using Laplace transform to solve heat equation

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Along the whole positive x-axis, we have an heat-conducting rod, the surface of which is . The initial temperature of the rod is 0 . Determine the temperature function $u(x,\,t)$ when at the time $t=0$

(a) the head $x=0$ of the rod is set permanently to the constant temperature;

(b) through the head $x=0$ one directs a constant heat flux.

The heat equation in one dimension reads

$$u''_{xx}(x,\,t) \;=\; \frac{1}{c^2}\,u'_t(x,\,t). \tag{1}$$

In this we have

$$\text{(a)}\;\begin{cases}\text{boundary conditions} & u(\infty,\,t)=0, \qquad u(0,\,t)=u_0,\\ \text{initial conditions} & u(x,\,0)=0, \qquad \underbrace{u'_t(x,\,0)=0}_{\text{for } x>0}\end{cases}$$

and

$$\text{(b)}\;\begin{cases}\text{boundary conditions} & u(\infty,\,t)=0, \quad u'_x(0,\,t)=-k,\\ \text{initial conditions} & u(x,\,0)=0, \quad \underbrace{u'_t(x,\,0)=0}_{\text{for } x>0}.\end{cases}$$

For solving (1), we first form its Laplace transform (see the table of Laplace transforms)

$$U''_{xx}(x,\,s) \;=\; \frac{1}{c^2}[s\,U(x,\,s)-u(x,\,0)],$$

which is a ordinary linear differential equation

$$U''_{xx}(x,\,s) \;=\; \left(\frac{\sqrt{s}}{c}\right)^2 U(x,\,s)$$

of `http://planetmath.org/ODE`order two. Here, s is only a parametre, and the general solution of the equation is

$$U(x,\,s) \;=\; C_1e^{\frac{\sqrt{s}}{c}x}+C_2e^{-\frac{\sqrt{s}}{c}x}$$

(see `http://planetmath.org/SecondOrderLinearODEWithConstantCoefficients`this entry). Since

$$U(\infty,\,s)\;=\;\int_0^\infty e^{-st}u(\infty,\,t)\,dt\;=\;\int_0^\infty 0\,dt\;\equiv\;0,$$

we must have $C_1 = 0$. Thus the Laplace transform of the solution of (1) is in both cases (a) and (b)

$$U(x,\,s)\;=\;C_2e^{-\frac{\sqrt{s}}{c}x}. \tag{2}$$

For (a), the second boundary condition implies $U(0,\,s) = \dfrac{u_0}{s}$. But by (2) we must have $U(0,\,s) = C_2\!\cdot\!1$, whence we infer that $C_2 = \dfrac{u_0}{s}$. Accordingly,

$$U(x,\,s)\;=\;u_0\cdot\frac{1}{s}e^{-\frac{x}{c}\sqrt{s}},$$

which corresponds to the solution function

$$u(x,\,t)\;:=\;u_0\;\mathrm{erfc}\frac{x}{2c\sqrt{t}}$$

of the heat equation (1).

For (b), the second boundary condition says that $U'_x(0,\,s) = -\dfrac{k}{s}$, and since (2) implies that $U'_x(x,\,s) = -\frac{\sqrt{s}}{c}C_2e^{-\frac{\sqrt{s}}{c}x}$, we can infer that now

$$C_2\;=\;\frac{ck}{s\sqrt{s}}.$$

Thus

$$U(x,\,s)\;=\;\frac{ck}{s\sqrt{s}}e^{-\frac{x}{c}\sqrt{s}},$$

which corresponds to

$$u(x,\,t)\;:=\;k\left[2c\sqrt{\frac{t}{\pi}}e^{-\frac{x^2}{4c^2t}}-x\;\mathrm{erfc}\frac{x}{2c\sqrt{t}}\right].$$