telegraph equation

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Both the electric voltage and the satisfy the *telegraph equation*

$$f''_{xx} - af''_{tt} - bf'_t - cf = 0, \tag{1}$$

where x is distance, t is time and a, b, c are non-negative constants. The equation is a generalised form of the wave equation.

If the initial conditions are $f(x,\,0) = f'_t(x,\,0) = 0$ and the boundary conditions $f(0,\,t) = g(t)$, $f(\infty,\,t) = 0$, then the Laplace transform of the solution function $f(x,\,t)$ is

$$F(x,\,s) = G(s)e^{-x\sqrt{as^2+bs+c}}. \tag{2}$$

In the special case $b^2 - 4ac = 0$, the solution is

$$f(x,\,t) = e^{-\frac{bx}{2\sqrt{a}}}g(t - x\sqrt{a})H(t - x\sqrt{a}). \tag{3}$$

Justification of (2). Transforming the partial differential equation (1) (x may be regarded as a parametre) gives

$$F''_{xx}(x,\,s) - a[s^2F(x,\,s) - sf(x,\,0) - f'_t(x,\,0)] - b[sF(x,\,s) - f(x,\,0)] - cF(x,\,s) = 0,$$

which due to the initial conditions simplifies to

$$F''_{xx}(x,\,s) = (\underbrace{as^2 + bs + c}_{K^2})F(x,\,s).$$

The solution of this ordinary differential equation is

$$F(x,\,s) = C_1e^{Kx} + C_2e^{-Kx}.$$

Using the latter boundary condition, we see that

$$F(\infty,\,s) = \int_0^\infty e^{-st}f(\infty,\,t)\,dt \equiv 0,$$

whence $C_1 = 0$. Thus the former boundary condition implies

$$C_2 = F(0,\,s) = \mathcal{L}\{g(t)\} = G(s).$$

So we obtain the equation (2).

Justification of (3). When the `http://planetmath.org/QuadraticFormula`discriminant of the quadratic equation $as^2+bs+c=0$ vanishes, the `http://planetmath.org/Equation`roots coincide to $s=-\frac{b}{2a}$, and $as^2+bs+c=a(s+\frac{b}{2a})^2$. Therefore (2) reads

$$F(x,\,s)=G(s)a^{-x\sqrt{a}(s+\frac{b}{2a})}=e^{-\frac{bx}{2\sqrt{a}}}e^{-x\sqrt{a}s}G(s).$$

According to the delay theorem, we have

$$\mathcal{L}^{-1}\{e^{-ks}G(s)\}=g(t-k)H(t-k).$$

Thus we obtain for $\mathcal{L}^{-1}\{F(x,\,s)\}$ the expression of (3).