Fourier series of function of bounded variation

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Owner	pahio (2872)
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If the real function f is of bounded variation on the interval $[-\pi,\,+\pi]$, then its Fourier series expansion

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{1}$$

with the http://planetmath.org/FourierCoefficientscoefficients

$$\begin{cases}a_n & =\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\\ b_n & =\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx\end{cases} \tag{2}$$

converges at every point of the interval. The sum of the series is at the interior points x equal to the arithmetic mean of the http://planetmath.org/OneSidedLimitleft-sided and the right-sided limit of f at x and at the end-points of the interval equal to $\frac{1}{2}\left(\lim_{x\to-\pi+}f(x)+\lim_{x\to+\pi-}f(x)\right)$.

Remark 1. Because of the periodicity of the terms of the terms, the expansion (1) converges for all real values of x and it represents a periodic function with the period 2π.

Remark 2. If the function f is of bounded variation, instead of $[-\pi,\,+\pi]$, on the interval $[-p,\,+p]$ the equations (1) and (2) may be converted via the change of variable $x:=\frac{pt}{\pi}$ to

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi t}{p}+b_n\sin\frac{n\pi t}{p}) \tag{3}$$

and

$$\begin{cases}a_n & =\frac{1}{p}\int_{-p}^{p}f(t)\cos\frac{n\pi t}{p}\,dt\\ b_n & =\frac{1}{p}\int_{-p}^{p}f(t)\sin\frac{n\pi t}{p}\,dt.\end{cases} \tag{4}$$

Correspondingly, the sum of (3) at the points of $[-p,\,+p]$ is expressed via the left-sided and righr-sided limits of $f(t)$.