solution of equations by divided difference interpolaton

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Divided diference interpolation can be used to obtain approximate solutions to equations and to invert functions numerically. The idea is that, given an equation $f(y) = x$ which we want to solve for y, we first take several numbers $y_1, \ldots, y_n$ and compute $x_1 \ldots x_n$ as $x_i = f(y_i)$. Then we compute the divided differences of the y_i's regarded as functions of the x_i's and form the divided difference series. Substituting x in this series provides an approximation to y.

To illustrate how this works, we will examine the transcendental equation $x + e^{-x} = 2$. We note that $2 + e^{-2} = 2.13533$ and $1.5 + e^{-1.5} = 1.72313$, so there will be a solution between 1.5 and 2, likely closer to 2 than 1.5. Therefore, as our values of the y_i's, we shall take 1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1. We now tabulate $x_i = y_i + e^{-y_i}$ for those values:

y_i	x_i
1.5	1.72313
1.6	1.80190
1.7	1.88268
1.8	1.96530
1.9	2.04957
2.0	2.13533
2.1	2.22246

Next, we form a divided difference table of the y_i's as a function of the x_i's:

1.72313	1.50000						
		1.26952					
1.80190	1.60000		−0.19799				
		1.23793		0.12082			
1.88268	1.70000		−0.16873		−0.039609		
		1.21036		0.10789		0.091553	
1.96530	1.80000		−0.14201		−0.077347		−0.13457
		1.18666		0.08210		0.024360	
2.04957	1.90000		−0.12127		−0.067102		
		1.16604		0.05930			
2.13533	2.00000		−0.10602				
		1.14771					
2.22246	2.10000						

From this table, we form the series

$$\begin{aligned}1.50000 &+ 1.26952(x-1.72313) - 0.19799(x-1.72313)(x-1.80190)\\ &+ 0.12082(x-1.72313)(x-1.80190)(x-1.88268)\\ &- 0.039609(x-1.72313)(x-1.80190)(x-1.88268)(x-1.96530)\\ &+ 0.091553(x-1.72313)(x-1.80190)(x-1.88268)(x-1.96530)(x-2.04957)\\ &- 0.13457(x-1.72313)(x-1.80190)(x-1.88268)(x-1.96530)(x-2.04957)(x-2.13533)\end{aligned}$$

Substituting 2.00000 for x, we obtain 1.84140. Given that

$$1.84140 + e^{-1.84140} < 2 < 1.84141 + e^{-1.84141},$$

this answer is correct to all 5 decimal places.

In the presentation above, we tacitly assumed that there was a solution to our equation and focussed our attention on finding that answer numerically. To complete the treatment we will now show that there indeed exists a unique solution to the equation $x + e^{-x} = 2$ in the interval $(0, \infty)$.

Existence follows from the intermediate value theorem. As noted above,

$$1.5 + e^{-1.5} < 2 < 3 + e^{-2}.$$

Since $x + e^{-x}$ depends continuously on x, it follows that there exists $x \in (1.5, 2)$ such that $x + e^{-x} = 2$.

As for uniqueness, note that the derivative of $x + e^{-x}$ is $1 - e^{-x}$. When $x > 0$, we have $e^{-x} < 1$, or $1 - e^{-x} > 0$. Hence, $x + e^{-x}$ is a strictly increaing function of x, so there can be at most one x such that $x + e^{-x} = 2$.