Riemann-Lebesgue lemma

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. Let $f:[a,b]\to\mathbb{C}$ be a measurable function. If f is L^1 integrable, that is to say if the Lebesgue integral of $|f|$ is finite, then

$$\int_a^b f(x)e^{inx}\,dx \to 0, \quad as \quad n\to\pm\infty.$$

The above result, commonly known as the Riemann-Lebesgue lemma, is of basic importance in harmonic analysis. It is equivalent to the assertion that the Fourier coefficients $\hat{f}_n$ of a periodic, integrable function $f(x)$, tend to 0 as $n\to\pm\infty$.

The proof can be organized into 3 steps.

Step 1. An elementary calculation shows that

$$\int_I e^{inx}\,dx \to 0, \quad as \quad n\to\pm\infty$$

for every interval $I\subset[a,b]$. The proposition is therefore true for all step functions with `http://planetmath.org/SupportOfFunction`support in $[a,b]$.

Step 2. By the monotone convergence theorem, the proposition is true for all positive functions, integrable on $[a,b]$.

Step 3. Let f be an arbitrary measurable function, integrable on $[a,b]$. The proposition is true for such a general f, because one can always write

$$f = g - h,$$

where g and h are positive functions, integrable on $[a,b]$.