minimality property of Fourier coefficients

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Let f be a Riemann integrable periodic real function with period 2π and n a positive integer. Among all "trigonometric polynomials"

$$\varphi(x) := \frac{\alpha_0}{2}+\sum_{j=1}^{n}(\alpha_j \cos jx + \beta_j \sin jx),$$

the polynomial with the coefficients α_j and β_j being the Fourier coefficients

$$\alpha_j = a_j := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos jx\, dx$$

and

$$\beta_j = b_j := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin jx\, dx$$

for the Fourier series of f produces the minimal value of the mean square deviation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)-\varphi(x)]^2\, dx.$$

Proof. For any fixed number n, it's a question of giving the least value to the definite integral

$$m := \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x)-\frac{\alpha_0}{2}-\sum_{j=1}^{n}(\alpha_j \cos jx + \beta_j \sin jx)\right]^2 dx \quad (\geqq 0). \qquad (1)$$

Expanding m and integrating termwise yields

$$\begin{aligned} m \;=\; & \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^2\, dx - \frac{\alpha_0}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx \\ & - \frac{1}{\pi}\sum_{j=1}^{n}\alpha_j\int_{-\pi}^{\pi} f(x)\cos jx\, dx - \frac{1}{\pi}\sum_{j=1}^{n}\beta_j\int_{-\pi}^{\pi} f(x)\sin jx\, dx + \frac{1}{2\pi}\frac{\alpha_0^2}{4}\int_{-\pi}^{\pi} dx \\ & + \frac{1}{2\pi}\sum_{j=1}^{n}\alpha_j^2\int_{-\pi}^{\pi}\cos^2 jx\, dx + \frac{1}{2\pi}\sum_{j=1}^{n}\beta_j^2\int_{-\pi}^{\pi}\sin^2 jx\, dx \\ & + \frac{\alpha_0}{2\pi}\sum_{j=1}^{n}\alpha_j\int_{-\pi}^{\pi}\cos jx\, dx + \frac{\alpha_0}{2\pi}\sum_{j=1}^{n}\beta_j\int_{-\pi}^{\pi}\sin jx\, dx + \frac{1}{\pi}\sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_j\beta_k\int_{-\pi}^{\pi}\cos jx\ \sin kx\, dx \\ & + \frac{1}{\pi}\sum_{j=1}^{n}\sum_{k\neq j}\alpha_j\alpha_k\int_{-\pi}^{\pi}\cos jx\ \cos kx\, dx + \frac{1}{\pi}\sum_{j=1}^{n}\sum_{k\neq j}\beta_j\beta_k\int_{-\pi}^{\pi}\sin jx\ \sin kx\, dx. \end{aligned}$$

Here, we have the Fourier coefficients

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \;=\; a_0, \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos jx\,dx \;=\; a_j, \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin jx\,dx \;=\; b_j.$$

Furthermore,

$$\int_{-\pi}^{\pi}\cos^2 jx\,dx = \int_{-\pi}^{\pi}\sin^2 jx\,dx \;=\; \pi, \qquad \int_{-\pi}^{\pi}\cos jx\,\sin kx\,dx \;=\; 0$$

and

$$\int_{-\pi}^{\pi}\cos jx\,\cos kx\,dx = \int_{-\pi}^{\pi}\sin jx\,\sin kx\,dx \;=\; 0 \quad \text{for } \; k \neq j.$$

Using all these we can write

$$m \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx - \frac{\alpha_0 a_0}{2} - \sum_{i=1}^{n}(\alpha_i a_i + \beta_i b_i) + \frac{a_0^2}{4} + \frac{1}{2}\sum_{i=1}^{n}(\alpha_i^2 + \beta_i^2).$$

Adding and subtracting still the sum $\frac{a_0^2}{4} + \frac{1}{2}\sum_{i=1}^{n}(a_i^2 + b_i^2)$ yields finally the form

$$m \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx - \frac{a_0^2}{4} - \frac{1}{2}\sum_{i=1}^{n}(a_i^2 + b_i^2) + \frac{1}{4}(\alpha_0 - a_0)^2 + \frac{1}{2}\sum_{i=1}^{n}[(\alpha_i - a_i)^2 + (\beta_i - b_i)^2].$$

The three first addends of this sum do not depend on the choice of the quantities α_i and β_i. The other addends are non-negative, and their sum is minimal, equal 0, when

$$\alpha_i = a_i, \quad \beta_i = b_i \quad \forall i.$$

Accordingly, the mean square deviation m, i.e. (1), is minimal when one uses the Fourier coefficients. Q.E.D.

References

[1] N. Piskunov: *Diferentsiaal- ja integraalarvutus kõrgematele tehnilistele õppeasutustele.* Kirjastus Valgus, Tallinn (1966).