bounded linear extension of an operator

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Defines	continuous extension of a normed algebra homomorphism

0.1 Bounded Linear Extension

Let X and Y be normed vector spaces and denote by $\widetilde{X}$ and $\widetilde{Y}$ their completions.

Theorem 1 - Every bounded linear operator $T : X \longrightarrow Y$ can be extended to a bounded linear operator $\widetilde{T} : \widetilde{X} \longrightarrow \widetilde{Y}$. Moreover, this extension is unique and $\|T\| = \|\widetilde{T}\|$.

In particular, if Y is a Banach space and $S \subseteq X$ is a (not necessarily `http://planetmath.org/ClosedSet`closed) subspace of X, an operator $T : S \longrightarrow Y$ has an extension $\widetilde{T} : \overline{S} \longrightarrow Y$ to $\overline{S}$ (the `http://planetmath.org/Closure`closure of S), which is unique and such that $\|T\| = \|\widetilde{T}\|$.

0.2 Functorial Property of the Extension

The extension of bounded linear operators between two normed vector spaces to their completions is functorial. More precisely, let **NVec** be the category of normed vector spaces (whose `http://planetmath.org/Category`morphisms are the bounded linear operators) and **Ban** the categroy of Banach spaces (whose are also the bounded linear operators). We have that

Theorem 2 - The completion $\widetilde{\ } :$ **NVec** $\longrightarrow$ **Ban**, which associates each normed vector space X with its completion $\widetilde{X}$ and each bounded linear operator T with its extension $\widetilde{T}$, is a covariant functor.

This, in particular, implies that $\widetilde{T_1 T_2} = \widetilde{T_1}\widetilde{T_2}$.

0.3 Extensions in Spaces with Additional Structure

When the normed vector spaces X and Y have some additional structure (for example, when X and Y are normed algebras) it is interesting to know if the (unique) extension of a morphism $T : X \longrightarrow Y$ preserves the additional structure. The following theorem states that this indeed the case for normed algebras or normed *-algebras.

Theorem 3 - If X and Y be normed vector spaces that are also normed algebras (normed *-algebras) and $T : X \longrightarrow Y$ is a bounded homomorphism (bounded *-homomorphism), then the unique bounded linear extension $\widetilde{T}$ of T is also an homomorphism (*-homomorphism).

Thus, completion is also a covariant functor from the category of normed algebras (normed *-algebras) to category of Banach algebras (Banach *-algebras).