Fuchsian singularity

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Defines	irregular singular point
Defines	irregular singularity
Defines	Hamburger equation

Suppose that D is an open subset of $\mathbb{C}$ and that the n functions $c_k\colon D \to \mathbb{C}, \quad k = 0, \ldots, n-1$ are meromorphic. Consider the ordinary differential equation

$$\frac{d^n w}{dz^n} + \sum_{k=0}^{n-1} c_k(z) \frac{d^k w}{dz^k} = 0$$

A point $p \in D$ is said to be a *regular singular point* or a *Fuchsian singular point* of this equation if at least one of the functions c_k has a pole at p and, for every value of k between 0 and n, either c_k is regular at p or has a pole of order not greater than $n-k$.

If p is a Fuchsian singular point, then the differential equation may be rewritten as a system of first order equations

$$\frac{dv_i}{dz} = \frac{1}{z} \sum_{j=1}^{n} b_{ij}(z) v_j(z)$$

in which the coefficient functions b_{ij} are analytic at z. This fact helps explain the restiction on the orders of the poles of the c_k's.

If an equation has a Fuchsian singularity, then the solution can be expressed as a Frobenius series in a neighborhood of this point.

A singular point of a differential equation which is not a regular singular point is known as an irregular singular point.

Examples

The Bessel equation

$$w'' + \frac{1}{z} w' + \frac{z^2 - 1}{z^2} w = 0$$

has a Fuchsian singularity at $z = 0$ since the coefficient of w' has a pole of order 1 and the coefficient of w has a pole of order 2.

On the other hand, the *Hamburger equation*

$$w'' + \frac{2}{z} w' + \frac{z^2 - 1}{z^4} w = 0$$

has an irregular singularity at $z = 0$ since the coefficient of w has a pole of order 4.