cone

Canonical name	Cone1
Date of creation	2013-03-22 15:32:58
Last modified on	2013-03-22 15:32:58
Owner	matte (1858)
Last modified by	matte (1858)
Numerical id	16
Author	matte (1858)
Entry type	Definition
Classification	msc 46-00
Related topic	ProperCone
Related topic	GeneralizedFarkasLemma
Defines	blunt cone
Defines	pointed cone
Defines	salient cone
Defines	cone with vertex
Defines	wedge
Defines	proper cone
Defines	generating

Definition 1. *Suppose V is a real (or complex) vector space with a subset C.*

1. *If $\lambda C \subset C$ for any real $\lambda > 0$, then C is called a* **cone**.
2. *If the origin belongs to a cone, then the cone is said to be* **pointed**. *Otherwise, the cone is* **blunt**.
3. *A pointed cone is* **salient**, *if it contains no 1-dimensional vector subspace of V.*
4. *If $C - x_0$ is a cone for some x_0 in V, then C is a* **cone with vertex** *at x_0.*
5. *A convex pointed cone is called a* **wedge**.
6. *A* **proper cone** *is a convex cone C with vertex at 0, such that $C \cap (-C) = \{0\}$. A slightly more specific definition of a proper cone is this* `http://planetmath.org/ProperCone`*entry, but it requires the vector space to be topological.*
7. *A cone C is said to be* **generating** *if $V = C - C$. In this case, V is said to be* **generated by** *C.*

Examples

1. In $\mathbb{R}$, the set $x > 0$ is a blunt cone.
2. In $\mathbb{R}$, the set $x \geq 0$ is a pointed salient cone.
3. Suppose $x \in \mathbb{R}^n$. Then for any $\varepsilon > 0$, the set

   $$C = \bigcup \{ \lambda B_x(\varepsilon) \mid \lambda > 0 \}$$

   is an open cone. If $|x| < \varepsilon$, then $C = \mathbb{R}^n$. Here, $B_x(\varepsilon)$ is the open ball at x with radius ε.
4. In a normed vector space, a blunt cone C is completely determined by the intersection of C with the unit sphere.

Properties

1. The union and intersection of a collection of cones is a cone. In other words, the set of cones forms a complete lattice.

2. The complement of a cone is a cone. This means that the complete lattice of cones is also a complemented lattice.

3. A cone C is convex iff $C + C \subseteq C$.

   Proof. If C is convex and $a, b \in C$, then $\frac{1}{2}a, \frac{1}{2}b \in C$, so their sum, being the convex combination of a, b, is in C, and therefore $a + b = 2(\frac{1}{2}a + \frac{1}{2}b) \in C$ also. Conversely, suppose a cone C satisfies $C + C \subseteq C$, and $a, b \in C$. Then $\lambda a, (1 - \lambda)b \in C$ for $\lambda > 0$ (the case when $\lambda = 0$ is obvious). Therefore their sum is also in C. $\square$

4. A cone containing 0 is a cone with vertex at 0. As a result, a wedge is a cone with vertex at 0.

5. The only cones that are subspaces at the same time are wedges.

References

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