Green's function

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Some general preliminary considerations

Let (Ω, μ) be a bounded measure space and $\mathcal{F}(\Omega)$ be a linear function space of bounded functions defined on Ω, i.e. $\mathcal{F}(\Omega) \subset \mathcal{L}^\infty(\Omega)$. We would like to note two types of functionals from the dual space $(\mathcal{F}(\Omega))^*$, which will be used here:

1. Each function $g(x) \in \mathcal{L}^1(\Omega)$ defines a functional $\varphi \in (\mathcal{F}(\Omega))^*$ in the following way:
$$\varphi(f) = \int_\Omega g(x)\, f(x)\, d\mu.$$
Such functional we will call *regular* functional and function g — its *generator*.

2. For each $x \in \Omega$, we will consider a functional $\delta_x \in (\mathcal{F}(\Omega))^*$ defined as follows:
$$\delta_x(f) = f(x). \tag{1}$$
Since generally, we can not speak about values at the point for functions from $(L)^\infty$, in the following, we assume some regularity for functions from considered spaces, so that (**??**) is correctly defined.

Necessary notations and motivation

Let (Ω_x, μ_x), (Ω_y, μ_y) be some bounded measure spaces; $\mathcal{F}(\Omega_x), \mathcal{G}(\Omega_y)$ be some linear function spaces. Let $A : \mathcal{F}(\Omega_x) \to \mathcal{G}(\Omega_y)$ be a linear operator which has a well-defined inverse $A^{-1} : \mathcal{G}(\Omega_y) \to \mathcal{F}(\Omega_x)$.

Consider an operator equation:

$$Af = g \tag{2}$$

where $f \in \mathcal{F}(\Omega_x)$ is unknown and $g \in \mathcal{G}(\Omega_y)$ is given. We are interested to have an integral representation for solution of (**??**). For this purpose we write:
$$f(x) = \delta_x(f) = \delta_x(A^{-1}(g)) = [\,(A^{-1})^*\delta_x\,](g).$$

Definition of Green's function

If $\forall x \in \Omega_x$ the functional $(A^{-1})^* \delta_x$ is regular with generator $G(\cdot, y) \in \mathcal{L}^1(\Omega_y)$, then G is called **Green's function of operator** A and solution of (**??**) admits the following integral representation:

$$f(x) = \int\limits_{\Omega_y} G(x, y)\, g(y)\, d\mu_y$$