generating function of Legendre polynomials

Canonical name	GeneratingFunctionOfLegendrePolynomials
Date of creation	2015-03-08 20:29:06
Last modified on	2015-03-08 20:29:06
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	17
Author	pahio (2872)
Entry type	Result
Classification	msc 33B99
Classification	msc 30D10
Classification	msc 30B10
Related topic	GeneratingFunctionOfHermitePolynomials
Related topic	GeneratingFunctionOfLaguerrePolynomials
Related topic	VariantOfCauchyIntegralFormula

For finding the generating function

$$F(t) \;=\; \sum_{n=0}^{\infty} P_n(z)t^n$$

of the sequence of the Legendre polynomials

$$\begin{aligned}
P_0(z) &= 1\\
P_1(z) &= z\\
P_2(z) &= \tfrac{1}{2}(3z^2-1)\\
P_3(x) &= \tfrac{1}{2}(5z^3-3z)\\
P_4(z) &= \tfrac{1}{8}(35z^4-30z^2+3)\\
P_5(z) &= \tfrac{1}{8}(63z^5-70z^3+15z)\\
\ldots & \quad \ldots
\end{aligned}$$

we have to present $P_n(z)$ as the general coefficient of Taylor series in t, i.e. as the nth derivative of some $F(t)$ in the origin, divided by the factorial $n!$. The Cauchy integral formula offers the chance to implement that.

Starting from the `http://planetmath.org/node/11983`Rodrigues formula of Legendre polynomials, we may write

$$P_n(z) \;=\; \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2–1)^n \;=\; \frac{1}{2^n n!}\frac{n!}{2i\pi}\oint_c \frac{(\zeta^2-1)^n}{(\zeta-z)^{n+1}}d\zeta \;=\; \frac{1}{2i\pi}\oint_c \left(\frac{1}{2}\frac{\zeta^2-1}{\zeta-z}\right)^n \frac{d\zeta}{\zeta-z},$$

where the contour c runs anticlockwise once around the point z. The change of variable

$$\frac{\zeta^2-1}{2(\zeta-z)} \;=\; \frac{1}{t}, \qquad d\zeta \;=\; \frac{zt-1-\sqrt{1-zt+t^2}}{t^2\sqrt{1-zt+t^2}}dt$$

gives

$$P_n(z) \;=\; -\frac{1}{2i\pi}\oint_{c'} \frac{dt}{t^n t\sqrt{1-zt+t^2}}$$

where t must go round the origin clockwise, but in

$$P_n(z) \;=\; \frac{1}{n!}\cdot\frac{n!}{2i\pi}\oint_{c'} \frac{dt}{\sqrt{1-zt+t^2}\cdot(t-0)^{n+1}}$$

anticlockwise. This is, by Cauchy integral formula again,

$$P_n(z) \;=\; \frac{1}{n!}\left[\frac{d^n}{dt^n}\frac{1}{\sqrt{1-zt+t^2}}\right]_{t=0}.$$

This means that

$$F(t) \;:=\; \frac{1}{\sqrt{1-zt+t^2}}$$

is the searched generating function of the Legendre polynomials:

$$\frac{1}{\sqrt{1-zt+t^2}} \;=\; P_0(z) + P_1(z)t + P_2(z)t^2 + P_3(z)t^3 + \ldots$$

Cf. the generating function of the Bessel functions.