Proof of Fekete's subadditive lemma

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Owner	Filipe (28191)
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Author	Filipe (28191)
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If there is a m such that $a_m = -\infty$, then, by subadditivity, we have $a_n = -\infty$ for all $n > m$. Then, both sides of the equality are $-\infty$, and the theorem holds. So, we suppose that $a_n \in \mathbf{R}$ for all n. Let $L = \inf_n \frac{a_n}{n}$ and let B be any number greater than L. Choose $k \geq 1$ such that

$$\frac{a_k}{k} < B$$

For $n > k$, we have, by the division algorithm there are integers p_n and q_n such that $n = p_n k + q_n$, and $0 \leq q_n \leq k - 1$. Applying the definition of subadditivity many times we obtain:

$$a_n = a_{p_n k + q_n} \leq a_{p_n k} + a_{q_n} \leq p_n a_k + a_{q_n}$$

So, dividing by n we obtain:

$$\frac{a_n}{n} \leq \frac{p_n k}{n} \frac{a_k}{k} + \frac{a_{q_n}}{n}$$

When n goes to infinity, $\frac{p_n k}{n}$ converges to 1 and $\frac{a_{q_n}}{n}$ converges to zero, because the numerator is bounded by the maximum of a_i with $0 \leq i \leq k - 1$. So, we have, for all $B > L$:

$$L \leq \lim_n \frac{a_n}{n} \leq \frac{a_k}{k} < B$$

Finally, let B go to L and we obtain

$$L = \inf_n \frac{a_n}{n} = \lim_n \frac{a_n}{n}$$