Laplace transform of $\frac{f(t)}{t}$

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Suppose that the quotient

$$\frac{f(t)}{t} \;:=\; g(t)$$

is http://planetmath.org/LaplaceTransformLaplace-transformable. It follows easily that also $f(t)$ is such. According to the http://planetmath.org/LaplaceTransformOf entry, we may write

$$\mathcal{L}^{-1}\left\{G'(s)\right\} \;=\; -t\,g(t) \;=\; -f(t) \;=\; \mathcal{L}^{-1}\left\{-F(s)\right\}.$$

Therefore

$$G'(s) \;=\; -F(s),$$

whence

$$G(s) \;=\; -F^{(-1)}(s)+C \tag{1}$$

where $F^{(-1)}(s)$ means any antiderivative of $F(s)$. Since each Laplace transformed function vanishes in the infinity $s = \infty$ and thus $G(\infty) = 0$, the equation (1) implies

$$C \;=\; F^{(-1)}(\infty)$$

and therefore

$$G(s) \;=\; F^{(-1)}(\infty)-F^{(-1)}(s) \;=\; \int_s^\infty F(u)\,du.$$

We have obtained the result

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} \;=\; \int_s^\infty F(u)\,du. \tag{2}$$

Application. By the table of Laplace transforms, $\mathcal{L}\left\{\sin t\right\} \;=\; \dfrac{1}{s^2+1}$. Accordingly the formula (2) yields

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\,du = \Big/_{\!s}^{\!\infty} \arctan u \;=\; \frac{\pi}{2}-\arctan s \;=\; \operatorname{arccot} s.$$

Thus we have

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} \;=\; \operatorname{arccot} s \;=\; \arctan\frac{1}{s}. \tag{3}$$

This result is derived in the entry Laplace transform of sine integral in two other ways.