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order n constant coefficient differential equations and matrix exponential

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Let P be a degree $n > 0$ monic complex polynomial in one indeterminate, let f be a continuous function on the real line, let k be an integer varying from 0 to $n-1$, and let y_k be a complex number. The solution to the ODE

$$P(d/dt)\ y = f(t), \quad y^{(k)}(0) = y_k \tag{1}$$

is

$$y(t) = \sum\ y_k\ g_k(t) + \int_0^t g_{n-1}(t-x)\ f(x)\ dx, \tag{2}$$

where $g_k(t)$ is the coefficient of z^k in the product of $P(z)$ by the singular part of

$$\frac{e^{tz}}{P(z)} \quad .$$

Moreover, if A is a complex square matrix annihilated by P, then

$$e^{tA} = \sum\ g_k(t)\ A^k. \tag{3}$$

(**??**) into

$$Y' - B\,Y = f(t)\ v, \quad Y(0) = Y_0 \tag{4}$$

by putting $Y_k := y^{(k)}$, $Y_{0k} := y_k$, and by letting B be the transpose companion matrix of P, and v the last vector of the canonical basis of $\mathbb{C}^n$. The solution to (**??**) is

$$Y(t) = e^{tB}\ Y_0 + \int_0^t\ f(x)\ e^{(t-x)B}\ v\ dx.$$

There is a unique n-tuple of functions h_k such that e^{tA} is the sum of the $h_k(t)\,A^k$ whenever A is a complex square matrix annihilated by P. The first line of B^k being the $(k+1)$-th vector of the canonical basis of $\mathbb{C}^n$ (for $0 \leq k < n$), we obtain

$$y(t) = \sum\ y_k\ h_k(t) + \int_0^t h_{n-1}(t-x)\ f(x)\ dx,$$

so that the proof of (**??**) and (**??**) boils down to verifying

$$h_k(t) = g_k(t).$$

a real value of t, let $G \in \mathbb{C}[X]$ be the sum of the $g_k(t)\,X^k$, form the entire function

$$\varphi(z) = \frac{e^{tz} - G(z)}{P(z)} \quad ,$$

multiply the above equality by $P(z)$, and replace z by A.