Fourier sine and cosine series

Canonical name	FourierSineAndCosineSeries
Date of creation	2013-03-22 15:42:20
Last modified on	2013-03-22 15:42:20
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	26
Author	pahio (2872)
Entry type	Topic
Classification	msc 42A32
Classification	msc 42A20
Classification	msc 42A16
Classification	msc 26A06
Related topic	SubstitutionNotation
Related topic	IntegralsOfEvenAndOddFunctions
Related topic	CosineAtMultiplesOfStraightAngle
Related topic	ExampleOfFourierSeries
Related topic	DoubleSeries
Related topic	UniquenessOfFourierExpansion
Related topic	DeterminationOfFourierCoefficients
Related topic	TwoDimensionalFourierTransforms
Defines	Fourier sine series
Defines	Fourier cosine series
Defines	sine series
Defines	cosine series
Defines	half-interval
Defines	Fourier double sine series
Defines	Fourier double cosine series

One sees from the formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

of the coefficients a_n and b_n for the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

of the Riemann integrable real function f on the interval $[-\pi,\,\pi]$, that

- $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx,\;\; b_n = 0\;\;\forall n$ if f is an even function;
- $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx,\;\; a_n = 0\;\;\forall n$ if f is an odd function.

Thus the Fourier series of an even function mere cosine and of an odd function mere sine . This concerns the whole interval $[-\pi,\,\pi]$. So e.g. one has on this interval

$$x \equiv 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - + \cdots\right).$$

Remark 1. On the *half-interval* $[0,\,\pi]$ one can in any case expand each Riemann integrable function f both to a cosine series and to a sine series, irrespective of how it is defined for the negative half-interval or is it defined here at all.

Remark 2. On an interval $[-p,\,p]$, instead of $[-\pi,\,\pi]$, the Fourier coefficients of the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right)$$

have the expressions

- $a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}\,dx,\;\; b_n = 0\;\;\forall n\;$ if f is an even function;
- $b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx,\;\; a_n = 0\;\;\forall n\;$ if f is an odd function.

Example. Expand the http://planetmath.org/IdentityMapidentity function $x \mapsto x$ to a Fourier cosine series on $[0,\,\pi]$.

This odd function may be replaced with the even function $f : x \mapsto |x|$. Then we get

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \pi$$

and, integrating by parts,

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx = \frac{2}{\pi}\left[\,\Big/_{\!0}^{\pi} x\frac{\sin nx}{n} - \int_0^\pi \frac{\sin nx}{n}\,dx\right] = \frac{2}{\pi}\;\Big/_{\!0}^{\pi}\frac{\cos nx}{n^2} = \frac{2}{\pi n^2}((-1)^n\text{–}1));$$

this equals to $-\dfrac{4}{\pi n^2}$ if n is an odd integer, but vanishes for each even n. Thus we obtain on the interval $[0,\,\pi]$ the cosine series

$$x \;\equiv\; \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right).$$

Chosing here $x := 0$ one gets the result

$$\frac{\pi^2}{8} \;=\; 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

(cf. the entry on http://planetmath.org/node/11010Dirichlet eta function at 2).

Fourier double series. The Fourier sine and cosine series introduced in Remark 1 on the half-interval $[0,\,\pi]$ for a function of one real variable may be generalized for e.g. functions of two real variables on a rectangle $\{(x,\,y) \in \mathbb{R}^2 \,\vdots\, 0 \le x \le a,\, 0 \le y \le b\}$:

$$f(x,\,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{1}$$

$$f(x,\,y) = \frac{d_{00}}{4} + \frac{1}{2}\sum_{l=1}^{\infty}\left(d_{l0}\cos\frac{l\pi x}{a} + d_{0l}\cos\frac{l\pi y}{b}\right) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} d_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{2}$$

The coefficients of the *Fourier double sine series* (1) are got by the double integral

$$c_{mn} = \frac{4}{ab}\int_0^a \int_0^b f(x,\,y)\,\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

where $m = 1,\,2,\,3,\,\ldots$ and $n = 1,\,2,\,3,\,\ldots$ The coefficients of the *Fourier double cosine series* (2) are correspondingly

$$d_{mn} = \frac{4}{ab}\int_0^a \int_0^b f(x,\,y)\,\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\,dx\,dy$$

where $m = 0,\,1,\,2,\,\ldots$ and $n = 0,\,1,\,2,\,\ldots$

Note. One can use in the double series of (1) and (2) also the *diagonal summing*, e.g. for the double sine series as follows:
$c_{11}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + \left(c_{12}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b} + c_{21}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}\right) + \left(c_{13}\sin\frac{\pi x}{a}\sin\frac{3\pi y}{b} + c_{22}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b} + c_{31}\sin\frac{3\pi x}{a}\sin\right.$
$\ldots$

References

[1] K. Väisälä: *Matematiikka IV.* Hand-out Nr. 141. Teknillisen korkeakoulun ylioppilaskunta, Otaniemi, Finland (1967).