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orthogonal polynomials

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1 Orthogonal Polynomials

Polynomials of degree n are analytic functions that can be written in the form

$$p_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

They can be differentiated and integrated for any value of x , and are fully determined by the $n + 1$ coefficients $a_0 \dots a_n$. For this simplicity they are frequently used to approximate more complicated or unknown functions. In approximations, the necessary degree n of the polynomial is not normally defined by criteria other than the quality of the approximation.

Using polynomials as defined above tends to lead into numerical difficulties when determining the a_i , even for small values of n . It is therefore customary to stabilize results numerically by using orthogonal polynomials over an interval $[a, b]$, defined with respect to a positive weighting function $W(x) > 0$ by

$$\int_a^b p_n(x)p_m(x)W(x)dx = 0 \text{ for } n \neq m$$

Orthogonal polynomials are obtained in the following way: define the scalar product.

$$(f, g) = \int_a^b f(x)g(x)W(x)dx$$

between the functions f and g , where $W(x)$ is a weight factor. Starting with the polynomials $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2$, etc., from the Gram-Schmidt decomposition one obtains a sequence of orthogonal polynomials $q_0(x), q_1(x), \dots$, such that $(q_m, q_n) = N_n\delta_{mn}$. The normalization factors N_n are arbitrary. When all N_i are equal to one, the polynomials are called orthonormal.

Some important orthogonal polynomials are:

a	b	$W(x)$	name
-1	1	1	Legendre polynomials
-1	1	$(1 - x^2)^{-1/2}$	Chebyshev polynomials
$-\infty$	∞	e^{-x^2}	Hermite polynomials

Orthogonal polynomials of successive orders can be expressed by a recurrence relation

$$p_n = (A_n + B_n x)p_{n-1} + C_n p_{n-2}$$

This relation can be used to compute a finite series

$$a_0 p_0 + a_1 p_1 + \cdots + a_n p_n$$

with arbitrary coefficients a_i , without computing explicitly every polynomial p_j (Horner's Rule).

Chebyshev polynomials $T_n(x)$ are also orthogonal with respect to discrete values x_i :

$$\sum_i T_n(x_i) T_m(x_i) = 0 \text{ for } n < m \leq M$$

where the x_i depend on M .

For more information, see [?, ?].

References

- [1] Originally from The Data Analysis Briefbook
(<http://rkb.home.cern.ch/rkb/titleA.html><http://rkb.home.cern.ch/rkb/titleA.html>)
- [2] M. Abramowitz and I.A. Stegun (Eds.), Handbook of Mathematical Functions, National Bureau of Standards, Dover, New York, 1974.
- [3] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, Numerical Recipes in C, Second edition, Cambridge University Press, 1995. (The same book exists for the Fortran language). There is also an Internet version which you can work from.