Hermite equation

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The linear differential equation

$$\frac{d^2 f}{dz^2} - 2z\frac{df}{dz} + 2nf \;=\; 0,$$

in which n is a real , is called the *Hermite equation*. Its general solution is $f := Af_1{+}Bf_2$ with A and B arbitrary and the functions f_1 and f_2 presented as

$$f_1(z) \;:=\; z + \tfrac{2(1-n)}{3!}z^3 + \tfrac{2^2(1-n)(3-n)}{5!}z^5 + \tfrac{2^3(1-n)(3-n)(5-n)}{7!}z^7 + \ldots,$$

$$f_2(z) \;:=\; 1 + \tfrac{2(-n)}{2!}z^2 + \tfrac{2^2(-n)(2-n)}{4!}z^4 + \tfrac{2^3(-n)(2-n)(4-n)}{6!}z^6 + \ldots$$

It's easy to check that these power series satisfy the differential equation. The coefficients b_ν in both series obey the recurrence

$$b_\nu \;=\; \frac{2(\nu-2-n)}{\nu(nu-1)}b_{\nu\text{-}2}.$$

Thus we have the `http://planetmath.org/RadiusOfConvergence`radii of convergence

$$R \;=\; \lim_{\nu\to\infty}\left|\frac{b_{\nu-2}}{b_\nu}\right| \;=\; \lim_{\nu\to\infty}\frac{\nu}{2}\cdot\frac{1-1/\nu}{1-(n+2)/\nu} \;=\; \infty.$$

Therefore the series converge in the whole complex plane and define entire functions.

If the n is a non-negative integer, then one of f_1 and f_2 is simply a polynomial function. The polynomial solutions of the Hermite equation are usually normed so that the highest `http://planetmath.org/PolynomialRing`degree is $(2z)^n$ and called the Hermite polynomials.