generating function of Hermite polynomials

Canonical name	GeneratingFunctionOfHermitePolynomials
Date of creation	2013-03-22 19:05:25
Last modified on	2013-03-22 19:05:25
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	9
Author	pahio (2872)
Entry type	Derivation
Classification	msc 33E30
Classification	msc 33B99
Classification	msc 26C05
Classification	msc 26A09
Classification	msc 12D99
Related topic	OrthogonalPolynomials
Related topic	ExampleOfFindingTheGeneratingFunction
Related topic	GeneratingFunctionOfLaguerrePolynomials
Related topic	VariantOfCauchyIntegralFormula

We start from the definition of Hermite polynomials via their http://planetmath.org/node/11
formula

$$H_n(z) \;:=\; (-1)^n e^{z^2}\frac{d^n}{dz^n}e^{-z^2} \qquad (n \;=\; 0,\, 1,\, 2,\, \ldots). \tag{1}$$

The consequence

$$f^{(n)}(z) \;=\; \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}\;d\zeta \tag{2}$$

of http://planetmath.org/node/1150Cauchy integral formula allows to write (1) as the complex integral

$$H_n(z) \;=\; (-1)^n\frac{n!}{2i\pi}\oint_C \frac{e^{z^2-\zeta^2}}{(\zeta-z)^{n+1}}\,d\zeta,$$

where C is any contour around the point z and the direction is anticlockwise. The http://planetmath.org/node/11373substitution $z-\zeta := t$ here yields

$$H_n(z) \;=\; \frac{n!}{2i\pi}\oint_{C'} \frac{e^{z^2-(z-t)^2}}{t^{n+1}}\,dt,$$

where the contour C' goes round the origin. Accordingly, by (2) we can infer that

$$H_n(z) \;=\; \left[\frac{d^n}{dt^n}e^{z^2-(z-t)^2}\right]_{t=0},$$

whence we have found the generating function

$$e^{z^2-(z-t)^2} \;=\; \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!}$$

of the Hermite polynomials.