topological conjugation

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Defines	topologically semiconjugate
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Defines	topological equivalence

Let X and Y be topological spaces, and let $f\colon X \to X$ and $g\colon Y \to Y$ be continuous functions. We say that f is **topologically semiconjugate** to g, if there exists a continuous surjection $h\colon Y \to X$ such that $fh = hg$. If h is a homeomorphism, then we say that f and g are **topologically conjugate**, and we call h a **topological conjugation** between f and g.

Similarly, a flow φ on X is topologically semiconjugate to a flow ψ on Y if there is a continuous surjection $h\colon Y \to X$ such that $\varphi(h(y), t) = h\psi(y, t)$ for each $y \in Y$, $t \in \mathbb{R}$. If h is a homeomorphism then ψ and φ are topologically conjugate.

0.1 Remarks

Topological conjugation defines an equivalence relation in the space of all continuous surjections of a topological space to itself, by declaring f and g to be related if they are topologically conjugate. This equivalence relation is very useful in the theory of dynamical systems, since each class contains all functions which share the same dynamics from the topological viewpoint. In fact, orbits of g are mapped to homeomorphic orbits of f through the conjugation. Writing $g = h^{-1}fh$ makes this fact evident: $g^n = h^{-1}f^nh$. Speaking informally, topological conjugation is a "change of coordinates" in the topological sense.

However, the analogous definition for flows is somewhat restrictive. In fact, we are requiring the maps $\varphi(\cdot, t)$ and $\psi(\cdot, t)$ to be topologically conjugate for each t, which is requiring more than simply that orbits of φ be mapped to orbits of ψ homeomorphically. This motivates the definition of **topological equivalence**, which also partitions the set of all flows in X into classes of flows sharing the same dynamics, again from the topological viewpoint.

We say that ψ and φ are **topologically equivalent**, if there is an homeomorphism $h : Y \to X$, mapping orbits of ψ to orbits of φ homeomorphically, and preserving orientation of the orbits. This means that:

1. $h(\mathcal{O}(y, \psi)) = \{h(\psi(y, t)) : t \in \mathbb{R}\} = \{\varphi(h(y), t) : t \in \mathbb{R}\} = \mathcal{O}(h(y), \varphi)$ for each $y \in Y$;

2. for each $y \in Y$, there is $\delta > 0$ such that, if $0 < |s| < t < \delta$, and if s is such that $\varphi(h(y), s) = h(\psi(y, t))$, then $s > 0$.