vibrating string with variable density

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The unidimensional wave's problem may be stated as

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad x \in (0,l), \quad t > 0,$$

with initial conditions

$$\begin{cases} u(x,0) = f(x), \\ \frac{\partial u}{\partial t}(x,0) = g(x), \end{cases}$$

and boundary conditions

$$\begin{cases} u(0,t) = 0, \\ u(l,t) = 0, \end{cases}$$

which may be specialized to a string's motion if we physically interpret $c^2(x) = T_0/\rho(x)$ as the ratio between the string's initial tension and its linear density. We will discuss the free string's vibrations (i.e. $f(x,t) \equiv 0$) given by the string's problem

$$\frac{\partial^2 u}{\partial t^2} - (1+x)^2\frac{\partial^2 u}{\partial x^2} = 0, \qquad x \in (0,1), \quad t > 0, \tag{1}$$

initial conditions

$$\begin{cases} u(x,0) = f(x), & \text{initial string's form,} \\ \frac{\partial u}{\partial t}(x,0) = 0, & \text{string starts from the rest,} \end{cases}$$

and boundary conditions

$$\begin{cases} u(0,t) = 0, & \text{string's left end fixed,} \\ u(1,t) = 0, & \text{string's right end fixed.} \end{cases}$$

Without loss of generality, we assume unitary the natural undeformed string's length. The solution of this problem approaches to a string's motion whose linear density is proportional to $(1+x)^{-2}$. The *method of separation of variables* (i.e. $u(x,t) = X(x)T(t)$) gives the equations

$$X'' + \frac{\lambda}{(1+x)^2}X = 0, \tag{2}$$

with boundary conditions $X(0) = X(1) = 0$, and

$$T'' + \lambda T = 0, \tag{3}$$

with initial conditions $T(0) = 1$, $T'(0) = 0$. In these equations, λ is a constant parameter.
In (2), we are dealing with a Sturm-Liouville problem. To find out the eigenvalues, one searches the solution of (2) on the form $X(x) = (1+x)^a$, as we realize that (2) outcomes the associated characteristic equation

$$a(a-1)+\lambda = 0. \qquad \text{That is,} \qquad a = \frac{1}{2}(1 \pm \sqrt{1-4\lambda}).$$

In order to satisfying $X(0) = 0$, let us choose

$$X(x) = (1+x)^{\frac{1}{2}(1+\sqrt{1-4\lambda})} - (1+x)^{\frac{1}{2}(1-\sqrt{1-4\lambda})}.$$

Thus, the boundary condition $X(1) = 0$ becomes

$$2^{\frac{1}{2}(1+\sqrt{1-4\lambda})} - 2^{\frac{1}{2}(1-\sqrt{1-4\lambda})} = 0, \quad \text{or} \quad 2^{\sqrt{1-4\lambda}} = 1.$$

We next study all the possible cases for the eigenvalue λ in the last above equation.

1. $\lambda < 1/4$. Then $\sqrt{1-4\lambda}$ is real, and the equation does not have solution.

2. $\lambda = 1/4$. Then the pair of solutions, above indicated, will not be independent. Indeed the functions $(1+x)^{1/2}$ and $(1+x)^{1/2}\log(1+x)$ are linearly independent solutions of (2), in $(0,1)$. Nevertheless, although the last one satisfies the boundary condition at $x = 0$, it does not vanish at $x = 1$. Hence, $\lambda = 1/4$ is not an eigenvalue.

3. $\lambda > 1/4$. Then $\sqrt{1-4\lambda}$ is imaginary. We may even get two solutions by setting $(i = \sqrt{-1})$

$$Z(x) = X_1(x) + iX_2(x) = (1+x)^{\frac{1}{2}(1+i\sqrt{4\lambda-1})} = (1+x)^{\frac{1}{2}} e^{\frac{1}{2}i\sqrt{4\lambda-1}\log(1+x)}$$

$$= (1+x)^{\frac{1}{2}} \left\{ \cos\left(\sqrt{\lambda - \frac{1}{4}}\log(1+x)\right) + i\sin\left(\sqrt{\lambda-\frac{1}{4}}\log(1+x)\right)\right\},$$

being the real and imaginary parts of $Z(x)$ two linearly independent solutions. For satisfying $X(0) = 0$ one sets

$$X(x) = (1+x)^{\frac{1}{2}} \sin\left(\sqrt{\lambda-\frac{1}{4}}\log(1+x)\right),$$

then the boundary condition $X(1) = 0$ gives

$$2^{\frac{1}{2}} \sin\left(\sqrt{\lambda - \frac{1}{4}} \log 2\right) = 0.$$

Therefore, $\sqrt{\lambda - 1/4}\log 2$ must be an integral multiple of π, i.e. $\sqrt{\lambda - 1/4}\log 2 = n\pi$, or

$$\lambda \;\mapsto\; \lambda_n = \frac{n^2\pi^2}{\log^2 2} + \frac{1}{4}, \qquad n \in \mathbb{Z}^+.$$

To these eigenvalues correspond the eigenfunctions

$$X_n(x) = (1+x)^{\frac{1}{2}} \sin\left(n\pi \frac{\log(1+x)}{\log 2}\right),$$

and these form a *complete system*, whenever we impose to $f(x)$ certain conditions that we shall see later. Moreover, $f(x)$ may be expanded in Fourier series

$$f(x) \;\sim\; \sum_{n=1}^{\infty} c_n X_n(x),$$

where

$$c_n = \frac{\int_0^1 f(x)(1+x)^{-\frac{3}{2}} \sin\left(n\pi \frac{\log(1+x)}{\log 2}\right) dx}{\int_0^1 (1+x)^{-1} \sin^2\left(n\pi \frac{\log(1+x)}{\log 2}\right) dx} = \frac{2}{\log 2}\int_0^1 f(x)(1+x)^{-\frac{3}{2}} \sin\left(n\pi \frac{\log(1+x)}{\log 2}\right) dx.$$

The completeness above mentioned and the Fourier series converges absolutely and uniformly to $f(x)$ in $(0,1)$, only if $f(x) \in \mathcal{PC}^1(0,1)$, $f(0) = f(1) = 0$ and $\int_0^1 f'^2(x)dx$ is *finite*. [1].
On the other hand, for satisfying (3) and its initial conditions, we choose the eigenfunction

$$T(t) \;\mapsto\; T_n(t) = \cos\sqrt{\lambda_n}\, t.$$

Thus, a solution of (1) is given by

$$u_n(x,t) = X_n(x)T_n(t) = (1+x)^{\frac{1}{2}} \sin\left(n\pi \frac{\log(1+x)}{\log 2}\right) \cos\sqrt{\lambda_n}\, t,$$

[1] A result due to Green-Parseval-Schwarz (GPS) and Bessel's inequality

and the general solution of (1) may be determined as a linear (infinite) combination of these eigenfunctions, that is

$$u(x,t) \ \sim \ \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t).$$

So that, the string's problem (1) has the formal solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n (1+x)^{\frac{1}{2}} \sin\left(n\pi \frac{\log(1+x)}{\log 2}\right) \cos\left(\sqrt{\frac{n^2\pi^2}{\log^2 2} + \frac{1}{4}}\, t\right). \qquad (4)$$

This series converges uniformly, and hence satisfies the initial and boundary conditions, as the series for $f(x)$ converges uniformly. However, in order to assure continuous derivatives and the partial differential equation (1) to be satisfied, we need suppose that the series for $f''(x)$ converges uniformly, i.e. we must suppose that $f(x)$ to be *regular* [2], that $f(0) = f(1) = f''(0) = f''(1) = 0$, and that $\int_0^1 f'''^{\,2}(x)dx$ to be finite. [3].

[2] i.e. $f(x) \in \mathcal{C}^2(0,1)$.
[3] By GPS, again