examples on how to find Taylor series from other known series

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In this section we present numerous examples that provide a number of useful procedures to find new Taylor series from Taylor series that we already know. However, we are only worried about "computing" and we don't worry (for now) about the convergence of the series we find.

We know "by heart" the following series:

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
(1+x)^p &= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots + \frac{p(p-1)\cdot\ldots\cdot(p-(n-1))}{n!}x^n + \ldots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}
\end{aligned}
$$

Remark 1. *Recall that the first three have radius of convergence $R = \infty$ but for the last two $R = 1$.*

Example 1. Find the Taylor series about $x = 0$ for $\sin(x^2)$. If we try to take derivatives then we soon realize that consecutive derivatives get extremely hard to compute. However, one can do a simple trick. Since we know the Taylor series for $\sin(x)$ we can evaluate it at x^2:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \ldots = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots$$

One can also use the Σ notation:

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(2n+1)}}{(2n+1)!}.$$

Example 2. Find the Taylor series about $x = 0$ for e^{-x^2} (this is a very important function, for example in probability theory). Again, we use the simple Taylor series of e^x:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3!} + \ldots = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \ldots$$

Using the sigma notation we obtain:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

Example 3. Series can also be multiplied by x. For example, we find the Taylor series for xe^x:

$$xe^x = x(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \ldots)$$

or

$$xe^x = x\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

Example 4. Series can also be divided by x **provided** that the result has **only** non-negative exponents. For example, we find the Taylor series for $\frac{\ln(1+x)}{x}$:

$$\frac{\ln(1+x)}{x} = \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots$$

or

$$\frac{\ln(1+x)}{x} = \frac{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}}{x} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n-1}}{n}$$

Example 5. As well, we can multiply two Taylor series (term by term). Suppose we want to find the Taylor polynomial of degree 3 about $x = 0$ of $e^x \cos x$. Then we can multiply the respective Taylor polynomials of degree 3 of e^x and $\cos x$ and disregard any term higher than 3:

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2}\right)$$

$$= \left(1 - \frac{x^2}{2}\right) + \left(x - \frac{x^3}{2}\right) + \left(\frac{x^2}{2}\right) + \left(\frac{x^3}{6}\right) + \dots$$

Since every other term in the product is of degree higher than 3 we disregard them. Thus:

$$T_3(x) = \left(1 - \frac{x^2}{2}\right) + \left(x - \frac{x^3}{2}\right) + \left(\frac{x^2}{2}\right) + \left(\frac{x^3}{6}\right) = 1 + x - \frac{x^3}{2} + \frac{x^3}{6} = 1 + x - \frac{x^3}{3}.$$

Example 6. Find the first three terms of the Taylor series for $\sqrt{1 + 2\sin x}$. Since $\sqrt{1+x} = (1+x)^{1/2}$ we know that:

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots$$

Thus:

$$\sqrt{1 + 2\sin x} = 1 + \frac{2\sin x}{2} - \frac{(2\sin x)^2}{8} + \dots$$

Moreover, $\sin x = x - \frac{x^3}{3!} + \dots$:

$$\sqrt{1 + 2\sin x} = 1 + \frac{2(x - \frac{x^3}{3!} + \dots)}{2} - \frac{(2(x - \frac{x^3}{3!} + \dots))^2}{8} + \dots$$

By disregarding other than the first term in $\sin x$ we obtain the first three terms of the series are:

$$\sqrt{1 + 2\sin x} = 1 + \frac{2x}{2} - \frac{(2x)^2}{8} + \dots = 1 + x - \frac{x^2}{2} + \dots$$

Example 7 (Differentiation). Notice that we can deduce the series of $\sin x$ from the series for $\cos x$ by differentiating. Indeed $\frac{d}{dx}\cos x = -\sin x$ and we differentiate (term by term) the Taylor series of $\cos x$ we obtain the Taylor series of $\sin x$ (DO IT!).

Another example. Let us deduce the Taylor series of $\frac{1}{(1-x)^2}$. Notice that $\frac{d}{dx}\frac{1}{(1-x)} = \frac{1}{(1-x)^2}$. Since:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

by deriving both sides we obtain:

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \dots = \sum_{n=0}^{\infty}(n+1)x^n$$

and if we derive again we obtain:

$$\frac{2}{(1-x)^3} = 2 + 6x + 12x^2 \dots$$

Thus,

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \dots = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n.$$

Example 8 (Integration). Finally, we will deduce the Taylor series for $\arctan x$ using integration (term by term). Notice that:

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

Moreover, since $1/(1-x) = 1 + x + x^2 + x^3 + \dots$ by substituting x by $-x^2$ we obtain:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Thus, the Taylor series of $\arctan x$ can be constructed integrating the previous one:

$$\arctan x = \int \frac{1}{1+x^2} dx = \int (1 - x^2 + x^4 - x^6 + \dots) dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

In Σ notation:

$$\arctan x = \int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Example 9. As an application of the previous example, we compute π. Indeed:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

converges between $-1 \leq x \leq 1$ and in particular

$$\arctan 1 = \frac{\pi}{4}$$

by the definition of $\tan x$ and $\arctan x$. Thus:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

and

$$\pi = 4 \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \right) = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \right)$$

For example, if one adds up to the $1/9$ term, one obtains the approximation $\pi \approx 3.33$. Unfortunately, the convergence is very slow. If you want to have about m correct digits then you have to add about $10^m/2$ terms. For example, if you add $10^3/2 = 500$ terms we get $3.143588\dots$.