existence and uniqueness of solution to Cauchy problem

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Let

$$\begin{cases} \dot{\mathbf{x}} = F(\mathbf{x}, t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

be a Cauchy problem, where $F : U \to \mathbb{R}$ is

- a continuous function of $n + 1$ variables defined in a neighborhood $U \subseteq \mathbb{R}^{n+1}$ of $(\mathbf{x}_0, t_0)$
- Lipschitz continuous with respect to the first n variables (i.e. with respect to $\mathbf{x}$).

Then there exists a unique solution $\mathbf{x} : I \to \mathbb{R}^n$ of the Cauchy problem, defined in a neighborhood $I \subseteq \mathbb{R}$ of t_0.

Proof

Solving the Cauchy problem is equivalent to solving the following integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} F(\mathbf{x}(\tau), \tau)\mathrm{d}\tau$$

Let X be the set of continuous functions $\mathbf{f} : [t_0 - \delta, t_0 + \delta] \to B(\mathbf{x}_0, \epsilon)$. We'll assume ϵ to be chosen such that the $B(\mathbf{x}_0, \epsilon) \subseteq U$ [1]. In this ball, therefore, F is Lipschitz continuous with respect to the first n variable, in other words, there exists a real number L such that

$$F(\mathbf{x}, t) - F(\mathbf{y}, t) \leq L\|\mathbf{x} - \mathbf{y}\|$$

for all points $\mathbf{x}, \mathbf{y}$ sufficiently near to $\mathbf{x}_0$.

Now let's define the mapping $T : X \to X$ as follows

$$T\mathbf{x} : t \mapsto \mathbf{x}_0 + \int_{t_0}^{t} F(\mathbf{x}(\tau), \tau)\mathrm{d}\tau$$

We make the following observations about T.

1. Since F is continuous, $\|F\|$ attains a maximum value M on the compact set $B(\mathbf{x}_0, \epsilon) \times [t_0 \pm \delta]$. But by hypothesis, $\|\mathbf{x}(t) - \mathbf{x}_0\| \leq \epsilon$, hence

   $$\|\mathbf{x}(t) - \mathbf{x}_0\| \leq \int_{t_0}^{t} \|F(\mathbf{x}(\tau), \tau)\|\mathrm{d}\tau \leq M(t - t_0) \leq M\delta$$

   for all $t \in [t_0 \pm \delta]$.

[1] $B(\mathbf{x}_0, \epsilon)$ denotes the closed ball $\{\mathbf{x} : \|\mathbf{x}_0 - \mathbf{x}\| \leq \epsilon\}$

2. The Lipschitz continuity of F yields

$$\|T\mathbf{x}(t) - T\mathbf{y}(t)\| \leq \int_{t_0}^{t} \|F(\mathbf{x}(\tau), \tau) - F(\mathbf{y}(\tau), \tau)\| \mathrm{d}\tau \leq \int_{t_0}^{t} L\|\mathbf{x}(\tau) - \mathbf{y}(\tau)\| \mathrm{d}\tau \leq L\delta d_\infty(\mathbf{x}, \mathbf{y})$$

If we choose $\delta < \min\{1/L, \epsilon/M\}$ these conditions ensure that

- $T(X) \subseteq X$, i.e. T doesn't send us outside of X.
- T is a contraction mapping with respect to the uniform convergence metric d_∞ on X, i.e. there exists $\lambda \in \mathbb{R}$ such that for all $\mathbf{x}, \mathbf{y} \in X$,

$$d_\infty(T\mathbf{x}, T\mathbf{y}) \leq \lambda d_\infty(\mathbf{x}, \mathbf{x})$$

In particular, the second point allows us to apply Banach's theorem and define

$$\mathbf{x}^\star = \lim_{k \to \infty} T^k \mathbf{x}_0$$

to find the unique fixed point of T in X, i.e. the unique function which solves

$$T\mathbf{x} = \mathbf{x} \text{ in other words } \mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} F(\mathbf{x}(\tau), \tau) \mathrm{d}\tau$$

and which therefore locally solves the Cauchy problem.