equation $y'' = f(x)$

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A simple special case of the second order linear differential equation with constant coefficients is

$$\frac{d^2y}{dx^2} = f(x) \tag{1}$$

where f is continuous. We obtain immediately $\dfrac{dy}{dx} = C_1 + \int f(x)\,dx$,

$$y = C_1x + C_2 + \int\left(\int f(x)\,dx\right)dx. \tag{2}$$

A particular solution $y(x)$ of (1) satisfying the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0$$

is obtained more simply by integrating (1) twice between the http://planetmath.org/DefiniteIn x_0 and x, thus getting

$$y(x) = y_0 + y'_0\cdot(x-x_0) + \int_{x_0}^{x}\left(\int_{x_0}^{x} f(x)\,dx\right)dx.$$

But here, the two first addends are the first terms of the Taylor polynomial of $y(x)$, expanded by the powers of $x-x_0$, whence the double integral is the corresponding http://planetmath.org/RemainderVariousFormulasremainder term

$$\int_{x_0}^{x} y''(x)(x-t)\,dt = \int_{x_0}^{x} f(t)(x-t)\,dt.$$

Hence the particular solution can be written with the simple integral as

$$y(x) = y_0 + y'_0\cdot(x-x_0) + \int_{x_0}^{x} f(t)(x-t)\,dt. \tag{3}$$

The result may be generalised for the n^{th} http://planetmath.org/ODEorder differential equation

$$\frac{d^ny}{dx^n} = f(x) \tag{4}$$

with corresponding n initial conditions:

$$y(x) = y_0 + y'_0\cdot(x-x_0) + \frac{y''_0}{2!}(x-x_0)^2 + \ldots + \frac{y_0^{(n-1)}}{(n-1)!}(x-x_0)^{n-1} + \frac{1}{(n-1)!}\int_{x_0}^{x} f(t)(x-t)^{n-1}\,dt. \tag{5}$$