Riccati equation

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The nonlinear differential equation

$$\frac{dy}{dx} \;=\; f(x)+g(x)y+h(x)y^2 \tag{1}$$

is called *Riccati equation.* If $h(x)\equiv 0,$ it is a question of a linear differential equation; if $f(x)\equiv 0,$ of a Bernoulli equation. There is no general method for integrating explicitely the equation (1), but via the substitution

$$y \;:=\; -\frac{w'(x)}{h(x)w(x)}$$

one can convert it to a homogeneous linear differential equation with non-constant coefficients.

If one can find a particular solution $y_0(x),$ then one can easily verify that the substitution

$$y \;:=\; y_0(x)+\frac{1}{w(x)} \tag{2}$$

converts (1) to

$$\frac{dw}{dx}+[g(x){+}2h(x)y_0(x)]\,w+h(x) \;=\; 0, \tag{3}$$

which is a linear differential equation of first order with respect to the function $w = w(x).$

Example. The Riccati equation

$$\frac{dy}{x} \;=\; 3+3x^2y-xy^2 \tag{4}$$

has the particular solution $y := 3x.$ Solve the equation.

We substitute $y := 3x+\frac{1}{w(x)}$ to (4), getting

$$\frac{dw}{dx}-3x^2w-x \;=\; 0.$$

For solving this `http://planetmath.org/LinearDifferentialEquationOfFirstOrder`first order equation we can put $w = uv,$ $w' = uv'+u'v,$ writing the equation as

$$u\cdot(v'-3x^3v)+u'v \;:=\; x, \tag{5}$$

where we choose the value of the expression in parentheses equal to 0:

$$\frac{dv}{dx} - 3x^2 v \;=\; 0$$

After separation of variables and integrating, we obtain from here a solution $v = e^{x^3}$, which is set to the equation (5):

$$\frac{du}{dx} e^{x^3} \;=\; x$$

Separating the variables yields

$$du \;=\; \frac{x}{e^{x^3}}\, dx$$

and integrating:

$$u \;=\; C + \int x e^{-x^3}\, dx.$$

Thus we have

$$w \;=\; w(x) \;=\; uv \;=\; e^{x^3}\left[C + \int x e^{-x^3}\, dx\right],$$

whence the general solution of the Riccati equation (4) is

$$y \;=\; 3x + \frac{e^{-x^3}}{C + \int x e^{-x^3}\, dx}.$$

It may be proved that if one knows three different solutions of Riccati equation (1), the each other solution may be expresses as a rational function of them.