Legendre polynomial

Canonical name	LegendrePolynomial
Date of creation	2013-03-22 15:12:10
Last modified on	2013-03-22 15:12:10
Owner	mathcam (2727)
Last modified by	mathcam (2727)
Numerical id	14
Author	mathcam (2727)
Entry type	Definition
Classification	msc 33C45
Related topic	OrthogonalPolynomials
Defines	Rodrigues's Formula
Defines	Legendre's Differential Equation

The *Legendre polynomials* are a set of polynomials $\{P_n\}_{n=0}^{\infty}$ each of order n that satisfy Legendre's ODE:

$$\frac{d}{dx}[(1-x^2)P_n'(x)] + n(n+1)P_n(x) = 0.$$

Alternatively P_n is an eigenfunction of the self-adjoint differential operator $\frac{d}{dx}(1-x^2)\frac{d}{dx}$ with eigenvalue $-n(n+1)$.

The Legendre polynomials are also known as Legendre functions of the first kind.

By Sturm-Liouville theory, this means they're orthogonal over some interval with some weight function. In fact it can be shown that they're orthogonal on $[-1, 1]$ with weight function $W(x) = 1$. As with any set of orthogonal polynomials, this can be used to generate them (up to normalization) by Gram-Schmidt orthogonalization of the monomials $\{x^i\}$. The normalization used is $\langle P_n \| P_n \rangle = 2/(2n+1)$, which makes $P_n(\pm 1) = (\pm 1)^n$

Rodrigues's Formula (which can be generalized to some other polynomial sets) is a sometimes convenient form of P_n in terms of derivatives:

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$$

The first few explicitly are:

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2-1) \\
P_3(x) &= \frac{1}{2}(5x^3-3x) \\
P_4(x) &= \frac{1}{8}(35x^4-30x^2+3) \\
&\ldots
\end{aligned}$$

As all orthogonal polynomials do, these satisfy a three-term recurrence relation:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - (n)P_{n-1}(x)$$

The Legendre functions of the second kind also satisfy the Legendre ODE but are not regular at the origin.

Related are the associated Legendre functions, and spherical harmonics.