# every Hilbert space has an orthonormal basis

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**Theorem -** Every Hilbert space $H \neq \{0\}$ has an orthonormal basis.

***Proof :*** As could be expected, the proof makes use of Zorn's Lemma. Let $\mathcal{O}$ be the set of all orthonormal sets of $H$. It is clear that $\mathcal{O}$ is non-empty since the set $\{x\}$ is in $\mathcal{O}$, where $x$ is an element of $H$ such that $\|x\| = 1$.

The elements of $\mathcal{O}$ can be ordered by inclusion, and each chain $\mathcal{C}$ in $\mathcal{O}$ has an upper bound, given by the union of all elements of $\mathcal{C}$. Thus, Zorn's Lemma assures the existence of a maximal element $B$ in $\mathcal{O}$. We claim that $B$ is an orthonormal basis of $H$.

It is clear that $B$ is an orthonormal set, as it belongs to $\mathcal{O}$. It remains to see that the linear span of $B$ is dense in $H$.

Let $\overline{\operatorname{span} B}$ denote the closure of the span of $B$. Suppose $\overline{\operatorname{span} B} \neq H$. By the orthogonal decomposition theorem we know that

$$H = \overline{\operatorname{span} B} \oplus (\overline{\operatorname{span} B})^{\perp}$$

Thus, we conclude that $(\overline{\operatorname{span} B})^{\perp} \neq \{0\}$, i.e. there are elements which are `http://planetmath.org/OrthogonalVectors`orthogonal to $\overline{\operatorname{span} B}$. This contradicts the maximality of $B$ since, by picking an element $y \in (\overline{\operatorname{span} B})^{\perp}$ with $\|y\| = 1$, $B \cup \{y\}$ would belong belong to $\mathcal{O}$ and would be greater than $B$.

Hence, $\overline{\operatorname{span} B} = H$, and this finishes the proof. $\square$