continuous linear mapping

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Defines	bounded linear transform
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If $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ are normed vector spaces, a linear mapping $T : V_1 \to V_2$ is continuous if it is continuous in the metric induced by the norms.

If there is a nonnegative constant c such that $\|T(x)\|_2 \leq c\|x\|_1$ for each $x \in V_1$, we say that T is . This should not be confused with the usual terminology referring to a bounded function as one that has bounded range. In fact, bounded linear mappings usually have unbounded ranges.

The expression *bounded linear mapping* is often used in functional analysis to refer to continuous linear mappings as well. This is because the two definitions are equivalent:

If T is bounded, then $\|T(x) - T(y)\|_2 = \|T(x-y)\|_2 \leq c\|x-y\|_1$, so T is a Lipschitz function. Now suppose T is continuous. Then there exists $r > 0$ such that $\|T(x)\|_2 \leq 1$ when $\|x\|_1 \leq r$. For any $x \in V_1$, we then have

$$\frac{r}{\|x\|_1}\|T(x)\|_2 = \|T\left(\frac{r}{\|x\|_1}x\right)\|_2 \leq 1,$$

hence $\|T(x)\|_2 \leq r\|x\|_1$; so T is bounded.

It can be shown that a linear mapping between two topological vector spaces is continuous if and only if it is `http://planetmath.org/Continuous`continuous at 0 [?].

References

[1] W. Rudin, *Functional Analysis*, McGraw-Hill Book Company, 1973.