stable manifold

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Let X be a topological space, and $f\colon X \to X$ a homeomorphism. If p is a fixed point for f, the **stable and unstable sets** of p are defined by

$$W^s(f,p) = \{q \in X : f^n(q) \xrightarrow[n\to\infty]{} p\},$$
$$W^u(f,p) = \{q \in X : f^{-n}(q) \xrightarrow[n\to\infty]{} p\},$$

respectively.

If p is a periodic point of least period k, then it is a fixed point of f^k, and the stable and unstable sets of p are

$$W^s(f,p) = W^s(f^k,p)$$
$$W^u(f,p) = W^u(f^k,p).$$

Given a neighborhood U of p, the **local stable and unstable sets** of p are defined by

$$W^s_{\mathrm{loc}}(f,p,U) = \{q \in U : f^n(q) \in U \text{ for each } n \geq 0\},$$
$$W^u_{\mathrm{loc}}(f,p,U) = W^s_{\mathrm{loc}}(f^{-1},p,U).$$

If X is metrizable, we can define the stable and unstable sets for any point by

$$W^s(f,p) = \{q \in U : d(f^n(q), f^n(p)) \xrightarrow[n\to\infty]{} 0\},$$
$$W^u(f,p) = W^s(f^{-1},p),$$

where d is a metric for X. This definition clearly coincides with the previous one when p is a periodic point.

When K is an invariant subset of X, one usually denotes by $W^s(f,K)$ and $W^u(f,K)$ (or just $W^s(K)$ and $W^u(K)$) the stable and unstable sets of K, defined as the set of points $x \in X$ such that $d(f^n(x), K) \to 0$ when $x \to \infty$ or $-\infty$, respectively.

Suppose now that X is a compact smooth manifold, and f is a $\mathcal{C}^k$ diffeomorphism, $k \geq 1$. If p is a hyperbolic periodic point, the stable manifold theorem assures that for some neighborhood U of p, the local stable and unstable sets are $\mathcal{C}^k$ embedded disks, whose tangent spaces at p are E^s and E^u (the stable and unstable spaces of $Df(p)$), respectively; moreover, they vary continuously (in certain sense) in a neighborhood of f in the $\mathcal{C}^k$ topology of $\mathrm{Diff}^k(X)$ (the space of all $\mathcal{C}^k$ diffeomorphisms from X to itself). Finally, the stable and unstable sets are $\mathcal{C}^k$ injectively immersed disks. This is why they are commonly called **stable and unstable** manifolds. This result is also valid for nonperiodic points, as long as they lie in some hyperbolic set (stable manifold theorem for hyperbolic sets).