derivation of wave equation

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Let a string of matter be tightened between the points $x = 0$ and $x = p$ of the x-axis and let the string be made vibrate in the xy-plane. Let the of the string be the constant σ. We suppose that the amplitude of the vibration is so small that the tension $\vec{T}$ of the string can be regarded to be constant.

The position of the string may be represented as a function

$$y \;=\; y(x,\, t)$$

where t is the time. We consider an element dm of the string situated on a tiny interval $[x,\, x+dx]$; thus its mass is $\sigma\, dx$. If the angles the vector $\vec{T}$ at the ends x and $x+dx$ of the element forms with the direction of the x-axis are α and β, then the scalar force $\vec{F}$ of all on dm (the gravitation omitted) are

$$F_x \;=\; -T\cos\alpha + T\cos\beta, \quad F_y \;=\; -T\sin\alpha + T\sin\beta.$$

Since the angles α and β are very small, the ratio

$$\frac{F_x}{F_y} \;=\; \frac{\cos\beta - \cos\alpha}{\sin\beta - \sin\alpha} \;=\; \frac{-2\sin\frac{\beta-\alpha}{2}\sin\frac{\beta+\alpha}{2}}{2\sin\frac{\beta-\alpha}{2}\cos\frac{\beta+\alpha}{2}},$$

having the expression $-\tan\frac{\beta+\alpha}{2}$, also is very small. Therefore we can omit the horizontal component F_x and think that the vibration of all elements is strictly vertical. Because of the smallness of the angles α and β, their sines in the expression of F_y may be replaced with their tangents, and accordingly

$$F_y \;=\; T\cdot(\tan\beta - \tan\alpha) \;=\; T\,[y'_x(x+dx,\, t) - y'_x(x,\, t)] \;=\; T\,y''_{xx}(x,\, t)\,dx,$$

the last form due to the mean-value theorem.

On the other hand, by Newton the force equals the mass times the acceleration:

$$F_y \;=\; \sigma\, dx\, y''_{tt}(x,\, t)$$

Equating both expressions, dividing by $T\,dx$ and denoting $\sqrt{\frac{T}{\sigma}} = c$, we obtain the partial differential equation

$$y''_{xx} \;=\; \frac{1}{c^2}y''_{tt} \tag{1}$$

for the equation of the vibrating string.

But the equation (1) don't suffice to entirely determine the vibration. Since the end of the string are immovable,the function $y(x,\,t)$ has in to satisfy the boundary conditions

$$y(0,\,t) \;=\; y(p,\,t) \;=\; 0 \tag{2}$$

The vibration becomes completely determined when we know still e.g. at the beginning $t = 0$ the position $f(x)$ of the string and the initial velocity $g(x)$ of the points of the string; so there should be the initial conditions

$$y(x,\,0) \;=\; f(x), \quad y'_t(x,\,0) \;=\; g(x). \tag{3}$$

The equation (1) is a special case of the general wave equation

$$\nabla^2 u \;=\; \frac{1}{c^2}u''_{tt} \tag{4}$$

where $u = u(x,\,y,\,z,\,t)$. The equation (4) rules the spatial waves in $\mathbb{R}$. The number c can be shown to be the velocity of propagation of the wave motion.

References

[1] K. Väisälä: *Matematiikka IV*. Handout Nr. 141. Teknillisen korkeakoulun ylioppilaskunta, Otaniemi, Finland (1967).