# proof of bounded linear functionals on $L^p(\mu)$



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If $(X, \mathfrak{M}, \mu)$ is a $\sigma$-finite measure-space and $p, q$ are http://planetmath.org/ConjugateIndexH conjugates with $p < \infty$, then we show that $L^q$ is isometrically isomorphic to the dual space of $L^p$.

For any $g \in L^q$, define the linear map

$$\Phi_g \colon L^p \to \mathbb{C},\ f \mapsto \Phi_g(f) = \int f g \, d\mu.$$

This is a bounded linear map with operator norm $\|\Phi_g\| = \|g\|_q$ (see http://planetmath.org/LpNor norm is dual to $L^q$), so the map $g \mapsto \Phi_g$ gives an isometric embedding from $L^q$ to the dual space of $L^p$. It only remains to show that it is onto.

So, suppose that $\Phi \colon L^p \to \mathbb{C}$ is a bounded linear map. It needs to be shown that there is a $g \in L^q$ with $\Phi = \Phi_g$. As http://planetmath.org/AnySigmaFiniteMeasureIs $\sigma$-finite measure is equivalent to a probability measure, there is a bounded $h > 0$ such that $\int h \, d\mu = 1$. Let $\tilde{\Phi} \colon L^\infty \to \mathbb{C}$ be the bounded linear map given by $\tilde{\Phi}(f) = \Phi(hf)$. Then, there is a $g_0 \in L^1$ such that

$$\Phi(hf) = \tilde{\Phi}(f) = \int f g_0 \, d\mu$$

for every $f \in L^\infty$ (see http://planetmath.org/BoundedLinearFunctionalsOnLinftymubounded linear functionals on $L^\infty$). Set $g = h^{-1} g_0$ and, for any $f \in L^p$, let $f_n$ be the sequence

$$f_n = f 1_{\{|h^{-1} f| < n\}}.$$

As $h^{-1} f_n \in L^\infty$,

$$\|f_n g\|_1 = \|h^{-1} f_n g_0\|_1 = \Phi(\operatorname{sign}(f g_0) f_n) \leq \|\Phi\| \|f_n\|_p.$$

Letting $n$ tend to infinity, dominated convergence says that $f_n \to f$ in the $L^p$-norm, so Fatou's lemma gives

$$\|fg\|_1 \leq \liminf_{n \to \infty} \|f_n g\|_1 \leq \|\Phi\| \|f\|_p.$$

In particular, $\|g\|_q \leq \|\Phi\|$ (see http://planetmath.org/LpNormIsDualToLq$L^p$-norm is dual to $L^q$), so $g \in L^q$. As $|f_n g| \leq |fg|$ are in $L^1$, dominated convergence finally gives

$$\int f g \, d\mu = \lim_{n \to \infty} \int f_n g \, d\mu = \lim_{n \to \infty} \Phi(f_n) = \Phi(f)$$

so $\Phi_g = \Phi$ as required.