Taylor formula remainder: various expressions

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Let $f:\mathbb{R}\to\mathbb{R}$ be an $n+1$ times differentiable function, and let $T_{n,a}(x)$ its n^{th} -degree Taylor polynomial;

Then the following expressions for the remainder $R_{n,a}(x)=f(x)-T_{n,a}(x)$ hold:

1)

$$R_{n,a}(x)=\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt$$

(Integral form)

2)

$$R_{n,a}(x)=\frac{f^{(n+1)}(\eta)}{n!p}(x-\eta)^{n-p+1}(x-a)^p$$

for a $\eta(x)\in(a,x)$ and $\forall p>0$ (Schlömilch form)

3)

$$R_{n,a}(x)=\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a)$$

for a $\xi(x)\in(a,x)$ (Cauchy form)

4)

$$R_{n,a}(x)=\frac{f^{(n+1)}(\vartheta)}{(n+1)!}(x-a)^{n+1}$$

for a $\vartheta(x)\in(a,x)$ (Lagrange form)

Moreover the following result holds for the integral of the remainder from the center point a to an arbitrary point b:

5)

$$\int_a^b R_{n,a}(x)dx=\int_a^b \frac{f^{(n+1)}(x)}{(n+1)!}(b-x)^{n+1}dx$$

Proof:

1) Let's proceed by induction.

$n=0$. $\int_a^x f^{'}(t)dt=f(x)-f(a)=R_{0,a}(x)$, since $T_{0,a}(x)=f(a)$.

Let's take it for true that $R_{n-1,a}(x)=\frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}dt$, and let's compute $\int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt$ by parts.

$$\begin{aligned}
& \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt = \\
= \quad & \frac{f^{(n)}(t)}{n!}(x-t)^n|_a^x + \int_x^a \frac{f^{(n)}(t)}{n!} n(x-t)^{n-1} dt \\
= \quad & -\frac{f^{(n)}(a)}{n!}(x-a)^n + \int_x^a \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} dt \\
= \quad & R_{n-1,a}(x) - \frac{f^{(n)}(a)}{n!}(x-a)^n \\
= \quad & f(x) - T_{n-1,a} - \frac{f^{(n)}(a)}{n!}(x-a)^n \\
= \quad & f(x) - T_{n,a}(x) = R_{n,a}(x).
\end{aligned}$$

2) Let's write the remainder in the integral form this way:

$$R_{n,a}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^{n-p+1}(x-t)^{p-1} dt$$

Now, since $(x-t)^{p-1}$ doesn't change sign between a and x, we can apply the `http://planetmath.org/IntegralMeanValueTheorem`integral Mean Value theorem. So a point $\eta(x) \in (a,x)$ exists such that

$$\begin{aligned}
& R_{n,a}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^{n-p+1}(x-t)^{p-1} dt \\
= \quad & \frac{f^{(n+1)}(\eta)}{n!}(x-\eta)^{n-p+1}\int_a^x (x-t)^{p-1} dt \\
= \quad & \frac{f^{(n+1)}(\eta)}{n!p}(x-\eta)^{n-p+1}(x-a)^p
\end{aligned}$$

(Note that the condition $p > 0$ is needed to ensure convergence of the integral)

3) and 4) are obtained from Schlömilch form by plugging in $p = 1$ and $p = n+1$ respectively.

5) Let's start from the right-end side:

$$
\begin{aligned}
&\int_a^b \frac{f^{(n+1)}(x)}{(n+1)!}(b-x)^{n+1}dx = \\
= \quad & \frac{f^{(n)}(x)}{(n+1)!}(b-x)^{n+1}|_a^b + \int_a^b \frac{f^{(n)}(x)}{(n+1)!}(n+1)(b-x)^n dx \\
= \quad & -\frac{f^{(n)}(a)}{(n+1)!}(b-a)^{n+1} + \int_a^b \frac{f^{(n)}(x)}{n!}(b-x)^n dx \\
= \quad & \ldots = -\sum_{k=0}^{n} \frac{f^{(k)}(a)}{(k+1)!}(b-a)^{k+1} + \int_a^b f(x)dx \\
= \quad & -\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \int_a^b (x-a)^k dx + \int_a^b f(x)dx \\
= \quad & \int_a^b \left(-\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + f(x) \right) dx = \int_a^b R_{n,a}(x)dx.
\end{aligned}
$$

Note:

1) The proof of the integral form could also be stated as follow:

Let

$\phi(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$

Then $\phi(x) = f(x)$ and $\phi(a) = T_{n,a}(x)$, so that $R_{n,a}(x) = \phi(x) - \phi(a) = \int_a^x \phi'(t)dt$.

Let's now compute $\phi'(t)$.

$$
\begin{aligned}
&\phi'(t) = \sum_{k=0}^{n} \frac{1}{k!}\left[f^{(k+1)}(t)(x-t)^k - f^{(k)}(t)k(x-t)^{k-1}\right] \\
= \quad & \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} \\
= \quad & \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!}(x-t)^k \\
= \quad & \frac{f^{(n+1)}(t)}{n!}(x-t)^n.
\end{aligned}
$$

2) From the integral form of the remainder it is possible to obtain the entire Taylor formula; indeed, repeatly integrating by parts, one gets:

$$
\begin{aligned}
& \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt = \frac{f^{(n)}(t)}{n!}(x-t)^n|_a^x + \int_a^x \frac{f^{(n)}(t)}{n!} n(x-t)^{n-1} dt \\
= & \ -\frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} dt \\
= & \ -\frac{f^{(n)}(a)}{n!}(x-a)^n - \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \int_a^x \frac{f^{(n-1)}(t)}{(n-2)!}(x-t)^{n-2} dt \\
= & \ ... = -\frac{f^{(n)}(a)}{n!}(x-a)^n - \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} - ... - \frac{f^{(n-k+1)}(a)}{(n-k+1)!}(x-a)^{n-k+1} + \int_a^x \frac{f^{(n-k+1)}}{(n-k} \\
& \ ... \\
= & \ -\sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f'(t) dt \\
= & \ -\sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + f(x) - f(a) \\
= & \ -\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + f(x).
\end{aligned}
$$

that is

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_{n,a}(x).
\end{aligned}
$$