applying generating function

Canonical name	ApplyingGeneratingFunction
Date of creation	2013-03-22 19:06:58
Last modified on	2013-03-22 19:06:58
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Last modified by	pahio (2872)
Numerical id	11
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Entry type	Example
Classification	msc 33B99
Classification	msc 33C45
Classification	msc 26C05
Classification	msc 26A42
Related topic	AreaUnderGaussianCurve

The generating function of a function sequence carries information common to the members of the sequence. It may be utilised for deriving various properties, such as recurrence relations, orthogonality properties etc. We take as example

$$e^{2zt-t^2} \;=\; \sum_{n=0}^\infty \frac{H_n(z)}{n!}t^n, \tag{1}$$

the http://planetmath.org/node/11980generating function of the of Hermite polynomials, and derive from it a recurrence relation and the http://planetmath.org/Orthon formula.

1. First we form the partial derivative with respect to t of both of (1):

$$(2z-2t)e^{2zt-t^2} \;=\; \sum_{m=1}^\infty \frac{H_m(z)}{(m-1)!}t^{m-1}$$

Here we substitute (1) to the left hand side and rewrite the right hand side, getting

$$\sum_{n=0}^\infty \frac{2zH_n(z)}{n!}t^n - \sum_{n=1}^\infty \frac{2H_{n-1}(z)}{(n-1)!}t^n \;=\; \sum_{n=0}^\infty \frac{H_{n+1}(z)}{n!}t^n,$$

where we can compare the coefficients of t^n:

$$\frac{2zH_n}{n!} - \frac{2H_{n-1}}{(n-1)!} \;=\; \frac{H_{n+1}}{n!} \qquad (n = 1,\,2,\,\ldots)$$

Thus we have gotten the recurrence relation

$$H_{n+1}(z) \;=\; 2zH_n(z) - 2nH_{n-1}(z) \qquad (n = 1,\,2,\,\ldots). \tag{2}$$

Differentiating (1) partially with respect to z enables respectively to find a formula expressing the derivative $H_n'(z)$ via the themselves.

2. We copy the equation (1) twice in the forms

$$\sum_{n=0}^\infty \frac{H_n(x)}{n!}t^n \;=\; e^{2xt-t^2}, \quad \sum_{n=0}^\infty \frac{H_n(x)}{n!}u^n \;=\; e^{2xu-u^2},$$

multiply these with each other and by e^{-x^2} and then integrate the obtained equation termwise over $\mathbb{R}$:

$$\begin{aligned}
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)\,dx\right)\frac{t^m u^n}{m!n!} &= \int_{-\infty}^{\infty} e^{-x^2}e^{2xt-t^2}e^{2xu-u^2}\,dx \\
&= \int_{-\infty}^{\infty} e^{2x(t+u)-t^2-u^2-x^2}\,dx \\
&= \int_{-\infty}^{\infty} e^{-[(t+u)^2-2(t+u)x+x^2]+2tu}\,dx \\
&= e^{2tu}\int_{-\infty}^{\infty} e^{-[x-(t+u)]^2}\,dx \\
&= e^{2tu}\int_{-\infty}^{\infty} e^{-y^2}\,dy \\
&= e^{2tu}\sqrt{\pi} \\
&= \sum_{\jmath=0}^{\infty}\sqrt{\pi}\frac{2^j t^j u^j}{j!} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\pi}}{n!}\cdot 2^n\delta_{mn}\right)t^m u^n
\end{aligned}$$

Thus we can infer that

$$\frac{\int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)\,dx}{m!n!} = \frac{\sqrt{\pi}}{n!}\cdot 2^n\delta_{mn},$$

which implies the orthonormality relation

$$\int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)\,dx = 2^m m!\,\delta_{mn}\sqrt{\pi}. \tag{3}$$

Cf. Hermite polynomials.