$|\langle Tv,v\rangle|\leq\mu\|v\|^2$ for all v implies $\|T\|\leq\mu$

Canonical name	vertlangleTvvranglevertleqmuVertVVert2ForAllVImpliesVertTVertleqmu
Date of creation	2013-03-22 15:25:33
Last modified on	2013-03-22 15:25:33
Owner	Gorkem (3644)
Last modified by	Gorkem (3644)
Numerical id	16
Author	Gorkem (3644)
Entry type	Theorem
Classification	msc 46C05

Theorem. *Let H be a unitary space, T be a self-adjoint linear operator and $\mu \geq 0$. If $|\langle Tv, v\rangle| \leq \mu\|v\|^2$ for all $v \in H$ then T is a bounded operator and $\|T\| \leq \mu$.*

Proof. We will show that $\|Tv\| \leq \mu\|v\|$ for all $v \in H$. This is trivial if $\|Tv\|$ or $\|v\|$ is zero, so assume they are not. Let λ be any positive number.

$$\begin{aligned}
\|Tv\|^2 &= \langle Tv, Tv\rangle \\
&= \frac{1}{4}\left[\left\langle T\left(\lambda v + \frac{1}{\lambda}Tv\right), \left(\lambda v + \frac{1}{\lambda}Tv\right)\right\rangle - \left\langle T\left(\lambda v - \frac{1}{\lambda}Tv\right), \left(\lambda v - \frac{1}{\lambda}Tv\right)\right\rangle\right] \\
&\leq \frac{\mu}{4}\left[\left\|\lambda v + \frac{1}{\lambda}Tv\right\|^2 + \left\|\lambda v - \frac{1}{\lambda}Tv\right\|^2\right] \\
&\leq \frac{\mu}{2}\left[\lambda^2\|v\|^2 + \frac{1}{\lambda^2}\|Tv\|^2\right]
\end{aligned}$$

Now if we put $\lambda^2 = \dfrac{\|Tv\|}{\|v\|}$ we get $\|Tv\|^2 \leq \mu\|Tv\|\|v\|$ hence $\|Tv\| \leq \mu\|v\|$. $\square$

Reference:

F. Riesz and B. Sz-Nagy, *Functional Analysis*, F. Ungar Publishing, 1955, chap VI.