equivalence of definitions of C^*-algebra

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In this entry, we will prove that the definitions of C^* algebra given in the main entry are equivalent.

Theorem 1. *A Banach algebra A with an antilinear involution $*$ such that $\|a\|^2 \leq \|a^*a\|$ for all $a \in A$ is a C^*-algebra.*

Proof. It follows from the product inequality $\|ab\| \leq \|a\|\|b\|$ that

$$\|a\|^2 \leq \|a^*a\| \leq \|a^*\|\|a\|.$$

Therefore, $\|a\| \leq \|a^*\|$. Putting a^* for a, we also have $\|a^*\| \leq \|a^{**}\| = \|a\|$. Thus, the involution is an isometry: $\|a\| = \|a^*\|$. So now,

$$\|a\|^2 \leq \|a^*a\| \leq \|a\|^2.$$

Hence, $\|a^*a\| = \|a\|^2$. □

Theorem 2. *A Banach algebra A with an antilinear involution $*$ such that $\|a^*a\| = \|a^*\|\|a\|$ is a C^*-algebra.*