# Liouville's theorem

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| Canonical name | LiouvillesTheorem |
| Date of creation | 2013-03-22 15:14:55 |
| Last modified on | 2013-03-22 15:14:55 |
| Owner | Koro (127) |
| Last modified by | Koro (127) |
| Numerical id | 20 |
| Author | Koro (127) |
| Entry type | Theorem |
| Classification | msc 34A34 |

Let

$$\dot{x} = f(x) \tag{1}$$

be a autonomous ordinary differential equation in $\mathbb{R}^n$ defined by a smooth vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and the Jacobian of $f$ is denoted $\frac{\partial f}{\partial x}$. Also let $\Phi_t(x)$ be the `http://planetmath.org/Flow2`flow associated with (**??**). Let

$$V(t) = \int_{\Phi_t(D)} dx$$

be the volume of the image of $D$ under this flow after a time $t$.

**Theorem 1** (Liouville's theorem)**.** *If $D \subseteq \mathbb{R}^n$ is a bounded measurable domain. Then*

$$\dot{V}(t) = \int_{\Phi_t(D)} \operatorname{div} f(x) dx$$

*Proof.* Let $V(t)$ be defined as above then

$$\begin{aligned} V(t_0 + h) &= \int_{\Phi_{t_0+h}(D)} dy \\ &= \int_{\Phi_h(\Phi_{t_0}(D))} dy \\ &= \int_{\Phi_{t_0}(D)} \det\left(\frac{\partial \Phi_h}{\partial x}(x)\right) dx. \end{aligned}$$

We claim that, for $x \in \Phi_{t_0}(D)$,

$$\frac{\partial \Phi_t}{\partial x}(x) = I + t\frac{\partial f}{\partial x}(x) + o(t)$$

as $t \to 0$.

In fact,

$$\Phi_t(x) = x + \int_0^t f(\Phi_s(x)) ds,$$

and by the Leibniz integral rule

$$\frac{\partial \Phi_t}{\partial x}(x) = I + \int_0^t \frac{\partial}{\partial x} f(\Phi_s(x)) ds,$$

so that

$$\frac{\partial}{\partial t}\frac{\partial \Phi_t}{\partial x}(x) = \frac{\partial}{\partial x} f(\Phi_t(x))$$

and evaluating at $t = 0$ we get

$$\frac{\partial}{\partial t}\frac{\partial \Phi_t}{\partial x}(x)\Big|_{t=0} = \frac{\partial}{\partial x} f(\Phi_0(x)) = \frac{\partial f}{\partial x}(x).$$

Our claim follows from this and from the definition of derivative.

Hence

$$\begin{aligned}
\det\left(\frac{\partial \Phi_t}{\partial x}(x)\right) &= \det\left(I + t\frac{\partial f}{\partial x}(x)\right) + o(t) \\
&= \prod_{i=1}^{n}(1 + \frac{\partial f_i}{\partial x_i}(x)) + o(t) \\
&= 1 + t\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + o(t) \\
&= 1 + t\,\mathrm{div}\ f(x) + o(t)
\end{aligned}$$

as $t \to 0$. It follows that

$$V(t_0 + h) = \int_{\Phi_{t_0}(D)} 1 + h\,\mathrm{div}\ f(x) + o(h)dx$$

and

$$\begin{aligned}
\dot{V}(t_0) &= \lim_{h\to 0}\frac{V(t_0+h) - V(t_0)}{h} \\
&= \frac{\int_{\Phi_{t_0}(D)} 1 + h\,\mathrm{div}\ f(x) + o(h)dx - V(t_0)}{h} \\
&= \frac{V(t_0) + h\int_{\Phi_{t_0}(D)} \mathrm{div}\ f(x)dx + o(h) - V(t_0)}{h} \\
&= \int_{\Phi_{t_0}(D)} \mathrm{div}\ f(x)dx + \lim_{h\to 0}\frac{o(h)}{h} \\
&= \int_{\Phi_{t_0}(D)} \mathrm{div}\ f(x)dx.
\end{aligned}$$

□

**Corollary 1.** *The flow of an* ***http://planetmath.org/HamiltonianEquations****Hamiltonian system preserves volume.*

*Proof.* It follows directly since the vector field of an Hamiltonian system has divergence equal to zero. Hence $\dot{V} = 0$ implies that the volume is constant. $\square$

# References


[TG] Teschl, Gerald: Ordinary Differential Equations and Dynamical Systems. `http://www.mat.univie.ac.at/ gerald/ftp/book-ode/index.html`http://www.mat.univie
ode/index.html, 2004.