Ornstein-Weiss lemma

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Let G be a group. For a fixed $K \subseteq G$, define the *K-boundary* of $U \subseteq G$ as

$$\partial_K U = \{g \in G \mid Kg \cap U, Kg \cap (G \setminus U) \neq \emptyset\} \ . \tag{1}$$

Let $\mathcal{PF}(G)$ be the set of finite subsets of G. Call a *Følner net* for G a net $\mathcal{X} = \{X_i\}_{i \in \mathcal{I}} \subseteq \mathcal{PF}(G)$, $\mathcal{I}$ being a directed set, such that for every *finite* $K \subseteq G$,

$$\lim_{i \in \mathcal{I}} \frac{|\partial_K X_i|}{|X_i|} = 0 \ , \tag{2}$$

where the limit is taken in the sense of directed sets. Recall that G has a Følner net if and only if G is amenable.

Theorem 1 (Ornstein-Weiss lemma) *Let G be an amenable group and $F : \mathcal{PF}(G) \to \mathbb{R}$ a subadditive, right-invariant function, that is:*

1. *For any two finite subsets U, V of G,*

$$F(U \cup V) \leq F(U) + F(V) \ . \tag{3}$$

2. *For any $g \in G$ and finite $U \subseteq G$,*

$$F(Ug) = F(U) \ . \tag{4}$$

Then for any Følner net $\mathcal{X} = \{X_i\}_{i \in \mathcal{I}}$ on G, the limit

$$L = \lim_{i \in \mathcal{I}} \frac{F(X_i)}{|X_i|} \tag{5}$$

exists, and does not depend on the choice of $\mathcal{X}$.

The Ornstein-Weiss lemma allows to prove variants of Birkhoff's ergodic theorem for actions of amenable groups, rather than only those generated by an invertible, measure invariant map. Moreover, it shares several similarities with Fekete's lemma on subadditive functions over the positive integers, although it is not a complete counterpart. In fact, putting $X_n = \{1, \ldots, n\}$ determines a Følner sequence on $\mathbb{Z}$; however, if $f : \mathbb{N} \to [0, \infty)$ is subadditive, then $F(U) = f(|U|)$ is right-invariant, but not necessarily subadditive. (Counterexample: $f(n) = n \bmod 2$, $U = \{1, 2\}$, $V = \{2, 3\}$.)

References

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