ODE types reductible to the variables separable case

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There are certain of non-linear ordinary differential equations of http://planetmath.org/ODEfi
order which may by a suitable substitution be to a form where one can http://planetmath.org/SeparationOfVariablesseparate the variables.

I. So-called homogeneous differential equation

This means the equation of the form

$$X(x,\,y)dx+Y(x,\,y)dy = 0,$$

where X and Y are two homogeneous functions of the same http://planetmath.org/Homogeneous
Therefore, if the equation is written as

$$\frac{dy}{dx} = -\frac{X(x,\,y)}{Y(x,\,y)},$$

its right hand side is a homogeneous function of degree 0, i.e. it depends only on the ratio $y\!:\!x$, and has thus the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right). \tag{1}$$

Accordingly, if this ratio is constant, then also $\frac{dy}{dx}$ is constant; thus all lines $\frac{y}{x} = \text{constant}$ are isoclines of the family of the integral curves which intersect any such line isogonally.

We can infer as well, that if one integral curve is represented by $x = x(t)$, $y = y(t)$, then also $x = Cx(t)$, $y = Cy(t)$ an integral curve for any constant C. Hence the integral curves are homothetic with respect to the origin; therefore some people call the equation (1) a *similarity equation.*

For generally solving the equation (1), make the substitution

$$\frac{y}{x} := t; \quad y = tx; \quad \frac{dy}{dx} = t+x\frac{dt}{dx}.$$

The equation takes the form

$$t+x\frac{dt}{dx} = f(t) \tag{2}$$

which shows that any http://planetmath.org/Equationroot t_ν of the equality $f(t) = t$ gives a singular solution $y = t_\nu x$. The variables in (2) may be :

$$\frac{dx}{x} = \frac{dt}{f(t)-t}$$

Thus one obtains $\ln|x| = \int \frac{dt}{f(t)-t} + \ln C$, whence the general solution of the homogeneous differential equation (1) is in a parametric form

$$x = Ce^{\int \frac{dt}{f(t)-t}}, \quad y = Cte^{\int \frac{dt}{f(t)-t}}.$$

II. Equation of the form $\boldsymbol{y' = f(ax+by+c)}$

It's a question of the equation

$$\frac{dy}{dx} = f(ax+by+c), \tag{3}$$

where a, b and c are given constants. If $ax+by$ is constant, then $\frac{dy}{dx}$ is constant, and we see that the lines $ax+by = \text{constant}$ are isoclines of the intgral curves of (3).

Let

$$ax+by+c := u \tag{4}$$

be a new variable. It changes the equation (3) to

$$\frac{du}{dx} = a + bf(u). \tag{5}$$

Here, one can see that the real zeros u of the right hand side yield lines (4) which are integral curves of (3), and thus we have singular solutions. Moreover, one can separate the variables in (5) and integrate, obtaining x as a function of u. Using still (4) gives also y. The general solution is

$$x = \int \frac{du}{a+bf(u)} + C, \quad y = \frac{1}{b}\left(u - c - a\int \frac{du}{a+bf(u)} - aC\right).$$

Example. In the nonlinear equation

$$\frac{dy}{dx} = (x-y)^2,$$

which is of the type II, one cannot separate the variables x and y. The substitution $x - y := u$ converts it to

$$\frac{du}{dx} = 1 - u^2,$$

where one can separate the variables. Since the right hand side has the zeros $u = \pm 1,$ the given equation has the singular solutions y given by $x - y = \pm 1.$ Separating the variables x and u, one obtains

$$dx = \frac{du}{1-u^2},$$

whence

$$x = \int \frac{du}{(1+u)(1-u)} = \frac{1}{2}\int \left(\frac{1}{1+u} + \frac{1}{1-u}\right) du = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C.$$

Accordingly, the given differential equation has the parametric solution

$$x = \ln\sqrt{\left|\frac{1+u}{1-u}\right|} + C, \quad y = \ln\sqrt{\left|\frac{1+u}{1-u}\right|} - u + C.$$

References

[1] E. Lindelöf: *Differentiali- ja integralilasku III 1.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1935).