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bound on matrix differential equation

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Suppose that A and Z are two square matrices dependent on a parameter which satisfy the differential equation

$$Z'(t) = A(t)Z(t)$$

withh initial condition $Z(0) = I$. Letting $\| \cdot \|$ denote the matrix operator norm, we will show that, if $\|A(t)\| \le C$ for some constant C when $0 \le t \le R$, then

$$\|Z(t) - I\| \le C \left(e^{Ct} - 1\right)$$

when $0 \le t \le R$.

We begin by applying the product inequality for the norm, then employing the triangle inequality (both in the sum and integral forms) after expressing Z as the integral of its derivative:

$$\begin{aligned}
\|Z'(t)\| &\le \|A(t)\| \|Z(t)\| \\
&\le C\|Z(t)\| \\
&= C \left\| I + \int_0^t ds\, Z'(s) \right\| \\
&\le C\|I\| + C \int_0^t ds\, \|Z'(s)\| \\
&\le C + C \int_0^t ds\, \|Z'(s)\|
\end{aligned}$$

For convenience, let us define $f(t) = \int_0^t ds\, \|Z'(s)\|$. Then we have $f'(t) \le C + Cf(t)$ according to the foregoing derivation. By the product rule,

$$\frac{d}{dt}\left(e^{-Ct} f(t)\right) = e^{-Ct}(f'(t) - Cf(t)).$$

Since $f'(t) - Cf(t) \le C$, we have

$$\frac{d}{dt}\left(e^{-Ct} f(t)\right) \le Ce^{-Ct}.$$

Taking the integral from 0 to t of both sides and noting that $f(0) = 0$, we have

$$e^{-Ct} f(t) \le C\left(1 - e^{-Ct}\right).$$

Multiplying both sides by e^{Ct} and recalling the definition of f, we conclude

$$\int_0^t ds\, \|Z'(s)\| \le C\left(e^{Ct} - 1\right).$$

Finally, by the triangle inequality,

$$\|Z(t) - I\| = \left\| \int_0^t ds \, Z(s) \right\| \leq \int_0^t ds \, \|Z(s)\|.$$

Combining this with the inequality derived in the last paragraph produces the answer:

$$\|Z(t) - I\| \leq C \left(e^{Ct} - 1 \right).$$