generating function of Laguerre polynomials

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We start from the definition of Laguerre polynomials via their http://planetmath.org/node/1
formula

$$L_n(z) \;:=\; e^z\frac{d^n}{dz^n}e^{-z}z^n \qquad (n \;=\; 0,\,1,\,2,\,\ldots). \tag{1}$$

The consequence

$$f^{(n)}(z) \;=\; \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}\;d\zeta \tag{2}$$

of http://planetmath.org/node/1150Cauchy integral formula allows to write (1) as the complex integral

$$L_n(z) \;=\; \frac{n!}{2i\pi}\oint_C \frac{e^z e^{-\zeta}}{(\zeta-z)^{n+1}}\,d\zeta \;=\; \frac{n!}{2i\pi}\oint_C \frac{e^{z-\zeta}\,d\zeta}{(1-\frac{z}{\zeta})^n(\zeta-z)},$$

where C is any contour around the point z and the direction is anticlockwise. The http://planetmath.org/node/11373substitution

$$\zeta-z \;:=\; \frac{zt}{1-t}, \quad \zeta \;=\; \frac{z}{1-t}, \quad t \;=\; 1-\frac{z}{\zeta} \quad d\zeta \;=\; \frac{z\,dt}{(1-t)^2}$$

here yields

$$L_n(z) \;=\; \frac{n!}{2i\pi}\oint_{C'} \frac{e^{-\frac{zt}{1-t}}\,z\,dt}{(1-t)^2t^n\cdot\frac{zt}{1-t}} \;=\; \frac{n!}{2i\pi}\oint_{C'} \frac{e^{-\frac{zt}{1-t}}\,dt}{(1-t)t^{n+1}}$$

where the contour C' goes round the origin. Accordingly, by (2) we can infer that

$$L_n(z) \;=\; \left[\frac{d^n}{dt^n}\frac{e^{-\frac{zt}{1-t}}}{1-t}\right]_{t=0},$$

whence we have found the generating function

$$\frac{e^{-\frac{zt}{1-t}}}{1-t} \;=\; \sum_{n=0}^{\infty}\frac{L_n(z)}{n!}t^n$$

of the Laguerre polynomials.