Chebyshev polynomial

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Defines	Chebyshev polynomial of first kind
Defines	Chebyshev polynomial of second kind

The *Chebyshev polynomials of first kind* are defined by the simple formula

$$T_n(x) = \cos(nt),$$

where $x = \cos t$.

It is an example of a *trigonometric polynomial*.

This can be seen to be a polynomial by expressing $\cos(kt)$ as a polynomial of $\cos(t)$, by using the formula for cosine of angle-sum:

$$\begin{aligned}
\cos(1t) &= \cos(t) \\
\cos(2t) &= \cos(t)\cos(t) - \sin(t)\sin(t) = 2(\cos(t))^2 - 1 \\
\cos(3t) &= 4(\cos(t))^3 - 3\cos(t) \\
&\vdots
\end{aligned}$$

So we have

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
&\vdots
\end{aligned}$$

These polynomials obey the recurrence relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

for $n = 1, 2, \ldots$

Related are the *Chebyshev polynomials of the second kind* that are defined as

$$U_{n-1}(\cos t) = \frac{\sin(nt)}{\sin(t)},$$

which can similarly be seen to be polynomials through either a similar process as the above or by the relation $U_{n-1}(t) = nT_n'(t)$.

The first few are:

$$U_0(x) = 1$$

$$
\begin{aligned}
U_1(x) &= 2x \\
U_2(x) &= 4x^2 - 1 \\
U_3(x) &= 8x^3 - 4x \\
&\vdots
\end{aligned}
$$

The same recurrence relation also holds for U:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

for $n = 1, 2, \ldots$.