Rayleigh quotient

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Definition. *The* **Rayleigh quotient***, $R_{\mathbf{A}}$, to the Hermitian matrix* $\mathbf{A}$ *is defined as*

$$R_{\mathbf{A}}(\mathbf{x}) = \frac{\mathbf{x}^H \mathbf{A} \mathbf{x}}{\mathbf{x}^H \mathbf{x}}, \quad \mathbf{x} \neq \mathbf{0},$$

where $\mathbf{x}^H$ *is the Hermitian conjugate of* $\mathbf{x}$*.*

The importance of this quantity (in fact, the reason Rayleigh first introduced it) is that its critical values are the eigenvectors of A and the values of the quotient at these special vectors are the corresponding eigenvalues. This observation leads to the *variational method* for computing the spectrum of a positive matrix (either exactly or approximately). Namely, one first minimizes the Rayleigh quotient over the whole vector space. This gives the lowest eigenvalue and corresponding eigenvector. Next, one restricts attention to the orthogonal complement of the eigenvector found in the first step and minimizes over this subspace. That produces the next lowest eigenvalue and corresponding eigenvector. One can continue this process recursively. At each step, one minimizes the Rayleigh quotient over the subspace orthogonal to all the vectors found in the preceding steps to find another eigenvalue and its corresponding eigenvector.

This concept of Rayleigh quotient also makes sense in the more general setting when A is a Hermitian operator on a Hilbert space. Furthermore, it is possible to make use of the Rayleigh-Ritz method in cases where the operator has a discrete spectrum bounded from below, such as the Laplace operator on a compact domain. This method is often employed in practise because, in physical applications, one is oftentimes interested in only the lowest eigenvalue or perhaps the first few lowest eigenvalues and not so concerned with the rest of the spectrum.