pre-injectivity

Canonical name	Preinjectivity
Date of creation	2013-03-22 19:22:04
Last modified on	2013-03-22 19:22:04
Owner	Ziosilvio (18733)
Last modified by	Ziosilvio (18733)
Numerical id	4
Author	Ziosilvio (18733)
Entry type	Definition
Classification	msc 37B15
Classification	msc 68Q80
Defines	mutually erasable patterns (cellular automaton)

Let $X = \prod_{i \in I} X_i$ be a Cartesian product. Call two elements $x, y \in X$ *almost equal* if the set $\Delta(x, y) = \{i \in I \mid x_i \neq y_i\}$ is finite. A function $f : X \to X$ is said to be *pre-injective* if it sends distinct almost equal elements into distinct elements, *i.e.*, if $0 < |\Delta(x, y)| < \infty$ implies $f(x) \neq f(y)$.

If X is finite, pre-injectivity is the same as injectivity; in general, the latter implies the former, but not the other way around. Moreover, it is not true in general that a composition of pre-injective functions is itself pre-injective.

A cellular automaton is said to be pre-injective if its global function is. As cellular automata send almost equal configurations into almost equal configurations, the composition of two pre-injective cellular automata is pre-injective.

Pre-injectivity of cellular automata can be characterized via mutually erasable patterns. Given a finite subset E of G, two patterns $p_1, p_2 : E \to Q$ are *mutually erasable* (briefly, m.e.) for a cellular automaton $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ on G if for any two configurations $c_1, c_2 : G \to Q$ such that $c_i|_E = p_i$ and $c_1|_{G \setminus E} = c_2|_{G \setminus E}$ one has $F_\mathcal{A}(c_1) = F_\mathcal{A}(c_2)$.

Lemma 1 *For a cellular automaton $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$, the following are equivalent.*

1. *$\mathcal{A}$ has no mutually erasable patterns.*

2. *$\mathcal{A}$ is pre-injective.*

Proof. It is immediate that the negation of point **??** implies the negation of point **??**. So let $c_1, c_2 : G \to Q$ be two distinct almost equal configurations such that $F_\mathcal{A}(c_1) = F_\mathcal{A}(c_2)$: it is not restrictive to suppose that $\mathcal{N}$ is symmetric (*i.e.*, if $x \in \mathcal{N}$ then $x^{-1} \in \mathcal{N}$) and e, the identity element of G, belongs to $\mathcal{N}$. Let Δ be a finite subset of G such that $c_1|_{G \setminus \Delta} = c_2|_{G \setminus \Delta}$, and let

$$E = \Delta\mathcal{N}^2 = \{g \in G \mid \exists z \in \Delta, u, v \in \mathcal{N} \mid g = zuv\} \ : \qquad (1)$$

we shall prove that $p_1 = c_1|_E$ and $p_2 = c_2|_E$ are mutually erasable. (They surely are distinct, since $\Delta \subseteq E$.)

So let $\gamma_1, \gamma_2 : G \to Q$ satisfy $\gamma_i|_E = p_i$ and $\gamma_2|_{G \setminus E} = p_i = \gamma_2|_{G \setminus E}$. Let $z \in G$. If $z \in \Delta\mathcal{N}$, then $F_\mathcal{A}(\gamma_1)(z) = F_\mathcal{A}(\gamma_2)(z)$, because by construction $\gamma_i|_{z\mathcal{N}} = c_i|_{z\mathcal{N}}$; if $z \in G \setminus \Delta\mathcal{N}$, then $F_\mathcal{A}(\gamma_1)(z) = F_\mathcal{A}(\gamma_2)(z)$ as well, because by construction $\gamma_1|_{z\mathcal{N}} = \gamma_2|_{z\mathcal{N}}$. Since γ_1 and γ_2 are arbitrary, p_1 and p_2 are mutually erasable. $\square$

References

Ceccherini-Silberstein, T. and Coornaert, M. (2010) *Cellular Automata and Groups.* Springer Verlag.