# solving the wave equation due to D. Bernoulli

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A string has been strained between the points $(0, 0)$ and $(p, 0)$ of the $x$-axis. The vibration of the string in the $xy$-plane is determined by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2} \tag{1}$$

satisfied by the ordinates $u(x, t)$ of the points of the string with the abscissa $x$ on the time $t \,(\geqq 0)$. The boundary conditions are thus

$$u(0, t) = u(p, t) = 0.$$

We suppose also the initial conditions

$$u(x, 0) = f(x), \quad u'_t(x, 0) = g(x)$$

which give the initial position of the string and the initial velocity of the points of the string.

For trying to separate the variables, set

$$u(x, t) := X(x)T(t).$$

The boundary conditions are then $X(0) = X(p) = 0$, and the partial differential equation (1) may be written

$$c^2 \cdot \frac{X''}{X} = \frac{T''}{T}. \tag{2}$$

This is not possible unless both sides are equal to a same constant $-k^2$ where $k$ is positive; we soon justify why the constant must be negative. Thus (2) splits into two ordinary linear differential equations of second order:

$$X'' = -\left(\frac{k}{c}\right)^2 X, \quad T'' = -k^2 T \tag{3}$$

The solutions of these are, as is well known,

$$\begin{cases} X = C_1 \cos \frac{kx}{c} + C_2 \sin \frac{kx}{c} \\ T = D_1 \cos kt + D_2 \sin kt \end{cases} \tag{4}$$

with integration constants $C_i$ and $D_i$.

But if we had set both sides of (2) equal to $+k^2$, we had got the solution $T = D_1 e^{kt} + D_2 e^{-kt}$ which can not present a vibration. Equally impossible would be that $k = 0$.

Now the boundary condition for $X(0)$ shows in (4) that $C_1 = 0$, and the one for $X(p)$ that

$$C_2 \sin \frac{kp}{c} = 0.$$

If one had $C_2 = 0$, then $X(x)$ were identically 0 which is naturally impossible. So we must have

$$\sin \frac{kp}{c} = 0,$$

which implies

$$\frac{kp}{c} = n\pi \quad (n \in \mathbb{Z}_+).$$

This means that the only suitable values of $k$ satisfying the equations (3), the so-called eigenvalues, are

$$k = \frac{n\pi c}{p} \quad (n = 1,\, 2,\, 3,\, \ldots).$$

So we have infinitely many solutions of (1), the eigenfunctions

$$u = XT = C_2 \sin \frac{n\pi}{p} x \left[ D_1 \cos \frac{n\pi c}{p} t + D_2 \sin \frac{n\pi c}{p} t \right]$$

or

$$u = \left[ A_n \cos \frac{n\pi c}{p} t + B_n \sin \frac{n\pi c}{p} t \right] \sin \frac{n\pi}{p} x$$

$(n = 1,\, 2,\, 3,\, \ldots)$ where $A_n$'s and $B_n$'s are for the time being arbitrary constants. Each of these functions satisfy the boundary conditions. Because of the linearity of (1), also their sum series

$$u(x,\, t) := \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi c}{p} t + B_n \sin \frac{n\pi c}{p} t \right) \sin \frac{n\pi}{p} x \tag{5}$$

is a solution of (1), provided it converges. It fulfils the boundary conditions, too. In order to also the initial conditions would be fulfilled, one must have

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{p} x = f(x),$$

$$\sum_{n=1}^{\infty} B_n \frac{n\pi c}{p} \sin \frac{n\pi}{p} x = g(x)$$

on the interval $[0,\, p]$. But the left sides of these equations are the Fourier sine series of the functions $f$ and $g$, and therefore we obtain the expressions for the coefficients:

$$A_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi x}{p}\, dx,$$

$$B_n = \frac{2}{n\pi c} \int_0^p g(x) \sin \frac{n\pi x}{p}\, dx.$$

# References


[1] K. Väisälä: *Matematiikka IV.* Hand-out Nr. 141. Teknillisen korkeakoulun ylioppilaskunta, Otaniemi, Finland (1967).