proof of Bohr-Mollerup theorem

Canonical name	ProofOfBohrMollerupTheorem
Date of creation	2013-03-22 13:18:14
Last modified on	2013-03-22 13:18:14
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Last modified by	lieven (1075)
Numerical id	6
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Entry type	Proof
Classification	msc 33B15
Defines	Gauß's product

We prove this theorem in two stages: first, we establish that the gamma function satisfies the given conditions and then we prove that these conditions uniquely determine a function on $(0,\infty)$.

By its definition, $\Gamma(x)$ is positive for positive x. Let $x, y > 0$ and $0 \leq \lambda \leq 1$.

$$
\begin{aligned}
\log\Gamma(\lambda x + (1-\lambda)y) &= \log\int_0^\infty e^{-t}t^{\lambda x+(1-\lambda)y-1}dt \\
&= \log\int_0^\infty (e^{-t}t^{x-1})^\lambda (e^{-t}t^{y-1})^{1-\lambda}dt \\
&\leq \log((\int_0^\infty e^{-t}t^{x-1}dt)^\lambda(\int_0^\infty e^{-t}t^{y-1}dt)^{1-\lambda}) \\
&= \lambda\log\Gamma(x) + (1-\lambda)\log\Gamma(y)
\end{aligned}
$$

The inequality follows from Hölder's inequality, where $p = \frac{1}{\lambda}$ and $q = \frac{1}{1-\lambda}$.

This proves that Γ is log-convex. Condition 2 follows from the definition by applying integration by parts. Condition 3 is a trivial verification from the definition.

Now we show that the 3 conditions uniquely determine a function. By condition 2, it suffices to show that the conditions uniquely determine a function on $(0, 1)$.

Let G be a function satisfying the 3 conditions, $0 \leq x \leq 1$ and $n \in \mathbb{N}$.

$n + x = (1-x)n + x(n+1)$ and by log-convexity of G, $G(n+x) \leq G(n)^{1-x}G(n+1)^x = G(n)^{1-x}G(n)^x n^x = (n-1)!n^x$.

Similarly $n+1 = x(n+x)+(1-x)(n+1+x)$ gives $n! \leq G(n+x)(n+x)^{1-x}$.

Combining these two we get

$$n!(n+x)^{x-1} \leq G(n+x) \leq (n-1)!n^x$$

and by using condition 2 to express $G(n+x)$ in terms of $G(x)$ we find

$$a_n := \frac{n!(n+x)^{x-1}}{x(x+1)\ldots(x+n-1)} \leq G(x) \leq \frac{(n-1)!n^x}{x(x+1)\ldots(x+n-1)} =: b_n.$$

Now these inequalities hold for every positive integer n and the terms on the left and right side have a common limit ($\lim_{n\to\infty}\frac{a_n}{b_n} = 1$) so we find this determines G.

As a corollary we find another expression for Γ.

For $0 \leq x \leq 1$,

$$\Gamma(x) = \lim_{n \to \infty} \frac{n! n^x}{x(x+1)\dots(x+n)}.$$

In fact, this equation, called Gauß's product, goes for the whole complex plane minus the negative integers.