natural symmetry of the Lorenz equation

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The Lorenz equation has a natural symmetry defined by

$$(x, y, z) \mapsto (-x, -y, z). \tag{1}$$

To verify that (**??**) is a symmetry of an ordinary differential equation (Lorenz equation) there must exist a 3×3 matrix which commutes with the differential equation. This can be easily verified by observing that the symmetry is associated with the matrix R defined as

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Let

$$\dot{\mathbf{x}} = f(\mathbf{x}) = \begin{bmatrix} \sigma(y-x) \\ x(\tau - z) - y \\ xy - \beta z \end{bmatrix} \tag{3}$$

where $f(\mathbf{x})$ is the Lorenz equation and $\mathbf{x}^T = (x, y, z)$. We proceed by showing that $Rf(\mathbf{x}) = f(R\mathbf{x})$. Looking at the left hand side

$$\begin{aligned} Rf(\mathbf{x}) &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma(y-x) \\ x(\tau - z) - y \\ xy - \beta z \end{bmatrix} \\ &= \begin{bmatrix} \sigma(x-y) \\ x(z-\tau) + y \\ xy - \beta z \end{bmatrix} \end{aligned}$$

and now looking at the right hand side

$$\begin{aligned} f(R\mathbf{x}) &= f(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}) \\ &= f(\begin{bmatrix} -x \\ -y \\ z \end{bmatrix}) \\ &= \begin{bmatrix} \sigma(x-y) \\ x(z-\tau) + y \\ xy - \beta z \end{bmatrix}. \end{aligned}$$

Since the left hand side is equal to the right hand side then (**??**) is a symmetry of the Lorenz equation.