# getting Taylor series from differential equation

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If a given function $f$ satisfies a differential equation, the Taylor series of $f$ can sometimes be obtained easily.

Let

$$f(x) = \sin(m \arcsin x),$$

where $m$ is a non-zero , be an example (`http://planetmath.org/Cf`cf. the cyclometric functions). We form the derivatives

$$f'(x) = \frac{m}{\sqrt{1-x^2}} \cos(m \arcsin x),$$

$$f''(x) = -\frac{m^2}{1-x^2} \sin(m \arcsin x) + \frac{mx}{(1-x^2)\sqrt{1-x^2}} \cos(m \arcsin x),$$

which show that $f$ satisfies the differential equation

$$(1-x^2)f'' - xf' + m^2 f = 0.$$

Differentiating this repeatedly gives the equations

$$(1-x^2)f''' - 3xf'' + (m^2-1)f' = 0,$$

$$(1-x^2)f^{(4)} - 5xf''' + (m^2-4)f'' = 0,$$

and so on. Using the sum of odd numbers $1+3+5+\cdots+(2n-1) = n^2$ and induction on $n$ yields the recurrence relation

$$(1-x^2)f^{(n+2)} - (2n+1)xf^{(n+1)} + (m^2-n^2)f^{(n)} = 0.$$

Plugging in $x = 0$ yields

$$f^{(n+2)}(0) = (n^2-m^2)f^{(n)}(0) \quad (n = 0,\, 1,\, 2,\, ...).$$

Since $f'(0) = m,$ we have that

$$f^{(2n+1)}(0) = m(1^2-m^2)(3^2-m^2)\ldots((2n-1)^2-m^2),$$

whereas all even derivatives of $f$ vanish at $x = 0$. (Note that $f$ is an odd function.) Thus, we obtain the Taylor of $f$:

$$\sin(m \arcsin x) = \frac{m}{1!}x + \frac{m(1^2-m^2)}{3!}x^3 + \frac{m(1^2-m^2)(3^2-m^2)}{5!}x^5 + \cdots$$

By the ratio test, this series converges for $|x| < 1$.

## References

[1] Ernst Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset* I. WSOY. Helsinki (1950).