explicit formula for divided differences

Canonical name	ExplicitFormulaForDividedDifferences
Date of creation	2013-03-22 14:41:16
Last modified on	2013-03-22 14:41:16
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Numerical id	21
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Entry type	Definition
Classification	msc 39A70

Theorem 1. *The n-th divided difference of a function f can be written explicitly as*

$$\Delta^n f[x_0, \dots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod\limits_{\substack{0 \le j \le n \\ j \neq i}} (x_i - x_j)}$$

Proof. We will proceed by recursion on n. When $n = 1$, the formula to be proven reduces to

$$\Delta^1 f[x_0, x_1] = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0},$$

which agrees with the definition of $\Delta^1 f$.

To prove that this is correct when $n > 1$, one needs to check that it the recurrence relation for divided differences.

$$\sum_{\substack{0\le i\le n+1\\ i\neq 0}} \frac{f(x_i)}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i\\ j\neq 0}}(x_i-x_j)} - \sum_{\substack{0\le i\le n+1\\ i\neq 1}} \frac{f(x_i)}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i\\ j\neq 1}}(x_i-x_j)}$$

$$= \frac{f(x_1)}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i\\ j\neq 0}}(x_1-x_j)} - \frac{f(x_0)}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i\\ j\neq 1}}(x_0-x_j)} +$$

$$\sum_{2\le i\le n+1} f(x_i)\left(\frac{1}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i\\ j\neq 0}}(x_i-x_j)} - \frac{1}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i\\ j\neq 1}}(x_i-x_j)}\right)$$

$$= -\frac{(x_0-x_1)f(x_0)}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i}}(x_0-x_j)} + \frac{(x_1-x_0)f(x_1)}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i}}(x_1-x_j)} +$$

$$\sum_{2\le i\le n+1} f(x_i)\left(\frac{x_i-x_0}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i}}(x_i-x_j)} - \frac{x_i-x_1}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i}}(x_i-x_j)}\right)$$

$$= \frac{(x_1-x_0)f(x_0)}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i}}(x_0-x_j)} + \frac{(x_1-x_0)f(x_1)}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i}}(x_1-x_j)} + (x_1-x_0)\sum_{2\le i\le n+1} \frac{(x_1-x_0)f(x_i)}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i}}(x_i-x_j)}$$

$$= (x_1-x_0)\sum_{0\le i\le n+1} \frac{f(x_i)}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i}}(x_i-x_j)}$$

Thus, we see that, if

$$\Delta^n f[x_0,\ldots,x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod\limits_{\substack{0\le j\le n\\ j\neq i}}(x_i-x_j)},$$

then

$$\Delta^{n+1} f[x_0,\ldots,x_{n+1}] = \sum_{i=0}^{n+1} \frac{f(x_i)}{\prod\limits_{\substack{0\le j\le n+1\\ j\neq i}}(x_i-x_j)}.$$

Hence, by induction, the formula holds for all n. $\square$

This formula may be phrased another way by introducing the polynomials p_n defined as

$$p_n(x) = \prod_{i=0}^{n} (x - x_i).$$

We may write

$$\Delta^n f[x_0, \ldots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{p'(x_i)}.$$

Either form of the explicit formula makes it obvious that divided differences are symmetric functions of $x_0, x_1, \ldots$.