Wronskian determinant

Canonical name	WronskianDeterminant
Date of creation	2013-03-22 12:22:59
Last modified on	2013-03-22 12:22:59
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	13
Author	rspuzio (6075)
Entry type	Definition
Classification	msc 34-00
Synonym	Wronskian
Related topic	GrammianDeterminant

Given functions $f_1, f_2, \ldots, f_n$, then the *Wronskian determinant* (or simply the Wronskian) $W(f_1, f_2, f_3, \ldots, f_n)$ is the determinant of the square matrix

$$W(f_1, f_2, f_3, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1' & f_2' & f_3' & \cdots & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_n'' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

where $f^{(k)}$ indicates the kth derivative of f (not exponentiation).

The Wronskian of a set of functions F is another function, which is zero over any interval where F is linearly dependent. Just as a set of vectors is said to be linearly dependent when there exists a non-trivial linear relation between them, a set of functions $\{f_1, f_2, f_3, \ldots, f_n\}$ is also said to be dependent over an interval I when there exists a non-trivial linear relation between them, i.e.,

$$a_1 f_1(t) + a_2 f_2(t) + \cdots + a_n f_n(t) = 0$$

for some $a_1, a_2, \ldots, a_n$, not all zero, at any $t \in I$.

Therefore the Wronskian can be used to determine if functions are independent. This is useful in many situations. For example, if we wish to determine if two solutions of a second-order differential equation are independent, we may use the Wronskian.

Examples Consider the functions x^2, x, and 1. Take the Wronskian:

$$W = \begin{vmatrix} x^2 & x & 1 \\ 2x & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = -2$$

Note that W is always non-zero, so these functions are independent everywhere. Consider, however, x^2 and x:

$$W = \begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix} = x^2 - 2x^2 = -x^2$$

Here $W = 0$ only when $x = 0$. Therefore x^2 and x are independent except at $x = 0$.

Consider $2x^2+3$, x^2, and 1:

$$W = \begin{vmatrix} 2x^2+3 & x^2 & 1 \\ 4x & 2x & 0 \\ 4 & 2 & 0 \end{vmatrix} = 8x - 8x = 0$$

Here W is always zero, so these functions are always dependent. This is intuitively obvious, of course, since

$$2x^2 + 3 = 2(x^2) + 3(1)$$

Given n linearly independant functions $f_1, f_2, \ldots, f_n$, we can use the Wronskian to construct a linear differential equation whose solution space is exactly the span of these functions. Namely, if g satisfies the equation

$$W(f_1, f_2, f_3, \ldots, f_n, g) = 0,$$

then $g = a_1 f_1(t) + a_2 f_2(t) + \cdots + a_n f_n(t)$ for some choice of $a_1, a_2, \ldots, a_n$.

As a simple illustration of this, let us consider polynomials of at most second order. Such a polynomial is a linear combination of 1, x, and x^2. We have

$$W(1, x, x^2, g(x)) = \begin{vmatrix} 1 & x & x^2 & g(x) \\ 0 & 1 & 2x & g'(x) \\ 0 & 0 & 2 & g''(x) \\ 0 & 0 & 0 & g'''(x) \end{vmatrix} = 2g'''(x).$$

Hence, the equation is $g'''(x) = 0$ which indeed has exactly polynomials of degree at most two as solutions.