Laplace transform of a Gaussian function

Canonical name	LaplaceTransformOfAGaussianFunction
Date of creation	2013-03-22 16:03:21
Last modified on	2013-03-22 16:03:21
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Last modified by	perucho (2192)
Numerical id	5
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Entry type	Application
Classification	msc 42-01

We evaluate the Laplace transform [1]

$$\mathcal{L}\{e^{-t^2}\} = \int_0^\infty e^{-st} e^{-t^2}\,dt = F(s). \tag{1}$$

In fact,

$$\mathcal{L}\{e^{-t^2}\} = \int_0^\infty e^{-(t^2+2\frac{s}{2}t+\frac{s^2}{4}-\frac{s^2}{4})}\,dt = e^{\frac{s^2}{4}}\int_0^\infty e^{-(t+\frac{s}{2})^2}\,dt.$$

By making the change of variable $t+\frac{s}{2}=u$, we have (by the second equality in (1), the variable on operator's argument is immaterial)

$$\mathcal{L}\{e^{-t^2}\} = e^{\frac{s^2}{4}}\int_{\frac{s}{2}}^\infty e^{-u^2}\,du.$$

That is,

$$\mathcal{L}\{e^{-t^2}\} = F(s) = \frac{\sqrt{\pi}}{2} e^{\frac{s^2}{4}} \operatorname{erfc}\left(\frac{s}{2}\right),$$

where $\operatorname{erfc}(\cdot)$ is the complementary error function. Its path of integration is subject to the restriction $\arg(u) \to \theta$, with $|\theta| \leq \pi/4$ as $u \to \infty$ along the path, with equality only if $\Re(u^2)$ remains bounded to the left.

[1] cf. *Gaussian function*, wikipedia.org