Young's theorem

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Let $f, g : \mathbb{R}^n \to \mathbb{R}$. Recall that the *convolution* of f and g at x is

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

provided the integral is defined.

The following result is due to William Henry Young.

Theorem 1 *Let $p, q, r \in [1, \infty]$ satisfy*

$$\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1 \tag{1}$$

with the convention $1/\infty = 0$. Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$. Then:

1. *The function $y \mapsto f(x-y)g(y)$ belongs to $L^1(\mathbb{R}^n)$ for almost all x.*
2. *The function $x \mapsto (f * g)(x)$ belongs to $L^r(\mathbb{R}^n)$.*
3. *There exists a constant $c = c_{p,q} \leq 1$, depending on p and q but not on f or g, such that*
$$\|f * g\|_r \leq c \cdot \|f\|_p \cdot \|g\|_q$$

Observe the analogy with the similar result with convolution replaced by ordinary (pointwise) product, where the requirement is $1/p + 1/q = 1/r$—*i.e.*, $1/p + 1/q - 1/r = 0$—instead of (**??**). The cases

1. $1/p + 1/q = 1$, $r = \infty$
2. $p = 1$, $q \in [1, \infty)$, $r = q$

are the most widely known; for these we provide a proof, supposing $c_{p,q} = 1$. We shall use the following facts:

- If $x \mapsto f(x), x \mapsto g(x)$ are measurable, then $(x, y) \mapsto f(x-y)g(y)$ is measurable.
- For any x, if $f \in L^p$, then $y \mapsto f(x-y)$ belongs to L^p as well, and its L^p-norm is the same as f's.
- For any y, if $f \in L^p$, then $x \mapsto f(x-y)$ belongs to L^p as well, and its L^p-norm is the same as f's.

Proof of case **??**.

Suppose $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ with $1/p + 1/q = 1$. Then

$$\left| \int f(x-y)g(y)dy \right| \leq \int |f(x-y)g(y)|dy \leq \|f\|_p \|g\|_q .$$

This holds for all $x \in \mathbb{R}^n$, therefore $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$ as well.

Proof of case **??**.

First, suppose $q = 1$. We may suppose f and g are Borel measurable: if they are not, we replace them with Borel measurable functions $\tilde{f}$ and $\tilde{g}$ which are equal to f and g, respectively, outside of a set of Lebesgue measure zero; apply the theorem to $\tilde{f}$, $\tilde{g}$, and $\tilde{f} * \tilde{g}$; and deduce the theorem for f, g, and $f * g$. By Tonelli's theorem,

$$\int \left(\int |f(x-y)g(y)|dy \right) dx = \int \left(\int |f(x-y)|dx \right) |g(y)|dy = \|f\|_1 \|g\|_1 ,$$

thus the function $(x, y) \mapsto f(x-y)g(y)$ belongs to $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. By Fubini's theorem, the function $y \mapsto f(x-y)g(y)$ belongs to $L^1(\mathbb{R}^n)$ for almost all x, and $x \mapsto (f * g)(x)$ belongs to $L^1(\mathbb{R}^n)$; plus,

$$\|f * g\|_1 \leq \int \int |f(x-y)g(y)|dydx = \|f\|_1 \|g\|_1 .$$

Suppose now $q > 1$; choose q' so that $1/q + 1/q' = 1$. By the argument above, $y \mapsto |f(x-y)| \cdot |g(y)|^q$ belongs to L^1 for almost all x: for those x, put $u(y) = |f(x-y)|^{1/q'}, v(y) = |f(x-y)|^{1/q}|g(y)|$. Then $u \in L^{q'}$ and $v \in L^q$ with $1/q' + 1/q = 1$, so $uv \in L^1$ and $\|uv\|_1 \leq \|u\|_{q'} \|v\|_q$: but $uv = |f(x-y)g(y)|$, so point **??** of the theorem is proved. By Hölder's inequality,

$$\left| \int f(x-y)g(y)dy \right| \leq \int |f(x-y)g(y)|dy \leq \|f\|_1^{1/q'} \left(\int |f(x-y)| \cdot |g(y)|^q dy \right)^{1/q} :$$

but we know that $|f| * |g|^q \in L^1$, so $f * g \in L^q$ and point **??** is also proved. Finally,

$$\|f * g\|_q^q \leq \|f\|_1^{q/q'} \| |f| * |g|^q \|_1 \leq \|f\|_1^{q/q'} \|f\|_1 \|g\|_q^q = \|f\|_1^{1+q/q'} \|g\|_q^q :$$

but $1/q + 1/q' = 1$ means $q + q' = qq'$ and thus $1 + q/q' = q$, so that point **??** is also proved.

References

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