relative of cosine integral

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For determining of the value of the improper integral

$$I(a) \;:=\; \int_0^\infty \frac{\cos ax^2-\cos ax}{x}\,dx \qquad (a>0),$$

related to the cosine integral, we think it as a function of the parametre a which we denote by t. Then we can take the Laplace transform (see the integration with respect to a parametre in the table of Laplace transforms):

$$\mathcal{L}\{I(t)\} \;=\; \mathcal{L}\{\int_0^\infty (\cos tx^2-\cos tx)\frac{dx}{x}\} \;=\; \int_0^\infty \left(\frac{s}{s^2\!+\!x^4}-\frac{s}{s^2\!+\!x^2}\right)\frac{dx}{x}$$

Splitting the fractional expressions to `http://planetmath.org/node/5812`partial fractions and integrating give

$$\begin{aligned}\mathcal{L}\{I(t)\} \;&=\; \frac{1}{s}\int_0^\infty \left(\frac{1}{x}-\frac{x^3}{s^2\!+\!x^4}-\frac{1}{x}+\frac{x}{s^2\!+\!x^2}\right)dx\\ &=\; \frac{1}{s}\Big/_{x=0}^{\infty}\left[\frac{1}{2}\ln(s^2\!+\!x^2)-\frac{1}{4}\ln(s^2\!+\!x^4)\right]\\ &=\; \frac{1}{4}\Big/_{x=0}^{\infty}\ln\frac{(s^2\!+\!x^2)^2}{s^2\!+\!x^4} \;=\; -\frac{\ln s}{2s}.\end{aligned}$$

As seen in the `http://planetmath.org/node/10588`table of Laplace transforms, the gotten expression is the Laplace transform of $\dfrac{\gamma+\ln t}{2} \;=\; I(t)$ (N.B. $\mathcal{L}\{1\}=\dfrac{1}{s}$), and thus we have the result

$$I(a) \;=\; \frac{\gamma+\ln a}{2}.$$