Parseval equality

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0.1 Parseval's Equality

Theorem. – If $\{e_j\colon\ j \in J\}$ is an orthonormal basis of an Hilbert space H, then for every $x \in H$ the following equality holds:

$$\|x\|^2 = \sum_{j \in J} |\langle x, e_j \rangle|^2.$$

The above theorem is a more sophisticated form of Bessel's inequality (where the inequality is in fact an equality). The difference is that for Bessel's inequality it is only required that the set $\{e_j : j \in J\}$ is an orthonormal set, not necessarily an orthonormal basis.

0.2 Parseval's Theorem

Applying Parseval's equality on the Hilbert space `http://planetmath.org/LpSpace`$L^2([-\pi, \pi])$, of square integrable functions on the interval $[-\pi, \pi]$, with the orthonormal basis consisting of trigonometric functions, we obtain

Theorem (Parseval's theorem). – Let f be a Riemann square integrable function from $[-\pi, \pi]$ to $\mathbb{R}$. The following equality holds

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx = \frac{(a_0^f)^2}{2} + \sum_{k=1}^{\infty} [(a_k^f)^2 + (b_k^f)^2],$$

where a_0^f, a_k^f, b_k^f are the Fourier coefficients of the function f.

The function f can be a Lebesgue-integrable function, if we use the Lebesgue integral in place of the Riemann integral.