injective C^*-algebra homomorphism is isometric

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Theorem - Let $\mathcal{A}$ and $\mathcal{B}$ be `http://planetmath.org/CAlgebra`C^*-algebras and $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ an injective *-homomorphism. Then $\|\Phi(x)\| = \|x\|$ and $\sigma(\Phi(x)) = \sigma(x)$ for every $x \in \mathcal{A}$, where $\sigma(y)$ denotes the spectrum of the element y.

Proof: It suffices to prove the result for unital C^*-algebras, since the general case follows directly by considering the minimal unitizations of $\mathcal{A}$ and $\mathcal{B}$. So we assume that $\mathcal{A}$ and $\mathcal{B}$ are unital and we will denote their identity elements by e, being clear from context which one is being used.

Let us first prove the second part of the theorem for normal elements $x \in \mathcal{A}$. It is clear that $\sigma(\Phi(x)) \subseteq \sigma(x)$ since if $x - \lambda e$ invertible for some $\lambda \in \mathcal{C}$, then so is $\Phi(x) - \lambda e = \Phi(x - \lambda e)$. Suppose the inclusion is strict, then there is a non-zero function $f \in C(\sigma(x))$ whose restriction to $\sigma(\Phi(x))$ is zero (here $C(\sigma(x))$ denotes the C^*-algebra of continuous functions $\sigma(x) \longrightarrow \mathbb{C}$). Thus we have, by the continuous functional calculus, that $f(x) \neq 0$ and also that

$$\Phi(f(x)) = f(\Phi(x)) = 0$$

by the continuous functional calculus and the result on `http://planetmath.org/CAlgebraHomomor` entry. Thus, we conclude that Φ is not injective and which is a contradiction. Hence we must have $\sigma(\Phi(x)) = \sigma(x)$.

Let $R_\sigma(z)$ denote the spectral radius of the element z. From the `http://planetmath.org/NormA` and spectral radius relation in C^*-algebras we know that, for an arbitrary element $x \in \mathcal{A}$, we have that

$$\|x\|^2 = R_\sigma(x^* x)$$

Since the element x^*x is normal, from the preceding paragraph it follows that $R_\sigma(x^*x) = R_\sigma(\Phi(x^*x))$, and hence we conclude that

$$\|x\|^2 = R_\sigma(x^*x) = R_\sigma\big(\Phi(x)^*\Phi(x)\big) = \|\Phi(x)\|^2$$

i.e. $\|\Phi(x)\| = \|x\|$.

Since Φ is isometric, $\Phi(\mathcal{A})$ is closed *-subalgebra of $\mathcal{B}$, i.e. $\Phi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}$, and it is isomorphic to $\mathcal{A}$. Using the spectral invariance theorem we conclude that $\sigma(x) = \sigma(\Phi(x))$ for every $x \in \mathcal{A}$. $\square$