maximal ideals of the algebra of continuous functions on a compact set

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Owner	asteroid (17536)
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Author	asteroid (17536)
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Synonym	character space of the algebra of continuous functions on a compact set

Let X be a compact Hausdorff space and $C(X)$ the Banach algebra of continuous functions $X \longrightarrow \mathbb{C}$ (with the sup norm).

In this entry we are interested in identifying the maximal ideals and the character space of $C(X)$. Since $C(X)$ is a Banach algebra with an identity element, there is a bijective correspondence between the character space of $C(X)$ and the set of maximal ideals of this algebra, given by

$$\phi \longleftrightarrow Ker\ \phi$$

Hence, by identifying the character space of $C(X)$ we are able to identify its maximal ideals.

Theorem 1 - Let Δ be the character space of $C(X)$. For each $x \in X$ let $ev_x \in \Delta$ be the point-evaluation at x, i.e.

$$ev_x(f) = f(x)\ , \qquad\qquad f \in C(X)$$

Then the mapping $x \longmapsto ev_x$ is an homeomorphism between Δ and X.

Thus, the character space of $C(X)$ is homeomorphic to X via point-evaluations.

Now, the maximal ideals of $C(X)$ correspond to the kernels of the point-evaluation functions. The kernel of ev_x, the point-evaluation at x, is just

$$\{f \in C(X) : f(x) = 0\}$$

i.e., the functions that vanish at x.

Thus, each maximal ideal of $C(X)$ is just the set of functions that vanish in a given point.

0.1 Generalization to locally compact Hausdorff spaces

Now, let X be a locally compact Hausdorff space and $C_0(X)$ the space of continuous functions $X \longrightarrow \mathbb{C}$ that vanish at infinity.

There is a generalization of Theorem 1 above that allows one to identify the character space of $C_0(X)$, but since this algebra is not unital unless X is compact, we cannot identify its maximal ideals by the above method.

Theorem 2- Let Δ be the character space of $C_0(X)$. For each $x \in X$ let $ev_x \in \Delta$ be the point-evaluation at x, i.e.

$$ev_x(f) = f(x) \, , \qquad\qquad f \in C_0(X)$$

Then the mapping $x \longmapsto ev_x$ is an homeomorphism between Δ and X.

Thus, the character space of $C_0(X)$ is also homeomorphic to X via point-evaluations.