Taylor's formula in Banach spaces

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Let U be an open subset of a real Banach space X. If $f\colon U \to \mathbb{R}$ is differentiable $n+1$ times on U, it may be expanded by Taylor's formula:

$$f(x) = f(a) + \mathrm{D}\, f(a) \cdot h + \frac{1}{2!}\, \mathrm{D}^2\, f(a) \cdot h^2 + \cdots + \frac{1}{n!}\, \mathrm{D}^n\, f(a) \cdot h^n + R_n(x)\,, \quad (1)$$

with the following expressions for the remainder term $R_n(x)$:

$$R_n(x) = \frac{1}{n!}\, \mathrm{D}^{n+1}\, f(\eta) \cdot (x-\eta)^n h \qquad \text{Cauchy form of remainder}$$

$$R_n(x) = \frac{1}{(n+1)!}\, \mathrm{D}^{n+1}\, f(\xi) \cdot h^{n+1} \qquad \text{Lagrange form of remainder}$$

$$R_n(x) = \frac{1}{n!} \int_0^1 \mathrm{D}^{n+1}\, f(a+th) \cdot ((1-t)h)^n h\, dt \qquad \text{integral form of remainder}$$

Here a and x must be points of U such that the line segment between a and x lie inside U, h is $x-a$, and the points ξ and η lie on the same line segment, strictly between a and x.

The kth Fréchet derivative of f at a is being denoted by $\mathrm{D}^k\, f(a)$, to be viewed as a multilinear map $X^k \to \mathbb{R}$. The $\cdot h^k$ notation means to evaluate a multilinear map at $(h, \dots, h)$.

1 Remainders for vector-valued functions

If Y is a Banach space, we may also consider Taylor expansions for $f\colon U \to Y$. Formula (**??**) takes the same form, but the Cauchy and Lagrange forms of the remainder will not be exact; they will only be bounds on $R_n(x)$. That is, for $f\colon U \to Y$,

$$\|R_n(x)\| \le \frac{1}{n!}\, \left\|\mathrm{D}^{n+1}\, f(\eta) \cdot (x-\eta)^n h\right\| \qquad \text{Cauchy form of remainder}$$

$$\|R_n(x)\| \le \frac{1}{(n+1)!}\, \left\|\mathrm{D}^{n+1}\, f(\xi) \cdot h^{n+1}\right\| \qquad \text{Lagrange form of remainder}$$

It is not hard to find counterexamples if we attempt to remove the norm signs or change the inequality to equality in the above formulas.

However, the integral form of the remainder continues to hold for $Y \neq \mathbb{R}$, although strictly speaking it only applies if the integrand is *integrable*. The integral form is also applicable when X and Y are complex Banach spaces.

Mean Value Theorem

The Mean Value Theorem can be obtained as the special case $n = 0$ with the Lagrange form of the remainder: for $f\colon U \to Y$ differentiable,

$$\|f(x) - f(a)\| \leq \|\mathrm{D}\, f(\xi) \cdot (x - a)\| \tag{2}$$

If $Y = \mathbb{R}$, then the norm signs may be removed from (**??**), and the inequality replaced by equality.

Formula (**??**) also holds under the much weaker hypothesis that f only has a directional derivative along the line segment from a to x.

Weaker bounds for the remainder

If $f\colon U \to Y$ is only differentiable n times *at* a, then we cannot quantify the remainder by the $n + 1$th derivative, but it is still true that

$$R_n(x) = o(\|x - a\|^n) \text{ as } x \to a. \tag{3}$$

Finite-dimensional case

If $X = \mathbb{R}^m$ and $Y = \mathbb{R}$, D^k has the following expression in terms of coordinates:

$$\mathrm{D}^k\, f(a) \cdot (\xi_1, \ldots, \xi_k) = \sum_{i_1,\ldots,i_k} \frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}}\, \xi_1^{i_1} \cdots \xi_k^{i_k}\,,$$

where each i_j runs from $1, \ldots, m$ in the sum.

If we collect the equal mixed partials (assuming that they are continuous) then

$$\frac{1}{k!}\, \mathrm{D}^k\, f(a) \cdot h^k = \sum_{|J|=k} \frac{1}{J!} \frac{\partial^{|J|} f}{\partial x^J} h^J\,,$$

where J is a multi-index of m components, and each component J_i indicates how many times the derivative with respect to the ith coordinate should be taken, and the exponent that the ith coordinate of h should be raised to in the monomial h^J. The multi-index J runs through all combinations such that $J_1 + \cdots + J_m = |J| = k$ in the sum. The notation $J!$ means $J_1! \cdots J_m!$.

All this is more easily assimilated if we remember that $\mathrm{D}^k\, f(a) \cdot h^k$ is supposed to be a polynomial of degree k. Also $|J|!/J!$ is just the multinomial coefficient.

Taylor series

If $\lim_{n\to\infty} R_n(x) = 0$, then we may write

$$f(x) = f(a) + \mathrm{D}\, f(a) \cdot h + \frac{1}{2!}\,\mathrm{D}^2\, f(a) \cdot h^2 + \cdots \tag{4}$$

as a convergent infinite series. Elegant as such an expansion is, it is not seen very often, for the reason that higher order Fréchet derivatives, especially in infinite-dimensional spaces, are often difficult to calculate.

But a notable exception occurs if a function f is defined by a convergent "power series"

$$f(x) = \sum_{k=0}^{\infty} M_k \cdot (x-a)^k \tag{5}$$

where $\{M_k : k = 0, 1, \dots\}$ is a family of continuous symmetric multilinear functions $X^k \to Y$. In this case, the series (**??**) is the Taylor series for f at a.

References

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[3] Michael Spivak. *Calculus*, third edition. Publish or Perish, 1994.