Banach *-algebra representation

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Entry type	Definition
Classification	msc 46H15
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Defines	subrepresentation
Defines	cyclic representation
Defines	cyclic vector
Defines	nondegenerate representation
Defines	topologically irreducible
Defines	algebrically irreducible
Defines	direct sum of representations
Defines	unitarily equivalent

Definition:

A **representation** of a Banach *-algebra $\mathcal{A}$ is a *-homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}(H)$ of $\mathcal{A}$ into the *-algebra of bounded operators on some Hilbert space H.

The set of all representations of $\mathcal{A}$ on a Hilbert space H is denoted $rep(\mathcal{A}, H)$.

Special kinds of representations:

- A **subrepresentation** of a representation $\pi \in rep(\mathcal{A}, H)$ is a representation $\pi_0 \in rep(\mathcal{A}, H_0)$ obtained from π by restricting to a closed $\pi(\mathcal{A})$-`http://planetmath.org/InvariantSubspace`invariant subspace [1] $H_0 \subseteq H$.

- A representation $\pi \in rep(\mathcal{A}, H)$ is said to be **nondegenerate** if one of the following equivalent conditions hold:

  1. $\pi(x)\xi = 0 \quad \forall x \in \mathcal{A} \implies \xi = 0$, where $\xi \in H$.
  2. H is the closed linear span of the set of vectors $\pi(\mathcal{A})H := \{\pi(x)\xi : x \in \mathcal{A}, \xi \in H\}$

- A representation $\pi \in rep(\mathcal{A}, H)$ is said to be **topologically irreducible** (or just) if the only closed $\pi(\mathcal{A})$-invariant of H are the trivial ones, $\{0\}$ and H.

- A representation $\pi \in rep(\mathcal{A}, H)$ is said to be **algebrically irreducible** if the only $\pi(\mathcal{A})$-invariant of H (not necessarily closed) are the trivial ones, $\{0\}$ and H.

- Given two representations $\pi_1 \in rep(\mathcal{A}, H_1)$ and $\pi_2 \in rep(\mathcal{A}, H_2)$, the of π_1 and π_2 is the representation $\pi_1 \oplus \pi_2 \in rep(\mathcal{A}, H_1 \oplus H_2)$ given by $\pi_1 \oplus \pi_2(x) := \pi_1(x) \oplus \pi_2(x), \quad x \in \mathcal{A}$.

  More generally, given a family $\{\pi_i\}_{i \in I}$ of representations, with $\pi_i \in rep(\mathcal{A}, H_i)$, their is the representation $\bigoplus_{i \in I} \pi_i \in rep(\mathcal{A}, \bigoplus_{i \in I} H_i)$, in the direct sum of Hilbert spaces $\bigoplus_{i \in I} H_i$, such that $\left(\bigoplus_{i \in I} \pi_i\right)(x) := \bigoplus_{i \in I} \pi_i(x)$ is the `http://planetmath.org/DirectSumOfBoundedOperatorsOnHilbertSpace` sum of the family of bounded operators $\{\pi_i(x)\}_{i \in I}$.

[1] by a $\pi(\mathcal{A})$- we a subspace which is invariant under every operator $\pi(a)$ with $a \in \mathcal{A}$

- Two representations $\pi_1 \in rep(\mathcal{A}, H_1)$ and $\pi_2 \in rep(\mathcal{A}, H_2)$ of a Banach *-algebra $\mathcal{A}$ are said to be **unitarily equivalent** if there is a unitary $U : H_1 \longrightarrow H_2$ such that

$$\pi_2(a) = U\pi_1(a)U^* \qquad \forall a \in \mathcal{A}$$

- A representation $\pi \in rep(\mathcal{A}, H)$ is said to be if there exists a vector $\xi \in H$ such that the set

$$\pi(A)\,\xi := \{\pi(a)\,\xi : a \in \mathcal{A}\}$$

is `http://planetmath.org/Dense`dense in H. Such a vector is called a **cyclic vector** for the representation π.

Linked file: http://aux.planetmath.org/files/objects/9843/BanachAlgebraRepresentation.pdf