spectral invariance theorem (for C^*-algebras)

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Synonym	spectral invariance theorem
Synonym	invariance of the spectrum of C^*-subalgebras
Defines	invertibility in C^*-subalgebras

The spectral permanence theorem (entry) relates the spectrums $\sigma_{\mathcal{B}}(x)$ and $\sigma_{\mathcal{A}}(x)$ of an element $x \in \mathcal{B} \subseteq \mathcal{A}$ relatively to the Banach algebras $\mathcal{B}$ and $\mathcal{A}$.

For http://planetmath.org/CAlgebraC^*-algebras the situation is quite .

Spectral invariance theorem - Suppose $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{B} \subseteq \mathcal{A}$ a C^*-subalgebra that contains the identity of $\mathcal{A}$. Then for every $x \in \mathcal{B}$ one has

$$\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x).$$

The spectral invariance theorem is a straightforward corollary of the next more general theorem about invertible elements in C^*-subalgebras.

Theorem - Let $x \in \mathcal{B} \subset \mathcal{A}$ be as above. Then x is invertible in $\mathcal{B}$ if and only if x invertible in $\mathcal{A}$.

Proof :

- $(\Longrightarrow)$

  If x is invertible in $\mathcal{B}$ then it is clearly invertible in $\mathcal{A}$.

- $(\Longleftarrow)$

  If x is invertible in $\mathcal{A}$, then so is $y = x^* x$. Thus, $0 \notin \sigma_{\mathcal{A}}(y)$.

  Since y is http://planetmath.org/InvolutaryRingself-adjoint, $\sigma_{\mathcal{A}}(y) \subseteq \mathbb{R}$ (see this http://planetmath.org/SpecialElementsInACAlgebraAndTheirSpectralProp and so $\mathbb{C} - \sigma_{\mathcal{A}}(y)$ has no http://planetmath.org/Boundedbounded connected components.

  By the http://planetmath.org/SpectralPermanenceTheoremspectral permanence theorem we must have $\sigma_{\mathcal{B}}(y) = \sigma_{\mathcal{A}}(y)$. Hence, $0 \notin \sigma_{\mathcal{B}}(y)$, i.e. y is invertible in $\mathcal{B}$.

  It follows that $x^{-1} = (x^* x)^{-1} x^* = y^{-1} x^* \in \mathcal{B}$, i.e. x is invertible in $\mathcal{B}$. $\square$