Hermite polynomials

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The polynomial solutions of the Hermite differential equation, with n a non-negative integer, are usually normed so that the highest http://planetmath.org/PolynomialF is $(2z)^n$ and called the *Hermite polynomials* $H_n(z)$. The Hermite polynomials may be defined explicitly by

$$H_n(z) \;:=\; (-1)^n e^{z^2}\frac{d^n}{dz^n}e^{-z^2}, \tag{1}$$

since this is a polynomial having the highest $(2z)^n$ and satisfying the Hermite equation. The equation (1) is the Rodrigues's formula for Hermite polynomials. Using the Faà di Bruno's formula, one gets from (1) also

$$H_n(x) \;=\; (-1)^n \sum_{m_1+2m_2=n} \frac{n!}{m_1!m_2!}(-1)^{m_1+m_2}(2x)^{m_1}.$$

The first six Hermite polynomials are
$H_0(z) \;\equiv\; 1,$
$H_1(z) \;\equiv\; 2z,$
$H_2(z) \;\equiv\; 4z^2-2,$
$H_3(z) \;\equiv\; 8z^3-12z,$
$H_4(z) \;\equiv\; 16z^4-48z^2+12,$
$H_5(z) \;\equiv\; 32z^5-160z^3+120z,$
and the general is

$$H_n(z) \;\equiv\; (2z)^n - \frac{n(n-1)}{1!}(2z)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2z)^{n-4} - + \ldots$$

Differentiating this termwise gives

$$H_n'(z) \;=\; 2n\left[(2z)^{n-1} - \frac{(n-1)(n-2)}{1!}(2z)^{n-3} + \frac{(n-1)(n-2)(n-3)(n-4)}{2!}(2z)^{n-5} - + \ldots\right],$$

i.e.

$$H_n'(z) \;=\; 2nH_{n-1}(z). \tag{2}$$

The Hermite polynomials are sometimes scaled to such ones $\mathrm{He_n}$ which obey the differentiation rule

$$\mathrm{He}_n'(z) \;=\; n\mathrm{He}_{n-1}(z). \tag{3}$$

Such Hermite polynomials form an Appell sequence.

We shall now show that the Hermite polynomials form an http://planetmath.org/Orthogonal set on the interval $(-\infty,\,\infty)$ with the http://planetmath.org/OrthogonalPolynomialsweight factor e^{-x^2}. Let $m<n$; using (1) and http://planetmath.org/IntegrationByPartsintegrating by parts we get

$$\begin{aligned}
(-1)^n\int_{-\infty}^{\infty}H_m(x)H_n(x)e^{-x^2}\,dx \;&=\; \int_{-\infty}^{\infty}H_m(x)\frac{d^ne^{-x^2}}{dx^n}\,dx \\
&=\; \Big/_{-\infty}^{\infty} H_m(x)\frac{d^{n-1}e^{-x^2}}{dx^{n-1}} - \int_{-\infty}^{\infty}H'_m(x)\frac{d^{n-1}e^{-x^2}}{dx^{n-1}}\,dx.
\end{aligned}$$

The substitution portion here equals to zero because e^{-x^2} and its derivatives vanish at $\pm\infty$. Using then (2) we obtain

$$\int_{-\infty}^{\infty}H_m(x)H_n(x)e^{-x^2}\,dx \;=\; 2(-1)^{1+n}m\int_{-\infty}^{\infty}H_{m-1}(x)\frac{d^{n-1}e^{-x^2}}{dx^{n-1}}\,dx.$$

Repeating the integration by parts gives the result

$$\begin{aligned}
\int_{-\infty}^{\infty}H_m(x)H_n(x)e^{-x^2}\,dx \;&=\; 2^m(-1)^{m+n}m!\int_{-\infty}^{\infty}H_0(x)\frac{d^{n-m}e^{-x^2}}{dx^{n-m}}\,dx \\
&=\; 2^m(-1)^{m+n}m!\;\Big/_{-\infty}^{\infty}\frac{d^{n-m-1}e^{-x^2}}{dx^{n-m-1}} \;=\; 0,
\end{aligned}$$

whereas in the case $m=n$ the result

$$\int_{-\infty}^{\infty}(H_n(x))^2e^{-x^2}\,dx \;=\; 2^n(-1)^{2n}n!\int_{-\infty}^{\infty}e^{-x^2}\,dx \;=\; 2^nn!\sqrt{\pi}$$

(see area under Gaussian curve). The results that the functions $x\mapsto \frac{H_n(x)}{\sqrt{2^nn!\sqrt{\pi}}}e^{-\frac{x^2}{2}}$ form an orthonormal set on $(-\infty,\,\infty)$.

The Hermite polynomials are used in the quantum mechanical treatment of a harmonic oscillator, the wave functions of which have the form

$$\xi \;\mapsto\; \Psi_n(\xi) \;=\; C_nH_n(\xi)e^{-\frac{\xi^2}{2}}.$$