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Sobolev space

Canonical name	SobolevSpace
Date of creation	2013-03-22 14:54:55
Last modified on	2013-03-22 14:54:55
Owner	paolini (1187)
Last modified by	paolini (1187)
Numerical id	10
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Entry type	Definition
Classification	msc 46E35
Synonym	Sobolev function
Related topic	WeakDerivative

We define the Sobolev spaces of functions $W^{m,p}(\Omega)$ where Ω is an open subset of $\mathbf{R}^n$, $m \geq 0$ is an integer and $p \in [1, +\infty]$.

The spaces $W^{0,p}(\Omega)$ are simply defined to be the spaces $L^p(\Omega)$ of Lebesgue p-summable functions. We then define the space $W^{m,p}(\Omega)$ to be the space of functions $u \in L^p(\Omega)$ which have weak derivatives $g = (g_1, \dots, g_n)$ such that $g_i \in W^{m-1,p}(\Omega)$.

The space $W^{m,p}$ turns out to be a Banach space when endowed with the norm

$$\|u\|_{W^{m,p}} = \sum_{k=0}^{m} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \left[\int_\Omega \left| \frac{\partial^k u(x)}{\partial x_{i_1} \cdots \partial x_{i_k}} \right|^p dx \right]^{\frac{1}{p}}$$

i.e. the sum of the L^p norms of u and of all weak derivatives of u up to the m-th order.

Of particular interest are the spaces $H^m(\Omega) := W^{m,2}(\Omega)$ which turn out to be Hilbert spaces with the scalar product given by

$$(u, v)_{H^m(\Omega)} = \sum_{k=0}^{m} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \int_\Omega \frac{\partial^k u(x)}{\partial x_{i_1} \cdots \partial x_{i_k}} \frac{\partial^k v(x)}{\partial x_{i_1} \cdots \partial x_{i_k}} \, dx.$$