time-dependent example of heat equation

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The initial temperature (at $t = 0$) of a plate

$$A = \{(x,\, y) \in \mathbb{R}^2\colon\ 0 < x < a,\ 0 < y < b\}$$

in the xy-plane is given by the function $f = f(x,\, y)$. The faces of the plate are supposed completely isolating. After the $t = 0$ the boundaries of A are held in the temperature 0. Determine the temperature function

$$u = u(x,\, y,\, t)$$

on A (where t is the time).

Since it's a question of a two-dimensional heat , the heat equation gets the form

$$\nabla^2 u \equiv u''_{xx} + u''_{yy} = \frac{1}{c^2} u'_t. \tag{1}$$

One have to find for (1) a solution function u which satisfies the initial condition

$$u(x,\, y,\, 0) = f(x,\, y) \ \text{ in } \ A \tag{2}$$

and the boundary condition

$$u(x,\, y,\, t) = 0 \ \text{ on boundary of } \ A \ \text{ for } t > 0. \tag{3}$$

For finding a simple solution of the differential equation (1) we try the form

$$u(x,\, y,\, t) := X(x)Y(y)T(t), \tag{4}$$

whence the boundary condition reads

$$X(0) = Y(0) = X(a) = Y(b) = 0. \tag{5}$$

Substituting (4) in (1) and dividing this equation by XYT give the form

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2} \cdot \frac{T'}{T}. \tag{6}$$

It's easily understood that such a condition requires that the both addends of the left side and the right side ought to be constants:

$$\frac{X''}{X} = -k_1^2,\ \frac{Y''}{Y} = -k_2^2,\ \frac{1}{c^2} \cdot \frac{T'}{T} = -k^2, \tag{7}$$

where $k^2 = k_1^2 + k_2^2$. We soon explain why these constants are negative. Because the equations (7) may be written

$$X'' = -k_1^2 X, \quad Y'' = -k_2^2 Y, \quad T' = -k^2 c^2 T,$$

the general solutions of these ordinary differential equations are

$$\begin{cases} X = C_1 \cos k_1 x + D_1 \sin k_1 x, \\ Y = C_2 \cos k_2 y + D_2 \sin k_2 y, \\ T = Ce^{-k^2 c^2 t}. \end{cases} \tag{8}$$

Now we remark that if the right side of the third equation (7) were $+k^2$, then we had $T = Ce^{k^2 c^2 t}$ which is impossible, since such a T and along with this also the temperature $u = XYT$ would ascend infinitely when $t \to \infty$. And since, by symmetry, the right sides the two first equations (7) must have the same sign, also they must by (6) be negative.

The two first boundary conditions (5) imply by (8) that $C_1 = C_2 = 0$, and then the two last conditions (5) require that

$$D_1 \sin k_1 a = 0, \quad D_2 \sin k_2 b = 0.$$

If we had $D_1 = 0$ or $D_2 = 0$, then X or Y would vanish identically, which cannot occur. Thus we have

$$\sin k_1 a = 0 \text{ and } \sin k_2 b = 0,$$

whence only the eigenvalues

$$\begin{cases} k_1 = \frac{m\pi}{a} \quad (m = 1,\, 2,\, 3,\, \ldots) \\ k_2 = \frac{n\pi}{b} \quad (n = 1,\, 2,\, 3,\, \ldots) \end{cases}$$

are possible for the obtained X and Y. Considering the equation $k^2 = k_1^2 + k_2^2$ we may denote

$$q_{mn} := k^2 c^2 = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] c^2 \tag{9}$$

for all $m,\, n \in \mathbb{Z}_+$.

Altogether we have infinitely many solutions

$$u_{mn} = XYT = CD_1 D_2 e^{-q_{mn} t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = c_{mn} e^{-q_{mn} t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

of the equation (1), where the coefficients c_{mn} are, for the present, arbitrary constants. These solutions fulfil the boundary condition (3). The sum of the solutions, i.e. the double series

$$u(x,\, y,\, t) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} e^{-q_{mn}t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \tag{10}$$

provided it converges, is also a solution of the linear differential equation (1) and fulfils the boundary condition. In order to fulfil also the initial condition (2), one must have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} e^{-q_{mn}t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = f(x,\, y).$$

But this equation presents the Fourier double sine series of $f(x,\, y)$ in the rectangle A, and therefore we have the expression

$$c_{mn} := \frac{4}{ab} \int_0^a \int_0^b f(x,\, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy \tag{11}$$

for the coefficients.

The result of calculating the solution of our problem is the temperature function (10) with the formulae (9) and (11).

References

[1] K. Väisälä: *Matematiikka IV.* Hand-out Nr. 141. Teknillisen korkeakoulun ylioppilaskunta, Otaniemi, Finland (1967).