partial fraction series for digamma function

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Theorem 1

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{z+k} \right) = -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{z+k} \right)$$

Proof: Start with

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-1} e^{z/k},$$

so

$$\ln \Gamma(z) = -\gamma z - \ln z + \sum_{k=1}^{\infty} \left(-\ln\left(1 + \frac{z}{k}\right) + \frac{z}{k} \right)$$

and thus, taking derivatives,

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \left(-\frac{1/k}{1 + \frac{z}{k}} + \frac{1}{k} \right) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{z+k} \right)$$

The second formula follows after rearranging terms (the rearrangement is legal since we are simply exchanging adjacent terms, so partial sums remain the same).