Laplace transform of periodic functions

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Let $f(t)$ be periodic with the positive http://planetmath.org/PeriodicFunctionsperiod p. Denote by $H(t)$ the Heaviside step function. If now

$$g(t) \;:=\; f(t)H(t)-f(t-p)H(t-p),$$

then it follows

$$g(t) \;=\; \begin{cases} f(t) & \text{for } \; 0 < t < p, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

By the http://planetmath.org/DelayTheoremparent entry, the Laplace transform of g is

$$G(s) \;=\; F(s)-e^{-ps}F(s),$$

whence

$$F(s) \;=\; \frac{G(s)}{1-e^{-ps}} \;=\; \frac{1}{1-e^{-ps}}\int_0^\infty e^{-st}g(t)\,dt \;=\; \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt.$$

Thus we have the rule

$$\mathcal{L}\{f(t)\} \;=\; \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt \qquad \text{(period } p\text{)}. \tag{2}$$

On the contrary, if $f(t)$ is antiperiodic with positive antiperiod p, then the function

$$g(t) \;:=\; f(t)H(t)+f(t-p)H(t-p)$$

also has the property (1). Analogically with the preceding procedure, one may derive the rule

$$\mathcal{L}\{f(t)\} \;=\; \frac{1}{1+e^{-ps}}\int_0^p e^{-st}f(t)\,dt \qquad \text{(antiperiod } p\text{)}. \tag{3}$$