Wirtinger's inequality

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Theorem: Let $f\colon \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π, which is continuous and has a continuous derivative throughout $\mathbb{R}$, and such that

$$\int_0^{2\pi} f(x) = 0 \; . \tag{1}$$

Then

$$\int_0^{2\pi} f'^2(x)dx \geq \int_0^{2\pi} f^2(x)dx \tag{2}$$

with equality if and only if $f(x) = a\cos x + b\sin x$ for some a and b (or equivalently $f(x) = c\sin(x+d)$ for some c and d).

Proof: Since Dirichlet's conditions are met, we can write

$$f(x) = \frac{1}{2}a_0 + \sum_{n\geq 1}(a_n \sin nx + b_n \cos nx)$$

and moreover $a_0 = 0$ by (**??**). By Parseval's identity,

$$\int_0^{2\pi} f^2(x)dx = \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

and

$$\int_0^{2\pi} f'^2(x)dx = \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2)$$

and since the summands are all ≥ 0, we get (**??**), with equality if and only if $a_n = b_n = 0$ for all $n \geq 2$.

Hurwitz used Wirtinger's inequality in his tidy 1904 proof of the isoperimetric inequality.