example of solving a functional equation

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Let's determine all twice differentiable real functions f which satisfy the functional equation

$$f(x+y)\cdot f(x-y) = [f(x)]^2-[f(y)]^2 \qquad (1)$$

for all real values of x and y.

Substituting first $y = 0$ in (1) we see that $f(x)^2 = f(x)^2 - f(0)^2$ or $f(0) = 0$. The substitution $x = 0$ gives $f(y)f(-y) = -f(y)^2$, whence $f(-y) = -f(y)$. So f is an odd function.

We differentiate both sides of (1) with respect to y and the result with respect to x:

$$f'(x+y)f(x-y)-f'(x-y)f(x+y) = -2f(y)f'(y)$$

$$f''(x+y)f(x-y)+f'(x-y)f'(x+y)-f''(x-y)f(x+y)-f'(x+y)f'(x-y) = 0$$

The result is simplified to $f''(x+y)f(x-y) = f''(x-y)f(x+y)$, i.e.

$$f''(x+y)/f(x+y) = f''(x-y)/f(x-y).$$

Denoting $x+y := u$, $x-y := v$ we obtain the equation

$$\frac{f''(u)}{f(u)} = \frac{f''(v)}{f(v)}$$

for all real values of u and v. This is not possible unless the proportion $\frac{f''(u)}{f(u)}$ has a on u. Thus the homogeneous linear differential equation $f''(t)/f(t) = \pm k^2$ or

$$f''(t) = \pm k^2 f(t),$$

with k some , is valid.

There are three cases:

1. $k = 0$. Now $f''(t) \equiv 0$ and consequently $f(t) \equiv Ct$. If one especially C equal to 1, the solution is the `http://planetmath.org/IdentityMap`identity function $f : t \mapsto t$. This yields from (1) the well-known "memory formula"

$$(x+y)(x-y) = x^2-y^2.$$

2. $f''(t) = -k^2 f(t)$ with $k \neq 0$. According to the oddness one obtains for the general solution the sine function $f : t \mapsto C \sin kt$. The special case $C = k = 1$ means in (1) the

$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$

which is easy to verify by using the `http://planetmath.org/AdditionFormula`addition and subtraction formulae of sine.

3. $f''(t) = k^2 f(t)$ with $k \neq 0$. According to the oddness we obtain for the general solution the `http://planetmath.org/HyperbolicFunctions`hyperbolic sine function $f : t \mapsto C \sinh kt$. The special case $C = k = 1$ gives from (1) the

$$\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y.$$

The solution method of (1) is due to andik and perucho.