exact differential equation

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Let R be a region in $\mathbb{R}^2$ and let the functions $X\colon R \to \mathbb{R}$, $Y\colon R \to \mathbb{R}$ have continuous partial derivatives in R. The first order differential equation

$$X(x,\,y) + Y(x,\,y)\frac{dy}{dx} \;=\; 0$$

or

$$X(x,\,y)dx + Y(x,\,y)dy \;=\; 0 \tag{1}$$

is called an *exact differential equation*, if the condition

$$\frac{\partial X}{\partial y} \;=\; \frac{\partial Y}{\partial x} \tag{2}$$

is true in R.

By (2), the left hand side of (1) is the total differential of a function, there is a function $f\colon R \to \mathbb{R}$ such that the equation (1) reads

$$d\,f(x,\,y) \;=\; 0,$$

whence its general integral is

$$f(x,\,y) \;=\; C.$$

The solution function f can be calculated as the line integral

$$f(x,\,y) \;:=\; \int_{P_0}^{P} [X(x,\,y)\,dx + Y(x,\,y)\,dy] \tag{3}$$

along any curve γ connecting an arbitrarily chosen point $P_0 = (x_0,\,y_0)$ and the point $P = (x,\,y)$ in the region R (the integrating factor is now $\equiv 1$).

Example. Solve the differential equation

$$\frac{2x}{y^3}\,dx + \frac{y^2-3x^2}{y^4}\,dy \;=\; 0.$$

This equation is exact, since

$$\frac{\partial}{\partial y}\frac{2x}{y^3} \;=\; -\frac{6x}{y^4} \;=\; \frac{\partial}{\partial x}\frac{y^2-3x^2}{y^4}.$$

If we use as the integrating way the broken line from $(0,\,1)$ to $(x,\,1)$ and from this to $(x,\,y)$, the integral (3) is simply

$$\int_0^x \frac{2x}{1^3}\,dx+\int_1^y \frac{y^2-3x^2}{y^4}\,dy \;=\; \frac{x^2}{y^3}-\frac{1}{y}+1 \;=\; x^2-\frac{1}{y}+\frac{x^2}{y^3}+1-x^2=\frac{x^2}{y^3}-\frac{1}{y}+1.$$

Thus we have the general integral

$$\frac{x^2}{y^3}-\frac{1}{y} \;=\; C$$

of the given differential equation.