Laplace transform of sine integral

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0.1 Derivation of $\mathcal{L}\{\mathrm{Si}\, t\}$

If one performs the change of integration variable

$$u = tx, \qquad du = t\,dx$$

in the defining `http://planetmath.org/DefiniteIntegral`integral

$$\mathrm{Si}\, t = \int_0^t \frac{\sin u}{u}\,du,$$

of the sine integral function, one obtains

$$\mathrm{Si}\, t = \int_0^1 \frac{\sin tx}{tx} t\,dx = \int_0^1 \frac{\sin tx}{x}\,dx,$$

getting `http://planetmath.org/UpperLimit`limits. We know (see the entry Laplace transform of sine and cosine) that

$$\mathcal{L}\left\{\frac{\sin tx}{x}\right\} = \frac{1}{s^2+x^2}.$$

This transformation formula can be integrated with respect to the parametre x:

$$\mathcal{L}\left\{\int_0^1 \frac{\sin tx}{x}\,dx\right\} = \int_0^1 \frac{1}{s^2+x^2}\,dx = \frac{1}{s} \Big/_{x=0}^{1} \arctan\frac{x}{s} = \frac{1}{s}\arctan\frac{1}{s}.$$

Thus we have the transformation formula of the sinus integralis:

$$\mathcal{L}\left\{\mathrm{Si}\, t\right\} = \frac{1}{s}\arctan\frac{1}{s}. \tag{1}$$

0.2 Laplace transform of sinc function

By the formula $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - \lim_{x\to 0+} f(x)$ of the `http://planetmath.org/LaplaceTransform` entry, we obtain as consequence of (1), that

$$\mathcal{L}\left\{\frac{d}{dt}\mathrm{Si}\, t\right\} = s\cdot\frac{1}{s}\arctan\frac{1}{s} - \mathrm{Si}\, 0,$$

i.e.

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \arctan\frac{1}{s}. \tag{2}$$

The formula (2) may be determined also directly using the definition of Laplace transform. Take an additional parametre a to the defining integral

$$\mathcal{L}\{\frac{\sin t}{t}\} = \int_0^\infty e^{-st}\,\frac{\sin t}{t}\,dt$$

by setting

$$\int_0^\infty e^{-st}\,\frac{\sin at}{t}\,dt := \varphi(a).$$

Now we have the derivative $\varphi'(a) = \int_0^\infty e^{-st}\cos at\,dt,$ where one can partially integrate twice, getting

$$\varphi'(a) = \int_0^\infty e^{-st}\cos at\,dt = \frac{1}{s} - \frac{a^2}{s^2}\int_0^\infty e^{-st}\cos at\,dt.$$

Thus we solve

$$\int_0^\infty e^{-st}\cos at\,dt = \frac{\frac{1}{s}}{1+\left(\frac{a}{s}\right)^2} = \varphi'(a),$$

and since $\varphi(0) = 0$, we obtain $\varphi(a) = \arctan\dfrac{a}{s}$. This yields

$$\int_0^\infty e^{-st}\,\frac{\sin t}{t}\,dt = \varphi(1) = \arctan\frac{1}{s},$$

i.e. the formula (2).

Formula (2) is derived `http://planetmath.org/LaplaceTransformOfFracftt`there in a third way.