proof of multiplication formula for gamma function

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Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
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Author	rspuzio (6075)
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Define the function f as

$$f(z) = \frac{n^{nz} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right)}{\Gamma(nz)}$$

By the functional equation of the gamma function,

$$f(z+1) = \frac{n^n n^{nz} \left(\prod_{m=0}^{n-1} \Gamma\left(z + \frac{m}{n}\right)\right) \prod_{k=0}^{n-1} \left(z + \frac{k}{n}\right)}{\Gamma(nz) \prod_{k=0}^{n-1} (nz+k)} = f(z)$$

Hence f is a periodic function of z. However, for large values of z, we can apply the Stirling approximation formula to conclude

$$f(z) = \frac{(2\pi)^{n/2} n^{nz} \prod_{k=0}^{n-1} \left[e^{-z-k/n}(z+k/n)^{z+k/n-1/2} + O(e^{-z}(z+k/n)^{z+k/n-3/2})\right]}{(2\pi)^{1/2} e^{-nz}(nz)^{nz-1/2} + O(e^{-nz}(nz)^{nz-3/2})} =$$

$$(2\pi)^{(n-1)/2} n^{1/2} \frac{\prod_{k=0}^{n-1} \left[e^{-k/n}(z+k/n)^{z+k/n-1/2} + O((z+k/n)^{z+k/n-3/2})\right]}{z^{nz-1/2} + O(z^{nz-3/2})} =$$

$$(2\pi)^{(n-1)/2} n^{1/2} \frac{z^{1/2} \prod_{k=0}^{n-1} \left[e^{-k/n} \left(1 + \frac{k}{nz}\right)^{z+k/n-1/2} z^{k/n-1/2} + O((z+k/n)^{k/n-3/2})\right]}{1 + O(z^{-1})}$$

Note that

$$\prod_{k=0}^{n-1} e^{-k/n} = e^{-\sum_{k=0}^{n-1} k/n} = e^{(1-n)/2}$$

$$z^{1/2} \prod_{k=0}^{n-1} z^{k/n-1/2} = z^{1/2} z^{\sum_{k=0}^{n-1}(k/n-1/2)} = z^{1/2+(n-1)/2-n/2} = 1$$

Also,

$$\left(1 + \frac{k}{nz}\right)^z = e^{k/n} + O(z^{-1})$$

Hence, $f(z) = (2\pi)^{(n-1)/2} n^{1/2} + O(z^{-1})$. Now, the only way for a function to be periodic and have a definite limit is for that function to be constant. Therefore, $f(z) = (2\pi)^{(n-1)/2} n^{1/2}$. Writing out the definition of f and rearranging gives the multiplication formula.