derivative as parameter for solving differential equations

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The solution of some differential equations of the forms $x = f(\frac{dy}{dx})$ and $y = f(\frac{dy}{dx})$ may be expressed in a parametric form by taking for the parameter the derivative

$$p := \frac{dy}{dx}. \tag{1}$$

I. Consider first the equation

$$x = f(\frac{dy}{dx}), \tag{2}$$

for which we suppose that $p \mapsto f(p)$ and its derivative $p \mapsto f'(p)$ are continuous and $f'(p) \neq 0$ on an interval $[p_1,\, p_2]$. It follows that on the interval, the function $p \mapsto f(p)$ changes monotonically from $f(p_1) := x_1$ to $f(p_2) := x_2$, whence conversely the equation

$$x = f(p) \tag{3}$$

defines from $[p_1,\, p_2]$ onto $[x_1,\, x_2]$ a bijection

$$p = g(x) \tag{4}$$

which is continuously differentiable. Thus on the interval $[x_1,\, x_2]$, the differential equation (2) can be replaced by the equation

$$\frac{dy}{dx} = g(x), \tag{5}$$

and therefore, the solution of (2) is

$$y = \int g(x)\, dx + C. \tag{6}$$

If we cannot express $g(x)$ in a , we take p as an independent variable through the substitution (3), which maps $[x_1,\, x_2]$ bijectively onto $[p_1,\, p_2]$. Then (6) becomes a function of p, and by the chain rule,

$$\frac{dy}{dp} = g(f(p))f'(p) = pf'(p).$$

Accordingly, the solution of the given differential equation may be presented on $[p_1,\, p_2]$ as

$$
\begin{cases}
x = f(p), \\
y = \displaystyle\int p\, f'(p)\, dp + C.
\end{cases}
\tag{7}
$$

II. With corresponding considerations, one can write the solution of the differential equation

$$
y = f(\frac{dy}{dx}) \;:=\; f(p),
\tag{8}
$$

where p changes on some interval $[p_1,\, p_2]$ where $f(p)$ and $f'(p)$ are continuous and $p \cdot f'(p) \neq 0$, in the parametric presentation

$$
\begin{cases}
x = \displaystyle\int \frac{f'(p)}{p}\, dp + C, \\
y = f(p).
\end{cases}
\tag{9}
$$

III. The procedures of **I** and **II** may be generalised for the differential equations of $x = f(y,\, p)$ and $y = f(x,\, p)$; let's consider the former one.

In

$$
x \;=\; f(y,\, p)
\tag{10}
$$

we regard y as the independent variable and differentiate with respect to it:

$$
\frac{dx}{dy} \;=\; \frac{1}{p} \;=\; f'_y(y,\, p) + f'_p(y,\, p)\frac{dp}{dy}.
$$

Supposing that the partial derivative $f'_p(y,\, p)$ does not vanish identically, we get

$$
\frac{dp}{dy} \;=\; \frac{\frac{1}{p} - f'_y(y,\, p)}{f'_p(y,\, p)} \;:=\; g(y,\, p).
\tag{11}
$$

If $p = p(y,\, C)$ is the general solution of (11), we obtain the general solution of (10):

$$
x \;=\; f(y,\, p(y, C))
\tag{12}
$$