Laplace transform of cosine and sine

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We start from the easily formula

$$e^{\alpha t} \curvearrowleft \frac{1}{s-\alpha} \qquad (s>\alpha), \tag{1}$$

where the curved from the Laplace-transformed function to the original function. Replacing α by $-\alpha$ we can write the second formula

$$e^{-\alpha t} \curvearrowleft \frac{1}{s+\alpha} \qquad (s>-\alpha). \tag{2}$$

Adding (1) and (2) and dividing by 2 we obtain (remembering the linearity of the Laplace transform)

$$\frac{e^{\alpha t}+e^{-\alpha t}}{2} \curvearrowleft \frac{1}{2}\left(\frac{1}{s-\alpha}+\frac{1}{s+\alpha}\right),$$

i.e.

$$\mathcal{L}\{\cosh \alpha t\} = \frac{s}{s^2-\alpha^2}. \tag{3}$$

Similarly, subtracting (1) and (2) and dividing by 2 give

$$\mathcal{L}\{\sinh \alpha t\} = \frac{a}{s^2-\alpha^2}. \tag{4}$$

The formulae (3) and (4) are valid for $s>|\alpha|$.

There are the hyperbolic identities

$$\cosh it = \cos t, \qquad \frac{1}{i}\sinh it = \sin t$$

which enable the transition from hyperbolic to trigonometric functions. If we choose $\alpha := ia$ in (3), we may calculate

$$\cos at = \cosh iat \curvearrowleft \frac{s}{s^2-(ia)^2} = \frac{s}{s^2+a^2},$$

the formula (4) analogously gives

$$\sin at = \frac{1}{i}\sinh iat \curvearrowleft \frac{1}{i}\cdot\frac{ia}{s^2-(ia)^2} = \frac{a}{s^2+a^2}.$$

Accordingly, we have derived the Laplace transforms

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \tag{5}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \tag{6}$$

which are true for $s>0$.