example of Cauchy multiplication rule

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Let us form the Taylor expansion of $e^x \sin y$ starting from the known Taylor expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots,$$

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + - \ldots$$

and multiplying these series with Cauchy multiplication rule. As power series, both series are absolutely convergent for all real (and complex) values of x and y. The rule gives immediately the series

$$y+(-\frac{y^3}{3!}+xy)+(\frac{y^5}{5!}-\frac{xy^3}{3!}+\frac{x^2y}{2!})+(-\frac{y^7}{7!}+\frac{xy^5}{5!}-\frac{x^2y^3}{2!3!}+\frac{x^3y}{3!})+(\frac{y^9}{9!}-\frac{xy^7}{7!}+\frac{x^2y^5}{2!5!}-\frac{x^3y^3}{3!3!}+\frac{x^4y}{4!})+\ldots \tag{1}$$

The parenthesis expressions here seem a bit irregular, but we can regroup and rearrange the terms in new parentheses:

$$e^x \sin y \;=\; y + \frac{xy}{1!1!} + \left(\frac{x^2y}{2!1!} - \frac{y^3}{3!}\right) + \left(\frac{x^3y}{3!1!} - \frac{xy^3}{1!3!}\right) + \left(\frac{x^4y}{4!1!} - \frac{x^2y^3}{2!3!} + \frac{y^5}{5!}\right) + \ldots \tag{2}$$

It's clear that the last series precisely the same terms as the preceding one. The regrouping and the rearranging of the terms is allowable, since also (1) is converges absolutely. In fact, if one would multiply the series of e^x with the series $y+\frac{y^3}{3!}+\frac{y^5}{5!}+\ldots$ of $\sinh y$ (which converges absolutely $\forall x \in \mathbb{R}$), one would get the series like (1) but all signs "+"; by the Cauchy multiplication rule this series converges especially for each positive x and y, in which case it is a series with positive terms; hence (1) is absolutely convergent.

The form (2) can be obtained directly from the `http://planetmath.org/TaylorSeries`Taylor series formula.