orthogonal decomposition theorem

Canonical name	OrthogonalDecompositionTheorem
Date of creation	2013-03-22 17:32:34
Last modified on	2013-03-22 17:32:34
Owner	asteroid (17536)
Last modified by	asteroid (17536)
Numerical id	4
Author	asteroid (17536)
Entry type	Theorem
Classification	msc 46A99
Synonym	closed subspaces of Hilbert spaces are complemented

Theorem - Let X be an Hilbert space and $A \subseteq X$ a closed subspace. Then the http://planetmath.org/Complimentaryorthogonal complement of A, denoted $A^{\perp}$, is a topological complement of A. That means $A^{\perp}$ is closed and

$$X = A \oplus A^{\perp} \ .$$

Proof :

- $A^{\perp}$ is closed :

  This follows easily from the continuity of the inner product. If a sequence (x_n) of elements in $A^{\perp}$ converges to an element $x_0 \in X$, then

  $$\langle x_0, a \rangle = \langle \lim_{n\to\infty} x_n, a \rangle = \lim_{n\to\infty} \langle x_n, a \rangle = 0 \quad \text{for every} a \in A$$

  which implies that $x_0 \in A^{\perp}$.

- $X = A \oplus A^{\perp}$:

  Since X is http://planetmath.org/Completecomplete and A is closed, A is a subspace of X. Therefore, for every $x \in X$, there exists a best approximation of x in A, which we denote by $a_0 \in A$, that satisfies $x - a_0 \in A^{\perp}$ (see this http://planetmath.org/BestApproximationInInnerProductSpacesentry).

  This allows one to write x as a sum of elements in A and $A^{\perp}$

  $$x = a_0 + (x - a_0)$$

  which proves that

  $$X = A + A^{\perp} \ .$$

  Moreover, it is easy to see that

  $$A \cap A^{\perp} = \{0\}$$

  since if $y \in A \cap A^{\perp}$ then $\langle y, y \rangle = 0$, which means $y = 0$.

  We conclude that $X = A \oplus A^{\perp}$. $\square$