Schur's condition for a matrix to be a bounded operator on l^2

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Theorem 0.1 *Let B be a matrix defined on $T \times T$ for some countable set T. If there exists a positive number C such that*

$$\sum_{t\in T} |b(s,t)| < C \text{ for all } s \quad \text{and} \quad \sum_{s\in T} |b(s,t)| < C \text{ for all } t,$$

then B is a bounded operator on $l^2(T)$ with its operator norm $\|B\|$ less than or equal to C.

Proof. Let x be a sequence in $l^2(T)$. We have

$$\begin{aligned}
\|Bx\|_2^2 &= \sum_{s\in T} \left| \sum_{t\in T} b(s,t)x(t) \right|^2 \\
&\le \sum_{s\in T} \left[\sum_{t\in T} \sqrt{|b(s,t)|} \left(\sqrt{|b(s,t)|}|x(t)| \right) \right]^2 \\
&\le \sum_{s\in T} \left[\left(\sum_{t\in T} |b(s,t)| \right) \left(\sum_{t\in T} |b(s,t)||x(t)|^2 \right) \right] \\
&\le C \sum_{s\in T} \sum_{t\in T} |b(s,t)||x(t)|^2 \\
&\le C \sum_{t\in T} |x(t)|^2 \sum_{s\in T} |b(s,t)| \\
&\le C^2 \sum_{t\in T} |x(t)|^2.
\end{aligned}$$

Therefore we have $\|Bx\|_2 \le C \|x\|_2$ for all $x \in l^2(T)$, hence $\|B\| \le C$. $\square$