proof that L^p spaces are complete

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Let's prove completeness for the classical Banach spaces, say $L^p[0,1]$ where $p \geq 1$.

Since the case $p = \infty$ is elementary, we may assume $1 \leq p < \infty$. Let $[f_.] \in (L^p)^{\mathbf{N}}$ be a Cauchy sequence. Define $[g_0] := [f_0]$ and for $n > 0$ define $[g_n] := [f_n - f_{n-1}]$. Then $[\sum_{n=0}^N g_n] = [f_N]$ and we see that

$$\sum_{n=0}^{\infty} \|g_n\| = \sum_{n=0}^{\infty} \|f_n - f_{n-1}\| \leq ??? < \infty.$$

Thus it suffices to prove that etc.

It suffices to prove that each absolutely summable series in L^p is summable in L^p to some element in L^p.

Let $\{f_n\}$ be a sequence in L^p with $\sum_{n=1}^{\infty} \|f_n\| = M < \infty$, and define functions g_n by setting $g_n(x) = \sum_{k=1}^{n} |f_k(x)|$. From the Minkowski inequality we have

$$\|g_n\| \leq \sum_{k=1}^{n} \|f_k\| \leq M.$$

Hence

$$\int g_n^p \leq M^p.$$

For each x, $\{g_n(x)\}$ is an increasing sequence of (extended) real numbers and so must converge to an extended real number $g(x)$. The function g so defined is measurable, and, since $g_n \geq 0$, we have

$$\int g^p \leq M^p$$

by Fatou's Lemma. Hence g^p is integrable, and $g(x)$ is finite for almost all x.

For each x such that $g(x)$ is finite the series $\sum_{k=1}^{\infty} f_k(x)$ is an absolutely summable series of real numbers and so must be summable to a real number $s(x)$. If we set $s(x) = 0$ for those x where $g(x) = \infty$, we have defined a function s which is the limit almost everywhere of the partial sums $s_n = \sum_{k=1}^{n} f_k$. Hence s is measurable. Since $|s_n(x)| \leq g(x)$, we have $|s(x)| \leq g(x)$. Consequently, s is in L^p and we have

$$|s_n(x) - s(x)|^p \leq 2^p \, [g(x)]^p.$$

Since $2^p g^p$ is integrable and $|s_n(x) - s(x)|^p$ converges to 0 for almost all x, we have

$$\int |s_n - s|^p \to 0$$

by the Lebesgue Convergence Theorem. Thus $\|s_n - s\|^p \to 0$, whence $\|s_n - s\| \to 0$. Consequently, the series $\{f_n\}$ has in L^p the sum s.

References

Royden, H. L. *Real analysis. Third edition.* Macmillan Publishing Company, New York, 1988.