derivation of wave equation from Maxwell's equations

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Maxwell was the first to note that Ampère's Law does not satisfy conservation of charge (his corrected form is given in Maxwell's equation). This can be shown using the equation of conservation of electric charge:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

Now consider Faraday's Law in differential form:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Taking the curl of both sides:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (-\frac{\partial \mathbf{B}}{\partial t})$$

The right-hand side may be simplified by noting that

$$\nabla \times (\frac{\partial \mathbf{B}}{\partial t}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

Recalling Ampère's Law,

$$-\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Therefore

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

The left hand side may be simplified by the following vector identity:

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}$$

Hence

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Applying the same analysis to Ampére's Law then substituting in Faraday's Law leads to the result

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Making the substitution $\mu_0 \epsilon_0 = 1/c^2$ we note that these equations take the form of a transverse wave travelling at constant speed c. Maxwell evaluated the constants μ_0 and ϵ_0 according to their known values at the time and concluded that c was approximately equal to 310,740,000 ms^{-1}, a value within 3% of today's results!