normed algebra

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Defines	normed ring
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A ring A is said to be a *normed ring* if A possesses a norm $\|\cdot\|$, that is, a non-negative real-valued function $\|\cdot\|: A \to \mathbb{R}$ such that for any $a, b \in A$,

1. $\|a\| = 0$ iff $a = 0$,
2. $\|a + b\| \leq \|a\| + \|b\|$,
3. $\|-a\| = \|a\|$, and
4. $\|ab\| \leq \|a\|\|b\|$.

Remarks.

- If A contains the multiplicative identity 1, then $0 < \|1\| \leq \|1\|\|1\|$ and so $1 \leq \|1\|$.
- However, it is usually required that in a normed ring, $\|1\| = 1$.
- $\|\cdot\|$ defines a metric d on A given by $d(a, b) = \|a - b\|$, so that A with d is a metric space and one can set up a topology on A by defining its subbasis a collection of $B(a, r) := \{x \in A \mid d(a, x) < r\}$ called *open balls* for any $a \in A$ and $r > 0$. With this definition, it is easy to see that $\|\cdot\|$ is continuous.
- Given a sequence $\{a_n\}$ of elements in A, we say that a is a limit point of $\{a_n\}$, if
$$\lim_{n\to\infty} \|a_n - a\| = 0.$$
By the triangle inequality, a, if it exists, is unique, and so we also write
$$a = \lim_{n\to\infty} a_n.$$
- In addition, the last condition ensures that the ring multiplication is continuous.

An algebra A over a field k is said to be a *normed algebra* if

1. A is a normed ring with norm $\|\cdot\|$,
2. k is equipped with a valuation $|\cdot|$, and
3. $\|\alpha a\| = |\alpha|\|a\|$ for any $\alpha \in k$ and $a \in A$.

Remarks.

- Alternatively, a normed algebra A can be defined as a normed vector space with a multiplication defined on A such that multiplication is continuous with respect to the norm $\|\cdot\|$.
- Typically, k is either the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$, and A is called a *real normed algebra* or a *complex normed algebra* correspondingly.
- A normed algebra that is complete with respect to the norm is called Banach algebra (the underlying field must be complete and algebraically closed), paralleling with the analogy with a Banach space versus a normed vector space.
- Normed rings and normed algebras are special cases of the more general notions of a topological ring and a *topological algebra*, the latter of which is defined as a topological ring over a field such that the scalar multiplication is continuous.

References

[1] M. A. Naimark: *Normed Rings*, Noordhoff, (1959).

[2] C. E. Rickart: *General Theory of Banach Algebras*, Van Nostrand, 1960.