Pontryagin duality

Canonical name	PontryaginDuality
Date of creation	2013-03-22 17:42:42
Last modified on	2013-03-22 17:42:42
Owner	asteroid (17536)
Last modified by	asteroid (17536)
Numerical id	7
Author	asteroid (17536)
Entry type	Theorem
Classification	msc 43A40
Classification	msc 22B05
Classification	msc 22D35
Synonym	Pontrjagin duality
Synonym	Pontriagin duality
Related topic	DualityInMathematics
Defines	Pontryagin dual
Defines	Pontrjagin dual
Defines	Pontriagin dual
Defines	dual of an abelian group
Defines	character

1 Pontryagin dual

Let G be a locally compact abelian http://planetmath.org/TopologicalGroupgroup and $\mathbb{T}$ the http://planetmath.org/NTorus1-torus, i.e. the unit circle in $\mathbb{C}$.

Definition - A continuous homomorphism $G \longrightarrow \mathbb{T}$ is called a **character** of G. The set of all characters is called the **Pontryagin dual** of G and is denoted by $\hat{G}$.

Under pointwise multiplication $\hat{G}$ is also an abelian group. Since $\hat{G}$ is a group of functions we can make it a topological group under the compact-open topology (topology of convergence on compact sets).

2 Examples

- $\hat{\mathbb{Z}} \cong \mathbb{T}$, via $n \mapsto z^n$ with $z \in \mathbb{T}$.
- $\hat{\mathbb{T}} \cong \mathbb{Z}$, via $z \mapsto z^n$ with $n \in \mathbb{Z}$.
- $\hat{\mathbb{R}} \cong \mathbb{R}$, via $t \mapsto e^{ist}$ with $s \in \mathbb{R}$.

3 Properties

The following are some important of the dual group:

Theorem - Let G be a locally compact abelian group. We have that

- $\hat{G}$ is also locally compact.
- $\hat{G}$ is second countable if and only if G is second countable.
- $\hat{G}$ is compact if and only if G is discrete.
- $\hat{G}$ is discrete if and only if G is compact.
- $(\widehat{\oplus_{i\in J} G_i}) \cong \oplus_{i \in J} \hat{G}_i$ for any finite set J. This isomorphism is natural.

4 Pontryagin duality

Let $f : G \longrightarrow H$ be a continuous homomorphism of locally compact abelian groups. We can associate to it a canonical map $\hat{f} : \hat{H} \longrightarrow \hat{G}$ defined by

$$\hat{f}(\phi)\,(s) := \phi(f(s))\;, \qquad\qquad \phi \in \hat{H},\; s \in G$$

This canonical construction preserves identity mappings and compositions, i.e. the dualization process $\hat{}$ is a functor:

Theorem - The dualization $\hat{} : \mathbf{LcA} \longrightarrow \mathbf{LcA}$ is a contravariant functor from the category of locally compact abelian groups to itself.

5 Isomorphism with the second dual

Although in general there is not a canonical identification of G with its dual $\hat{G}$, there is a natural isomorphism between G and its dual's dual $\hat{\hat{G}}$:

Theorem - The map $G \longrightarrow \hat{\hat{G}}$ defined by $s \mapsto \hat{\hat{s}}$, where $\hat{\hat{s}}(\phi) := \phi(s)$, is a natural isomorphism between G and $\hat{\hat{G}}$.

6 Applications

The study of dual groups allows one to visualize Fourier series, Fourier transforms and discrete Fourier transforms from a more abstract and unified viewpoint, providing the for a general definition of Fourier transform. Thus, dual groups and Pontryagin duality are the of the of abstract abelian harmonic analysis.