anti-cone

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Let X be a real vector space, and Φ be a subspace of linear functionals on X.

For any set $S \subseteq X$, its *anti-cone* S^+, with respect to Φ, is the set

$$S^+ = \{\phi \in \Phi \colon \phi(x) \geq 0\,,\ \text{for all } x \in S\}\,.$$

The anti-cone is also called the *dual cone*.

Usage

The anti-cone operation is generally applied to subsets of X that are themselves cones. Recall that a cone in a real vector space generalize the notion of linear inequalities in a finite number of real variables. The dual cone provides a natural way to transfer such inequalities in the original vector space to its dual space. The concept is useful in the theory of duality.

The set Φ in the definition may be taken to be any subspace of the algebraic dual space X^*. The set Φ often needs to be restricted to a subspace smaller than X^*, or even the continuous dual space X', in order to obtain the nice closure and reflexivity properties below.

Basic properties

Property 1. *The anti-cone is a convex cone in* Φ.

Proof. If $\phi(x)$ is non-negative, then so is $t\phi(x)$ for $t > 0$. And if $\phi_1(x), \phi_2(x) \geq 0$, then clearly $(1-t)\phi_1(x) + t\phi_2(x) \geq 0$ for $0 \leq t \leq 1$. □

Property 2. *If $K \subseteq X$ is a cone, then its anti-cone K^+ may be equivalently characterized as:*

$$K^+ = \{\phi \in \Phi \colon \phi(x) \text{ over } x \in K \text{ is bounded below}\}\,.$$

Proof. It suffices to show that if $\inf_{x \in K} \phi(x)$ is bounded below, then it is non-negative. If it were negative, take some $x \in K$ such that $\phi(x) < 0$. For any $t > 0$, the vector tx is in the cone K, and the function value $\phi(tx) = t\phi(x)$ would be arbitrarily large negative, and hence unbounded below. □

Topological properties

Assumptions. Assume that Φ separates points of X. Let X have the weak topology generated by Φ, and let Φ have the weak-* topology generated by X; this makes X and Φ into Hausdorff topological vector spaces.

Vectors $x \in X$ will be identified with their images $\hat{x}$ under the natural embedding of X in its double dual space.

The pairing (X, Φ) is sometimes called a dual pair; and (Φ, X), where X is identified with its image in the double dual, is also a dual pair.

Property 3. *S^+ is weak-* closed.*

Proof. Let $\{\phi_\alpha\} \subseteq \Phi$ be a net converging to ϕ in the weak-* topology. By definition, $\hat{x}(\phi_\alpha) = \phi_\alpha(x) \geq 0$. As the functional $\hat{x}$ is continuous in the weak-* topology, we have $\hat{x}(\phi_\alpha) \to \hat{x}(\phi) \geq 0$. Hence $\phi \in S^+$. $\square$

Property 4. $\overline{S}^+ = S^+$.

Proof. The inclusion $\overline{S}^+ \subseteq S^+$ is obvious. And if $\phi(x) \geq 0$ for all $x \in S$, then by continuity, this holds true for $x \in \overline{S}$ too — so $\overline{S}^+ \supseteq S^+$. $\square$

Properties involving cone inclusion

Property 5 (Farkas' lemma)**.** *Let $K \subseteq X$ be a weakly-closed convex cone. Then $x \in K$ if and only if $\phi(x) \geq 0$ for all $\phi \in K^+$.*

Proof. That $\phi(x) \geq 0$ for $\phi \in K^+$ and $x \in K$ is just the definition. For the converse, we show that if $x \in X \setminus K$, then there exists $\phi \in K^+$ such that $\phi(x) < 0$.

If $K = \emptyset$, then the desired $\phi \in \Phi = K^+$ exists because Φ can separate the points x and 0. If $K \neq \emptyset$, by the hyperplane separation theorem, there is a $\phi \in \Phi$ such that $\phi(x) < \inf_{y \in K} \phi(y)$. This ϕ will automatically be in K^+ by Property **??**. The zero vector is the weak limit of ty, as $t \searrow 0$, for any vector y. Thus $0 \in K$, and we conclude with $\inf_{y \in K} \phi(y) \leq 0$. $\square$

Property 6. *$K^{++} = \overline{K}$ for any convex cone K. (The anti-cone operation on K^+ is to be taken with respect to X.)*

Proof. We work with $\overline{K}$, which is a weakly-closed convex cone. By Property **??**, $x \in \overline{K}$ if and only if $\phi(x) \geq 0$ for all $\phi \in \overline{K}^+ = K^+$. But by definition of the second anti-cone, $\hat{x} \in (K^+)^+$ if and only if $\phi(x) = \hat{x}(\phi) \geq 0$ for all $\phi \in K^+$. $\square$

Property 7. *Let K and L be convex cones in X, with K weakly closed. Then $K^+ \subseteq L^+$ if and only if $K \supseteq L$.*

Proof.

$$K^+ \subseteq L^+ \implies K = \overline{K} = K^{++} \supseteq L^{++} = \overline{L} \supseteq L \implies K^+ \subseteq L^+ . \qquad \square$$

References

[1] B. D. Craven and J. J. Kohila. "Generalizations of Farkas' Theorem." *SIAM Journal on Mathematical Analysis*. Vol. 8, No. 6, November 1977.

[2] David G. Luenberger. *Optimization by Vector Space Methods*. John Wiley & Sons, 1969.