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relative of exponential integral

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Let a and b be positive numbers. We want to calculate the value of the improper integral

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx \tag{1}$$

related to the exponential integral.

The value may be found e.g. by utilising the derivative of the integral

$$I(y) \;:=\; \int_0^\infty e^{-xy}\cdot\frac{e^{-ax}-e^{-bx}}{x}\,dx$$

which can be formed by http://planetmath.org/DifferentiationUnderIntegralSigndifferentia under the integral sign:

$$\begin{aligned}
I'(y) &= \int_0^\infty e^{-xy}(-x)\frac{e^{-ax}-e^{-bx}}{x}\,dx\\
&= \int_0^\infty \left(e^{-(y+b)x}-e^{-(y+a)x}\right)\,dx\\
&= \Big/_{\!x=0}^{\infty} \left(\frac{e^{-(y+b)x}}{-(y+b)}-\frac{e^{-(y+a)x}}{-(y+a)}\right)\\
&= \frac{1}{y+b}-\frac{1}{y+a}
\end{aligned}$$

Thus,

$$I(y) \;=\; \ln(y+b)-\ln(y+a) \;=\; \ln\frac{y+b}{y+a},$$

and the integral (1) has the value $I(0)=\ln\dfrac{b}{a}$.

There is another method via Laplace transforms. By the table of Laplace transforms, we have

$$\mathcal{L}\{e^{-at}-e^{-bt}\} \;=\; \frac{1}{s+a}-\frac{1}{s+b}$$

and therefore

$$\mathcal{L}\{\frac{e^{-at}-e^{-bt}}{t}\} \;=\; \int_s^\infty \left(\frac{1}{u+a}-\frac{1}{u+b}\right)\,du \;=\; \Big/_{\!u=s}^{\infty} \ln\frac{u+a}{u+b} \;=\; \ln\frac{s+b}{s+a},$$

i.e.

$$\int_0^\infty e^{-st} \cdot \frac{e^{-at} - e^{-bt}}{t}\, dt \;=\; \ln \frac{s+b}{s+a}.$$

Letting $s \to 0+$, this yields the equation

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx \;=\; \ln \frac{b}{a}.$$