possible orders of elliptic functions

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The order of a non-trivial elliptic function cannot be zero. This is a simple consequence of Liouville's theorem. Were the order of an elliptic function zero, then the function would have no poles. By definition, an elliptic function has no essential singularities and is doubly periodic. Hence, if the degree were zero, the function would be continuous everywhere and hence, being doubly periodic, would be bounded (since continuous functions on a compact domain (like the closure of the fundamental parallelogram) are bounded). By Liouville's theorem, this would imply that the function is constant.

The order of an elliptic function cannot be 1. This follows from the fact that the residues at the poles of an elliptic function within a fundamental parallelogram must sum to zero — if the function were of degree 1, it would have exactly one first-order pole in the fundamental parallelgram but any first-order pole must have a non-zero residue.

Any number greater than one is possible as the order of an elliptic function. As an example of an elliptic function of order two, we may take the Weierstass $\wp$-function, which has a single pole of order 2 in the fundamental domain. The n-th derivative of this function will have a single pole of order $n+2$ in the fundamental domain, hence be of order $n+2$, so is an example showing that, for every integer greater than 2, there exists an elliptic function having that integer as its order.