the only compact metric spaces that admit a positively expansive homeomorphism are discrete spaces

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Theorem. Let (X, d) be a compact metric space. If there exists a positively expansive homeomorphism $f\colon X \to X$, then X consists only of isolated points, i.e. X is finite.

Lemma 1. If (X, d) is a compact metric space and there is an expansive homeomorphism $f\colon X \to X$ such that every point is Lyapunov stable, then every point is asymptotically stable.

Proof. Let $2c$ be the expansivity constant of f. Suppose some point x is not asymptotically stable, and let δ be such that $d(x, y) < \delta$ implies $d(f^n(x), f^n(y)) < c$ for all $n \in \mathbb{N}$. Then there exist $\epsilon > 0$, a point y with $d(x, y) < \delta$, and an increasing sequence $\{n_k\}$ such that $d(f^{n_k}(y), f^{n_k}(x)) > \epsilon$ for each k By uniform expansivity, there is $N > 0$ such that for every u and v such that $d(u, v) > \epsilon$ there is $n \in \mathbb{Z}$ with $|n| < N$ such that $d(f^n(x), f^n(y)) > c$. Choose k so large that $n_k > N$. Then there is n with $|n| < N$ such that $d(f^{n+n_k}(x), f^{n+n_k}(y)) = d(f^n(f^{n_k}(x)), f^n(f^{n_k}(y))) > c$. But since $n+n_k > 0$, this contradicts the choce of δ. Hence every point is asymptotically stable.

Lemma 2 If (X, d) is a compact metric space and $f\colon X \to X$ is a homeomorphism such that every point is asymptotically stable and Lyapunov stable, then X is finite.

Proof. For each $x \in X$ let K_x be a closed neighborhood of x such that for all $y \in K_x$ we have $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$. We assert that $\lim_{n\to\infty} \operatorname{diam}(f^n(K_x)) = 0$. In fact, if that is not the case, then there is an increasing sequence of positive integers $\{n_k\}$, some $\epsilon > 0$ and a sequence $\{x_k\}$ of points of K_x such that $d(f^{n_k}(x), f^{n_k}(x_k)) > \epsilon$, and there is a subsequence $\{x_{k_i}\}$ converging to some point $y \in K_x$.

From the Lyapunov stability of y, we can find $\delta > 0$ such that if $d(y, z) < \delta$, then $d(f^n(y), f^n(z)) < \epsilon/2$ for all $n > 0$. In particular $d(f^{n_{k_i}}(x_{k_i}), f^{n_{k_i}}(y)) < \epsilon/2$ if i is large enough. But also $d(f^{n_{k_i}}(y), f^{n_{k_i}}(x)) < \epsilon/2$ if i is large enough, because $y \in K_x$. Thus, for large i, we have $d(f^{n_{k_i}}(x_{k_i}), f^{n_{k_i}}(x)) < \epsilon$. That is a contradiction from our previous claim.

Now since X is compact, there are finitely many points $x_1, \dots, x_m$ such that $X = \bigcup_{i=1}^m K_{x_i}$, so that $X = f^n(X) = \bigcup_{i=1}^m f^n(K_{x_i})$. To show that $X = \{x_1, \dots, x_m\}$, suppose there is $y \in X$ such that $r = \min\{d(y, x_i) : 1 \le i \le m\} > 0$. Then there is n such that $\operatorname{diam}(f^n(K_{x_i})) < r$ for $1 \le i \le m$ but since $y \in f^n(K_{x_i})$ for some i, we have a contradiction.

Proof of the theorem. Consider the sets $K_\epsilon = \{(x, y) \in X \times X : d(x, y) \ge \epsilon\}$ for $\epsilon > 0$ and $U = \{(x, y) \in X \times X : d(x, y) > c\}$, where $2c$ is the expansivity constant of f, and let $F\colon X \times X \to X \times X$ be the mapping given by $F(x, y) = (f(x), f(y))$. It is clear that F is a homeomorphism. By uniform expansivity, we know that for each $\epsilon > 0$ there is N_ϵ such that for all $(x, y) \in K_\epsilon$, there is $n \in \{1, \ldots, N_\epsilon\}$ such that $F^n(x, y) \in U$.

We will prove that for each $\epsilon > 0$, there is $\delta > 0$ such that $F^n(K_\epsilon) \subset K_\delta$ for all $n \in \mathbb{N}$. This is equivalent to say that every point of X is Lyapunov stable for f^{-1}, and by the previous lemmas the proof will be completed.

Let $K = \bigcup_{n=0}^{N_\epsilon} F^n(K_\epsilon)$, and let $\delta_0 = \min\{d(x, y) : (x, y) \in K\}$. Since K is compact, the minimum distance δ_0 is reached at some point of K; i.e. there exist $(x, y) \in K_\epsilon$ and $0 \leq n \leq N_\epsilon$ such that $d(f^n(x), f^n(y)) = \delta_0$. Since f is injective, it follows that $\delta_0 > 0$ and letting $\delta = \delta_0/2$ we have $K \subset K_\delta$.

Given $\alpha \in K - K_\epsilon$, there is $\beta \in K_\epsilon$ and some $0 < m \leq N_\epsilon$ such that $\alpha = F^m(\beta)$, and $F^k(\beta) \notin K_\epsilon$ for $0 < k \leq m$. Also, there is n with $0 < m < n \leq N_\epsilon$ such that $F^n(\beta) \in U \subset K_\epsilon$. Hence $m < N_\epsilon$, and $F(\beta) = F^{m+1}(\alpha) \in F^{m+1}(K_\epsilon) \subset K$; On the other hand, $F(K_\epsilon) \subset K$. Therefore $F(K) \subset K$, and inductively $F^n(K) \subset K$ for any $n \in \mathbb{N}$. It follows that $F^n(K_\epsilon) \subset F^n(K) \subset K \subset K_\delta$ for each $n \in \mathbb{N}$ as required.