proof of angle sum identities

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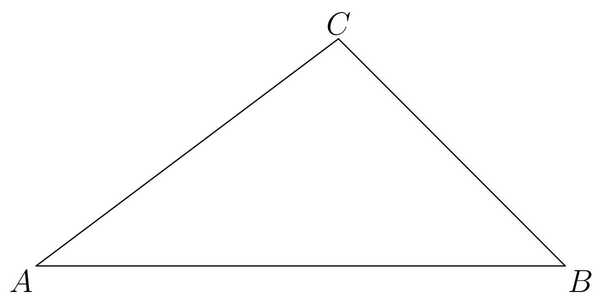
We will derive the angle sum identities for the various trigonometric functions here. We begin by deriving the identity for the sine by means of a geometric argument and then obtain the remaining identities by algebraic manipulation.

Theorem 1.

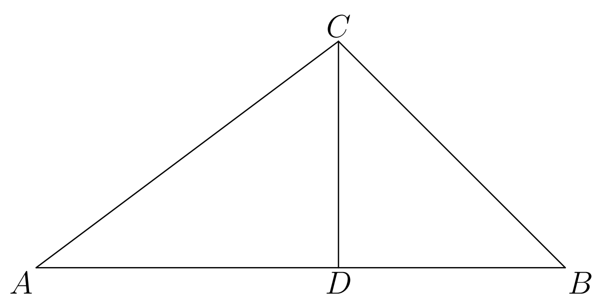
$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

Proof. Let us make the restrictions $0° < x < 90°$ and $0° < y < 90°$ for the time being. Then we may draw a triangle ABC such that $\angle CAB = x$ and $\angle ABF = y$:



Since the angles of a triangle add up to 180°, we must have $\angle BCA = 180° - x - y$, so we have $\sin(\angle BCA) = \sin(180° - x - y) = \sin(x + y)$.

We now draw perpendiculars two different ways in order to derive ratios. First, we drop a perpendicular AD from C to AB:

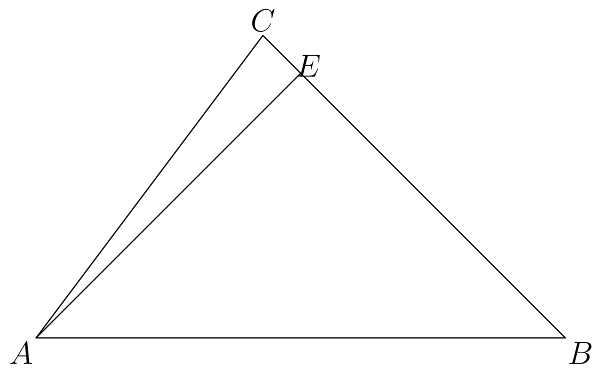


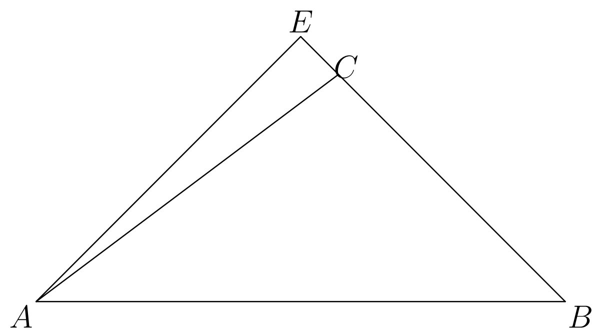
Since ACD and BCD are right triangles we have, by definition,

$$\cot(\angle CAB) = \overline{AD}/\overline{CD} \qquad \cot(\angle ABC) = \overline{BD}/\overline{CD} \qquad \sin(\angle CAB) = \overline{CD}/\overline{AC}.$$

Second, we draw a perpendicular AE form A to BC. Depending on whether $x + y < 90°$ or $x + y < 90°$ the point E will or will not lie between

B and C, as illustrated below. (There is also the case $x + y = 90°$, but it is trivial.)





Either way, ABE and ACE are right triangles, and we have, by definition,

$$\sin(\angle BCA) = \overline{AE}/\overline{AC} \qquad \sin(\angle ABC) = \overline{AE}/\overline{AB}.$$

Combining these ratios, we find that

$$\sin(\angle BCA)/\sin(\angle ABC) = \overline{AB}/\overline{AC}.$$

To finish deriving the sum identity, we manipulate the ratios derived above algebraically and use the fact that $\overline{AD} + \overline{BD} = \overline{AB}$:

$$\begin{aligned}
\sin(x+y) = \sin(\angle BCA) &= \overline{AB}\, \sin(\angle ABC)/\overline{AC} \\
&= (\overline{AD} + \overline{BD}) \sin(\angle ABC)/\overline{AC} \\
&= \overline{CD}\, (\cot(\angle CAB) + \cot(\angle ABC)) / \sin(\angle ABC)\overline{AC} \\
&= \sin(\angle CAB) \sin(\angle ABC) \left(\frac{\cos(\angle CAB)}{\sin(\angle CAB)} + \frac{\cos(\angle ABC)}{\sin(\angle ABC)} \right) \\
&= \sin(\angle CAB) \cos(\angle ABC) + \cos(\angle CAB) \sin(\angle ABC) \\
&= \sin(x) \cos(y) + \cos(x) \sin(y)
\end{aligned}$$

To lift the restriction on the range of x and y, we use the identities for complements and negatives of angles.

$\square$

Entry under construction