Cauchy initial value problem

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Let D be a subset of $\mathbb{R}^n \times \mathbb{R}$, (x_0, t_0) a point of D, and $f\colon D \to \mathbb{R}$ be a function.

We say that a function $x(t)$ is a solution to the Cauchy (or initial value) problem

$$\begin{cases} x'(t) = f(x(t), t) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

if

1. x is a differentiable function $x\colon I \to \mathbb{R}^n$ defined on a interval $I \subset \mathbb{R}$;
2. one has $(x(t), t) \in D$ for all $t \in I$ and $t_0 \in I$;
3. one has $x(t_0) = x_0$ and $x'(t) = f(x(t), t)$ for all $t \in I$.

We say that a solution $x\colon I \to \mathbb{R}^n$ is a *maximal solution* if it cannot be extended to a bigger interval. More precisely given any other solution $y\colon J \to \mathbb{R}^n$ defined on an interval $J \supset I$ and such that $y(t) = x(t)$ for all $t \in I$, one has $I = J$ (and hence x and y are the same function).

We say that a solution $x\colon I \to \mathbb{R}^n$ is a *global solution* if $D \subset= \mathbb{R}^n \times I$.

We say that a solution $x\colon I \to \mathbb{R}^n$ is *unique* if given any other solution $y\colon I \to \mathbb{R}^n$ one has $x(t) = y(t)$ for all $t \in I$ (i.e. x is the unique solution defined on the interval I).

0.1 Notation

Usually the differential equation in (1) is simply written as $x' = f(x, t)$. Also, depending on the topics, the name chosen for the function and for the variable, can change. Other common choices are $y' = f(y, t)$ or $y' = f(y, x)$. It is also common to write $\dot{x} = f(x, t)$ when the independent variable represents a time value.

0.2 Examples

1. The function $x(t) = \log t$ defined on $I = (0, +\infty)$ is the unique maximal solution to the Cauchy problem:

   $$\begin{cases} x'(t) = 1/t \\ x(1) = 0. \end{cases}$$

   In this case $f(x, t) = 1/t$, $D = \{(x, t)\colon t \neq 0\}$, $t_0 = 1$, $x_0 = 0$.

2. The function $x(t) = e^t$ is a global (and hence maximal), unique solution to the Cauchy problem:

$$\begin{cases} x'(t) = x(t) \\ x(0) = 1. \end{cases}$$

3. Consider the Cauchy problem

$$\begin{cases} x'(t) = \frac{3}{2}\sqrt[3]{x} \\ x(0) = 0. \end{cases}$$

The function $x(t) = 0$ defined on $I = \mathbb{R}$ is a global solution. However the function $y(t) = \sqrt{t^3}$ defined on $I = [0, +\infty)$ is also a solution and so are the functions

$$z(t) = \begin{cases} \sqrt{(t-c)^3} & \text{if } t \geq c \\ 0 & \text{if } t < c. \end{cases}$$

for every $c \geq 0$. So there are no unique solutions. Moreover y is not a maximal solution.