lecture notes on polynomial interpolation

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1 Summary of notation and terminonlogy

- *multi-evaluation mapping*: $\mathrm{ev}_{\mathbf{x}} : \mathcal{P}_m \to \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$. Here $\mathcal{P}_m$ is the vector space of polynomials of degree m or less.
- *interpolation mapping*: $\mathrm{pol}_{\mathbf{x}} : \mathbb{R}^n \to \mathcal{P}_{n-1}$, $\mathbf{x} \in \mathbb{R}^n$ where $x_1, \dots, x_n$ are distinct;
- *Vandermonde matrix and polynomial*;
- *overdetermined and underdetermined linear system;*
- *overdetermined and underdetermined interpolation problem*

2 The polynomial interpolation problem

Definition 1. Let $\mathbf{x} \in \mathbb{R}^n$ be given. Define $\mathrm{ev}_{\mathbf{x}} : \mathcal{P}_m \to \mathbb{R}^n$, the multi-evaluation mapping, to be the linear transformation given by

$$\mathrm{ev}_{\mathbf{x}} : p \to \begin{bmatrix} p(x_1) \\ \vdots \\ p(x_n) \end{bmatrix}, \quad p \in \mathcal{P}_m.$$

Problem 2 (Polynomial interpolation)**.** Given $(x_1, y_1), \dots, (x_n, y_n) \in \mathbb{R}^2$ to find a polynomial $p \in \mathcal{P}_m$ such that $p(x_i) = y_i$, $i = 1, \dots, n$. Equivalently, given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ to find the preimage $\mathrm{ev}_{\mathbf{x}}^{-1}(\mathbf{y})$.

Theorem 3. *Fix $\mathbf{x} \in \mathbb{R}^n$. The multi-evaluation mapping $\mathrm{ev}_{\mathbf{x}} : \mathcal{P}_{n-1} \to \mathbb{R}^n$ is an isomorphism if and only if the components $x_1, \dots, x_n$ are distinct.*

3 Vandermonde matrix, polynomial, and determinant

Definition 4. For a given $\mathbf{x} \in \mathbb{R}^n$, the following matrix

$$\mathsf{VM}(\mathbf{x}) = \mathsf{VM}(x_1, \dots, x_n) = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}$$

is called the *Vandermonde matrix*. The expression,

$$\mathsf{V}(\mathbf{x}) = \mathsf{V}(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} (x_j - x_i)$$

is called the Vandermonde polynomial. Note: we define $\mathsf{V}(x_1) = 1$; the usual convention is that the empty product is equal to 1.

Proposition 5. *The Vandermonde matrix is the transformation matrix of* $\mathrm{ev}_{\mathbf{x}}$ *with the monomial basis* $[1, x, \ldots, x^n]$ *as the input basis and the standard basis* $[\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}]$ *as the output basis.*

Theorem 6. *The Vandermonde polynomial gives the determinant of the Vandermonde matrix:*

$$\begin{vmatrix} 1 & x_1 & \ldots & x_1^{n-1} \\ 1 & x_2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \ldots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n+1} (x_j - x_i), \tag{1}$$

or more succinctly,

$$\det \mathsf{VM}(\mathbf{x}) = \mathsf{V}(\mathbf{x}).$$

Proof. We will prove formula (**??**) by induction on n. Note that

$$\mathsf{VM}(x_1) = \begin{bmatrix} 1 \end{bmatrix}, \quad \mathsf{V}(x_1) = 1.$$

Evidently then, the formula works for $n = 1$. Next, suppose that we believe the formula for a given n. We show that the formula is valid for $n + 1$. For $x_1, \ldots, x_n \in \mathbb{R}$ and a variable x, and consider the n^{th} degree polynomial

$$p(x) = \det \mathsf{VM}(x_1, \ldots, x_n, x) = \begin{vmatrix} 1 & x_1 & \ldots & x_1^{n-1} & x_1^n \\ 1 & x_2 & \ldots & x_2^{n-1} & x_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & \ldots & x_n^{n-1} & x_n^n \\ 1 & x & \ldots & x^{n-1} & x^n \end{vmatrix}$$

By the properties of determinants, $x_1, \ldots, x_n$ are roots of $p(x)$. Taking the cofactor expansion along the bottom row, we see that the coefficient of x^n is $\mathsf{V}(x_1, \ldots, x_n)$. Therefore,

$$p(x) = \mathsf{V}(x_1, \ldots, x_n)(x - x_1) \cdots (x - x_n) = \mathsf{V}(x_1, \ldots, x_n, x),$$

as was to be shown. $\square$

4 Lagrange interpolation formula

Let $x_1, \ldots, x_n \in \mathbb{R}$ be distinct. We know that $\mathrm{ev}_{\mathbf{x}} : \mathcal{P}_{n-1} \to \mathbb{R}^n$ is invertible. Let $\mathrm{pol}_{\mathbf{x}} : \mathbb{R}^n \to \mathcal{P}_{n-1}$ denote the inverse. In principle, this inverse is described by the inverse of the Vandermonde matrix. Is there another way to solve the interpolation problem? For $i = 1, \ldots, n$ let us define the polynomial

$$p_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

These $n-1$ degree polynomials have been "engineered" so that $p_i(x_i) = 1$ and so that $p_i(x_j) = 0$ for $i \neq j$

Theorem 7 (Lagrange interpolation formula)**.** *Let $x_1, \ldots, x_n \in \mathbb{R}$ be distinct. Then,*

$$\mathrm{pol}_{\mathbf{x}}(\mathbf{y}) = y_1 p_1 + \cdots + y_n p_n.$$

5 Underdetermined and overdetermined interpolation

Definition 8. Let $T : \mathcal{U} \to \mathcal{V}$ be a linear transformation of finite dimensional vector spaces. A linear problem is an equation of the form

$$T(\mathbf{u}) = \mathbf{v},$$

where $\mathbf{v} \in \mathcal{V}$ is given, and $\mathbf{u} \in \mathcal{U}$ is the unknown. To be more precise, the problem is to determine the preimage $T^{-1}(\mathbf{v})$ for a given $\mathbf{v} \in \mathcal{V}$. We say that the problem is overdetermined if $\dim \mathcal{V} > \dim \mathcal{U}$, i.e. if there are more equations than unknowns. The linear problem is said to be underdetermined if $\dim \mathcal{V} < \dim \mathcal{U}$, i.e., if there are more variables than equations.

Remark 9. By the rank-nullity theorem, an underdetermined linear problem will either be inconsistent, or will have multiple solutions. Thus, an underdetermined system arises when T is not one-to-one; i.e., the kernel $\ker(T)$ is non-trivial. An overdetermined linear system is inconsistent, unless $\mathbf{v}$ satisfies a number linear compatibility constraints, equations that describe the image $\mathrm{Im}(T)$. To put it another way, an overdetermined system arises when T is not onto.

Definition 10. Let $x_1, \dots, x_n \in \mathbb{R}$ be distinct and let $\mathrm{ev}_{\mathbf{x}} : \mathcal{P}_m \to \mathbb{R}^n$ be the corresponding multi-evaluation mapping. Let $\mathbf{y} \in \mathbb{R}^n$ be given. The linear equation

$$\mathrm{ev}_{\mathbf{x}}(p) = \mathbf{y}, \quad p \in \mathcal{P}_m$$

is called an underdetermined interpolation problem if $m \geq n$. If $m \leq n-2$, we call the above equation an overdetermined interpolation problem. Note that if $m = n-1$, the interpolation problem is "just right"; there is exactly one solution, namely the polynomial $\mathrm{pol}_{\mathbf{x}}(\mathbf{y})$ given by the Lagrange interpolation formula.

Proposition 11 (Underdetermined interpolation). *Let $x_1, \dots, x_n \in \mathbb{R}$ be distinct. Define*

$$q(x) = (x - x_1) \cdots (x - x_n)$$

to be the n^{th} degree polynomial with the x_i as its roots. Suppose that $m \geq n$. Then, the multi-evaluation mapping $\mathrm{ev}_{\mathbf{x}} : \mathcal{P}_m \to \mathbb{R}^n$ is not one-to-one. A basis for the kernel is given by $q(x), xq(x), \dots, x^{m-n}q(x)$. Let $y_1, \dots, y_n \in \mathbb{R}$ be given. The solution set to the interpolation problem $\mathrm{ev}_{\mathbf{x}}(p) = \mathbf{y}$, is given by

$$\mathrm{ev}_{\mathbf{x}}^{-1}(\mathbf{y}) = \{r + sq : s \in \mathcal{P}_{m-n}\},$$

where $r = \mathrm{pol}_{\mathbf{x}}(\mathbf{y})$ is the $n - 1^{st}$ degree polynomial given by the Lagrange interpolation formula. To put it another way, the general solution of the equation

$$\mathrm{ev}_{\mathbf{x}}(p) = \mathbf{y}$$

is given by

$$p = r + sq, \quad s \in \mathcal{P}_{n-m} \text{ free.}$$

Proposition 12 (Overdetermined interpolation). *Let $x_1, \dots, x_n \in \mathbb{R}$ be distinct. Suppose that $m \leq n - 2$. Then, the multi-evaluation mapping $\mathrm{ev}_{\mathbf{x}} : \mathcal{P}_m \to \mathbb{R}^n$ is not onto. If $m = n - 2$, then $\mathbf{y} \in \mathbb{R}^4$ belongs to $\mathrm{Im}(\mathrm{ev}_{\mathbf{x}})$ if and only if*

$$\begin{vmatrix} 1 & x_1 & \dots & x_1^m & y_1 \\ 1 & x_2 & \dots & x_2^m & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_{n-1} & \dots & x_{n-1}^m & y_{n-1} \\ 1 & x_n & \dots & x_n^m & y_n \end{vmatrix} = 0$$

More generally, for any $m \le n-2$*, the interpolation problem* $\mathrm{ev}_{\mathbf{x}}(p) = \mathbf{y},\ \mathbf{y} \in \mathbb{R}^n$ *has a solution if and only if* $\operatorname{rank} M(\mathbf{x}, \mathbf{y}) = m + 1$ *where*

$$M(\mathbf{x},\mathbf{y}) = \begin{bmatrix} 1 & x_1 & \dots & x_1^m & y_1 \\ 1 & x_2 & \dots & x_2^m & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_{m+1} & \dots & x_{m+1}^m & y_{m+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \dots & x_n^m & y_n \end{bmatrix}$$

Remark 13. The compatibility constraints on $y_1, \dots, y_n$ for an overdetermined system amount to the condition that $\mathbf{y}$ lie in the image of $\mathrm{ev}_{\mathbf{x}}$, or what is equivalent, belong to the column space of the corresponding transformation matrix. We can therefore obtain the constraint equations by row reducing the augmented matrix $M(\mathbf{x}, \mathbf{y})$ to echelon form. The back entries of the last $n - m - 1$ rows will hold the constraint equations. Equivalently, we can solve the interpolation problem for $(x_1, y_1), \dots, (x_m, y_m)$ to obtain a $p = y_1 p_1 + y_{m+1} p_{m+1} \in \mathcal{P}_m$. The additional equations

$$y_i = y_1 p_1(x_i) + \dots + y_{m+1} p_{m+1}(x_i),\ i = m+2, \dots, n$$

are the desired compatibility constraints on $y_1, \dots, y_n$.