criterion for a Banach *-algebra representation to be irreducible

Canonical name	CriterionForABanachalgebraRepresentationToBeIrreducible
Date of creation	2013-03-22 17:27:43
Last modified on	2013-03-22 17:27:43
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Last modified by	asteroid (17536)
Numerical id	9
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Entry type	Theorem
Classification	msc 46K10

Theorem - Let $\mathcal{A}$ be a Banach *-algebra, H an Hilbert space and I the identity operator in H. A http://planetmath.org/BanachAlgebraRepresentationrepresentation $\pi : \mathcal{A} \longrightarrow H$ is topologically irreducible if and only if $\pi(\mathcal{A})' = \mathbb{C}I$, i.e. if and only if the commutant of $\pi(\mathcal{A})$ consists of scalar multiples of the identity operator.

Proof : $(\Longrightarrow)$

As $\pi(\mathcal{A})$ is selfadjoint, $\pi(\mathcal{A})'$ is a von Neumann algebra.

Suppose $\pi(\mathcal{A})' \neq \mathbb{C}I$. Then the dimension of $\pi(\mathcal{A})'$ is greater than one.

It is known that von Neumann algebras of dimension greater than one contain non-trivial projections, so there is a projection $P \in \pi(\mathcal{A})'$ such that $P \neq 0$ and $P \neq I$.

As $P \in \pi(\mathcal{A})'$, P commutes with every operator $T \in \pi(\mathcal{A})$, that is $PT = TP$.

Thus *Ran* P is an invariant subspace of every $T \in \pi(\mathcal{A})$. Therefore π is not an irreducible representation.

$(\Longleftarrow)$

Conversely, suppose that π is not an irreducible representation. There exists a closed $\pi(\mathcal{A})$-invariant subspace different from $\{0\}$ and H.

Let P be the projection onto that closed invariant subspace.

Invariance can be expressed as: $\pi(a)P = P\pi(a)P$ for every $a \in \mathcal{A}$. It follows that

$$P\pi(a) = (\pi(a)^*P)^* = (\pi(a^*)P)^* = (P\pi(a^*)P)^* = P\pi(a^*)^*P = P\pi(a)P = \pi(a)P$$

for every $a \in \mathcal{A}$.

We conclude that P commutes with every element of $\pi(\mathcal{A})$, i.e. $P \in \pi(\mathcal{A})'$.

Thus $\pi(\mathcal{A})' \neq \mathbb{C}I$ $\square$