proof of quotients in C^*-algebras

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Proof: We have that $\mathcal{I}$ is http://planetmath.org/InvolutaryRingself-adjoint, since it is a closed ideal of a http://planetmath.org/CAlgebraC^*-algebra (see http://planetmath.org/ClosedIdealsInCAlgebrasAreSelfAdjointthis entry). Hence, the involution in $\mathcal{A}$ induces a well-defined involution in $\mathcal{A}/\mathcal{I}$ by $(x+\mathcal{I})^* := x^* + \mathcal{I}$.

Recall that, since $\mathcal{I}$ is closed, the quotient norm is indeed a norm in $\mathcal{A}/\mathcal{I}$ that makes $\mathcal{A}/\mathcal{I}$ a Banach algebra (see http://planetmath.org/QuotientsOfBanachAlgebrasthis entry). Thus we only have to prove the C^* to prove that $\mathcal{A}/\mathcal{I}$ is a C^*-algebra.

Recall that http://planetmath.org/CAlgebrasHaveApproximateIdentitiesC^*-algebras have approximate identities. Notice that $\mathcal{I}$ itself is a C^*-algebra and pick an approximate identity (e_λ) in $\mathcal{I}$ such that

- each e_λ is positive.
- $\|e_\lambda\| \leq 1$

We will only prove the case when $\mathcal{A}$ has an identity element e. For the non-unital case, one can consider $\mathcal{A}$ as a C^*-subalgebra of its minimal unitization and the same proof will still work.

Let $\|\cdot\|_q$ denote the quotient norm in $\mathcal{A}/\mathcal{I}$. We claim that for every $x \in \mathcal{A}$:

$$\|x+\mathcal{I}\|_q = \lim_\lambda \|x(e-e_\lambda)\| \tag{1}$$

We will prove the above equality as a lemma at the end of the entry. Assuming this result, it follows that for every $a \in \mathcal{A}$

$$\|x+\mathcal{I}\|_q^2 = \lim \|x(e-e_\lambda)\|^2 = \lim \|(e-e_\lambda)x^*x(e-e_\lambda)\| \leq \lim \|(e-e_\lambda)\| \|x^*x(e-e_\lambda)\|$$

Since each e_λ is positive and $\|e_\lambda\| \leq 1$ we know that its spectrum lies on the interval $[0,1]$. Hence $e-e_\lambda$ is also positive and its spectrum also lies on the interval $[0,1]$. Thus, $\|e-e_\lambda\| \leq 1$. Therefore:

$$\|x+\mathcal{I}\|_q^2 \leq \lim \|(e-e_\lambda)\| \|x^*x(e-e_\lambda)\| \leq \lim \|x^*x(e-e_\lambda)\| = \|x^*x+\mathcal{I}\|_q$$

Since $\mathcal{A}/\mathcal{I}$ is a Banach algebra, we also have $\|x^*x+\mathcal{I}\|_q \leq \|x+\mathcal{I}\|_q^2$ and so

$$\|x+\mathcal{I}\|_q^2 = \|x^*x+\mathcal{I}\|_q$$

which proves that $\mathcal{A}/\mathcal{I}$ is a C^*-algebra. $\square$

We now prove equality (1) as a lemma.

Lemma - Suppose $\mathcal{A}$ is a C^*-algebra with identity element e. Let $\mathcal{I} \subset \mathcal{A}$ be a closed ideal and (e_λ) be an approximate identity in $\mathcal{I}$ such that each e_λ is positive and $\|e_\lambda\| \leq 1$. Then

$$\|x + \mathcal{I}\|_q = \lim_\lambda \|x(e - e_\lambda)\|$$

for every x in $\mathcal{A}$.

Proof: Since $y(e - e_\lambda) \longrightarrow 0$ for every $y \in \mathcal{I}$ it follows that

$$\begin{aligned}
\limsup \|x(e - e_\lambda)\| &= \limsup \|x - xe_\lambda - y + ye_\lambda\| \\
&= \limsup \|(x - y)(e - e_\lambda)\| \\
&\leq \|x - y\|
\end{aligned}$$

Therefore, taking the infimum over all $y \in \mathcal{I}$ we obtain:

$$\limsup \|x(e - e_\lambda)\| \leq \inf_{y \in \mathcal{I}} \|x - y\| = \|x + \mathcal{I}\|_q$$

Also, since $xe_\lambda \in \mathcal{I}$,

$$\liminf \|x(e - e_\lambda)\| \geq \inf_{y \in \mathcal{I}} \|x - y\| = \|x + \mathcal{I}\|_q$$

and this proves the lemma. $\square$