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Korteweg - de Vries equation

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The *Korteweg - de Vries equation* is a partial differential equation defined as

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

where $u = u(x,t)$ and the subscripts indicate derivatives. This equation arises in hydrodynamics and was originally proposed to model waves in a canal. In addition to its practical applications, this equation is quite interesting as an object of mathematical study. It exhibits interesting soliton solutions, has a large algebra of conserved quantities, and can be solved using methods of inverse scattering.

1 Travelling Wave Solution

It is easy to exhibit a solution which describes a traveling wave. To find this solution, one substitutes the ansatz $u(x,t) = f(x - ct)$ [1] into the equation to obtain the following equation for f:

$$-cf' + 6ff' + f''' = 0$$

This equation can be written as

$$(-cf + 3f^2 + f'')' = 0$$

and, hence,

$$-cf + 3f^2 + f'' = k$$

for some constant k. Multiplying by f', we can repeat the same trick:

$$-cff' + 3f^2f' + f'f'' - kf' = (-cf^2/2 + f^3 + (f')^2 - kf)' = 0$$

hence

$$-cf^2/2 + f^3 + (f')^2 - kf = h$$

or

$$f' = \sqrt{kf + cf^2/2 - f^3}$$

This can be solved implicitly by an integral:

$$y = \int^{f(y)} \sqrt{kf + cf^2/2 - f^3}\, dy + C$$

[1]The most general form of a solution to the two-dimensional wave equation is $u(x,t) = f(x - ct) + f(x + ct)$ which describes waves propagating in both directions with velocity c. Here, we will look for a solution which only describes a wave propagatng in one direction.

Since this is an elliptic integral, the result is an elliptic function.

The solution obtained above is know as a *solitary wave*, or *soliton*. This term "solitary wave" refers to the fact that this solution describes a single wave pulse traveling with velocity c. Note that the amplitude of the pulse is determined by its velocity, unlike in the case of linear wave equations where the velocity of propagation does not depend upon the amplitude. There are also solutions which describe more than one solitary wave. In particular, there are solutions in which two of these waves collide and then re-emerge from the collision.

2 Conserved Currents

It is possible to exhibit integrals whose value is conserved under the evolution. To construct these quantities, we begin with vector fields. Suppose that u is a thrice differentiable function of x and t and consider the following vector fields:

$$\begin{aligned} v^{(1)} &= (u, 3u^2 + u_{xx}) \\ v^{(2)} &= (u^2, 4u^3 - u_x^2 + 2uu_{xx}) \end{aligned}$$

Computing their divergences and rearranging the result,

$$\begin{aligned} \frac{\partial v_t^{(1)}}{\partial t} + \frac{\partial v_t^{(1)}}{\partial x} &= u_t + 6uu_x + u_{xxx} \\ \frac{\partial v_t^{(2)}}{\partial t} + \frac{\partial v_t^{(2)}}{\partial x} &= 2u(u_t + 6uu_x + u_{xxx}). \end{aligned}$$

If u happens to satisfy our differential equation, then the divergence of these vector fields will go zero.

To obtain the conserved quantities, we will integrate them over a rectangle in the t, x plane with sides parallel to the x and t axes use Green's theorem .

$$
\begin{aligned}
\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} & dx\,(u_t(t,x) + 6u(t,x)u_x(t,x) + u_{xxx}(t,x)) \\
&= \int_{x_1}^{x_2} dx\,(3u^2(x,t_2) + u_{xx}(x,t_2)) + \int_{t_2}^{t_1} dt\, u(t,x_2) + \\
&\quad \int_{x_1}^{x_2} dx\,(3u^2(x,t_1) + u_{xx}(x,t_1)) + \int_{t_1}^{t_2} dt\, u(t,x_1) \\
&= 3\int_{x_1}^{x_2} dx\, u^2(x,t_2) - 3\int_{x_1}^{x_2} dx\, u^2(x,t_1) \\
&\quad + \int_{t_1}^{t_2} dt\, u(t,x_1) - \int_{t_1}^{t_2} dt\, u(t,x_2) + u_x(x_2,t_2) - u_x(x_1,t_2) - u_x(x_2,t_1) + u_x(x_1,t_1)
\end{aligned}
$$

$$
\begin{aligned}
\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} & dx\, 2u(u_t(t,x) + 6u(t,x)u_x(t,x) + u_{xxx}(t,x)) \\
&= \int_{x_1}^{x_2} dx\,(4u^3(x,t_2) - u_x^2(x,t_2) + 2u(x,t_2)u_{xx}(x,t_2)) + \int_{t_2}^{t_1} dt\, u^2(t,x_2) + \\
&\quad \int_{x_2}^{x_2} dx\,(4u^3(x,t_1) - u_x^2(x,t_1) + 2u(x,t_1)u_{xx}(x,t_1)) + \int_{t_1}^{t_2} dt\, u^2(t,x_1) \\
&= \int_{x_1}^{x_2} dx\,(4u^3(x,t_2) - 3u_x^2(x,t_2)) - \int_{x_1}^{x_2} dx\,(4u^3(x,t_1) - 3u_x^2(x,t_1)) \\
&\quad + \int_{t_1}^{t_2} dt\, u^2(t,x_1) - \int_{t_1}^{t_2} dt\, u^2(t,x_2) \\
&\quad + u(x_2,t_2)u_x(x_2,t_2) - u(x_1,t_2)u_x(x_1,t_2) - u(x_2,t_1)u_x(x_2,t_1) + u(x_1,t_1)u_x(x_1,t_1)
\end{aligned}
$$

We consider now several popular boundary conditions for our equation:

- **Periodic** Suppose that u is periodic in x with period p and satisfies the Kortwieg - de Vries equation. Then we integrals over t cancel against each other as do the endpoint terms, leaving us with

$$\int_{x_1}^{x_2} dx\, u^2(x,t_1) = \int_{x_1}^{x_2} dx\, u^2(x,t_2).$$

- **Whole line** Suppose that u satisfies the Kortwieg - de Vries equation and that u and its derivatives tend towards zero as $x \to \pm\infty$. Then, taking the limit as $x_1 \to -\infty$ and $x_2 \to +\infty$, we conclude that

$$\int_{-\infty}^{+\infty} dx\, u^2(x, t_1) = \int_{-\infty}^{+\infty} dx\, u^2(x, t_2).$$

- Finite interval We may also consider our differential equation on a finite interval and impose suitable boundary conditions at the endpoints. In this case, the integrals with respect to t do not cancel or automatically vanish as a consequence of the boundary conditions, so they will not, in general, give conserved quantities. Nevertheless, it is at least still possible to compute their value solely from the boundary data without having to know how the solution behaves in the interior.

Starting with the two vector fields given above, it is possible to generate more such vector fields and more conserved quantities. Since the Lie bracket of two vector fields with zero divergence also has zero divergence, we may