Dirac equation

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The Dirac equation is an equation derived by Paul Dirac in 1927 that describes relativistic spin 1/2 particles (fermions). It is given by:

$$(\gamma^\mu \partial_\mu - im)\psi = 0$$

The Einstein summation convention is used.

0.1 Derivation

Mathematically, it is interesting as one of the first uses of the spinor calculus in mathematical physics. Dirac began with the relativistic equation of total energy:

$$E = \sqrt{p^2c^2 + m^2c^4}$$

As Schrödinger had done before him, Dirac then replaced p with its quantum mechanical operator, $\hat{p} \Rightarrow i\hbar\nabla$. Since he was looking for a Lorentz-invariant equation, he replaced ∇ with the D'Alembertian or wave operator

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Note that some authors use $\Box^2$ for the D'Alembertian. Dirac was now faced with the problem of how to take the square root of an expression containing a differential operator. He proceeded to factorise the d'Alembertian as follows:

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = (a^0\frac{\partial}{\partial x} + a^1\frac{\partial}{\partial y} + a^2\frac{\partial}{\partial z} + a^3\frac{i}{c}\frac{\partial}{\partial t})^2$$

Multiplying this out, we find that:

$$(a^0)^2 = (a^1)^2 = (a^2)^2 = (a^3)^2 = 1$$

And

$$a^0a^1+a^1a^0 = a^0a^2+a^2a^0 = a^0a^3+a^3a^0 = a^1a^2+a^2a^1 = a^1a^3+a^3a^1 = a^2a^3+a^3a^2 = 0$$

Clearly these relations cannot be satisfied by scalars, so Dirac sought a set of four matrices which satisfy these relations. These are now known as the Dirac matrices, and are given as follows:

$$\gamma^0 = -ia^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = -ia^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = -ia^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = a^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

These matrices are also known as the generators of the special unitary group of order 4, i.e. the group of 4×4 matrices with unit determinant. Using these matrices, and switching to natural units ($\hbar = c = 1$) we can now obtain the Dirac equation:

$$(\gamma^\mu \partial_\mu - im)\psi = 0$$

0.2 Feynman slash notation

Richard Feynman developed the following convenient notation for terms involving Dirac matrices:

$$\gamma^\mu q_\mu := \not{q}$$

Using this notation, the Dirac equation is simply

$$(\not{\partial} - im)\psi = 0$$

0.3 Relationship with Pauli matrices

The Dirac matrices can be written more concisely as matrices of Pauli matrices, as follows:

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$$

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$