Heaviside formula

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Let $P(s)$ and $Q(s)$ be polynomials with the degree of the former less than the degree of the latter.

- If all complex `http://planetmath.org/Zero`zeroes a_1, a_2, $\ldots$, a_n of $Q(s)$ are simple, then

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{j=1}^{n}\frac{P(a_j)}{Q'(a_j)}e^{a_jt}. \tag{1}$$

- If the different zeroes $a_1, a_2, \ldots, a_n$ of $Q(s)$ have the multiplicities $m_1, m_2, \ldots, m_n$, respectively, we denote $F_j(s) := (s-a_j)^{m_j}P(s)/Q(s)$; then

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{j=1}^{n}e^{a_jt}\sum_{k=0}^{m_j-1}\frac{F_j^{(k)}(a_j)t^{m_j-1-k}}{k!(m_j-1-k)!}. \tag{2}$$

A special case of the *Heaviside formula* (1) is

$$\mathcal{L}^{-1}\left\{\frac{Q'(s)}{Q(s)}\right\} = \sum_{j=1}^{n}e^{a_jt}. \tag{3}$$

Example. Since the zeros of the binomial s^4+4a^4 are $s = (\pm1\pm i)a$, we can calculate by (3) as follows:

$$\mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4s^3}{s^4+4a^4}\right\} = \frac{1}{4}\sum_{\pm}e^{(\pm1\pm i)at} = \frac{e^{at}+e^{-at}}{2}\cdot\frac{e^{iat}+e^{-iat}}{2} = \cosh at\,\cos at$$

Proof of (1). Without hurting the generality, we can suppose that $Q(s)$ is monic. Therefore

$$Q(s) = (s-a_1)(s-a_2)\cdots(s-s_n).$$

For $j = 1, 2, \ldots, n,$ denoting

$$Q(s) := (s-a_j)Q_j(s),$$

one has $Q_j(a_j) \neq 0$. We have a partial fraction expansion of the form

$$\frac{P(s)}{Q(s)} = \frac{C_1}{s-a_1} + \frac{C_2}{s-a_2} + \ldots + \frac{C_n}{s-a_n} \tag{4}$$

with constants C_j. According to the linearity and the formula 1 of the `http://planetmath.org/LaplaceTransform`parent entry, one gets

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{j=1}^{n} C_j e^{a_j t}. \tag{5}$$

For determining the constants C_j, multiply (3) by $s-a_j$. It yields

$$\frac{P(s)}{Q_j(s)} = C_j + (s-a_j)\sum_{\nu\neq j}\frac{C_\nu}{s-a_\nu}.$$

Setting to this identity $s := a_j$ gives the value

$$C_j = \frac{P(a_j)}{Q_j(a_j)}. \tag{6}$$

But since $Q'(s) = \frac{d}{ds}((s-a_j)Q_j(s)) = Q_j(s)+(s-a_j)Q_j'(s)$, we see that $Q'(a_j) = Q_j(a_j)$; thus the equation (5) may be written

$$C_j = \frac{P(a_j)}{Q'(a_j)}. \tag{7}$$

The values (6) in (4) produce the formula (1).

References

[1] K. Väisälä: *Laplace-muunnos.* Handout Nr. 163. Teknillisen korkeakoulun ylioppilaskunta, Otaniemi, Finland (1968).