ordered vector space

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Let k be an ordered field. An *ordered vector space* over k is a vector space V that is also a poset at the same time, such that the following conditions are satisfied

1. for any $u, v, w \in V$, if $u \leq v$ then $u + w \leq v + w$,

2. if $0 \leq u \in V$ and any $0 < \lambda \in k$, then $0 \leq \lambda u$.

Here is a property that can be immediately verified: $u \leq v$ iff $\lambda u \leq \lambda v$ for any $0 < \lambda$.

Also, note that 0 is interpreted as the zero vector of V, not the bottom element of the poset V. In fact, V is both topless and bottomless: for if $\perp$ is the bottom of V, then $\perp \leq 0$, or $2\perp \leq \perp$, which implies $2\perp = \perp$ or $\perp = 0$. This means that $0 \leq v$ for all $v \in V$. But if $v \neq 0$, then $0 < v$ or $-v < 0$, a contradiction. V is topless follows from the implication that if $\perp$ exists, then $\top = -\perp$ is the top.

For example, any finite dimensional vector space over $\mathbb{R}$, and more generally, any (vector) space of real-valued functions on a given set S, is an ordered vector space. The natural ordering is defined by $f \leq g$ iff $f(x) \leq g(x)$ for every $x \in S$.

Properties. Let V be an ordered vector space and $u, v \in V$. Suppose $u \vee v$ exists. Then

1. $(u + w) \vee (v + w)$ exists and $(u + w) \vee (v + w) = (u \vee v) + w$ for any vector w.

   Proof. Let $s = (u \vee v) + w$. Then $u + w \leq s$ and $v + w \leq s$. For any upper bound t of $u + w$ and $v + w$, we have $u \leq t - w$ and $v \leq t - w$. So $u \vee v \leq t - w$, or $(u \vee v) + w \leq t$. So s is the least upper bound of $u + w$ and $v + w$. $\square$

2. $u \wedge v$ exists and $u \wedge v = (u + v) - (u \vee v)$.

   Proof. Let $s = (u + v) - (u \vee v)$. Since $u \leq u \vee v$, $-(u \vee v) \leq -u$, so $s \leq v$. Similarly $s \leq u$, so s is a lower bound of u and v. If $t \leq u$ and $t \leq v$, then $-u \leq -t$ and $-v \leq -t$, or $v \leq (u+v)-t$ and $u \leq (u+v)-t$, or $u \vee v \leq (u+v)-t$, or $t \leq (u+v)-(u \vee v) = s$. Hence s the greatest lower bound of u and v. $\square$

3. $\lambda u \vee \lambda v$ exists for any scalar $\lambda \in k$, and

(a) if $\lambda \geq 0$, then $\lambda u \vee \lambda v = \lambda(u \vee v)$

(b) if $\lambda \leq 0$, then $\lambda u \vee \lambda v = \lambda(u \wedge v)$

(c) if $u \neq v$, then the converse holds for (a) and (b).

Proof. Assume $\lambda \neq 0$ (clear otherwise). (a). If $\lambda > 0$, $u \leq u \vee v$ implies $\lambda u \leq \lambda(u \vee v)$. Similarly, $\lambda v \leq \lambda(u \vee v)$. If $\lambda u \leq t$ and $\lambda v \leq t$, then $u \leq \lambda^{-1}t$ and $v \leq \lambda^{-1}t$, hence $u \vee v \leq \lambda^{-1}t$, or $\lambda(u \vee v) \leq t$. Proof of (b) is similar to (a). (c). Suppose $\lambda u \vee \lambda v = \lambda(u \vee v)$ and $\lambda < 0$. Set $\gamma = -\lambda$. Then $\lambda u \vee \lambda v = \lambda(u \vee v) = -\gamma(u \vee v) = -(\gamma(u \vee v)) = -(\gamma u \vee \gamma v) = -((-\lambda u) \vee (-\lambda v)) = -(-(\lambda v \wedge \lambda u)) = \lambda v \wedge \lambda u$. This implies $\lambda u = \lambda v$, or $u = v$, a contradiction. □

Remarks.

- Since an ordered vector space is just an abelian po-group under $+$, the first two properties above can be easily generalized to a po-group. For this generalization, see this `http://planetmath.org/DistributivityInPoGroups`entry.

- A vector space V over $\mathbb{C}$ is said to be *ordered* if W is an ordered vector space over $\mathbb{R}$, where $V = W \oplus iW$ (V is the complexification of W).

- For any ordered vector space V, the set $V^+ := \{v \in V \mid 0 \leq v\}$ is called the *positive cone* of V. V^+ is clearly a convex set. Also, since for any $\lambda > 0$, $\lambda V^+ \subseteq V^+$, so V^+ is a convex cone. In addition, since $V^+ - \{0\}$ remains a cone, and $V^+ \cap (-V^+) = \{0\}$, V^+ is a proper cone.

- Given any vector space, a proper cone $P \subseteq V$ defiens a partial ordering on V, given by $u \leq v$ if $v - u \in P$. It is not hard to see that the partial ordering so defined makes V into an ordered vector space.

- So, there is a one-to-one correspondence between proper cones of V and partial orderings on V making V an ordered vector space.