bounded operator

Canonical name	BoundedOperator
Date of creation	2013-03-22 14:02:17
Last modified on	2013-03-22 14:02:17
Owner	bwebste (988)
Last modified by	bwebste (988)
Numerical id	19
Author	bwebste (988)
Entry type	Definition
Classification	msc 46B99
Related topic	VectorNorm

Definition [?]

1. Suppose X and Y are normed vector spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Further, suppose T is a linear map $T : X \to Y$. If there is a $C \in \mathbf{R}$ such that for all $x \in X$ we have

   $$\|Tx\|_Y \quad \leq \quad C\|x\|_X,$$

   then T is a **bounded operator**.

2. Let X and Y be as above, and let $T : X \to Y$ be a bounded operator. Then the **norm** of T is defined as the real number

   $$\|T\| := \sup\left\{ \frac{\|Tx\|_Y}{\|x\|_X} \,\middle|\, x \in X \setminus \{0\} \right\}.$$

   Thus the operator norm is the smallest constant $C \in \mathbf{R}$ such that

   $$\|Tx\|_Y \quad \leq \quad C\|x\|_X.$$

   Now for any $x \in X \setminus \{0\}$, if we let $y = x/\|x\|$, then linearity implies that

   $$\|Ty\|_Y = \left\| T\left(\frac{x}{\|x\|_X}\right)\right\|_Y = \frac{\|Tx\|_Y}{\|x\|_X}$$

   and thus it easily follows that

   $$\begin{aligned} \|T\| &= \sup\left\{ \frac{\|Tx\|_Y}{\|x\|_X} \,\middle|\, x \in X \setminus \{0\} \right\} = \sup\left\{ \|Ty\|_Y \,\middle|\, x \in X \setminus \{0\}, y = \frac{x}{\|x\|} \right\} \\ &= \sup\{\|Ty\|_Y | y \in X, \|y\| = 1\}. \end{aligned}$$

   In the special case when $X = \{\mathbf{0}\}$ is the zero vector space, any linear map $T : X \to Y$ is the zero map since $T(\mathbf{0}) = T(0\mathbf{0}) = 0T(\mathbf{0}) = 0$. In this case, we define $\|T\| := 0$.

3. To avoid cumbersome notational stuff usually one can simplify the symbols like $||x||_X$ and $||Tx||_Y$ by writing only $||x||$, $||Tx||$ since there is a little danger in confusing which is space about calculating norms.

0.0.1 TO DO:

1. The defined norm for mappings is a norm
2. Examples: identity operator, zero operator: see [**?**].
3. Give alternative expressions for norm of T.
4. Discuss boundedness and continuity

Theorem [**?**, **?**] Suppose $T : X \to Y$ is a linear map between normed vector spaces X and Y. If X is finite-dimensional, then T is bounded.

Theorem Suppose $T : X \to Y$ is a linear map between normed vector spaces X and Y. The following are equivalent:

1. T is continuous in some point $x_0 \in X$
2. T is uniformly continuous in X
3. T is bounded

Lemma Any bounded operator with a finite dimensional kernel and cokernel has a closed image.

Proof By Banach's isomorphism theorem.

References

[1] E. Kreyszig, *Introductory Functional Analysis With Applications*, John Wiley & Sons, 1978.

[2] G. Bachman, L. Narici, *Functional analysis*, Academic Press, 1966.