2m+\frac{9}{2}k^2\right) \quad & \int z^{2m+4}\sqrt{(1-z^2)(1-k^2z^2)}\,dz - \\
(4+2m)(1+k^2) \quad & \int z^{2m+2}\sqrt{(1-z^2)(1-k^2z^2)}\,dz + \\
& \int z^{2m}\sqrt{(1-z^2)(1-k^2z^2)} = \\
& x^{2k+1}(1-z^2)(1-k^2z^2)\sqrt{(1-z^2)(1-k^2z^2)} + C
\end{aligned}
$$

By repeated use of this identity, we may express any integral of the form $\int z^{2m}\sqrt{P(z)}\,dz$ as the sum of a linear combination of $\int z^2\sqrt{P(z)}\,dz$ and $\int \sqrt{P(z)}\,dz$ and the product of a polyomial and $\sqrt{P(z)}$.

Likewise, we can use integration by parts to simplify integrals of the form

$$
\int \frac{\sqrt{P(z)}}{(z-r)^n}\,dz
$$

Will finish later — saving in case of computer crash.