application of sine integral at infinity

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For finding the value of the improper integral

$$\int_0^\infty \frac{\sin ax}{x(1+x^2)}\,dx \;:=\; f(a) \qquad (a > 0) \tag{1}$$

we first use the `http://planetmath.org/PartialFractionsOfExpressions`partial fraction representation

$$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}.$$

Thus we may write

$$f(a) = \int_0^\infty \frac{\sin ax}{x}\,dx - \int_0^\infty \frac{x\sin ax}{1+x^2}\,dx.$$

But by the entry sine integral at infinity, the first integral equals $\frac{\pi}{2}$. When we check

$$f'(a) \;=\; \int_0^\infty \frac{\cos ax}{1+x^2}\,dx, \quad f''(a) \;=\; -\int_0^\infty \frac{x\sin ax}{1+x^2}\,dx,$$

we see that there is the linear differential equation

$$f(a) = \frac{\pi}{2} + f''(a) \tag{2}$$

i.e.

$$f'' - f \;=\; -\frac{\pi}{2},$$

satisfied by the sought function $a \mapsto f(a)$. We have the initial conditions

$$f(0) \;=\; \int_0^\infty 0\,dx \;=\; 0, \quad f'(0) \;=\; \int_0^\infty \frac{dx}{1+x^2} \;=\; \Big/_{\!0}^{\!\infty} \arctan x \;=\; \frac{\pi}{2}.$$

Therefore the general solution

$$f(a) \;=\; C_1e^a + C_2e^{-a} + \frac{\pi}{2}$$

of (2) requires that $C_1 = 0$, $C_2 = \frac{\pi}{2}$, and consequently the sought integral $f(a)$ has the value

$$\int_0^\infty \frac{\sin ax}{x(1+x^2)}\,dx \;=\; \frac{\pi}{2}(1-e^{-a}) \tag{3}$$