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Fourier transform

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Defines	first Parseval theorem

The *Fourier transform* $F(s)$ of a function $f(t)$ is defined as follows:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f(t)dt.$$

The Fourier transform exists if f is Lebesgue integrable on the whole real axis.

If f is Lebesgue integrable and can be divided into a finite number of continuous, monotone functions and at every point both one-sided limits exist, the Fourier transform can be inverted:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} F(s)ds.$$

Sometimes the Fourier transform is also defined without the factor $\frac{1}{\sqrt{2\pi}}$ in one direction, but therefore giving the transform into the other direction a factor $\frac{1}{2\pi}$. So when looking a transform up in a table you should find out how it is defined in that table.

The Fourier transform has some important properties, that can be used when solving differential equations. We denote the Fourier transform of f with respect to t in terms of s by $\mathcal{F}_t(f)$.

- $\mathcal{F}_t(af + bg) = a\mathcal{F}_t(f) + b\mathcal{F}_t(g)$,
  where a and b are constants.
- $\mathcal{F}_t\left(\frac{\partial}{\partial t} f\right) = is\mathcal{F}_t(f)$.
- $\mathcal{F}_t\left(\frac{\partial}{\partial x} f\right) = \frac{\partial}{\partial x}\mathcal{F}_t(f)$.
- We define the bilateral convolution of two functions f_1 and f_2 as:

  $$(f_1 * f_2)(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau)d\tau.$$

  Then the following equation holds:

  $$\mathcal{F}_t((f_1 * f_2)(t)) = \mathcal{F}_t(f_1) \cdot \mathcal{F}_t(f_2).$$

If $f(t)$ is some signal (maybe a wave) then the frequency domain of f is given as $\mathcal{F}_t(f)$. Rayleigh's theorem states that then the energy E carried by the signal f given by:

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

can also be expressed as:

$$E = \int_{-\infty}^{\infty} |\mathcal{F}_t(f)(s)|^2 ds.$$

In general we have:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\mathcal{F}_t(f)(s)|^2 ds,$$

also known as the *first Parseval theorem*.