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limits of natural logarithm

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The `http://planetmath.org/NaturalLogarithm`parent entry defines the natural logarithm as

$$\ln x \;=\; \int_1^x \frac{1}{t}\,dt \qquad (x>0) \tag{1}$$

and derives the

$$\ln xy \;=\; \ln x + \ln y$$

which implies easily by induction that

$$\ln a^n \;=\; n\ln a. \tag{2}$$

Basing on (1), we prove here the

Theorem. The function $x \mapsto \ln x$ is strictly increasing and continuous on $\mathbb{R}_+$. It has the limits

$$\lim_{x\to+\infty} \ln x \;=\; +\infty \quad \text{and} \quad \lim_{x\to 0+} \ln x \;=\; -\infty. \tag{3}$$

Proof. By the above definition, $\ln x$ is differentiable:

$$\frac{d}{dx}\ln x \;=\; \frac{1}{x} \;>\; 0$$

Accordingly, $\ln x$ is also continuous and strictly increasing.

Let M be an arbitrary positive number. We have $\ln 2 = \int_1^2 \frac{dt}{t} > 0$. There exists a positive integer n such that $n\ln 2 > M$ (see Archimedean property). By (2) we thus get $\ln 2^n > M$, and since $\ln x$ is strictly increasing, we see that

$$\ln x > M \quad \forall x > 2^n.$$

Hence the first limit assertion is true. Now $-M < 0$. If $x > 2^n$, then $\ln x > M$ and

$$0 \;<\; \frac{1}{x} \;<\; 2^{-n}, \qquad \ln\frac{1}{x} \;=\; \int_1^{\frac{1}{x}} \frac{dt}{t} \;=\; \int_x^1 \frac{du}{u} \;=\; -\ln x \;<\; -M$$

(`http://planetmath.org/SubstitutionForIntegration`substitution $xt := u$). From this we can infer the second limit assertion.