# Laplace transform of $t^n f(t)$

| | |
|---|---|
| Canonical name | LaplaceTransformOfTnft |
| Date of creation | 2013-03-22 18:05:49 |
| Last modified on | 2013-03-22 18:05:49 |
| Owner | pahio (2872) |
| Last modified by | pahio (2872) |
| Numerical id | 6 |
| Author | pahio (2872) |
| Entry type | Derivation |
| Classification | msc 44A10 |
| Related topic | TableOfLaplaceTransforms |

Let

$$F(s) := \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

A differentiation under the integral sign with respect to $s$ yields

$$F'(s) = -\int_0^\infty e^{-st} t f(t)\,dt = -\mathcal{L}\{tf(t)\}.$$

Differentiating again under the integral sign gives

$$F''(s) = +\int_0^\infty e^{-st} t^2 f(t)\,dt = \mathcal{L}\{t^2 f(t)\}.$$

One can continue similarly, and then we apparently have

$$F^{(n)}(s) = (-1)^n \int_0^\infty e^{-st} t^n f(t)\,dt = (-1)^n \mathcal{L}\{t^n f(t)\}. \tag{1}$$

If this equation is multiplied by $(-1)^n$, it gives the

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \tag{2}$$

which is true for $n = 1,\, 2,\, 3,\, \ldots$

**Application.** Evaluate the improper integral

$$I := \int_0^\infty t^3 e^{-t} \sin t\,dt.$$

By the `http://planetmath.org/LaplaceTransform`parent entry, we have $\mathcal{L}\{\sin t\} = \frac{1}{1+s^2}$. Using this and (2), we may write

$$\int_0^\infty t^3 e^{-st} \sin t\,dt = \mathcal{L}\{t^3 \sin t\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right) = \frac{24(s-s^3)}{(1+s^2)^4}.$$

The value of $I$ is obtained by substituting here $s = 1$:

$$I = \frac{24(1-1^3)}{(1+1^2)^4} = 0.$$