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Poisson summation formula

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Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function and let

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) dx, \quad \xi \in \mathbb{R}.$$

be its Fourier transform. The Poisson summation formula is the assertion that

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n). \tag{1}$$

whenever f is such that both of the above infinite sums are absolutely convergent.

Equation (**??**) is useful because it establishes a correspondence between Fourier series and Fourier integrals. To see the connection, let

$$g(x) = \sum_{n \in \mathbb{Z}} f(x+n), \quad x \in \mathbb{R},$$

be the periodic function obtained by pseudo-averaging[1] f relative to $\mathbb{Z}$ acting as the discrete group of translations on $\mathbb{R}$. Since f was assumed to be integrable, g is defined almost everywhere, and is integrable over $[0, 1]$ with

$$\|g\|_{L^1[0,1]} \leq \|f\|_{L^1(\mathbb{R})}.$$

Since f is integrable, we may interchange integration and summation to obtain

$$\hat{f}(k) = \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i k x} dx = \int_0^1 e^{-2\pi i k x} g(x) dx$$

for every $k \in \mathbb{Z}$. In other words, the restriction of the Fourier transform of f to the integers gives the Fourier coefficients of the averaged, periodic function g. Since we have assumed that the $\hat{f}(k)$ form an absolutely convergent series, we have that

$$g(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$$

in the sense of uniform convergence. Evaluating the above equation at $x = 0$, we obtain the Poisson summation formula (**??**).

[1] This terminology is at best a metaphor. The operation in question is not a genuine mean, in the technical sense of that word.