Laplace transform of Dirac delta

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The http://planetmath.org/DiracDeltaFunctionDirac delta δ can be interpreted as a linear functional, i.e. a linear mapping from a function space, consisting e.g. of certain real functions, to $\mathbb{R}$ (or $\mathbb{C}$), having the property

$$\delta[f] \;=\; f(0).$$

One may think this as the inner product

$$\langle f,\,\delta\rangle \;=\; \int_0^\infty f(t)\delta(t)\,dt$$

of a function f and another "function" δ, when the well-known

$$\int_0^\infty f(t)\delta(t)\,dt \;=\; f(0)$$

is true. Applying this to $f(t) := e^{-st}$, one gets

$$\int_0^\infty e^{-st}\delta(t)\,dt \;=\; e^{-0},$$

i.e. the Laplace transform

$$\mathcal{L}\{\delta(t)\} \;=\; 1. \tag{1}$$

By the delay theorem, this result may be generalised to

$$\mathcal{L}\{\delta(t-a))\} \;=\; e^{-as}.$$

When introducing some "nascent Dirac delta function", for example

$$\eta_\varepsilon(t) \;:=\; \begin{cases} \frac{1}{\varepsilon} & \text{for } \; 0 \leq t \leq \varepsilon,\\ 0 & \text{for } \qquad t > \varepsilon, \end{cases}$$

as an "approximation" of Dirac delta, we obtain the Laplace transform

$$\mathcal{L}\{\eta_\varepsilon(t)\} \;=\; \int_0^\infty e^{-st}\eta_\varepsilon(t)\,dt \;=\; \int_0^\varepsilon \frac{e^{-st}}{\varepsilon}\,dt + \int_\varepsilon^\infty e^{-st}\cdot 0\,dt \;=\; \frac{1}{\varepsilon}\int_0^\varepsilon e^{-st}\,dt \;=\; \frac{1-e^{-\varepsilon s}}{\varepsilon s}.$$

As the Taylor expansion shows, we then have

$$\lim_{\varepsilon\to 0+}\mathcal{L}\{\eta_\varepsilon(t)\} \;=\; 1,$$

being in accordance with (1).