ODE types solvable by two quadratures

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The second order ordinary differential equation

$$\frac{d^2y}{dx^2} = f\left(x,\, y,\, \frac{dy}{dx}\right) \tag{1}$$

may in certain special cases be solved by using two quadratures, sometimes also by reduction to a http://planetmath.org/ODEfirst order differential equation and a quadrature.

If the right hand side of (1) contains at most one of the quantities x, y and $\frac{dy}{dx}$, the general solution solution is obtained by two quadratures.

- The equation

$$\frac{d^2y}{dx^2} = f(x) \tag{2}$$

  is considered http://planetmath.org/EquationYFxhere.

- The equation

$$\frac{d^2y}{dx^2} = f(y) \tag{3}$$

  has as constant solutions all real roots of the equation $f(y) = 0$. The other solutions can be gotten from the normal system

$$\frac{dy}{dx} = z, \qquad \frac{dz}{dx} = f(y) \tag{4}$$

  of (3). Dividing the equations (4) we get now $\frac{dz}{dy} = \frac{f(y)}{z}$. By separation of variables and integration we may write

$$\frac{z^2}{2} = \int f(y)\,dy + C_1,$$

  whence the first equation of (4) reads

$$\frac{dy}{dx} = \sqrt{2\int f(y)\,dy + C_1}.$$

here the variables and integrating give the general integral of (3) in the form

$$\int \frac{dy}{\sqrt{2\int f(y)\,dy + C_1}} \;=\; x + C_2. \tag{5}$$

The http://planetmath.org/SolutionsOfOrdinaryDifferentialEquationintegration constant C_1 has an influence on the form of the integral curves, but C_2 only translates them in the direction of the x-axis.

- The equation

$$\frac{d^2y}{dx^2} \;=\; f(\frac{dy}{dx}) \tag{6}$$

is http://planetmath.org/Equivalent3equivalent with the normal system

$$\frac{dy}{dx} \;=\; z, \quad \frac{dz}{dx} \;=\; f(z). \tag{7}$$

If the equation $f(z) = 0$ has real roots $z_1,\, z_2,\, \ldots,$ these satisfy the latter of the equations (7), and thus, according to the former of them, the differential equation (6) has the solutions $y := z_1x + C_1,\;\; y := z_2x + C_2,\;\; \ldots$.

The other solutions of (6) are obtained by separating the variables and integrating:

$$x \;=\; \int \frac{dz}{f(z)} + C. \tag{8}$$

If this antiderivative is expressible in closed form and if then the equation (8) can be solved for z, we may write

$$z \;=\; \frac{dy}{dx} \;=\; g(x{-}C).$$

Accordingly we have in this case the general solution of the ODE (6):

$$y \;=\; \int g(x{-}C)\,dx + C'. \tag{9}$$

In other cases, we express also y as a function of z. By the chain rule, the normal system (7) yields

$$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = \frac{z}{f(z)},$$

whence

$$y = \int \frac{z\,dz}{f(z)} + C'.$$

Thus the general solution of (6) reads now in a parametric form as

$$x = \int \frac{dz}{f(z)} + C, \qquad y = \int \frac{z\,dz}{f(z)} + C'. \tag{10}$$

The equations 10 show that a translation of any integral curve yields another integral curve.