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proof of Hartman-Grobman theorem

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Lemma 1. *Let $A\colon E \to E$ be an hyperbolic isomorphism, and let φ and ψ be ε-Lipschitz maps from E to itself such that $\varphi(0) = \psi(0) = 0$. If ε is sufficiently small, then $A + \varphi$ and $A + \psi$ are topologically conjugate.*

Since A is hyperbolic, we have $E = E^s \oplus E^u$ and there is $\lambda < 1$ (possibly changing the norm of E by an equivalent box-type one), called the skewness of A, such that

$$\|A|_{E^s}\| < \lambda, \quad \|A^{-1}|_{E^u}\| < \lambda$$

and

$$\|x\| = \max\{\|x_s\|, \|x_u\|\}.$$

Let us denote by $(\tilde{E}, \|\cdot\|_0)$ the Banach space of all bounded, continuous maps from E to itself, with the norm of the supremum induced by the norm of E. The operator A induces a linear operator $\tilde{A} : \tilde{E} \to \tilde{E}$ defined by $(\tilde{A}u)(x) = A(u(x))$, which is also hyperbolic. In fact, letting $\tilde{E}^i$ be the set of all maps $u\colon \tilde{E} \to \tilde{E}$ whose range is contained in E^i (for $i = s, u$) we have that $\tilde{E} = \tilde{E}^s \oplus \tilde{E}^u$ is a hyperbolic splitting for $\tilde{A}$ with the same skewness as A.

From now on we denote the projection of x to E^i by x_i, and the restriction $A|_{E_i}\colon E^i \to E^i$ by A_i ($i = s, u$).

We will try to find a conjugation of the form $I + u$ where $u \in \tilde{E}$.

Proposition 1. *There exists $\varepsilon > 0$ such that if φ and ψ are ε-Lipschitz, then there is a unique $u \in \tilde{E}$ such that*

$$(I + u)(A + \varphi) = (A + \psi)(I + u).$$

Proof. We want to find u such that

$$A + \varphi + u(A + \varphi) = A + Au + \psi(I + u)$$

which is the same as

$$\varphi + u(A + \varphi) = Au + \psi(I + u).$$

This can be rewriten as

$$\begin{aligned} u_u &= A_u^{-1}(u_u(A + \varphi) + \varphi_u - \psi_u(I + u)) \\ u_s &= (A_s u_s + \psi_s(I + u) - \varphi_s)(A + \varphi)^{-1}, \end{aligned}$$

where we use the fact that by the Lipschitz inverse mapping theorem, if $\mathrm{Lip}(\varphi) < 1/\lambda \leq \|A^{-1}\|^{-1}$ (where λ is the skewness of A) then $A+\varphi$ is invertible with Lipschitz inverse.

Now define $\Gamma : \tilde{E} \to \tilde{E}$ by

$$\begin{aligned} \Gamma_s(u) &= (A_s u_s + \psi_s(I+u) - \varphi_s)(A+\varphi)^{-1} \\ \Gamma_u(u) &= A_u^{-1}(u_u(A+\varphi) + \varphi_u - \psi_u(I+u)) \end{aligned}$$

We assert that, if ε is small, Γ is a contraction. In fact,

$$\begin{aligned} \|\Gamma_s(u) - \Gamma_s(v)\|_0 &= \left\|(A_s(u_s - v_s) + \psi_s(I+u) - \psi_s(I+v))\,(A+\varphi)^{-1}\right\|_0 \\ &\leq \|\tilde{A}_s\| \cdot \|(u_s - v_s)(A+\varphi)^{-1}\|_0 + \|(\psi_s(I+u) - \psi_s(I+v))(A+\varphi)^{-1}\|_0 \\ &\leq \lambda \|u_s - v_s\|_0 + \varepsilon \|u - v\|_0 \\ &\leq (\lambda + \varepsilon)\|u - v\|_0 \end{aligned}$$

and

$$\begin{aligned} \|\Gamma_u(u) - \Gamma_u(v)\|_0 &= \left\|A_u^{-1}(u_u(A+\varphi) - v_u(A+\varphi) - \psi_u(I+u) + \psi_u(I+v))\right\|_0 \\ &\leq \|\tilde{A}_u^{-1}\| \cdot (\|u_u(A+\varphi) - v_u(A+\varphi)\|_0 + \|\psi_u(I+u) - \psi_u(I+v)\|_0) \\ &\leq \lambda\,(\|u_u - v_u\|_0 + \varepsilon\|u - v\|_0) \\ &\leq \lambda(1+\varepsilon)\|u - v\|_0. \end{aligned}$$

Thus, if $\varepsilon < \varepsilon_0 \doteq \min\{\lambda, (1-\lambda)/\lambda\}$, Γ has Lipschitz constant smaller than 1, so it is a contraction. Hence u exists and is unique. □

Proposition 2. *The map u from the previous proposition is a homeomorphism.*

Proof. Using the previous proposition with φ and ψ switched, we get a unique $v \in \tilde{E}$ such that

$$(I+v)(A+\psi) = (A+\varphi)(I+v).$$

It follows that

$$(I+v)(I+u)(A+\varphi) = (I+v)(A+\psi)(I+u) = (A+\varphi)(I+v)(I+u). \quad (1)$$

Also, the previous proposition with $\varphi = \psi$ implies that that there is a unique $w \in \tilde{E}$ such that

$$(I+w)(A+\varphi) = (A+\varphi)(I+w),$$

which obviously is $w = 0$. But since $(I + v)(I + u) = I + (u + v + uv)$ and $u + v + uv \in \tilde{E}$, (**??**) implies that $w = u + v + uv$ is a solution of the above equation, so that $u + v + uv = 0$ and $(I + v)(I + u) = I$. In a similar way, we see that $(I + u)(I + v) = I$. Hence $I + u$ is invertible, with continuous inverse. □

The two previous propositions prove the lemma.

Proposition 3. *If U is an open neighborhood of 0 and $f \colon U \to E$ is a $\mathcal{C}^\infty$ map with $f(0) = 0$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that $\varphi \doteq f - Df(0)$ is ε-Lipschitz in the ball $B(0, \delta)$.*

Proof. This is a direct consequence of the mean value inequality and the fact that $D\varphi$ is continuous and $D\varphi(0) = 0$. □

Proposition 4. *There is a constant k such that if $\varphi \colon \overline{B}(0, r) \to E$ is an ε-Lipschitz map, then there is a $k\varepsilon$-Lipschitz map $\tilde{\varphi} \colon E \to E$ which coincides with φ in $B(0, r/2)$.*

Proof. Let $\eta \colon \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ bump function: an infinitely differentiable map such that $\eta(x) = 1$ for $x < 1/2$ and $\eta(x) = 0$ for $x > 1$, with derivative bounded by M and $|\eta(x)\| \leq 1$ for all $x \in \mathbb{R}$. Now define $\tilde{\varphi}(x) = \varphi(x)\eta(\|x\|/r)$ (when $\varphi(x)$ is not defined, we assume that it is zero). If x and y are both in $B(0, r)$ then we have

$$\begin{aligned}
\|\tilde{\varphi}(x) - \tilde{\varphi}(y)\| &= \big\|\varphi(x)\eta(\|x\|/r) - \varphi(y)\eta(\|y\|/r)\big\| \\
&\leq \big\|(\varphi(x) - \varphi(y))\eta(\|x\|/r)\big\| + \big\|\varphi(y)(\eta(\|x\|/r) - \eta(\|y\|/r))\big\| \\
&\leq \varepsilon\|x - y\| + \|\varphi(y) - \varphi(0)\| \cdot \big\|\eta(\|x\|/r) - \eta(\|y\|/r)\big\| \\
&\leq \varepsilon\|x - y\| + \varepsilon\|y\|(M\|x - y\|/r) \\
&\leq (M + 1)\varepsilon\|x - y\|;
\end{aligned}$$

if x is in $B(0, r)$ and y is not, then

$$\|\tilde{\varphi}(x) - \tilde{\varphi}(y)\| = \|\tilde{\varphi}(x) - \tilde{\varphi}(y^*)\|,$$

where y^* is defined as $x + \tau(y - x)$ with

$$\tau = \sup\{t : x + t(y - x) \in E \setminus B(0, r)\}$$

This is true because $\tilde{\varphi}(y^*) = 0$. Also, $\|x - y^*\| = \tau\|x - y\| \leq \|x - y\|$; hence

$$\|\tilde{\varphi}(x) - \tilde{\varphi}(y)\| = \|\tilde{\varphi}(x) - \tilde{\varphi}(y^*)\| \leq (M + 1)\varepsilon\|x - y^*\| \leq (M + 1)\varepsilon\|x - y\|.$$

Finally, if both x and y are outside $B(0, r)$, then $\|\tilde{\varphi}(x) - \tilde{\varphi}(y)\| = 0 \leq (M+1)\|x - y\|$. Letting $k = M + 1$ we get the desired result. $\square$

Proof of the theorem. Taking the particular $\psi = 0$ in the lemma, we observe that there is $\varepsilon > 0$ such that for any ε-Lipschitz map φ, $Df(0)$ is conjugate to $\varphi + Df(0)$. Choose δ such that $f - Df(0)$ is ε/k-Lipschitz in $B(0, 2\delta)$. Let $\tilde{\varphi}$ be the ε-Lipschitz extension of $f - Df(0)$ to $B(0, \delta)$ obtained from the previous proposition. We have that $Df(0) + \tilde{\varphi}$ is conjugate to $Df(0)$. But for $x \in B(0, \delta)$ we have $Df(0) + \tilde{\varphi} = f$, so that f is locally conjugate to $Df(0)$.