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special cases of hypergeometric function

Canonical name	SpecialCasesOfHypergeometricFunction
Date of creation	2013-03-22 18:54:39
Last modified on	2013-03-22 18:54:39
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	8
Author	pahio (2872)
Entry type	Example
Classification	msc 33C05
Related topic	FrobeniusMethod
Related topic	IndexOfSpecialFunctions
Related topic	GettingTaylorSeriesFromDifferentialEquation

Many http://planetmath.org/ElementaryFunctionelementary and non-elementary transcendental functions may be expressed as special cases of the hypergeometric functions

$$F(a,\, b,\, c;\, x) \;=\; 1+\frac{ab}{1!c}x+\frac{a(a+1)b(b+1)}{2!c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}x^3+\ldots,$$

which are solutions of the hypergeometric equation

$$x(x-1)\frac{d^2y}{dx^2}+(c-(a+b+1))\frac{dy}{dx}-aby \;=\; 0.$$

For example:

- $(1+x)^n \;=\; F(-n,\, 1,\, 1;\, -x)$
- $\ln(1+x) \;=\; xF(1,\, 1,\, 2;\, -x)$
- $\ln\frac{1+x}{1-x} \;=\; 2xF(\frac{1}{2},\, 1,\, \frac{3}{2};\, x^2)$
- $\arcsin x \;=\; xF(\frac{1}{2},\, \frac{1}{2},\, \frac{3}{2};\, x^2)$
- $\arctan x \;=\; xF(\frac{1}{2},\, 1,\, \frac{3}{2};\, -x^2)$
- $\sin(m\arcsin x) \;=\; mxF(\frac{1+m}{2},\, \frac{1-m}{2},\, \frac{3}{2};\, x^2)$
- $\cos(m\arcsin x) \;=\; F(\frac{m}{2},\, -\frac{m}{2},\, \frac{1}{2};\, x^2)$
- $T_n(x) \;=\; F(n,\, -n,\, \frac{1}{2};\, \frac{1-x}{2})$ (Chebyshev polynomials)
- $P_n(x) \;=\; F(-n,\, n+1,\, 1;\, \frac{1-x}{2})$ (Legendre polynomials)
- $\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-x^2\sin^2\varphi}} \;=\; \frac{\pi}{2}F(\frac{1}{2},\, \frac{1}{2},\, 1;\, x^2)$ (complete elliptic integral of 1st kind)
- $\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{1-x^2\sin^2\varphi}\;d\varphi \;=\; \frac{\pi}{2}F(-\frac{1}{2},\, \frac{1}{2},\, 1;\, x^2)$ (complete elliptic integral of 2nd kind)