indefinite sum

Canonical name	IndefiniteSum
Date of creation	2013-03-22 17:35:14
Last modified on	2013-03-22 17:35:14
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	20
Author	CWoo (3771)
Entry type	Definition
Classification	msc 39A99
Related topic	FiniteDifference

Recall that the finite difference operator Δ defined on the set of functions $\mathbb{R} \to \mathbb{R}$ is given by

$$\Delta f(x) := f(x+1) - f(x).$$

The difference operator can be thought of as the discrete version of the derivative operator sending a function to its derivative (if it exists). With the derivative operation, there corresponds an inverse operation called the antiderivative, which, given a function f, finds its antiderivative F so that the derivative of F gives f. There is also a discrete analog of this inverse operation, and it is called the *indefinite sum*.

The *indefinite sum* of a function $f : \mathbb{R} \to \mathbb{R}$ is the set of functions

$$\{F : \mathbb{R} \to \mathbb{R} \mid \Delta F = f\}.$$

This set is often denoted by $\Delta^{-1} f$ or Σf, and any element in $\Delta^{-1} f$ is called an indefinite sum of f.

Remark. Like the indefinite integral, the indefinite sum Δ^{-1} is shift invariant. This means that for any $F \in \Delta^{-1} f$, then $F + c \in \Delta^{-1} f$ for any $c \in \mathbb{R}$. But, unlike the indefinite integral, the indefinite sum is also invariant by a shift of a periodic real function of period 1. Conversely, the difference of two indefinite sums of a function f is a periodic real function of period 1.

In the following discussion, we consider the indefinite sum of a function as a function.

Basic Properties

1. $\Delta\Delta^{-1} f = f$, and $\Delta^{-1}\Delta f = f$ modulo a real function of period 1.
2. Modulo a real number, and treating Δ^{-1} as an operator taking a function into a function, we see that Δ^{-1} is linear, that is,
   - $\Delta^{-1}(rf) = r\Delta^{-1} f$ for any $r \in \mathbb{R}$, and
   - $\Delta^{-1}(f+g) = \Delta^{-1} f + \Delta^{-1} g$.
3. If $F(x) = \Delta^{-1} f(x)$, then $F(x+a) = \Delta^{-1} f(x+a)$.
4. If $F = \Delta^{-1} f$, then we see that

$$\begin{aligned} F(a+1) - F(a) &= f(a), \\ F(a+2) - F(a+1) &= f(a+1), \\ &\vdots \\ F(x) - F(x-1) &= f(x-1). \end{aligned}$$

where $x - a$ is a positive integer. Summing these expressions, we get

$$F(x) - F(a) = \sum_{i=1}^{x-a} f(a + i - 1).$$

This is the discrete version of the fundamental theorem of calculus.

Below is a table of some basic functions and their indefinite sums (C is a real-valued periodic function with period 1):

$f(x)$	$\Delta^{-1} f(x)$	Comment
$r \in \mathbb{R}$	$rx + C$	
x	$\frac{x(x-1)}{2} + C$	
x^2	$\frac{x(x-1)(2x-1)}{6} + C$	
x^3	$\frac{x^2(x-1)^2}{4} + C$	
x^n	$T_n(x) + C$	See this `http://planetmath.org/SumOfPowers`link for detail
a^x	$\frac{a^x}{a-1} + C$	$a \neq 1$
$(x)_n$	$\frac{(x)_n}{n+1} + C$	$(x)_n$ is the falling factorial of degree n
$\binom{x}{n}$	$\binom{x}{n+1} + C$	$\binom{x}{n} := \frac{(x)_n}{n!}$
$\frac{1}{x}$	$\psi(x) + C$	$\psi(x)$ is the digamma function
$\ln x$	$\ln \Gamma(x) + C$	$\Gamma(x)$ is the gamma function
$\sin x$	$-\frac{\cos(x-1/2)}{2\sin(1/2)} + C$	
$\cos x$	$\frac{\sin(x-1/2)}{2\sin(1/2)} + C$	

References

[1] C. Jordan. *Calculus of Finite Differences*, third edition. Chelsea, New York (1965)