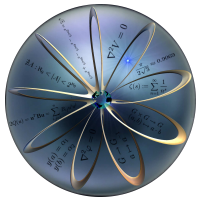

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examples of elliptic functions

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Examples of Elliptic Functions
Let $\Lambda \subset \mathbb{C}$ be a lattice generated by w_1, w_2. Let Λ^* denote $\Lambda - \{0\}$.

1. The *Weierstrass* $\wp$-function is defined by the series

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{w\in\Lambda^*} \frac{1}{(z-w)^2} - \frac{1}{w^2}$$

2. The derivative of the Weierstrass $\wp$-function is also an elliptic function

$$\wp'(z;\Lambda) = -2\sum_{w\in\Lambda^*} \frac{1}{(z-w)^3}$$

3. The *Eisenstein series of weight* $2k$ for Λ is the series

$$\mathcal{G}_{2k}(\Lambda) = \sum_{w\in\Lambda^*} w^{-2k}$$

   The Eisenstein series of weight 4 and 6 are of special relevance in the theory of elliptic curves. In particular, the quantities g_2 and g_3 are usually defined as follows:

$$g_2 = 60\cdot\mathcal{G}_4(\Lambda), \quad g_3 = 140\cdot\mathcal{G}_6(\Lambda)$$

Remark: The elliptic functions $\wp$, $\wp'$ and $\mathcal{G}_{2k}$ are related by the following important equation:

$$(\wp'(z;\Lambda))^2 = 4\wp(z;\Lambda)^3 - g_2(\Lambda)\wp(z;\Lambda) - g_3(\Lambda)$$

In particular, the previous equation provides an isomorphism between $\mathbb{C}/\Lambda$ and the elliptic curve $E : y^2 = 4x^3 - g_2x - g_3$ given by:

$$\mathbb{C}/\Lambda \to E, \quad z \mapsto (\wp(z;\Lambda), \wp'(z;\Lambda)).$$