application of logarithm series

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The integrand of the improper integral

$$I \;:=\; \int_0^1 \frac{\ln(1+x)}{x}dx \tag{1}$$

is not defined at the lower limit 0. However, from the Taylor series expansion

$$\ln(1+x) \;=\; x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+-\ldots \qquad (-1 < x \leqq 1)$$

of the natural logarithm we obtain the expansion of the integrand

$$\frac{\ln(1+x)}{x} \;=\; 1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+-\ldots \qquad (-1 < x < 0,\;\; 0 < x \leqq 1)$$

whence

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} \;=\; 1. \tag{2}$$

This implies that the integrand of (1) is bounded on the interval $[0,\,1]$ and also continuous, if we think that (2) defines its value at $x=0$. Accordingly, the integrand is Riemann integrable on the interval, and we can determine the improper integral by integrating termwise:

$$\begin{aligned} I \;&=\; \int_0^1 \left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+-\ldots\right)dx \\ &=\; \Big/_{\!0}^{1} \left(x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+-\ldots\right) \\ &=\; 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+-\ldots \end{aligned}$$

By the entry on `http://planetmath.org/ValueOfDirichletEtaFunctionAtS2`Dirichlet eta function at 2, the sum of the obtained series is $\eta(2)=\frac{\pi^2}{12}$. Thus we have the result

$$\int_0^1 \frac{\ln(1+x)}{x}dx \;=\; \frac{\pi^2}{12}. \tag{3}$$

Similarly, using the series

$$\ln(1-x) \;=\; -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots \qquad (-1 \leqq x < 1)$$

and the result in the entry http://planetmath.org/ValueOfTheRiemannZetaFunctionAtS2Riema
zeta function at 2, one can calculate that

$$\int_0^1 \frac{\ln(1-x)}{x} dx = -\frac{\pi^2}{6}. \tag{4}$$