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d’Alembert’s equation

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The first differential equation

$$y = \varphi(\frac{dy}{dx}) \cdot x + \psi(\frac{dy}{dx})$$

is called *d'Alembert's differential equation*; here φ and ψ some known differentiable real functions.

If we denote $\frac{dy}{dx} := p$, the equation is

$$y = \varphi(p) \cdot x + \psi(p).$$

We take p as a new variable and derive the equation with respect to p, getting

$$p - \varphi(p) = [x\varphi'(p) + \psi'(p)]\frac{dp}{dx}.$$

If the equation $p - \varphi(p) = 0$ has the roots $p = p_1$, p_2, ..., p_k, then we have $\frac{dp_\nu}{dx} = 0$ for all ν's, and therefore there are the special solutions

$$y = p_\nu x + \psi(p_\nu) \quad (\nu = 1, 2, ..., k)$$

for the original equation. If $\varphi(p) \not\equiv p$, then the derived equation may be written as

$$\frac{dx}{dp} = \frac{\varphi'(p)}{p - \varphi(p)}x + \frac{\psi'(p)}{p - \varphi(p)},$$

which linear differential equation has the solution $x = x(p, C)$ with the integration constant C. Thus we get the general solution of d'Alembert's equation as a parametric

$$\begin{cases} x = x(p, C), \\ y = \varphi(p)x(p, C) + \psi(p) \end{cases}$$

of the integral curves.