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symmetry of divided differences

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Theorem 1. *If $y_0, \dots, y_n$ is a permutation of $x_0, \dots, x_n$, then*

$$\Delta^n f[x_0, \dots, x_n] = \Delta^n f[y_0, \dots, y_n].$$

Proof. We proceed by induction. When $n = 1$, we have, from the definition,

$$\Delta^1 f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \Delta^1 f[x_1, x_0].$$

Since the only permutations of two elements are the identity and the transposition, we see that the first divided differrence is symmetric.

Now suppose that we already know that the n-th divided difference is symmetric under permutation of its arguments for some $n \geq 1$. We will prove that the $n+1$-st divided difference is also symmmetric under all permutations of its arguments.

The divided difference is symmetric under transposing x_0 with x_1:

$$\begin{aligned}
\Delta^{n+1} f[x_0, x_1, x_2, \dots, x_{n+1}] &= \frac{\Delta^n f[x_1, x_2, \dots x_{n+1}] - \Delta^n f[x_0, x_2, \dots x_{n+1}]}{x_1 - x_0} \\
&= \frac{\Delta^n f[x_0, x_2, \dots x_{n+1}] - \Delta^n f[x_1, x_2, \dots x_{n+1}]}{x_0 - x_1} \\
&= \Delta^{n+1} f[x_1, x_0, x_2, \dots, x_{n+1}]
\end{aligned}$$

The divided difference is symmetric under transposing x_1 with x_2:

$$\begin{aligned}
&\Delta^{n+1} f[x_0, x_1, x_2, \dots, x_{n+1}] \\
&\quad= \frac{\Delta^n f[x_1, x_2, \dots x_{n+1}] - \Delta^n f[x_0, x_2, \dots x_{n+1}]}{x_1 - x_0} \\
&\quad= \frac{\begin{array}{c}(x_2 - x_0)\left(\Delta^{n-1} f[x_2, x_3 \dots x_{n+1}] - \Delta^{n-1} f[x_1, x_3 \dots x_{n+1}]\right) \\ -(x_2 - x_1)\left(\Delta^{n-1} f[x_2, x_3 \dots x_{n+1}] - \Delta^{n-1} f[x_0, x_3 \dots x_{n+1}]\right)\end{array}}{(x_2 - x_1)(x_1 - x_0)(x_2 - x_0)} \\
&\quad= \frac{\begin{array}{rcr}(x_1 - x_0)\Delta^{n-1} f[x_2, x_3 \dots x_{n+1}] & - & (x_2 - x_0)\Delta^{n-1} f[x_1, x_3 \dots x_{n+1}] \\ & + & (x_2 - x_1)\Delta^{n-1} f[x_0, x_3 \dots x_{n+1}]\end{array}}{(x_2 - x_1)(x_1 - x_0)(x_2 - x_0)} \\
&\quad= \frac{\begin{array}{c}(x_1 - x_0)\left(\Delta^{n-1} f[x_1, x_3 \dots x_{n+1}] - \Delta^{n-1} f[x_2, x_3 \dots x_{n+1}]\right) \\ -(x_1 - x_2)\left(\Delta^{n-1} f[x_1, x_3 \dots x_{n+1}] - \Delta^{n-1} f[x_0, x_3 \dots x_{n+1}]\right)\end{array}}{(x_1 - x_2)(x_1 - x_0)(x_2 - x_0)} \\
&\quad= \frac{\Delta^n f[x_2, x_1, \dots x_{n+1}] - \Delta^n f[x_0, x_1, \dots x_{n+1}]}{x_1 - x_0} \\
&\quad= \Delta^{n+1} f[x_0, x_2, x_1, \dots, x_{n+1}]
\end{aligned}$$

The divided difference is symmetric under transposing x_k with x_{k+1} when $k > 1$:

$$\begin{aligned}
&\Delta^{n+1} f[x_0, x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{n+1}] \\
&\qquad = \frac{\Delta^n f[x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{n+1}] - \Delta^n f[x_0, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{n+1}]}{x_1 - x_0} \\
&\qquad = \frac{\Delta^n f[x_1, x_2, \ldots, x_{k+1}, x_k, \ldots, x_{n+1}] - \Delta^n f[x_0, x_2, \ldots, x_{k+1}, x_k, \ldots, x_{n+1}]}{x_1 - x_0} \\
&\qquad = \Delta^{n+1} f[x_0, x_1, x_2, \ldots, x_{k+1}, x_k, \ldots, x_{n+1}]
\end{aligned}$$

Since any permutation of $x_0, x_1, \ldots x_{n+1}$ can be genreated from the transpositions of x_k with x_{k+1} for k between 0 and n, it follows that $\Delta^{n+1} f[x_0, x_1, \ldots, x_k, x_{k+1}, \ldots, x_{n+1}]$ is symmetric under all permutaions of $x_0, x_1, \ldots x_{n+1}$. $\square$