orthogonality of Chebyshev polynomials

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By expanding the of de Moivre identity

$$\cos n\varphi \;=\; (\cos\varphi + i\sin\varphi)^n$$

to sum, one obtains as real part certain terms containing power products of $\cos\varphi$ and $\sin\varphi$, the latter ones only with even exponents. When these are expressed with cosines ($\sin^2\varphi = 1-\cos^2\varphi$), the real part becomes a polynomial T_n of degree n in the `http://planetmath.org/Argument2`argument $\cos\varphi$:

$$\cos n\varphi \;=\; T_n(\cos\varphi) \tag{1}$$

This can be written `http://planetmath.org/Equivalent3`equivalently

$$T_n(x) \;=\; \cos(n\arccos x). \tag{2}$$

It's a question of *Chebyshev polynomial of first kind* and of n (cf. special cases of hypergeometric function).

For showing the orthogonality of T_m and T_n we start from the integral $\displaystyle\int_0^\pi \cos m\varphi\cos n\varphi\,d\varphi$, which via the substitution

$$\cos\varphi \;:=\; x, \quad dx \;=\; -\sin\varphi\,d\varphi \;=\; -\sqrt{1-x^2}\,d\varphi$$

changes to

$$\int_0^\pi \cos m\varphi\,\cos n\varphi\,d\varphi \;=\; -\int_1^{-1} T_m(x)T_n(x)\frac{dx}{\sqrt{1-x^2}}. \tag{3}$$

The left of this equation is evaluated by using the product formula in the entry trigonometric identities:

$$\int_0^\pi \cos m\varphi\,\cos n\varphi\,d\varphi \;=\; \frac{1}{2}\int_0^\pi (\cos(m-n)\varphi + \cos(m+n)\varphi)\,d\varphi \;=\; \begin{cases} 0 & \text{for } m \neq n, \\ \frac{\pi}{2} & \text{for } m = n \neq 0. \end{cases}$$

By (3), we thus have

$$\int_{-1}^1 T_m(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} \;=\; \begin{cases} 0 & \text{for } m \neq n, \\ \frac{\pi}{2} & \text{for } m = n \neq 0, \end{cases}$$

which means the orthogonality of the polynomials $T_m(x)$ and $T_n(x)$ weighted by $\frac{1}{\sqrt{1-x^2}}$.

Any Riemann integrable real function f, defined on $[-1, 1]$, may be expanded to the series

$$f(x) \;=\; \frac{a_0}{2}T_0(x) + \sum_{j=1}^{\infty} a_j T_j(x),$$

where

$$a_j \;=\; \frac{2}{\pi}\int_{-1}^{1} f(x)T_j(x)\frac{dx}{\sqrt{1-x^2}} \qquad (j = 0,\, 1,\, 2,\, \ldots)$$

This concerns especially the polynomials $f(x) := x^n$, for which we obtain

$$\begin{aligned} x^n \;&=\; \cos^n\varphi \;=\; \cosh^n i\varphi \;=\; 2^{-n}(e^{i\varphi} + e^{-i\varphi}) \\ &=\; 2^{-n}\left[\binom{n}{0}(e^{ni\varphi} + e^{-ni\varphi}) + \binom{n}{1}(e^{(n-2)i\varphi} + e^{-(n-2)i\varphi}) + \ldots\right] \\ &=\; 2^{1-n}\left[\binom{n}{0}T_n(x) + \binom{n}{1}T_{n-2}(x) + \binom{n}{2}T_{n-4}(x) + \ldots\right]. \end{aligned}$$

(If n is even, the last term contains $T_0(x)$ but its coefficient is only a half of the middle number of the Pascal's triangle row in question.) Explicitly:

$$\begin{aligned} 1 \;&=\; T_0 \\ x \;&=\; T_1 \\ x^2 \;&=\; 2^{-1}(T_2 + T_0) \\ x^3 \;&=\; 2^{-2}(T_3 + 3T_1) \\ x^4 \;&=\; 2^{-3}(T_4 + 4T_2 + 3T_0) \\ x^5 \;&=\; 2^{-4}(T_5 + 5T_3 + 10T_1) \\ x^6 \;&=\; 2^{-5}(T_6 + 6T_4 + 15T_2 + 10T_0) \\ x^7 \;&=\; 2^{-6}(T_7 + 7T_5 + 21T_3 + 35T_1) \\ x^8 \;&=\; 2^{-7}(T_8 + 8T_6 + 28T_4 + 56T_2 + 36T_0) \\ x^9 \;&=\; 2^{-8}(T_9 + 9T_7 + 36T_5 + 84T_3 + 126T_1) \\ \cdots \;&\quad\; \cdots \end{aligned}$$

References

[1] Pentti Laasonen: *Matemaattisia erikoisfunktioita.* Handout No. 261. Teknillisen Korkeakoulun Ylioppilaskunta; Otaniemi, Finland (1969).