Garden of Eden

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Defines	orphan pattern (cellular automaton)
Defines	orphan pattern principle

A *Garden of Eden* (briefly, GoE) for a cellular automaton $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ on a group G is a configuration $c \in Q^G$ which is not in the image of the global function $F_{\mathcal{A}}$ of $\mathcal{A}$.

In other words, a Garden of Eden is a global situation which can be started from, but never returned to.

The finitary counterpart of a GoE configuration is an *orphan pattern*: a pattern which cannot be obtained by synchronous application of the local function f.

Of course, any cellular automaton with an orphan pattern also has a GoE configuration.

Lemma 1 (Orphan pattern principle) *If a cellular automaton with finite set of states Q has a* GoE *configuration, then it also has an orphan pattern.*

Proof. First, suppose that G is countable. Let $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ be a cellular automaton with no orphan pattern. Let $c : G \to Q$ be a configuration: we will prove that there is some $e : G \to Q$ such that $F_{\mathcal{A}}(e) = c$.

Let $G = \{g_n\}_{n \geq 0}$ be an enumeration of G: put $E_n = \{g_i \mid i \leq n\}$ and let $p_n : E_n \to Q$ be defined as $p_n = c|_{E_n}$. By hypothesis, none of the p_n's is an orphan, so there is a sequence of configurations $c_n : G \to Q$ satisfying $F_{\mathcal{A}}(c_n)|_{E_n} = p_n = c|_{E_n}$. It is easy to see that $\lim_{k \to \infty} F_{\mathcal{A}}(c_{n_k}) = c$. But if Q is finite, then Q^G is compact by Tychonoff's theorem, so there exista a subsequence $\{c_{n_k}\}_{k \geq 0}$ and a configuration $e : G \to Q$ satisfying $\lim_{k \to \infty} c_{n_k} = e$: Since $F_{\mathcal{A}}$ is continuous in the product topology, $F(e) = c$.

Let now G be arbitrary. Let H be the subgroup generated by the neighborhood index $\mathcal{N}$: since $\mathcal{N}$ is finite, H is countable. Let J be a set of representatives of the left cosets of H in G, so that $G = \bigsqcup_{j \in J} jH$. (Observe that we do *not* require that H is normal in G.) Call $\mathcal{A}_H$ the cellular automaton *on* H that has the same local description (set of states, neighborhood index, local function) as $\mathcal{A}$. Let $c : G \to Q$ be a Garden of Eden configuration for $\mathcal{A}$: then at least one of the configurations $c_j(h) = c(jh)$ must be a Garden of Eden for $\mathcal{A}_H$. By the discussion above, $\mathcal{A}_H$ must have an orphan pattern, which is also an orphan pattern for $\mathcal{A}$. $\square$