# example of use of Taylor's theorem

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In this entry we use Taylor's Theorem in the following form:

**Theorem 1** (Taylor's Theorem: Bounding the Error)**.** *Suppose $f$ and all its derivatives are continuous. If $T_n(x)$ is the $n$-th Taylor polynomial of $f(x)$ around $x = a$, then the error, or the difference between the real value of $f$ and the values of $T_n(x)$ is given by:*

$$|E_n(x)| = |f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

*where $M$ is the maximum value of $f^{(n+1)}$ (the $n+1$-th derivative of $f$) in the interval between $a$ and $x$.*

**Example 2.** Suppose we want to approximate $e$ using the Taylor polynomial of degree 4, $T_4(x)$, around $x = 0$ for the function $e^x$. We know that

$$T_4(x) = 1 + x + x^2/2 + x^3/3! + x^4/4!$$

so we are asking how close are $e$ and $T_4(1) = 1+1+1/2+1/6+1/24$. In order to use the formula in the theorem, we just need to find $M$, the maximum value of the 4th derivative of $e^x$ between $a = 0$ and $x = 1$. Since $f^{(4)} = e^x$ and $e^x$ is strictly increasing, the maximum in $(0, 1)$ happens at $x = 1$. Thus $M = e$ which is a number, say, less than 3. Therefore:

$$|E_4(1)| = |f(1)-T_4(1)| = |e-(1+1+1/2+1/6+1/24)| \leq \frac{M}{5!}|1-0|^5 = \frac{M}{5!} < \frac{3}{5!} = 0.025$$

Thus the approximation has an error of less than 0.025. Actually, if we use a calculator we obtain that the error is *exactly* 0.0099. But, of course, *the whole point of the theorem is not to use a calculator.*

**Example 3.** What Taylor polynomial $T_n(x)$ (what $n$) should we use to approximate $e$ within 0.0001? As above, we will be using the Taylor polynomial $T_n(x)$ for $e^x$, evaluated at $x = 1$. Thus, we want the error $|E_n(1)| < 0.0001$. Notice all derivatives are $e^x$ and the maximum happens at $x = 1$, where $e^1 = e$, so for all derivatives $M < 3$. Hence by the theorem:

$$|E_n(1)| = |f(1)-T_n(1)| = |e-(1+1+1/2+\ldots+1/n!)| \leq \frac{M}{(n+1)!}|1-0|^{n+1} = \frac{M}{(n+1)!} < \frac{3}{(n+1)!}$$

So we need $3/(n+1)! < 0.0001$. Try several values of $n$ until that is satisfied:

$$3/2 = 1.5,\ 3/3! = 0.5,\ 3/4! = 0.125,\ 3/5! = 0.025,\ 3/6! = 0.00416$$

$$3/7! = 0.00059, \quad 3/8! = 0.00007$$

Thus $n = 7$ should work. So we just need $T_7(x)$, or add $1+1+1/2+\ldots+1/7!$.