von Neumann double commutant theorem

Canonical name	VonNeumannDoubleCommutantTheorem
Date of creation	2013-03-22 18:40:27
Last modified on	2013-03-22 18:40:27
Owner	asteroid (17536)
Last modified by	asteroid (17536)
Numerical id	4
Author	asteroid (17536)
Entry type	Theorem
Classification	msc 46H35
Classification	msc 46K05
Classification	msc 46L10
Synonym	double commutant theorem
Synonym	bicommutant theorem
Synonym	von Neumann bicommutant theorem
Synonym	von Neumann density theorem

The von Neumann double commutant theorem is a remarkable result in the theory of self-adjoint algebras of operators on Hilbert spaces, as it expresses purely topological aspects of these algebras in terms of purely algebraic properties.

Theorem - von Neumann - Let H be a `http://planetmath.org/HilbertSpace`Hilbert space and $B(H)$ its algebra of bounded operators. Let $\mathcal{M}$ be a *-subalgebra of $B(H)$ that contains the identity operator. The following statements are equivalent:

1. $\mathcal{M} = \mathcal{M}''$, i.e. $\mathcal{M}$ equals its double commutant.
2. $\mathcal{M}$ is closed in the weak operator topology.
3. $\mathcal{M}$ is closed in the strong operator topology.

Thus, a purely topological property of a $\mathcal{M}$, as being closed for some operator topology, is equivalent to a purely algebraic property, such as being equal to its double commutant.

This result is also known as the *bicommutant theorem* or the *von Neumann density theorem*.