values of complex cosine

Canonical name	ValuesOfComplexCosine
Date of creation	2013-03-22 17:36:29
Last modified on	2013-03-22 17:36:29
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	16
Author	pahio (2872)
Entry type	Topic
Classification	msc 33B10
Classification	msc 30A99
Related topic	RealPart
Related topic	PropertiesOfQuadraticEquation
Related topic	TakingSquareRootAlgebraically
Related topic	ComplexLogarithm
Defines	period strip

Since the complex cosine function $z \mapsto \cos z$ has the prime period 2π, the cosine attains all of its possible values in one of its *period strips*, for example in the period strip

$$\{z \in \mathbb{C} \colon \; -\pi \leqq \operatorname{Re}(z) \; < \; \pi\}. \tag{1}$$

For finding out which values the cosine function can attain in a period strip, we solve the equation $\cos z = w$, where w is any complex number. Using `http://planetmath.org/ComplexSineAndCosine`Euler's formula

$$\cos z \; = \; \frac{e^{iz} + e^{-iz}}{2},$$

the equation may be written as

$$(e^{iz})^2 - 2we^{iz} + 1 \; = \; 0. \tag{2}$$

This is a quadratic equation in e^{iz}, whence we obtain the two `http://planetmath.org/Equationro`

$$e^{iz} \; = \; w \pm \sqrt{w^2 - 1}.$$

The product of the roots is 1, and therefore the roots are distinct from zero for all values of w. If we set

$$w + \sqrt{w^2 - 1} \; = \; re^{i\varphi} \quad (-\pi \leqq \varphi < \pi),$$

the other root is the inverse number

$$w - \sqrt{w^2 - 1} \; = \; \frac{1}{r}e^{-i\varphi}.$$

The solution of the equation

$$e^{iz} \; = \; re^{i\varphi}$$

is then obtained by taking the complex logarithm

$$z \; = \; z_1 \; = \; \frac{1}{i}\log(re^{i\varphi}) \; = \; \varphi - i\ln r + n \cdot 2\pi \quad (n \in \mathbb{Z}),$$

and the other solution of (2) is

$$z \; = \; z_2 \; = \; -\varphi + i\ln r + n \cdot 2\pi \quad (n \in \mathbb{Z}).$$

In the period strip (1), we have one solution z_1 and one solution z_2, both obtained with the value $n = 0$ (except z_2 in the case $\varphi = -\pi$ with $n = -1$). In (1), the points z_1 and z_2 are situated symmetrically with respect the origin. In the cases $w = 1$ and $w = -1$, the equation (2) has double roots $z = 0$ and $z = -\pi$, respectively; then we may say that z_1 and z_2 coincide. Anyhow, we have the

Theorem. In every period strip, cosine attains any complex value at two points.

Example. The solution of the equation $\cos z = 2$ is obtained from $e^{iz} = 2 \pm \sqrt{3}$. In the period strip (1) we get

$$z = \frac{1}{i}\log(2 \pm \sqrt{3}) = -i\ln(2 \pm \sqrt{3}) + 0 \cdot 2\pi.$$

Since $2 \pm \sqrt{3}$ are inverse numbers of each other, we have as result the purely imaginary numbers $z = \pm i\ln(2 + \sqrt{3})$.

From trigonometry, we know that the real zeros of cosine are the odd multiples of $\frac{\pi}{2}$; from these points, $\pm\frac{\pi}{2}$ belong to the period strip (1). Thus $\pm\frac{\pi}{2}$ are the only points of (1) where the cosine vanishes. Therefore, according to the preceding theorem, the well-known points

$$(2n+1)\frac{\pi}{2} \quad (n = 0, \pm 1, \pm 2, \ldots)$$

are the only zeros of the cosine function on the whole complex plane.

The values of complex cosine function may be transferred to the complex sine function by means of the complement formula

$$\sin z = \cos(\frac{\pi}{2} - z).$$

One can think all points of the z-plane to bear the corresponding value of cosine, and then one can translate the plane in the direction of the real axis the distance $\frac{\pi}{2}$; then the values of the sine have been placed to their correct . So one has transferred also the above properties of cosine to sine.

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References

[1] ERNST LINDELÖF: *Johdatus funktioteoriaan.* Second edition. Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1936).

[2] R. NEVANLINNA & V. PAATERO: *Funktioteoria.* Kustannusosakeyhtiö Otava, Helsinki (1963).