uniform expansivity

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Let (X, d) be a compact metric space and let $f\colon X \to X$ be an expansive homeomorphism.

Theorem (uniform expansivity). For every $\epsilon > 0$ and $\delta > 0$ there is $N > 0$ such that for each pair x, y of points of X such that $d(x, y) > \epsilon$ there is $n \in \mathbb{Z}$ with $|n| \le N$ such that $d(f^n(x), f^n(y)) > c - \delta$, where c is the expansivity constant of f.

Proof. Let $K = \{(x, y) \in X \times X : d(x, y) \ge \epsilon/2\}$. Then K is closed, and hence compact. For each pair $(x, y) \in K$, there is $n_{(x,y)} \in \mathbb{Z}$ such that $d(f^{n_{(x,y)}}(x), f^{n_{(x,y)}}(y)) \ge c$. Since the mapping $F\colon X \times X \to X \times X$ defined by $F(x, y) = (f(x), f(y))$ is continuous, F^{n_x} is also continuous and there is a neighborhood $U_{(x,y)}$ of each $(x, y) \in K$ such that $d(f^{n_{(x,y)}}(u), f^{n_{(x,y)}}(v)) < c - \delta$ for each $(u, v) \in U_{(x,y)}$. Since K is compact and $\{U_{(x,y)} : (x, y) \in K\}$ is an open cover of K, there is a finite subcover $\{U_{(x_i,y_i)} : 1 \le i \le m\}$. Let $N = \max\{|n_{(x_i,y_i)}| : 1 \le i \le m\}$. If $d(x, y) > \epsilon$, then $(x, y) \in K$, so that $(x, y) \in U_{(x_i,y_i)}$ for some $i \in \{1, \ldots, m\}$. Thus for $n = n_{(x_i,y_i)}$ we have $d(f^n(x), f^n(y)) < c - \delta$ and $|n| \le N$ as requred.