Cauchy principal part integral

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Definition [?, ?, ?] Let $C_0^\infty(\mathbb{R})$ be the set of smooth functions with compact support on $\mathbb{R}$. Then the *Cauchy principal part integral* (or, more in line with the notation, the *Cauchy principal value*) $\operatorname{p.v.}(\frac{1}{x})$ is mapping $\operatorname{p.v.}(\frac{1}{x}) : C_0^\infty(\mathbb{R}) \to \mathbb{C}$ defined as

$$\operatorname{p.v.}(\frac{1}{x})(u) = \lim_{\varepsilon\to 0+} \int_{|x|>\varepsilon} \frac{u(x)}{x} dx$$

for $u \in C_0^\infty(\mathbb{R})$.

Theorem The mapping $\operatorname{p.v.}(\frac{1}{x})$ is a http://planetmath.org/Distribution4distribution of first order. That is, $\operatorname{p.v.}(\frac{1}{x}) \in \mathcal{D}'^1(\mathbb{R})$.

(http://planetmath.org/OperatornamepvfraclxIsADistributionOfFirstOrderproof.)

0.0.1 Properties

1. The distribution $\operatorname{p.v.}(\frac{1}{x})$ is obtained as the limit ([?], pp. 250)

$$\frac{\chi_{n|x|}}{x} \to \operatorname{p.v.}(\frac{1}{x}).$$

   as $n \to \infty$. Here, χ is the characteristic function, the locally integrable functions on the left hand side should be interpreted as distributions (see http://planetmath.org/EveryLocallyIntegrableFunctionIsADistributionthis page), and the limit should be taken in $\mathcal{D}'(\mathbb{R})$. It should also be noted that $\operatorname{p.v.}(\frac{1}{x})$ can be represented by a proper integral as

$$\operatorname{p.v.}(\frac{1}{x})(u) = \int_0^\infty \frac{u(x) - u(-x)}{x},$$

   where we have used the fact that the integrand is continuous because of the differentiability at 0. In fact, this viewpoint can be used to somewhat vastly increase the set of functions for which this principal value is well-defined, such as functions that are integrable, satisfy a Lipschitz condition at 0, and whose behavior for large x makes the integral converge at infinity.

2. Let $\ln|t|$ be the distribution induced by the locally integrable function $\ln|t| : \mathbb{R} \to \mathbb{R}$. Then, for the http://planetmath.org/OperationsOnDistributionsdistribu derivative D, we have ([?], pp. 149)

$$D(\ln|t|) = \operatorname{p.v.}(\frac{1}{x}).$$

References

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[3] J. Rauch, *Partial Differential Equations*, Springer-Verlag, 1991.