proof of von Neumann double commutant theorem

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Lemma - Let H be a Hilbert space and $B(H)$ its algebra of bounded operators. Let $\mathcal{N}$ be a *-subalgebra of $B(H)$ that contains the identity operator and is closed in the strong operator topology. If $T \in \mathcal{N}''$, the double commutant of $\mathcal{N}$, then for each $x \in H$ there is an operator $A \in \mathcal{N}$ such that $\|(A-T)x\| < 1$.

: Let $\overline{\mathcal{N}x} \subseteq H$ be the closure of the subspace $\mathcal{N}x := \{Sx : S \in \mathcal{N}\}$. It is clear that $\mathcal{N}x$ is an invariant subspace for $\mathcal{N}$, hence so is its closure $\overline{\mathcal{N}x}$ (see `http://planetmath.org/InvariantSubspacesForSelfAdjointAlgebrasOfOperators`this entry, Proposition 5).

Let P be the orthogonal projection onto $\overline{\mathcal{N}x}$. Since $\overline{\mathcal{N}x}$ is invariant for $\mathcal{N}$, we have that $P \in \mathcal{N}'$ (see `http://planetmath.org/InvariantSubspacesForSelfAdjointAlgebra` entry, last theorem). Since $\mathcal{N}$ contains the identity operator, we know that x belongs to $\overline{\mathcal{N}x}$. Hence,

$$Tx = TPx = PTx$$

where the last equality comes from the fact that $T \in \mathcal{N}''$ and $P \in \mathcal{N}'$. Thus, we see that $Tx \in \overline{\mathcal{N}x}$, which implies that there exists an $A \in \mathcal{N}$ such that $\|Tx - Ax\| < 1$. $\square$

(1) $\Longrightarrow$ (2) Since $\mathcal{M}''$ is the commutant of some set, namely it is the commutant of $\mathcal{M}'$, it follows that $\mathcal{M}''$ is closed in the weak operator topology (see `http://planetmath.org/CommutantIsAWeakOperatorClosedSubalgebra`this entry). But we are assuming that $\mathcal{M} = \mathcal{M}''$, hence $\mathcal{M}$ is closed in the weak operator topology.

(2) $\Longrightarrow$ (3) This part is obvious since the weak operator topology is weaker than the strong operator topology.

(3) $\Longrightarrow$ (1) Suppose $\mathcal{M}$ is closed in the strong operator topology.

A subset of $B(H)$ is always contained in its double commutant, thus $\mathcal{M} \subseteq \mathcal{M}''$. So it remains to prove the opposite inclusion.

Let $T \in \mathcal{M}''$. We are going to prove that T belongs to the strong operator closure of $\mathcal{M}$, and since $\mathcal{M}$ is closed under this topology, it will follow that $T \in \mathcal{M}$.

Recall that the strong operator topology is the topology in $B(H)$ generated by the family of seminorms $\|\cdot\|_x, x \in H$ defined by $\|S\|_x := \|Sx\|$. A local base around T, in this topology, consists of sets of the form

$$V(x_1,\ldots,x_n;\,\epsilon) := \{S \in B(H) : \|(S-T)x_i\| \leq \epsilon, i = 1,\ldots,n\}\,, \qquad x_1,\ldots,x_n \in H, \epsilon > 0$$

We can however consider ϵ to be 1, since $V(x_1,\ldots,x_n;\,\epsilon) = V(\epsilon^{-1}x_1,\ldots,\epsilon^{-1}x_n;1)$.

For every $x_1,\ldots,x_n \in H$ we want to find $A \in \mathcal{M}$ such that $A \in V(x_1,\ldots,x_n;1)$, i.e. such that $\|(A-T)x_i\| < 1$, for each i.

Let $\widetilde{H}$ be the direct sum of Hilbert spaces $\widetilde{H} := \oplus_{i=1}^n H$. For every $A \in B(H)$ let $\widetilde{A} \in B(\widetilde{H})$ be the http://planetmath.org/DirectSumOfBoundedOperatorsOnHilbertSp
sum of bounded operators $\widetilde{A} := \oplus_{i=1}^n A$, i.e.

$$\widetilde{A}(y_1,\ldots,y_n) = (Ay_1,\ldots,Ay_n)\,, \qquad y_1,\ldots,y_n \in H$$

We have that $\mathcal{N} := \{\widetilde{A} : A \in \mathcal{M}\}$ is a *-algebra of bounded operators in $\widetilde{H}$.

Claim 1 - $\widetilde{T} \in \mathcal{N}''$.

The algebra $B(\widetilde{H})$ can be canonically identified with the algebra of $n \times n$ matrices with entries in $B(H)$, and $\mathcal{N}$ corresponds to the diagonal matrices with an element $A \in \mathcal{M}$ in the diagonal. Thus, it is easy to check that $\mathcal{N}'$ is precisely the set of matrices whose entries belong to $\mathcal{M}'$.

Since the http://planetmath.org/UnitMatrixunit matrices belong to $\mathcal{N}'$, it follows that $\mathcal{N}''$ consists solely of diagonal matrices with one element on the diagonal (see http://planetmath.org/CentralizerOfMatrixUnitsthis entry). It is easy to check that $\mathcal{N}''$ is precisely the set of diagonal matrices with one element of $\mathcal{M}''$ in the diagonal. Hence, we conclude that $\widetilde{T} \in \mathcal{N}''$, and Claim 1 is proved.

Now, we observe that $\mathcal{N}$ is a *-subalgebra of $B(\widetilde{H})$ that contains the identity operator. Since $\mathcal{M}$ is closed in the strong operator topology, it follows easily that $\mathcal{N}$ is also closed in the strong operator topology. Since $\widetilde{T} \in \mathcal{N}''$, Lemma 1 that for each $\widetilde{x} := (x_1,\ldots,x_n) \in \widetilde{H}$ there exists an operator $\widetilde{A} \in \mathcal{N}$ such that $\|(\widetilde{A}-\widetilde{T})\widetilde{x}\| < 1$. But this is implies that $\|(A-T)x_i\| < 1$ for each $1 \leq i \leq n$.

Thus, $T \in V(x_1,\ldots,x_n;1)$. Hence we conclude that T belongs to the operator closure of $\mathcal{M}$, but since $\mathcal{M}$ is closed under this topology, $T \in \mathcal{M}$.

We conclude that $\mathcal{M}'' = \mathcal{M}$. $\square$