method of integrating factors

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The *method of integrating factors* is in principle a means for solving ordinary differential equations of first . It has not great practical significance, but is theoretically important.

Let us consider a differential equation solved for the derivative y' of the unknown function and write the equation in the form

$$X(x,\,y)\,dx+Y(x,\,y)\,dy \;=\; 0. \tag{1}$$

We assume that the functions X and Y have continuous partial derivatives in a region R of $\mathbb{R}^2$.

If there is a solution of (1) which may be expressed in the form

$$f(x,\,y) \;=\; C$$

with f having continuous partial derivatives in R and with C an arbitrary constant, then it's not difficult to see that such an f satisfies the linear partial differential equation

$$X\frac{\partial f}{\partial y}-Y\frac{\partial f}{\partial x} \;=\; 0. \tag{2}$$

Conversely, every non-constant solution f of (2) gives also a solution $f(x,\,y) = C$ of (1). Thus, solving (1) and solving (2) are `http://planetmath.org/Equivalent3`equivalent tasks.

It's straightforward to show that if $f_0(x,\,y)$ is a non-constant solution of the equation (2), then all solutions of this equation are $F(f_0(x,\,y))$ where F is a freely chosen function with (mostly) continuous derivative.

The connection of the equations (1) and (2) may be presented also in another form. Suppose that $f(x,\,y) = C$ is any solution of (1). Then (2) implies the proportion equation

$$\frac{f'_x}{X} \;=\; \frac{f'_y}{Y}.$$

If we denote the common value of these two ratios by $\mu(x,\,y) = \mu$, then we have

$$f'_x \;=\; \mu X, \qquad f'_y \;=\; \mu Y.$$

This gives to the differential of the function f the expression

$$d\,f(x,\,y) \;=\; \mu(x,\,y)(X(x,\,y)\,dx+Y(x,\,y)\,dy).$$

We see that $\mu(x,\,y)$ is the *integrating factor* or *Euler multiplicator* of the given differential equation (1), i.e. the left hand side of (1) turns, when multiplied by $\mu(x,\,y)$, to an `http://planetmath.org/ExactDifferentialForm`exact differential.

Conversely, any integrating factor μ of (1), i.e. such that $\mu X\,dx+\mu Y\,dy$ is the differential of some function f, is easily seen to determine the solutions of the form $f(x,\,y)=C$ of (1). Altogether, solving the differential equation (1) is equivalent with finding an integrating factor of the equation.

When an integrating factor μ of (1) is available, the solution function f can be gotten from the line integral

$$f(x,\,y)\;=:\;\int_{P_0}^{P}[\mu(x,\,y)X(x,\,y)\,dx+\mu(x,\,y)Y(x,\,y)\,dy]$$

along any curve γ connecting an arbitrarily chosen point $P_0=(x_0,\,y_0)$ and the point $P=(x,\,y)$ in the region R.

Note. In general, it's very hard to find a suitable integrating factor. One special case where such can be found, is that X and Y are homogeneous functions of same `http://planetmath.org/HomogeneousFunction`degree: then the expression $\dfrac{1}{xX+yY}$ is an integrating factor.

Example. In the differential equation

$$(x^4+y^4)\,dx-xy^3\,dy\;=\;0$$

we see that $X=:x^4+y^4$ and $Y=:-xy^3$ both define a `http://planetmath.org/HomogeneousFunc` function of degree 4. Thus we have the integrating factor $\mu=:\dfrac{1}{x^5+xy^4-xy^4}=\dfrac{1}{x^5}$, and the left hand side of the equation

$$\left(\frac{1}{x}+\frac{y^4}{x^5}\right)\,dx-\frac{y^3}{x^4}\,dy\;=\;0$$

is an exact differential. We can integrate it along the broken line, first from $(1,\,0)$ to $(x,\,0)$ and then still to $(x,\,y)$, obtaining

$$f(x,\,y)\;=:\;\int_1^x\left(\frac{1}{x}+\frac{0^4}{x^5}\right)\,dx-\int_0^y\frac{y^3\,dy}{x^4}\;=\;\ln|x|-\frac{y^4}{4x^4}.$$

So the general solution of the given differential equation is

$$\ln|x| - \frac{y^4}{4x^4} \;=\; C.$$

References

[1] E. Lindelöf: *Differentiali- ja integralilasku III 1.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1935).