proof of Bohr-Mollerup theorem

Canonical name	ProofOfBohrMollerupTheorem
Date of creation	2013-03-22 14:53:39
Last modified on	2013-03-22 14:53:39
Owner	Andrea Ambrosio (7332)
Last modified by	Andrea Ambrosio (7332)
Numerical id	18
Author	Andrea Ambrosio (7332)
Entry type	Proof
Classification	msc 33B15

To show that the gamma function is logarithmically convex, we can examine the product representation:

$$\Gamma(x) = \frac{1}{x} e^{-\gamma x} \prod_{n=1}^{\infty} \frac{n}{x+n} e^{-x/n}$$

Since this product converges absolutely for $x > 0$, we can take the logarithm term-by-term to obtain

$$\log \Gamma(x) = -\log x - \gamma x - \sum_{n=1}^{\infty} \log\left(\frac{n}{x+n}\right) - \frac{x}{n}$$

It is justified to differentiate this series twice because the series of derivatives is absolutely and uniformly convergent.

$$\frac{d^2}{dx^2} \log \Gamma(x) = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2}$$

Since every term in this series is positive, Γ is logarithmically convex. Furthermore, note that since each term is monotonically decreasing, $\log \Gamma$ is a decreasing function of x. If $x > m$ for some integer m, then we can bound the series term-by-term to obtain

$$\frac{d^2}{dx^2} \log \Gamma(x) < \sum_{n=0}^{\infty} \frac{1}{(m+n)^2} = \sum_{n=m}^{\infty} \frac{1}{n^2}$$

Therefore, as $x \to \infty$, $d^2\Gamma/dx^2 \to 0$.

Next, let f satisfy the hypotheses of the Bohr-Mollerup theorem. Consider the function g defined as $e^{g(x)} = f(x)/\Gamma(x)$. By hypothesis 3, $g(1) = 0$. By hypothesis 2, $e^{g(x+1)} = e^{g(x)}$, so $g(x+1) = g(x)$. In other , g is periodic.

Suppose that g is not constant. Then there must exist points x_0 and x_1 on the real axis such that $g(x_0) \neq g(x_1)$. Suppose that $g(x_1) > g(x_0)$ for definiteness. Since g is periodic with period 1, we may assume without loss of generality that $x_0 < x_1 < x_0 + 1$. Let D_2 denote the second divided difference of g:

$$D_2 = \Delta_2(g; x_0, x_1, x_0+1) = -\frac{g(x_0)}{x_0 - x_1} + \frac{g(x_1)}{(x_1 - x_0)(x_1 - x_0 - 1)} - \frac{g(x_0+1)}{x_0 - x_1 + 1}$$

By our assumptions, $D_2 < 0$. By linearity,

$$D_2 = \Delta_2(\log f; x_0, x_1, x_0 + 1) - \Delta_2(\log \Gamma; x_0, x_1, x_0 + 1)$$

By periodicity, we have

$$D_2 = \Delta_2(\log f; x_0 + n, x_1 + n, x_0 + n + 1) - \Delta_2(\log \Gamma; x_0 + n, x_1 + n, x_0 + n + 1)$$

for every integer $n > 0$. However,

$$|\Delta_2(\log \Gamma; x_0 + n, x_1 + n, x_0 + n + 1)| < \max_{x_0+n \leq x \leq x_0+1} \frac{d^2}{dx^2} \log \Gamma(x)$$

As $n \to \infty$, the right hand side approaches zero. Hence, by choosing n sufficiently large, we can make the left-hand side smaller than $|D_2|/2$. For such an n,

$$\Delta_2(\log f; x_0 + n, x_1 + n, x_0 + n + 1) < 0$$

However, this contradicts hypothesis 1. Therefore, g must be constant. Since $g(0) = 0$, $g(x) = 0$ for all x, which implies that $e^0 = f(x)/\Gamma(x)$. In other words, $f(x) = \Gamma(x)$ as desired.