$(1 + 1/n)^n$ is an increasing sequence

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Theorem 1. *The sequence* $(1+1/n)^n$ *is increasing.*

Proof. To see this, rewrite $1+(1/n)=(1+n)/n$ and divide two consecutive terms of the sequence:

$$\begin{aligned}
\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^{n-1}} &= \frac{\left(\frac{n+1}{n}\right)^n}{\left(\frac{n}{n-1}\right)^{n-1}} \\
&= \left(\frac{(n-1)(n+1)}{n^2}\right)^{n-1}\frac{n+1}{n} \\
&= \left(1-\frac{1}{n^2}\right)^{n-1}\left(1+\frac{1}{n}\right)
\end{aligned}$$

Since $(1-x)^n \geq 1-nx$, we have

$$\begin{aligned}
\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^{n-1}} &\geq \left(1-\frac{n-1}{n^2}\right)\left(1+\frac{1}{n}\right) \\
&= 1+\frac{1}{n^3} \\
&> 1,
\end{aligned}$$

hence the sequence is increasing. □

Theorem 2. *The sequence* $(1+1/n)^{n+1}$ *is decreasing.*

Proof. As before, rewrite $1+(1/n)=(1+n)/n$ and divide two consecutive terms of the sequence:

$$\begin{aligned}
\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n-1}\right)^{n}} &= \frac{\left(\frac{n+1}{n}\right)^{n+1}}{\left(\frac{n}{n-1}\right)^{n}} \\
&= \left(\frac{(n-1)(n+1)}{n^2}\right)^{n}\frac{n+1}{n} \\
&= \left(1-\frac{1}{n^2}\right)^{n}\left(1+\frac{1}{n}\right)
\end{aligned}$$

Writing $1+1/n$ as $1+n/n^2$ and applying the inequality $1+n/n^2 \leq$

$(1+1/n^2)^n$, we obtain

$$\begin{aligned}
\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n-1}\right)^{n}} &\leq \left(1-\frac{1}{n^2}\right)^n \left(1+\frac{1}{n^2}\right)^n \\
&= \left(1-\frac{1}{n^4}\right)^n \\
&< 1,
\end{aligned}$$

hence the sequence is decreasing.

□

Theorem 3. *For all positive integers m and n, we have $(1+1/m)^m < (1+1/n)^{n+1}$.*

Proof. We consider three cases.

Suppose that $m = n$. Since $n > 0$, we have $1/n > 0$ and $1 < 1 + 1/n$. Hence, $(1+1/n)^n < (1+1/n)^{n+1}$.

Suppose that $m < n$. By the previous case, $(1+1/n)^n < (1+1/n)^{n+1}$. By theorem 1, $(1+1/m)^m < (1+1/n)^n$. Combining, $(1+1/m)^m < (1+1/n)^{n+1}$.

Suppose that $m > n$. By the first case, $(1+1/m)^m < (1+1/m)^{m+1}$ By theorem 2, $(1+1/m)^{m+1} < (1+1/n)^{n+1}$. Combining, $(1+1/m)^m < (1+1/n)^{n+1}$. □