almost periodic function (classical definition)

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A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *almost periodic* if, for every $\epsilon > 0$, there exists an a number $L_\epsilon > 0$ such that for every interval I of length L_ϵ there exists a number $\omega_I \in I$ such that

$$|f(x + \omega_I) - f(x)| < \epsilon$$

whenever $x \in \mathbb{R}$.

Intuition: we want the function to have an "approximate period". However, it is easy to write too weak condition. First, we want uniform estimate in x. If we allow ω to be small than the condition degenerates to uniform continuity. If we require a single ω, than the condition still is too weak (it allows pretty wide changes). For periodic function every multiple of a period is still a period. So, if the length of an interval is longer than the period, then the interval contains a period. The definition of almost periodic functions mimics the above property of periodic functions: every sufficiently long interval should contain an approximate period.

It is possible to generalize this notion. The range of the function can be taken to be a normed vector space — in the first definition, we merely need to replace the absolute value with the norm:

$$\|f(x + \omega) - f(x)\| < \epsilon$$

In the second definition, interpret uniform convergence as uniform convergence with respect to the norm. A common case of this is the case where the range is the complex numbers. It is worth noting that if the vector space is finite dimensional, a function is almost periodic if and only if each of its components with respect to a basis is almost periodic.

Also the domain may be taken to be a group G. A function is called almost periodic iff set of its translates is pre-compact (compact after completion). Equivalently, a continuous function f on a topological group G is almost periodic iff there is a compact group K, a continuous function g on K and a (continuous) homomorphism h form G to K such that f is the composition of g and h. The classical case described above arises when the group is the additive group of the real number field. Almost periodic functions with respect to groups play a role in the representation theory of non-compact Lie algebras. (In the compact case, they are trivial — all continuous functions are almost periodic.)

The notion of an almost periodic function should not be confused with the notion of quasiperiodic function.