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divided differences of powers

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In this entry, we will prove the claims about divided differences of polynomials. Because the divided difference is a linear operator, we can focus our attention on powers.

Theorem 1. *If $f(x) = x^n$ and $m \leq n$, then*

$$\Delta^m f[x_0, \cdots x_m] = \sum_{k_0+\cdots+k_m=n-m} x_0^{k_0} \cdots x_m^{k_m}.$$

If $m > n$, then $\Delta^m f[x_0, \cdots x_m] = 0$.

Proof. We proceed by induction. The formula is trivially true when $m = 0$. Assume that the formula is true for a certain value of m. Then we have

$$\begin{aligned}\Delta^{m+1} f[x_0, \cdots x_{m+1}] &= \frac{\Delta^m f[x_1, x_2 \cdots x_m] - \Delta^m f[x_0, x_2 \cdots x_m]}{x_1 - x_0} \\ &= \sum_{k+k_2+\cdots+k_m=n-m} \frac{(x_1^k - x_0^k) x_2^{k_2} \cdots x_m^{k_m}}{x_1 - x_0}\end{aligned}$$

Using the identity for the sum of a geometric series,

$$\frac{x_1^k - x_0^k}{x_1 - x_0} = \sum_{k_0+k_1=k-1} x_0^{k_0} x_1^{k_1},$$

this becomes

$$\begin{aligned}\Delta^{m+1} f[x_0, \cdots x_{m+1}] &= \sum_{k+k_2+\cdots+k_m=n-m} \sum_{k_0+k_1=k-1} x_0^{k_0} x_1^{k_1} x_2^{k_2} \cdots x_{m+1}^{k_{m+1}} \\ &= \sum_{k_0+k_1+k_2+\cdots+k_m=n-(m+1)} x_0^{k_0} x_1^{k_1} x_2^{k_2} \cdots x_{m+1}^{k_{m+1}}.\end{aligned}$$

Note that when $k = 0$, we have $x_1^k - x_0^k = 0$, which is consistent with the formula given above because, in that case, there are no solutions to $k_1 + k_2 = k$, so the sum is empty and, by convention, equals zero. Likewise, when $n = m$, then the only solution to $k_0 + \cdots + k_m = 0$ is $k_0 = \cdots = k_m = 0$, so the sum only consists of one term, $x_0^0 \cdots x_m^0 = 1$ so $\Delta^n f[x_0, \cdots x_n] = 1$, hence taking further differences produces zero. □