orthogonality of Laguerre polynomials

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We use the definition of Laguerre polynomials $L_n(x)$ via their http://planetmath.org/Rodrigu
formula

$$L_n(x) \ := \ e^x \frac{d^n}{dx^n}(x^n e^{-x}). \qquad (1)$$

The polynomials (1) themselves are not orthogonal to each other, but the expressions $e^{-\frac{x}{2}}L_n(x)$ $(n = 0,\, 1,\, 2, \ldots)$ are http://planetmath.org/OrthogonalPolynomialsorthog
on the interval from 0 to ∞, i.e. the polynomials are orthogonal with respect to the weighting function e^{-x} on that interval, as is seen in the following.

Let m be another nonnegative integer. We http://planetmath.org/IntegrationByPartsinteg
by parts m times in

$$\int_0^\infty e^{-x}x^m L_n(x)\,dx \ = \ \int_0^\infty x^m \frac{d^n}{dx^n}(x^n e^{-x})\,dx \ = \ (-1)^m m! \int_0^\infty \frac{d^{n-m}}{dx^{n-m}}(x^m e^{-x})\,dx.$$

When $m < n$, this yields

$$\int_0^\infty e^{-x}x^m L_n(x)\,dx \ = \ (-1)^m m! \Big/_{x=0}^{\infty} \frac{d^{n-m-1}}{dx^{n-m-1}}(x^m e^{-x}) \ = \ 0. \qquad (2)$$

and for $m = n$ it gives

$$\int_0^\infty e^{-x}x^m L_n(x)\,dx \ = \ (-1)^n n! \int_0^\infty x^n e^{-x}\,dx \ = \ (-1)^n (n!)^2. \qquad (3)$$

The result (2) implies, because $L_m(x)$ is a polynomial of degree m, that

$$\int_0^\infty e^{-x}L_m(x)L_n(x)\,dx \ = \ 0 \qquad (m \ < \ n),$$

whence also

$$\int_0^\infty e^{-x}L_m(x)L_n(x)\,dx \ = \ 0 \qquad (m \ \neq \ n). \qquad (4)$$

Thus the orthogonality has been shown. Therefore, since the leading term of $L_n(x)$ is $(-1)^n x^n$, we infer by (3) and (4) that

$$\int_0^\infty e^{-x}[L_n(x)]^2\,dx \ = \ (-1)^n \int_0^\infty e^{-x}x^n L_n(x)\,ds \ = \ (n!)^2,$$

so that the expressions $\frac{L_n(x)}{n!}$ form a system of orthonormal polynomials.

References

[1] H. Eyring, J. Walter, G. Kimball: *Quantum chemistry*. Eight printing. Wiley & Sons, New York (1958).