Laplace transform of integral

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On can show that if a real function $t \mapsto f(t)$ is http://planetmath.org/LaplaceTransformLap transformable, as well is $\displaystyle\int_0^t f(\tau)\,d\tau$. The latter is also continuous for $t > 0$ and by the http://planetmath.org/FundamentalTheoremOfCalculusNewton–Leibniz formula, has the derivative equal $f(t)$. Hence we may apply the formula for Laplace transform of derivative, obtaining

$$F(s) \;=\; \mathcal{L}\{f(t)\} \;=\; s\,\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} - \int_0^0 f(t)\,dt \;=\; s\,\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\},$$

i.e.

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} \;=\; \frac{F(s)}{s}. \tag{1}$$

Application. We start from the easily derivable rule

$$\frac{1}{s} \;\curvearrowright\; 1,$$

where the curved from the Laplace-transformed function to the original function. The formula (1) thus yields successively

$$\frac{1}{s^2} \;\curvearrowright\; \int_0^t 1\,d\tau \;=\; t,$$

$$\frac{1}{s^3} \;\curvearrowright\; \int_0^t \tau\,d\tau \;=\; \frac{t^2}{2!},$$

$$\frac{1}{s^4} \;\curvearrowright\; \int_0^t \frac{\tau^2}{2!}\,d\tau \;=\; \frac{t^3}{3!},$$

etc. Generally, one has

$$\frac{1}{s^n} \;\curvearrowright\; \frac{t^{n-1}}{(n-1)!} \quad \forall\, n \in \mathbb{Z}_+. \tag{2}$$