invertible elements in a Banach algebra form an open set

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Theorem - Let $\mathcal{A}$ be a Banach algebra with identity element e and $G(\mathcal{A})$ be the set of invertible elements in $\mathcal{A}$. Let $B_r(x)$ denote the open ball of radius r centered in x.

Then, for all $x \in G(\mathcal{A})$ we have that

$$B_{\|x^{-1}\|^{-1}}(x) \subseteq G(\mathcal{A})$$

and therefore $G(\mathcal{A})$ is open in $\mathcal{A}$.

Proof : Let $x \in G(\mathcal{A})$ and $y \in B_{\|x^{-1}\|^{-1}}(x)$. We have that

$$\|e-x^{-1}y\| = \|x^{-1}x-x^{-1}y\| = \|x^{-1}(x-y)\| \leq \|x^{-1}\|\|x-y\| < \|x^{-1}\|\|x^{-1}\|^{-1} = 1$$

So, by the `http://planetmath.org/NeumannSeriesInBanachAlgebras`Neumann series we conclude that $e-(e-x^{-1}y)$ is invertible, i.e. $x^{-1}y \in G(\mathcal{A})$.

As $G(\mathcal{A})$ is a group we must have $y \in G(\mathcal{A})$.

So $B_{\|x^{-1}\|^{-1}}(x) \subseteq G(\mathcal{A})$ and the theorem follows. $\square$