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d'Alembert and D. Bernoulli solutions of wave equation

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Let's consider the http://planetmath.org/WaveEquationd'Alembert's solution

$$u(x,\,t)\;:=\;\frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\,ds \tag{1}$$

of the wave equation in one dimension in the special case when the other initial condition is

$$u'_t(x,\,0)\;:=\;g(x)\;\equiv\;0. \tag{2}$$

We shall see that the solution is equivalent with the solution of D. Bernoulli.

We the given function f to the Fourier sine series on the interval $[0,\,p]$:

$$f(y)\;=\;\sum_{n=1}^{\infty}A_n\sin\frac{n\pi y}{p}\quad\text{with}\;\;A_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx\quad(n=1,\,2,\,\ldots)$$

Thus we may write

$$\begin{cases}f(x-ct)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{p}-\frac{n\pi ct}{p}\right)=\sum_{n=1}^{\infty}A_n\left(\sin\frac{n\pi x}{p}\cos\frac{n\pi ct}{p}-\cos\frac{n\pi x}{p}\sin\frac{n\pi ct}{p}\right),\\ f(x+ct)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{p}+\frac{n\pi ct}{p}\right)=\sum_{n=1}^{\infty}A_n\left(\sin\frac{n\pi x}{p}\cos\frac{n\pi ct}{p}+\cos\frac{n\pi x}{p}\sin\frac{n\pi ct}{p}\right).\end{cases}$$

Adding these equations and dividing by 2 yield

$$u(x,\,t)=\frac{1}{2}[f(x-ct)+f(x+ct)]=\sum_{n=1}^{\infty}A_n\cos\frac{n\pi ct}{p}\sin\frac{n\pi x}{p}, \tag{3}$$

which indeed is the http://planetmath.org/SolvingTheWaveEquationByDBernoullisolution of D. Bernoulli in the case $g(x)\equiv 0$.

Note. The solution (3) of the wave equation is especially in the special case where one has besides (2) the sine-formed initial condition

$$u(x,\,0)\;:=\;f(x)\;\equiv\;\sin\frac{\pi x}{p}. \tag{4}$$

Then $A_n=0$ for every n except 1, and one obtains

$$u(x,\,t)\;=\cos\frac{\pi ct}{p}\sin\frac{\pi x}{p}\,. \tag{5}$$

Remark. In the case of quantum systems one has http://planetmath.org/SchrodingersWave wave equation whose solutions are different from the above.