Laplace transform

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Let $f(t)$ be a function defined on the interval $[0, \infty)$. The *Laplace transform* of $f(t)$ is the function $F(s)$ defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

provided that the integral converges. [1] It suffices that f be defined when $t > 0$ and s can be complex. We will usually denote the Laplace transform of f by $\mathcal{L}\{f\}$. Some of the most common Laplace transforms are:

1. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}, \quad s > a$
2. $\mathcal{L}\{\cos(bt)\} = \dfrac{s}{s^2+b^2}, \quad s > 0$
3. $\mathcal{L}\{\sin(bt)\} = \dfrac{b}{s^2+b^2}, \quad s > 0$
4. $\mathcal{L}\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}, \quad s > 0,\ n > -1.$
5. $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - \lim_{x \to 0+} f(x)$

For more particular Laplace transforms, see the table of Laplace transforms.

Notice the Laplace transform is a linear transformation. It is worth noting that, if

$$\int_0^\infty e^{-st} |f(t)|\, dt < \infty$$

for some $s \in \mathbb{R}$, then $\mathcal{L}\{f\}$ is an analytic function in the complex half-plane $\{z \mid \Re z > s\}$.

Much like the Fourier transform, the Laplace transform has a convolution. However, the form of the convolution used is different.

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

[1] Depending on the definition of integral one is using, one may prefer to define the Laplace transform as $\lim_{x\to 0+} \int_x^\infty e^{-st} f(t)\, dt$

where

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,ds$$

and

$$\mathcal{L}\{fg\}(s) = \int_{c-i\infty}^{c+i\infty} \mathcal{L}\{f\}(z)\mathcal{L}\{g\}(s-z)\,dz$$

The most popular usage of the Laplace transform is to solve initial value problems by taking the Laplace transform of both sides of an ordinary differential equation; see the entry "`http://planetmath.org/ImageEquation`image equation".