topological group representation

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1 Finite Dimensional Representations

Let G be a topological group and V a finite-dimensional normed vector space. We denote by $GL(V)$ the general linear group of V, endowed with the topology coming from the operator norm.

Regarding only the group structure of G, recall that a representation of G in V is a group homomorphism $\pi : G \longrightarrow GL(V)$.

Definition - A **representation** of the topological group G in V is a continuous group homomorphism $\pi : G \longrightarrow GL(V)$, i.e. is a continuous representation of the abstract group G in V.

We have the following equivalent definitions:

- A representation of G in V is a group homomorphism $\pi : G \longrightarrow GL(V)$ such that the mapping $G \times V \longrightarrow V$ defined by $(g, v) \mapsto \pi(g)v$ is continuous.

- A representation of G in V is a group homomorphism $\pi : G \longrightarrow GL(V)$ such that, for every $v \in V$, the mapping $G \longrightarrow V$ defined by $g \mapsto \pi(g)v$ is continuous.

2 Representations in Hilbert Spaces

Let G be a topological group and H a Hilbert space. We denote by $B(H)$ the algebra of bounded operators endowed with the strong operator topology (this topology does not coincide with the norm topology unless H is finite-dimensional). Let $\mathcal{G}(H)$ the set of invertible operators in $B(H)$ endowed with the subspace topology.

Definition - A **representation** of the topological group G in H is a continuous group homomorphism $\pi : G \longrightarrow \mathcal{G}(H)$, i.e. is a continuous representation of the abstract group G in H.

We denote by $rep(G, H)$ the set of all representations of G in the Hilbert space H.

We have the following equivalent definitions:

- A representation of G in H is a group homomorphism $\pi : G \longrightarrow \mathcal{G}(H)$ such that the mapping $G \times H \longrightarrow H$ defined by $(g, v) \mapsto \pi(g)v$ is continuous.

- A representation of G in H is a group homomorphism $\pi : G \longrightarrow \mathcal{G}(H)$ such that, for every $v \in H$, the mapping $G \longrightarrow H$ defined by $g \mapsto \pi(g)v$ is continuous.

Remark - The 3rd definition is exactly the same as the 1st definition, just written in other .

3 Representations as G-modules

Recall that, for an abstract group G, it is the same to consider a representation of G or to consider a `http://planetmath.org/GModule`G-module, i.e. to each representation of G corresponds a G-module and vice-versa.

For a topological group G, representations of G satisfy some continuity . Thus, we are not interested in all G-modules, but rather in those which are compatible with the continuity conditions.

Definition - Let G be a topological group. A G**-module** is a normed vector space (or a Hilbert space) V where G acts continuously, i.e. there is a continuous action $\psi : G \times V \longrightarrow V$.

To give a representation of a topological group G is the same as giving a G-module (in the sense described above).

4 Special Kinds of Representations

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- Let $\pi \in rep(G, H)$. We say that a subspace $V \subseteq H$ is by π if V is invariant under every operator $\pi(s)$ with $s \in G$.

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- A of a representation $\pi \in rep(G, H)$ is a representation $\pi_0 \in rep(G, H_0)$ obtained from π by restricting to a closed subspace $H_0 \subseteq H$.

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- A representation $\pi \in rep(G, H)$ is said to be if the only closed subspaces of H are the trivial ones, $\{0\}$ and H.

- Two representations $\pi_1 : G \longrightarrow GL(V_1)$ and $\pi_2 : G \longrightarrow GL(V_2)$ of a topological group G are said to be **equivalent** if there exists an invertible linear transformation $T : V_1 \longrightarrow V_2$ such that for every $g \in G$ one has $\pi_1(g) = T^{-1}\pi_2(g)T$.

  The definition is similar for Hilbert spaces, by taking T as an invertible bounded linear operator.