proof of arithmetic-geometric-harmonic means inequality

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We can use the Jensen inequality for an easy proof of the arithmetic-geometric-harmonic means inequality.

Let $x_1, \ldots, x_n > 0$; we shall first prove that

$$\sqrt[n]{x_1 \cdot \ldots \cdot x_n} \leq \frac{x_1 + \ldots + x_n}{n}.$$

Note that log is a concave function. Applying it to the arithmetic mean of $x_1, \ldots, x_n$ and using Jensen's inequality, we see that

$$\begin{aligned}
\log\left(\frac{x_1 + \ldots + x_n}{n}\right) &\geq \frac{\log(x_1) + \ldots + \log(x_n)}{n} \\
&= \frac{\log(x_1 \cdot \ldots \cdot x_n)}{n} \\
&= \log \sqrt[n]{x_1 \cdot \ldots \cdot x_n}.
\end{aligned}$$

Since log is also a monotone function, it follows that the arithmetic mean is at least as large as the geometric mean.

The proof that the geometric mean is at least as large as the harmonic mean is the usual one (see "proof of arithmetic-geometric-harmonic means inequality").