localization for distributions

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Definition Suppose U is an open set in $\mathbb{R}^n$ and T is a distribution $T \in \mathcal{D}'(U)$. Then we say that T *vanishes* on an open set $V \subset U$, if the restriction of T to V is the zero distribution on V. In other words, T vanishes on V, if $T(v) = 0$ for all $v \in C_0^\infty(V)$. (Here $C_0^\infty(V)$ is the set of smooth function with compact support in V.) Similarly, we say that two distributions $S, T \in \mathcal{D}'(U)$ are *equal*, or *coincide* on V, if $S - T$ vanishes on V. We then write: $S = T$ on V.

Theorem[**?**, **?**] Suppose U is an open set in $\mathbb{R}^n$ and $\{U_i\}_{i\in I}$ is an open cover of U, i.e.,

$$U = \bigcup_{i \in I} U_i.$$

Here, I is an arbitrary index set. If S, T are distributions on U, such that $S = T$ on each U_i, then $S = T$ (on U).

Proof. Suppose $u \in \mathcal{D}(U)$. Our aim is to show that $S(u) = T(u)$. First, we have $\operatorname{supp} u \subset K$ for some compact $K \subset U$. `http://planetmath.org/YIsCompactIfAndOnlyIf` follows that there exist a finite collection of U_i:s from the open cover, say $U_1, \ldots, U_N$, such that $K \subset \cup_{i=1}^N U_i$. By a smooth partition of unity, there are smooth functions $\phi_1, \ldots, \phi_N : U \to \mathbb{R}$ such that

1. $\operatorname{supp} \phi_i \subset U_i$ for all i.
2. $\phi_i(x) \in [0, 1]$ for all $x \in U$ and all i,
3. $\sum_{i=1}^N \phi_i(x) = 1$ for all $x \in K$.

From the first property, and from a property for the `http://planetmath.org/SupportOfFunctions` of a function, it follows that $\operatorname{supp} \phi_i u \subset \operatorname{supp} \phi_i \cap \operatorname{supp} u \subset U_i$. Therefore, for each i, $S(\phi_i u) = T(\phi_i u)$ since S and T conicide on U_i. Then

$$S(u) = \sum_{i=1}^N S(\phi_i u) = \sum_{i=1}^N T(\phi_i u) = T(u),$$

and the theorem follows. $\square$

References

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[2] W. Rudin, *Functional Analysis*, McGraw-Hill Book Company, 1973.

[3] L. Hörmander, *The Analysis of Linear Partial Differential Operators I, (Distribution theory and Fourier Analysis)*, 2nd ed, Springer-Verlag, 1990.