minimal unitizations of algebras with additional structure

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Defines	minimal unitization of a topological algebra
Defines	minimal unitization of a Banach algebra
Defines	minimal unitization of a Banach-* algebra
Defines	minimal unitization of a C^*-algebra

Given a (non-unital) http://planetmath.org/Algebraalgebra there is a procedure to add an unit to it (http://planetmath.org/Unitizationparent entry). When the algebra has some additional structure (topological structure, for example), it is often useful to endow the same structure on the minimal unitization of the algebra.

All the algebras are to be considered non-unital.

0.1 Topological Algebras

Let $\mathcal{A}$ be a topological algebra algebra over a (topological) field $\mathbb{K}$. Let $\widetilde{\mathcal{A}}$ be its minimal unitization.

Then $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{K}$ is a topological algebra with the product topology.

0.2 Normed and Banach Algebras

Let $\mathcal{A}$ be a normed algebra over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) with norm $\|\cdot\|$. Let $\widetilde{\mathcal{A}}$ be its minimal unitization.

Then $\widetilde{\mathcal{A}}$ is a normed algebra under the norm $\|\cdot\|_u$:

$$\|a+\lambda\|_u = \|a\| + |\lambda|\,, \qquad a \in \mathcal{A}\,, \lambda \in \mathbb{K}$$

Moreover, if $\mathcal{A}$ is a Banach algebra, then $\widetilde{\mathcal{A}}$ is a Banach algebra with the norm $\|\cdot\|_u$.

0.3 *-algebras

Let $\mathcal{A}$ be a *-algebra over an http://planetmath.org/InvolutaryRinginvolutory field $\mathbb{K}$. Let $\widetilde{\mathcal{A}}$ be its minimal unitization.

Then $\widetilde{\mathcal{A}}$ is a *-algebra with involution given by:

$$(a+\lambda)^* = a^* + \overline{\lambda} \qquad a \in \mathcal{A},\ \lambda \in \mathbb{K}$$

0.4 Topological *-algebras, Normed *-algebras and Banach *-algebras

Let $\mathcal{A}$ be a topological *-algebra over $\mathbb{C}$. Let $\widetilde{\mathcal{A}}$ be its minimal unitization.

Then $\widetilde{\mathcal{A}}$ is a topological *-algebra with the product topology and the involution defined above.

Also, if $\mathcal{A}$ is a normed *-algebra (Banach -*algebra), then $\widetilde{\mathcal{A}}$ is also a normed *-algebra (Banach *-algebra) under the above involution and the norm $\|\cdot\|_u$.

0.5 C*-algebras

Let $\mathcal{A}$ be a `http://planetmath.org/CAlgebra`C^*-algebra with norm $\|\cdot\|$. Let $\widetilde{\mathcal{A}}$ be its minimal unitization.

Then $\widetilde{\mathcal{A}}$ is C^*-algebra under the norm $\|\cdot\|_L$:

$$\|a+\lambda\|_L = \sup_{\|b\|=1} \|ab+\lambda b\|\,, \qquad a \in \mathcal{A},\ \lambda \in \mathbb{C}$$

This norm comes from regarding elements of $\widetilde{\mathcal{A}}$ as left on $\mathcal{A}$. The norm $\|\cdot\|_L$ is to the norm $\|\cdot\|_u$.