properties of orthogonal polynomials

Canonical name	PropertiesOfOrthogonalPolynomials
Date of creation	2013-03-22 19:05:34
Last modified on	2013-03-22 19:05:34
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	12
Author	pahio (2872)
Entry type	Topic
Classification	msc 42C05
Classification	msc 33D45
Related topic	HilbertSpace
Related topic	TopicsOnPolynomials
Related topic	IndexOfSpecialFunctions
Related topic	OrthogonalityOfLaguerrePolynomials
Related topic	OrthogonalityOfChebyshevPolynomials
Defines	Rodrigues formula

A countable *system of orthogonal polynomials*

$$p_0(x),\, p_1(x),\, p_2(x),\, \ldots \tag{1}$$

on an interval $[a,\, b]$, where a inner product of two functions

$$(f,\, g) \;:=\; \int_a^b f(x)g(x)W(x)\,dx$$

is defined with respect to a weighting function $W(x)$, satisfies the `http://planetmath.org/Orthogo`
condition

$$(p_m,\, p_n) \;=\; 0 \quad \text{always when} \quad m \neq n.$$

One also requires that

$$\deg\left(p_n(x)\right) \;=\; n \quad \text{for all } n.$$

Such a system (1) may be used as basis for the vector space of functions defined on $[a,\, b]$, i.e. certain such functions f may be expanded as a `http://planetmath.org/FunctionSeries`series

$$f(x) \;=\; c_0p_0(x)+c_1p_1(x)+c_2p_2(x)+\ldots$$

where the coefficients c_n have the expression

$$c_n \;=\; \int_a^b f(x)p_n(x)W(x)\,dx.$$

Other properties

- The basis property of the system (1) comprises that any polynomial $P(x)$ of degree n can be uniquely expressed as a finite linear combination

$$P(x) \;=\; c_0p_0(x)+c_1p_1(x)+\ldots+c_np_n(x).$$

- Every member $p_n(x)$ of (1) is orthogonal to any polynomial $P(x)$ of degree less than n.

- There is a recurrence relation

$$p_{n+1}(x) \;=\; (a_n x + b_n)p_n(x) + c_n p_{n-1}(x)$$

enabling to determine a .

- The zeros of $p_n(x)$ are all real and belong to the open interval $(a,\, b)$; between two of those zeros there are always zeros of $p_{n+1}(x)$.

- The Sturm–Liouville differential equation

$$Q(x)p'' + L(x)p' + \lambda p \;=\; 0, \qquad (2)$$

where $Q(x)$ is a polynomial of at most degree 2 and $L(x)$ a linear polynomial, gives under certain conditions, as `http://planetmath.org/node/8719`solutions p a system of orthogonal polynomials $p_0,\, p_1,\, \ldots$ corresponding suitable values (eigenvalues) $\lambda_0,\, \lambda_1,\, \ldots$ of the parametre λ. Those satisfy the Rodrigues formula

$$p_n(x) \;=\; \frac{k_n}{W(x)}\frac{d^n}{dx^n}\left(W(x)[Q(x)]^n\right),$$

where k_n is a constant and

$$W(x) \;:=\; \frac{1}{Q(x)}e^{\int \frac{L(x)}{Q(x)}dx}.$$

The classical `http://planetmath.org/ChebyshevPolynomial`Chebyshev, `http://planetmath.org/HermitePolynomials`Hermite, `http://planetmath.org/Laguerr`
and Legendre polynomials all satisfy an equation (2).

[Not ready . . .]