logarithm series

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The derivative of $\ln(1+x)$ is $\dfrac{1}{1+x}$, which can be represented as the sum of geometric series:

$$\frac{1}{1+x} \;=\; 1-x+x^2-x^3+-\ldots \qquad \text{for} \quad -1<x<1.$$

Integrating both from 0 to x gives

$$\ln(1+x) \;=\; x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+-\ldots \qquad \text{for} \quad -1<x<1. \tag{1}$$

which is valid on the whole open interval of convergence $-1<x<1$ of this power series and in for $x=1$, as one may prove.

Replacing x with $-x$ in (1) yields the series

$$\ln(1-x) \;=\; -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots \qquad \text{for} \quad -1<x<1. \tag{2}$$

Subtracting (2) from (1) gives

$$\ln\frac{1+x}{1-x} \;=\; 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots\right) \tag{3}$$

which also is true for $-1<x<1$. Here the inner function of the logarithm attains all positive real values when $0<x<1$ (its http://planetmath.org/Graph2graph is a http://planetmath.org/Hyperbola2hyperbola with http://planetmath.org/AsymptotesOf
$x=1$ and $y=-1$). Thus, in principle, the series (3) can be used for calculating any values of http://planetmath.org/NaturalLogarithm2natural logarithm. For this purpose, one could denote

$$\frac{1+x}{1-x} \;:=\; t,$$

which implies

$$x \;=\; \frac{t-1}{t+1},$$

and accordingly

$$\ln t \;=\; 2\left[\frac{t-1}{t+1}+\frac{1}{3}\left(\frac{t-1}{t+1}\right)^3+\frac{1}{5}\left(\frac{t-1}{t+1}\right)^5+\ldots\right]. \tag{4}$$

For example,

$$\ln 3 \;=\; 2\left(\frac{1}{2}+\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}+\ldots\right).$$

The convergence of (4) is the slower the greater is t.