Von Neumann's ergodic theorem

Canonical name	VonNeumannsErgodicTheorem
Date of creation	2014-03-18 14:02:09
Last modified on	2014-03-18 14:02:09
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Numerical id	6
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Entry type	Theorem
Related topic	Birkhoff ergodic theorem

Let $U : H \to H$ be an isometry in a Hilbert space H. Consider the subspace $I(U) = \{v \in H : Uv = v\}$, called the space of invariant vectors. Denote by P the orthogonal projection over the subspace $I(U)$. Then,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} U^j(v) = P(v), \forall v \in H$$

This general theorem for Hilbert spaces can be used to obtain an ergodic theorem for the $L^2(\mu)$ space by taking H to be the $L^2(\mu)$ space, and U to be the composition operator (also called Koopman operator) associated to a transformation $f : M \to M$ that preserves a measure μ, i.e., $U_f(\psi) = \psi \circ f$, where $\psi : M \to \mathbf{R}$. The space of invariant functions is the set of functions ψ such that $\psi \circ f = \psi$ almost everywhere. For any $\psi \in L^2(\mu)$, the sequence:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi \circ f^j$$

converges in $L^2(\mu)$ to the orthogonal projection $\tilde{\psi}$ of the function ψ over the space of invariant functions.

The $L^2(\mu)$ version of the ergodic theorem for Hilbert spaces can be derived directly from the more general Birkhoff ergodic theorem, which asserts pointwise convergence instead of convergence in $L^2(\mu)$. Actually, from Birkhoff ergodic theorem one can derive a version of the ergodic theorem where convergence in $L^p(\mu)$ holds, for any $p > 1$.