generalized Riemann-Lebesgue lemma

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Lemma 1. *Let $h\colon \mathbb{R} \to \mathbb{C}$ be a bounded measurable function. If h satisfies the* averaging condition

$$\lim_{c \to +\infty} \frac{1}{c} \int_0^c h(t)\, dt = 0$$

then

$$\lim_{\omega \to \infty} \int_a^b f(t) h(\omega t)\, dt = 0$$

with $-\infty < a < b < +\infty$ for any $f \in L^1[a,b]$

Proof. Obviously we only need to prove the lemma when both h and f are real and $0 = a < b < \infty$.

Let $\mathbf{1}_{[a,b]}$ be the indicator function of the interval $[a,b]$. Then

$$\lim_{\omega \to \infty} \int_0^b \mathbf{1}_{[a,b]} h(\omega t)\, dt = \lim_{\omega \to \infty} \frac{1}{\omega} \int_0^{\omega b} h(t)\, dt = 0$$

by the hypothesis. Hence, the lemma is valid for indicators, therefore for step functions.

Now let C be a bound for h and choose $\epsilon > 0$. As step functions are dense in L^1, we can find, for any $f \in L^1[a,b]$, a step function g such that $\|f - g\|_1 < \epsilon$, therefore

$$\begin{aligned} \lim_{\omega \to \infty} \left| \int_a^b f(t) h(\omega t)\, dt \right| &\leqslant \lim_{\omega \to \infty} \int_a^b |f(t) - g(t)|\, |h(\omega t)|\; dt + \lim_{\omega \to \infty} \left| \int_a^b g(t) h(\omega t)\, dt \right| \\ &\leqslant \lim_{\omega \to \infty} C \|f - g\|_1 < C\epsilon \end{aligned}$$

because $\lim_{\omega \to \infty} \left| \int_a^b g(t) h(\omega t)\, dt \right| = 0$ by what we have proved for step functions. Since ϵ is arbitrary, we are done.

$\square$