Chebyshev equation

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Chebyshev's equation is the second order linear differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

where p is a real constant.

There are two independent solutions which are given as series by:

$$y_1(x) = 1 - \tfrac{p^2}{2!}x^2 + \tfrac{(p-2)p^2(p+2)}{4!}x^4 - \tfrac{(p-4)(p-2)p^2(p+2)(p+4)}{6!}x^6 + \cdots$$

and

$$y_2(x) = x - \tfrac{(p-1)(p+1)}{3!}x^3 + \tfrac{(p-3)(p-1)(p+1)(p+3)}{5!}x^5 - \cdots$$

In each case, the coefficients are given by the recursion

$$a_{n+2} = \frac{(n-p)(n+p)}{(n+1)(n+2)}a_n$$

with y_1 arising from the choice $a_0 = 1$, $a_1 = 0$, and y_2 arising from the choice $a_0 = 0$, $a_1 = 1$.

The series converge for $|x| < 1$; this is easy to see from the ratio test and the recursion formula above.

When p is a non-negative integer, one of these series will terminate, giving a polynomial solution. If $p \geq 0$ is even, then the series for y_1 terminates at x^p. If p is odd, then the series for y_2 terminates at x^p.

These polynomials are, up to multiplication by a constant, the Chebyshev polynomials. These are the only polynomial solutions of the Chebyshev equation.

(In fact, polynomial solutions are also obtained when p is a negative integer, but these are not new solutions, since the Chebyshev equation is invariant under the substitution of p by $-p$.)