corollary of Banach-Alaoglu theorem

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Corollary. *A Banach space $\mathscr{H}$ is isometrically isomorphic to a closed subspace of $C(X)$ for a compact Hausdorff space X.*

Proof. Let X be the unit ball $\mathcal{B}(\mathscr{H}^*)$ of $\mathscr{H}^*$. By the Banach-Alaoglu theorem it is compact in the weak-$*$ topology. Define the map $\Phi\colon \mathscr{H} \to C(X)$ by $(\Phi f)(\varphi) = \varphi(f)$. This is linear and we have for $f \in \mathscr{H}$:

$$\|\Phi(f)\|_\infty = \sup_{\varphi\in\mathcal{B}(\mathscr{H}^*)} |\Phi(f)(\varphi)| = \sup_{\varphi\in\mathcal{B}(\mathscr{H}^*)} |\varphi(f)| \leq \sup_{\varphi\in\mathcal{B}(\mathscr{H}^*)} \|\varphi\|\|f\| \leq \|f\|$$

With the Hahn-Banach theorem it follows that there is a $\varphi \in \mathcal{B}(\mathscr{H}^*)$ such that $\varphi(f) = \|f\|$. Thus $\|\Phi(f)\|_\infty = \|f\|$ and Φ is an isometric isomorphism, as required. $\square$