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Picard's theorem

Canonical name	PicardsTheorem
Date of creation	2013-03-22 14:59:57
Last modified on	2013-03-22 14:59:57
Owner	Daume (40)
Last modified by	Daume (40)
Numerical id	6
Author	Daume (40)
Entry type	Theorem
Classification	msc 34A12
Synonym	Picard-Lindelöf theorem
Related topic	ExistenceAndUniquenessOfSolutionOfOrdinaryDifferentialEquations

Theorem 1 (Picard's theorem [**?**]). *Let E be an open subset of $\mathbb{R}^2$ and a continuous function $f(x, y)$ defined as $f\colon E \to \mathbb{R}$. If $(x_0, y_0) \in E$ and f satisfies the Lipschitz condition in the variable y in E:*

$$|f(x, y) - f(x, y_1)| \leq M|y - y_1|$$

where M is a constant. Then the ordinary differential equation defined as

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition

$$y(x_0) = y_0$$

has a unique solution $y(x)$ on some interval $|x - x_0| \leq \delta$.

The above theorem is also named the *Picard-Lindelöf theorem* and can be generalized to a system of first order ordinary differential equations

Theorem 2 (generalization of Picard's theorem [**?**]). *Let E be an open subset of $\mathbb{R}^{n+1}$ and a continuous function $f(x, y_1, \dots, y_n)$ defined as $f = (f_1, \dots, f_n)\colon E \to \mathbb{R}^n$. If $(t_0, y_{10}, \dots, y_{n0}) \in E$ and f satisfies the Lipschitz condition in the variable $y_1, \dots, y_n$ in E:*

$$|f_i(x, y_1, \dots, y_n) - f_i(x, y'_1 \dots, y'_n)| \leq M \max_{1 \leq j \leq n} |y_j - y'_j|$$

where M is a constant. Then the system of ordinary differential equation defined as

$$\begin{aligned}\frac{dy_1}{dx} &= f_1(x, y_1, \dots, y_n)\\ &\vdots\\ \frac{dy_n}{dx} &= f_n(x, y_1, \dots, y_n)\end{aligned}$$

with the initial condition

$$y_1(x_0) = y_{10}, \dots, y_n(x_0) = y_{n0}$$

has a unique solution

$$y_1(x), \dots, y_n(x)$$

on some interval $|x - x_0| \leq \delta$.

see also:

- the entry existence and uniqueness of solution of ordinary differential equations

References

[KF] Kolmogorov, A.N. & Fomin, S.V.: Introductory Real Analysis, Translated & Edited by Richard A. Silverman. Dover Publications, Inc. New York, 1970.