inverse Laplace transform of derivatives

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It may be shown that the Laplace transform $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ is always differentiable and that its derivative can be formed by `http://planetmath.org/Differentia` under the integral sign, i.e. one has

$$F'(s) = \int_0^\infty \frac{\partial(e^{-st}f(t))}{\partial s}\,dt = \int_0^\infty e^{-st}(-t)f(t)\,dt.$$

This gives the rule

$$\mathcal{L}^{-1}\{F'(s)\} = -tf(t). \tag{1}$$

Applying (1) to $F'(s)$ instead of $F(s)$ gives

$$\mathcal{L}^{-1}\{F''(s)\} = t^2 f(t).$$

Continuing this way we can obtain the general rule

$$\mathcal{L}^{-1}\{F^{(n)}(s)\} = (-1)^n t^n f(t), \tag{2}$$

or equivalently

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \cdot \frac{d^n \mathcal{L}\{f(t)\}}{ds^n}, \tag{3}$$

for any $n = 1, 2, 3, \ldots$ (and of course for $n = 0$).

Example. Let's find the Laplace transform of the first kind and 0th Bessel function

$$J_0(t) := \sum_{m=0}^\infty \frac{(-1)^m}{(m!)^2}\left(\frac{t}{2}\right)^{2m},$$

which is the solution $y(t)$ of the Bessel's equation

$$ty''(t) + y'(t) + ty(t) = 0 \tag{4}$$

satisfying the initial condition $y(0) = 1$. The equation implies that $y'(0) = 0$.

By (3), the Laplace transform of the differential equation (4) is

$$-\frac{d\mathcal{L}\{y''(t)\}}{ds} + \mathcal{L}\{y'(t)\} - \frac{d\mathcal{L}\{y(t)\}}{ds} = 0.$$

Using here twice the rule 5 in the `http://planetmath.org/LaplaceTransform`parent entry gives us

$$-\frac{d(s^2Y(s)-s)}{ds}+sY(s)-1-\frac{dY(s)}{ds}=0,$$

which is simplified to

$$(s^2+1)\frac{dY}{ds}+sY=0,$$

i.e. to

$$\frac{dY}{Y}=-\frac{s\,ds}{s^2+1}.$$

Integrating this gives

$$\ln Y=-\frac{1}{2}\ln(s^2+1)+\ln C=\ln\frac{C}{\sqrt{s^2+1}},$$

i.e.

$$Y(s)=\frac{C}{\sqrt{s^2+1}}.$$

The initial condition enables to justify that the integration constant C must be 1. Thus we have the result

$$\mathcal{L}\{J_0(t)\}=\frac{1}{\sqrt{s^2+1}}.$$

References

[1] K. Väisälä: *Laplace-muunnos.* Handout Nr. 163. Teknillisen korkeakoulun ylioppilaskunta, Otaniemi, Finland (1968).