# proof of Ingham Inequality



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Let

$$f(x) = \sqrt{\frac{1}{2\pi}} \sum_{j=-n}^{n} c_j e^{it_j x}.$$

If $k(x)$ is a integrable function in $(-\infty, \infty)$, let

$$K(u) = \int_{-\infty}^{\infty} k(x) e^{ixu} dx.$$

It is easy to prove the equalities.

$$\begin{aligned} \int_{-\infty}^{\infty} k(x)|f(x)|^2 dx &= \sum_{j,h=-n}^{n} c_j \overline{c_h} K(t_j - t_h). \\ \int_{-\infty}^{\infty} k(x) f(x) e^{-ixt_h} dx &= \sum_{j=-n}^{n} c_j K(t_j - t_h). \end{aligned}$$

For the rest of the proof we make the following choices:

$$k(x) = \begin{cases} \cos(x/2), & |x| \leq \pi \\ 0, & |x| > \pi \end{cases}$$

Then, after computation, we have

$$K(u) = \frac{4\cos(\pi u)}{1 - 4u^2}.$$

Let $T = \pi$ and $\gamma = 1 + \delta$ ($\delta = \varepsilon/\pi > 0$). Since $|t_{j+1} - t_j| \geq \gamma$, we have that $|t_j - t_h| \geq |j - h|\gamma$. If we suppose that $h$ is fixed, we have

$$\begin{aligned} \sum_{j\neq h} |K(t_j - t_h)| &= \sum_{j\neq h} \left| \frac{4\cos\pi(t_j - t_h)}{1 - 4(t_j - t_h)^2} \right| \leq \sum_{j\neq h} \frac{4}{4(j-h)^2\gamma^2 - 1} \leq \\ &\leq \frac{4}{\gamma^2} \sum_{j\neq h} \frac{1}{4(j-h)^2 - 1} < \frac{8}{\gamma^2} \sum_{r=1}^{\infty} \frac{1}{4r^2 - 1} = \\ &= \frac{8}{2\gamma^2} \sum_{r=1}^{\infty} \left( \frac{1}{2r-1} - \frac{1}{2r+1} \right) = \frac{4}{\gamma^2} = \frac{K(0)}{\gamma^2}. \end{aligned}$$

But, since $2|c_j\overline{c_h}| \leq |c_j|^2 + |c_h|^2$ and $K(t_j - t_h) = K(t_h - t_j)$, we find

$$\begin{aligned}
\int_{-\infty}^{\infty} k(x)|f(x)|^2 dx &= \sum_{j,h=-n}^{n} c_j\overline{c_h}K(t_j - t_h) = \\
&= \sum_j |c_j|^2 K(0) + \sum_{j,h;\, j\neq h} \mathcal{O}(j,h)\frac{|c_j|^2 + |c_h|^2}{2}|K(t_j - t_h)| \\
&= \sum_j |c_j|^2 \left( K(0) + \sum_{h\neq j} \mathcal{O}(j,h)|K(t_j - t_h)| \right),
\end{aligned}$$

where $|\mathcal{O}(j,h)| \leq 1$. Using the definition of $k$ and the previous inequality, we have

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \geq \sum_j |c_j|^2 K(0)(1 - \frac{1}{\gamma^2}),$$

and we have obtained the conclusion.