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derivation of integral representations of Jacobi ϑ functions

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By rearranging the Fourier series of $\cos(ux)$, one obtains the series

$$\frac{\pi \cos(ux)}{2u \sin(\pi u)} = \frac{1}{2u^2} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{u^2 - n^2}$$

This equation which is valid for all real values of x such that $-\pi \le x \le \pi$ and all non-integral complex values of u. By comparison with the convergent series $\sum_{n=0}^{\infty} 1/n^2$, it follows that this series is absolutely convergent. Note that this series may be viewed as a Mittag-Leffler partial fraction expansion.

Let y be a positive real number. Multiply both by $2ue^{-yu^2}$ and integrate.

$$\int_{i-\infty}^{i+\infty} \frac{\pi \cos(ux) e^{-yu^2}}{\sin(\pi u)} \, dv = 2 \int_{i-\infty}^{i+\infty} e^{-yu^2} \left[\frac{1}{2u^2} + \sum_{n=0}^{\infty} (-1)^n \frac{\cos(nx)}{u^2 - n^2} \right] u \, du$$

Because of the exponential, the integrand decays rapidly as $u \to i \pm \infty$ provided that $\Re u > 0$, and hence the integral converges absolutely. Make a change of variables $v = u^2$

$$= \int_P e^{-yv} \left[\frac{1}{2v} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{v - n^2} \right] dv$$

The contour of integration P is a parabola in the complex v-plane, symmetric about the real axis with vertex at $v = -1$, which encloses the real axis. Its equation is $\Re v + 1 = 2(\Im v)^2$

Let S_m (m is an integer) be the straight line segment joining the points $v = (i + m + 1/2)^2$ and $v = (i - m - 1/2)^2$. Along this line segment, we may bound the integrand in absolute value as follows:

$$\left| \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{v - n^2} \right| \le \sum_{n=1}^{\infty} \frac{(-1)^n}{|v - n^2|} \le \sum_{n=1}^{\infty} \frac{(-1)^n}{|v_m - n^2|}$$

where $v_m = m^2 + m - 3/4$ is the point of intersection of S_m with the real axis. To proceed further, we break up the last summation into two parts.

Since the squares closest in absolute value to v_m are m^2 and $(m+1)^2 = m^2 + 2m + 1$, it follows that $|v_m - n^2| \ge |m - 3/4|$ for all m, n. Hence, we have

$$\sum_{i=1}^{2m} \frac{1}{|v_m - n^2|} \le \frac{2m}{m - 3/4} \le 8$$

When $n > 2m$, we have $n^2 \geq (2m+1)^2 = 4m^2+4m+1 > 4m^2+4m-3 = 4v_m$. Hence, $|n^2 - v_m| > 3n^2/4$ and

$$\sum_{n=2m+1}^{\infty} \frac{1}{|v_m - n^2|} < \frac{4}{3} \sum_{n=2m+1}^{\infty} \frac{1}{n^2} < \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi}{9}$$

Finally $1/(2v_m) < 1/2$ since $v_m > 1$ when $m \geq 1$. Also, $|e^{-yv}| = e^{-y\Re v} - e^{-yv_m} < e^{-ym^2}$. From these observations, we conclude that

$$\left| \int_{S_m} e^{-yv} \left[\frac{1}{2v} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{v - n^2} \right] dv \right| < e^{-ym^2} \left(1 + 8 + \frac{2\pi}{9} \right) \int_{S_m} dv = (4m+2) \left(9 + \frac{2\pi}{9} \right) e^{-ym^2}$$

Note that this quantity approaches 0 in the limit $m \to \infty$.

Let P_m be the arc of the parabola P bounded by the endpoints of S_m. Together, S_m and P_m form a closed contour which encloses poles of the integrand. Hence, by the residue theorem , we have

$$\int_{P_m} e^{-yv} \left[\frac{1}{2v} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{v - n^2} \right] dv + \int_{S_m} e^{-yv} \left[\frac{1}{2v} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{v - n^2} \right] dv =$$

$$2\pi i \sum_{n=1}^{m} (-1)^n \cos(nx) e^{-n^2 y}$$

Taking the limit $m \to \infty$ we obtain

$$\int_P e^{-yv} \left[\frac{1}{2v} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{v - n^2} \right] dv = 2\pi i \left(\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \cos(nx) e^{-n^2 y} \right)$$

Going back to the beginning of the proof, where the integral on the left hand side was expressed as an integral with respect to u, we obtain

$$\int_{i-\infty}^{i+\infty} \frac{\pi \cos(ux) e^{-yu^2}}{\sin(\pi u)} dv = 2\pi i \left(\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \cos(nx) e^{-n^2 y} \right)$$

Making a change of variables $x = 2z, y = -i\pi\tau$ and tidying up some, we obtain

$$\int_{i-\infty}^{i+\infty} \frac{\cos(2uz) e^{i\pi\tau u^2}}{\sin(\pi u)} dv = i \left(1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{i\pi n^2 \tau} \cos(2nz) \right) = i\vartheta_4(z|\tau)$$

Because of the initial assumption about the Fourier series, we only know that this formula is valid when τ is purely imaginary with strictly positive imaginary part and z is real and $\pi/2 < z < \pi/2$. However, we can use analytic continuation to extend the domain of its validity. On the one hand, the theta function on the right-hand side is analytic for all z and all τ such that $\Im\tau > 0$.

On the other hand, I claim that the integral on the left hand side is also an analytic function of z and τ whenever $\Im\tau > 0$. To validate this claim, we need to examine the behaviour of the integrand as $u \to i \pm \infty$. The contribution of the denominator is bounded;

$$\left| \frac{1}{\sin \pi u} \right| < c$$

for some constant c whenever $\Im u = 1$. The absolute value of the cosine in the numerator is easy to bound:

$$|\cos(2uz)| \leq e^{2|u|\,|z|}$$

To bound the remaining term, let us examine the argument of the exponential carefully:

$$\Im(\tau u^2) = 2\Re\tau\,\Re u + \Im\tau(\Re u)^2 - \Im\tau = \Im\tau \left(\left(\Re u + \frac{\Re\tau}{\Im\tau} \right)^2 - 1 - \left(\frac{\Re\tau}{\Im\tau} \right)^2 \right)$$

Therefore, if $|\Re u| > 1 + 3|\Re\tau|/(\Im\tau)$, it will be the case that $\Im(\tau u^2) \geq \Im\tau\,(\Re u)^2/9$, and so

$$\left| e^{i\pi\tau u^2} \right| = e^{-\pi\Im(\tau u^2)} \leq e^{-\pi\Im\tau\,(\Re u)^2/9}$$

Taken together, the estimates of the last paragraph imply that

$$\left| \int_{i+R}^{i+\infty} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)} \right| < c \int_{i+R}^{i+\infty} e^{2|u||z| - \pi\Im\tau\,(\Re u)^2/9}$$

when $R > 1 + 3|\Re\tau|/(\Im\tau)$. If we impose the further conditions

$$R > \frac{180|z|}{\pi\,\Im\tau} \qquad R^2 > \frac{180|z|}{\pi\,\Im\tau} \qquad ,$$

it will be the case that

$$2|u||z| - \pi\Im\tau\,(\Re u)^2/9 < 2\Re u\,|z| + 2|z| - \pi\Im\tau\,(\Re u)^2/9 <$$
$$\big(2\Re u\,|z| - \pi\Im\tau\,(\Re u)^2/180\big) + \big(2|z| - \pi\Im\tau\,(\Re u)^2/180\big) - \pi\Im\tau\,(\Re u)^2/10 <$$
$$-\pi\Im\tau\,(\Re u)^2/10 \qquad ,$$

and hence

$$\left|\int_{i+R}^{i+\infty} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}\,du\right| < c\int_{i+R}^{i+\infty} e^{-\pi\Im\tau\,(\Re u)^2/10}\,du < \frac{5c}{\pi\,\Im\tau} R e^{-\pi\Im\tau\,R^2/10} \qquad .$$

Likewise, under the same restriction on R,

$$\left|\int_{i-\infty}^{i-R} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}\,du\right| < c\int_{i+R}^{i+\infty} e^{-\pi\Im\tau\,(\Re u)^2/10}\,du < \frac{5c}{\pi\,\Im\tau} R e^{-\pi\Im\tau\,R^2/10} \qquad .$$

Since the contour of integration is compact and the integrand is analytic in a neighborhood of the contour,

$$\int_{i-R}^{i+R} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}\,du$$

will be an analytic function of z and τ. Suppose that z and τ are restricted to bounded regions of the complex plane and that, furthermore, $Im\tau$ is positive and bounded away from zero. Then the inequalities of the last paragraph imply that the integral converges uniformly as $R \to \infty$, and hence

$$\int_{i-\infty}^{i+\infty} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}\,du$$

is an analytic function of u and z in the domain $\Im\tau > 0$.

Thus, by the fundamental theorem of analytic continuation, we may conclude that

$$\int_{i-\infty}^{i+\infty} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}\,dv = i\left(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{i\pi n^2\tau}\cos(2nz)\right) = i\vartheta_4(z|\tau)$$

throughout this domain.

Finally, integral representations of the remaining three theta functions may be easily obtained from this one by adding the appropriate half-quasiperiods to z.