uncertainty theorem

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The uncertainty principle, as first formulated by Heisenberg, states that the product of the standard deviations of two *conjugated* variables, cannot be less than some minimum. This statement has been generalized to a precise mathematical theorem, in the frame of *wavelet* theory.

1 THE UNCERTAINTY THEOREM

Let $f(t)$ be a real function of the real variable, satisfying the L^2 condition (see below), and $F(\omega)$ its Fourier transform. The standard deviation Δ_t and Δ_ω of t and ω respectively, satisfy the following inequality:

$$\Delta_t \Delta_\omega \geq \frac{1}{2}$$

For this formula to make sense, Δ_t and Δ_ω must be precisely defined.

2 THE L^2 CONDITION

A real function $f(t)$ of the real variable t will be said to satisfy the L^2 condition if $f(t)$, $tf(t)$ and the derivative $f'(t)$ are all in L^2.
If $F(\omega)$ is its Fourier transform, $-i\omega F(\omega)$ is the transform of $f'(t)$. All the following functions belong to L^2:

$$f(t),\ f'(t),\ tf(t),\ F(\omega),\ \omega F(\omega)$$

The first three functions are just the definition and the two last ones result from Parseval's identity, recalled here in its integral form:

$$\int_{-\infty}^{\infty} \overline{U(\omega)} V(\omega) d\omega \ = \ 2\pi \int_{-\infty}^{\infty} \overline{u(t)} v(t) dt$$

$U(\omega)$ and $V(\omega)$ are the Fourier transforms of $u(t)$ and $v(t)$.

3 DEFINITIONS

$f(t)$ and $F(\omega)$ being in L^2, we may define their finite norms:

$$\| f \|^2 = \int_{-\infty}^{\infty} | f(t) |^2 \, dt \qquad\qquad \| F \|^2 = \int_{-\infty}^{\infty} | F(\omega) |^2 \, d\omega$$

By Parseval's identity, they are related:

$$|| f ||^2 = \frac{1}{2\pi} || F ||^2$$

We are now able to define the probability distributions $T(t)$ and $\Omega(\omega)$ for the "random" variables t and ω:

$$T(t) = \frac{| f(t) |^2}{|| f ||^2} \qquad \Omega(\omega) = \frac{| F(\omega) |^2}{|| F ||^2}$$

Since the L^2 integrals of T and Ω are 1, they are proper probability distributions. The *mean value* t_0 of t is defined the usual way:

$$t_0 = \int_{-\infty}^{\infty} tT(t)dt$$

Note that ω's mean value is always 0 because $f(t)$ is a real function. Finally, we have the standard deviations for the uncertainty theorem:

$$\Delta_t^2 = \int_{-\infty}^{\infty} T(t)(t - t_0)^2 dt \qquad \Delta_\omega^2 = \int_{-\infty}^{\infty} \Omega(\omega)(t - t_0)^2 d\omega$$

4 PROOF OF THE THEOREM

The heart of the proof is the Cauchy-Schwarz inequality in the L^2 Hilbert space: the product of the norms of two functions $u(t)$ and $v(t)$ is greater than, or equal to, the norm of their scalar product:

$$\int_{-\infty}^{\infty} | u(t) |^2 \, dt \int_{-\infty}^{\infty} | v(t) |^2 \, dt \quad \geq \quad | \int_{-\infty}^{\infty} \overline{u(t)} v(t) dt |^2$$

Equality occurs if, and only if, one of the functions is proportional to the other. For the two functions $u(t) = (t - t_0)f(t)$ and $v(t) = f'(t)$, we have therefore:

$$\int_{-\infty}^{\infty} (t - t_0)^2 f(t)^2 dt \int_{-\infty}^{\infty} f'(t)^2 dt \quad \geq \quad | \int_{-\infty}^{\infty} (t - t_0) f(t) f'(t) dt |^2$$

The integral at the right hand side can be integrated by parts. Using the definition of $|| f ||$:

$$\int_{-\infty}^{\infty} (t - t_0)^2 f(t)^2 dt \int_{-\infty}^{\infty} f'(t)^2 dt \quad \geq \quad \frac{1}{4} || f ||^4$$

But $|| \, f \, ||$ and $|| \, F \, ||$ are related by a 2π factor, so:

$$\int_{-\infty}^{\infty} (t - t_0)^2 f(t)^2 dt \int_{-\infty}^{\infty} f'(t)^2 dt \quad \geq \quad \frac{1}{8\pi} \, || \, f \, ||^2 || \, F \, ||^2$$

Applying Parseval's identity to the second integral of the left hand side, we get:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (t - t_0)^2 f(t)^2 dt \int_{-\infty}^{\infty} \omega^2 F(\omega)^2 d\omega \quad \geq \quad \frac{1}{8\pi} \, || \, f \, ||^2 || \, F \, ||^2$$

We have used the fact that the Fourier transform of $f'(t)$ if $-i\omega F(\omega)$.
Now, dividing both sides by the norms, and simplifying by the 2π factor, we get exactly the uncertainty theorem:

$$\int_{-\infty}^{\infty} (t - t_0)^2 T(t) dt \int_{-\infty}^{\infty} \omega^2 F(\omega)^2 d\omega \quad \geq \quad \frac{1}{4}$$

5 THE GAUSSIAN FUNCTION

The Cauchy-Schwarz inequality becomes an equality if, and only if, one of the functions is proportional to the other. In our case, this condition is expressed by $f'(t) = \lambda(t - t_0) f(t)$ where λ is a constant. This differential equation is readily solved: $f(t) = k e^{\lambda(t-t_0)^2}$. $f(t)$ must be in L^2 so that λ must be negative. Defining $\lambda = \frac{-1}{2\sigma^2}$, we get the gaussian function in its traditional form:

$$f(t) = e^{\frac{-(t-t_0)^2}{2\sigma^2}}$$

The constant k has been omitted because it cancels anyway in the probability distributions. The standard deviations are easily computed from their definitions:

$$\Delta_t = \frac{\sigma}{\sqrt{2}} \qquad \Delta_\omega = \frac{1}{\sigma\sqrt{2}}$$

Their product is $\frac{1}{2}$, independent of σ. There is no other function with this property.

References

[1] Roberto Celi *Time-Frequency visualization of helicopter noise* http://celi.umd.edu/Jour/NoisePaperColor.pdf

Despite its frightening title, this paper is mostly theoretical and it is the only place where I saw the uncertainty theorem clearly stated.

[2] Robi Polikar *The wavelet tutorial*
http://users.rowan.edu/ polikar/wavelets/wttutorial.html