asymptotic bounds for factorial

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Owner	stevecheng (10074)
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Stirling's formula furnishes the following asymptotic bound for the factorial:

$$n! = \sqrt{2\pi}\, n^{n+1/2} e^{-n} \left(1 + \Theta(\tfrac{1}{n})\right), \quad n \to \infty\,.$$

The following less precise bounds are useful for counting purposes in computer science:

$$n! = o(n^n) \tag{1}$$

$$n! = \omega(n^{\lambda n}) \qquad \text{for every } 0 \le \lambda < 1 \tag{2}$$

$$\log n! = \Theta(n \log n) \tag{3}$$

(o, ω, and Θ are Landau notation.)

Proof of the upper bound of equation (**??**). We must show that for every $\varepsilon > 0$, we have

$$n! \le \varepsilon\, n^n \quad \text{for large enough } n.$$

To see this, write

$$n! = \underbrace{\sqrt{2\pi}\, \frac{\sqrt{n}}{e^n} \left(1 + O(\tfrac{1}{n})\right)}\, n^n\,,$$

and observe that the middle quantity tends to 0 as $n \to \infty$, so it can be made smaller than any fixed ε by taking n large. □

Proof of the lower bound of equation (**??**). We are to show that for every $C > 0$, we have

$$n! \ge C\, n^{\lambda n} \quad \text{for large enough } n.$$

We write:

$$n! = \underbrace{\sqrt{2\pi n} \left(1 + O(\tfrac{1}{n})\right) \left(\frac{n}{e^{1/(1-\lambda)}}\right)^{(1-\lambda)n}}\, n^{\lambda n}\,,$$

and the middle expression can be made to be $\ge C$ by taking n to be large. □

Proof of bound for $\log n!$, *equation* (**??**). Showing $\log n! = O(n \log n)$ is simple:

$$\begin{aligned} n! &= n(n-1)\cdots(2)(1) \leq n^n \\ \log n! &\leq n \log n \end{aligned}$$

for all $n \geq 1$.

To show $\log n! = \Omega(n \log n)$, we can take equation (**??**) with the constants $C = 1$ and $\lambda = \frac{1}{2}$. Then

$$\log n! \geq \tfrac{1}{2}\, n \log n$$

for large enough n. In fact, it may be checked that $n \geq 1$ suffices. □

References

[1] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein. *Introduction to Algorithms*, second edition. MIT Press, 2001.

[2] Michael Spivak. *Calculus*, third edition. Publish or Perish, 1994.