proof of Bendixson's negative criterion

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Suppose that there exists a periodic solution called Γ which has a period of T and lies in E. Let the interior of Γ be denoted by D. Then by Green's Theorem we can observe that

$$\begin{aligned}
\iint_D \nabla \cdot \mathbf{f} \, \mathrm{d}x \, \mathrm{d}y &= \iint_D \frac{\partial \mathbf{X}}{\partial x} + \frac{\partial \mathbf{Y}}{\partial y} \, \mathrm{d}x \, \mathrm{d}y \\
&= \oint_\Gamma (\mathbf{X} \, \mathrm{d}y - \mathbf{Y} \, \mathrm{d}x) \\
&= \int_0^T (\mathbf{X}\dot{y} - \mathbf{Y}\dot{x}) \, \mathrm{d}t \\
&= \int_0^T (\mathbf{X}\mathbf{Y} - \mathbf{Y}\mathbf{X}) \, \mathrm{d}t \\
&= 0
\end{aligned}$$

Since $\nabla \cdot \mathbf{f}$ is not identically zero by hypothesis and is of one sign, the double integral on the left must be non zero and of that sign. This leads to a contradiction since the right hand side is equal to zero. Therefore there does not exists a periodic solution in the simply connected region E.