Fourier series in complex form and Fourier integral

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0.1 Fourier series in complex form

The Fourier series expansion of a Riemann integrable real function f on the interval $[-p,\,p]$ is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{p} + b_n\sin\frac{n\pi t}{p}\right), \tag{1}$$

where the coefficients are

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi t}{p}\,dt, \quad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi t}{p}\,dt. \tag{2}$$

If one expresses the cosines and sines via `http://planetmath.org/ComplexSineAndCosine`Euler formulas with `http://planetmath.org/ComplexExponentialFunction`exponential function, the series (1) attains the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi t}{p}}. \tag{3}$$

The coefficients c_n could be obtained of a_n and b_n, but they are comfortably derived directly by multiplying the equation (3) by $e^{-\frac{im\pi t}{p}}$ and integrating it from $-p$ to p. One obtains

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{\frac{-in\pi t}{p}}\,dt \qquad (n = 0,\,\pm1,\,\pm2,\,\ldots). \tag{4}$$

We may say that in (3), $f(t)$ has been dissolved to sum of *harmonics* (elementary waves) $c_n e^{\frac{in\pi t}{p}}$ with amplitudes c_n corresponding the frequencies n.

0.2 Derivation of Fourier integral

For seeing how the expansion (3) changes when $p \to \infty$, we put first the expressions (4) of c_n to the series (3):

$$f(t) = \sum_{n=-\infty}^{\infty} e^{\frac{in\pi t}{p}}\frac{1}{2p}\int_{-p}^{p} f(t)e^{\frac{-in\pi t}{p}}\,dt$$

By denoting $\omega_n := \frac{n\pi}{p}$ and $\Delta_n\omega := \omega_{n+1} - \omega_n = \frac{\pi}{p}$, the last equation takes the form

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{i\omega_n t} \Delta_n\omega \int_{-p}^{p} f(t) e^{-i\omega_n t}\, dt.$$

It can be shown that when $p \to \infty$ and thus $\Delta_n\omega \to 0$, the limiting form of this equation is

$$f(t) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} d\omega \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{5}$$

Here, $f(t)$ has been represented as a *Fourier integral.* It can be proved that for validity of the expansion (4) it suffices that the function f is piecewise continuous on every finite interval having at most a finite amount of extremum points and that the integral

$$\int_{-\infty}^{\infty} |f(t)|\, dt$$

converges.

For better to compare to the Fourier series (3) and the coefficients (4), we can write (5) as

$$f(t) \;=\; \int_{-\infty}^{\infty} c(\omega) e^{i\omega t} d\omega, \tag{6}$$

where

$$c(\omega) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{7}$$

0.3 Fourier transform

If we denote $2\pi c(\omega)$ as

$$F(\omega) \;=\; \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt, \tag{8}$$

then by (5),

$$f(t) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\, d\omega. \tag{9}$$

$F(\omega)$ is called the *Fourier transform* of $f(t)$. It is an integral transform and (9) its inverse transform.

N.B. that often one sees both the formula (8) and the formula (9) equipped with the same constant factor $\frac{1}{\sqrt{2\pi}}$ in front of the integral sign.

References

[1] K. Väisälä: *Laplace-muunnos.* Handout Nr. 163. Teknillisen korkeakoulun ylioppilaskunta, Otaniemi, Finland (1968).