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proof of Hahn-Banach theorem

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Consider the family of all possible extensions of f, i.e. the set $\mathcal{F}$ of all pairings (F, H) where H is a vector subspace of X containing U and F is a linear map $F\colon H \to K$ such that $F(u) = f(u)$ for all $u \in U$ and $|F(u)| \leq p(u)$ for all $u \in H$. $\mathcal{F}$ is naturally endowed with an partial order relation: given $(F_1, H_1), (F_2, H_2) \in \mathcal{F}$ we say that $(F_1, H_1) \leq (F_2, H_2)$ iff F_2 is an extension of F_1 that is $H_1 \subset H_2$ and $F_2(u) = F_1(u)$ for all $u \in H_1$. We want to apply Zorn's Lemma to $\mathcal{F}$ so we are going to prove that every chain in $\mathcal{F}$ has an upper bound.

Let (F_i, H_i) be the elements of a chain in $\mathcal{F}$. Define $H = \bigcup_i H_i$. Clearly H is a vector subspace of V and contains U. Define $F\colon H \to K$ by "merging" all F_i's as follows. Given $u \in H$ there exists i such that $u \in H_i$: define $F(u) = F_i(u)$. This is a good definition since if both H_i and H_j contain u then $F_i(u) = F_j(u)$ in fact either $(F_i, H_i) \leq (F_j, H_j)$ or $(F_j, H_j) \leq (F_i, H_i)$. Notice that the map F is linear, in fact given any two vectors $u, v \in H$ there exists i such that $u, v \in H_i$ and hence $F(\alpha u + \beta v) = F_i(\alpha u + \beta v) = \alpha F_i(u) + \beta F_i(v) = \alpha F(u) + \beta F(v)$. The so constructed pair (F, H) is hence an upper bound for the chain (F_i, H_i) because F is an extension of every F_i.

Zorn's Lemma then assures that there exists a maximal element $(F, H) \in \mathcal{F}$. To complete the proof we will only need to prove that $H = V$.

Suppose by contradiction that there exists $v \in V \setminus H$. Then consider the vector space $H' = H + Kv = \{u + tv\colon u \in H, \quad t \in K\}$ (H' is the vector space generated by H and v). Choose

$$\lambda = \sup_{x \in H} \{F(x) - p(x - v)\}.$$

We notice that given any $x, y \in H$ it holds

$$F(x) - F(y) = F(x - y) \leq p(x - y) = p(x - v + v - y) \leq p(x - v) + p(y - v)$$

i.e.

$$F(x) - p(x - v) \leq F(y) + p(y - v);$$

in particular we find that $\lambda < +\infty$ and for all $y \in H$ it holds

$$F(y) - p(y - v) \leq \lambda \leq F(y) + p(y - v).$$

Define $F'\colon H' \to K$ as follows:

$$F'(u + tv) = F(u) + t\lambda.$$

Clearly F' is a linear functional. We have

$$|F'(u+tv)| = |F(u)+t\lambda| = |t|\,|F(u/t)+\lambda|$$

and by letting $y = -u/t$ by the previous estimates on λ we obtain

$$F(u/t)+\lambda \leq F(u/t)+F(-u/t)+p(-u/t-v) = p(u/t+v)$$

and

$$F(u/t)+\lambda \geq F(u/t)+F(-u/t)-p(-u/t-v) = -p(u/t+v)$$

which together give

$$|F(u/t)+\lambda| \leq p(u/t+v)$$

and hence

$$|F'(u+tv)| \leq |t|p(u/t+v) = p(u+tv).$$

So we have proved that $(F', H') \in \mathcal{F}$ and $(F', H') > (F, H)$ which is a contradiction.