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every finite dimensional normed vector space is a Banach space

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Theorem 1. *Every finite dimensional normed vector space is a Banach space.*

Proof. Suppose $(V, \|\cdot\|)$ is the normed vector space, and $(e_i)_{i=1}^N$ is a basis for V. For $x = \sum_{j=1}^N \lambda_j e_j$, we can then define

$$\|x\|' = \sqrt{\sum_{j=1}^N |\lambda_j|^2}$$

whence $\|\cdot\|' \colon V \to \mathbb{R}$ is a norm for V. Since `http://planetmath.org/ProofThatAllNormsOnFinit` norms on a finite dimensional vector space are equivalent, there is a constant $C > 0$ such that

$$\frac{1}{C}\|x\|' \leq \|x\| \leq C\|x\|', \quad x \in V.$$

To prove that V is a Banach space, let $x_1, x_2, \ldots$ be a Cauchy sequence in $(V, \|\cdot\|)$. That is, for all $\varepsilon > 0$ there is an $M \geq 1$ such that

$$\|x_j - x_k\| < \varepsilon, \quad \text{for all} j, k \geq M.$$

Let us write each x_k in this sequence in the basis (e_j) as $x_k = \sum_{j=1}^N \lambda_{k,j} e_j$ for some constants $\lambda_{k,j} \in \mathbb{C}$. For $k, l \geq 1$ we then have

$$\begin{aligned}
\|x_k - x_l\| &\geq \frac{1}{C}\|x_k - x_l\|' \\
&\geq \frac{1}{C}\sqrt{\sum_{j=1}^N |\lambda_{k,j} - \lambda_{l,j}|^2} \\
&\geq \frac{1}{C}|\lambda_{k,j} - \lambda_{l,j}|
\end{aligned}$$

for all $j = 1, \ldots, N$. It follows that $(\lambda_{k,1})_{k=1}^\infty, \ldots, (\lambda_{k,N})_{k=1}^\infty$ are Cauchy sequences in $\mathbb{C}$. As $\mathbb{C}$ is complete, these converge to some complex numbers $\lambda_1, \ldots, \lambda_N$. Let $x = \sum_{j=1}^N \lambda_j e_j$.

For each $k = 1, 2, \ldots$, we then have

$$\begin{aligned}
\|x - x_k\| &\leq C\|x - x_k\|' \\
&\leq C\sqrt{\sum_{j=1}^N |\lambda_j - \lambda_{k,j}|^2}.
\end{aligned}$$

By taking $k \to \infty$ it follows that (x_j) converges to $x \in V$. $\square$