cellular automaton

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Owner	Ziosilvio (18733)
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Defines	neighborhood (cellular automaton)
Defines	neighborhood index (cellular automaton)
Defines	local function (cellular automaton)
Defines	global function (cellular automaton)

A d-dimensional *cellular automaton* is a triple $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ where:

1. Q is a nonempty set, called *alphabet* or *set of states*.
2. $\mathcal{N} = \{\nu_1, \ldots, \nu_{|\mathcal{N}|}\}$ is a *finite, nonempty* subset of $\mathbb{Z}^d$, called *neighborhood index*.
3. $f : Q^{\mathcal{N}} \to Q$ is the *local function*.

The local function of the cellular automaton $\mathcal{A}$ induces, by synchronous application at all points, a *global function* $F_{\mathcal{A}}$ on d-dimensional *configurations* $c : \mathbb{Z}^d \to Q$:

$$F_{\mathcal{A}}(c)(z) = f\left(c(z + \nu_1), \ldots, c(z + \nu_{|\mathcal{N}|})\right) = f\left(c|_{z+\mathcal{N}}\right) . \tag{1}$$

Observe that the global function is continuous in the product topology.

Cellular automata are a Turing-complete model of computation. In fact, given a Turing machine, it is straightforward to construct a cellular automaton that emulates it in real time. On the other hand, given a one-dimensional cellular automaton, there exists a Turing machine that emulates it in linear time with respect to the size of the input.

Cellular automata make good tools for *qualitative analysis*. In fact, given the description of a cellular automaton, it is straightforward to write a computer program that implements its features. Also, several phenomena in different fields of sciences — from physics to biology to sociology — can be described in terms of finite-range interactions between discrete agents.

On the other hand, retrieving the properties of the global dynamics from the local description is usually a very difficult task, and may depend on the dimension. As an example, reversibility of the global function is decidable in dimension 1 but not in dimension 2 (or higher).

The definition above can be generalized to the case of a generic group G acting by *translations* on the space Q^G of configurations. If

$$L_g(x) = g \cdot x \;\; \forall x \in G , \tag{2}$$

then the local function $f : Q^{\mathcal{N}} \to Q$ ($\mathcal{N}$ being a finite nonempty subset of G) induces $F_{\mathcal{A}} : Q^G \to Q^G$ via the relation

$$F_{\mathcal{A}}(c)(g) = f\left((c \circ L_g)|_{\mathcal{N}}\right) = f\left(c|_{g\mathcal{N}}\right) \;\; \forall c : G \to Q . \tag{3}$$

Observe that $L = \{L_g\}_{g \in G}$ is a *left* action, while $(c, g) \mapsto c \circ L_g$ is a *right* action. It is of course possible to only use left action by defining $F_{\mathcal{A}}(c)(g)$ as $f\left((c \circ L_{g^{-1}})|_{\mathcal{N}}\right)$ instead: since this is just a *reparametrization* of the family L, the bulk of the theory does not change.

More to come ...