proof of Heine-Cantor theorem

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We seek to show that $f : K \to X$ is continuous with K a compact metric space, then f is uniformly continuous. Recall that for $f : K \to X$, uniform continuity is the condition that for any $\varepsilon > 0$, there exists δ such that

$$d_K(x, y) < \delta \implies d_X(f(x), f(y)) < \epsilon$$

for all $x, y \in K$

Suppose K is a compact metric space, f continuous on K. Let $\epsilon > 0$. For each $k \in K$ choose δ_k such that $d(k, x) \leq \delta_k$ implies $d(f(k), f(x)) \leq \frac{\epsilon}{2}$. Note that the collection of balls $B(k, \frac{\delta_k}{2})$ covers K, so by compactness there is a finite subcover, say involving $k_1, \ldots, k_n$. Take

$$\delta = \min_{i=1,\ldots,n} \frac{\delta_{k_i}}{2}$$

Then, suppose $d(x, y) \leq \delta$. By the choice of $k_1, \ldots, k_n$ and the triangle inequality, there exists an i such that $d(x, k_i), d(y, k_i) \leq \delta_{k_i}$. Hence,

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f(k_i)) + d(f(y), f(k_i)) && (1) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} && (2) \end{aligned}$$

As x, y were arbitrary, we have that f is uniformly continuous.
This proof is similar to one found in Mathematical Principles of Analysis, Rudin.