hyperplane separation

Canonical name	HyperplaneSeparation
Date of creation	2013-03-22 17:19:01
Last modified on	2013-03-22 17:19:01
Owner	stevecheng (10074)
Last modified by	stevecheng (10074)
Numerical id	4
Author	stevecheng (10074)
Entry type	Theorem
Classification	msc 46A55
Classification	msc 49J27
Classification	msc 46A20
Synonym	separating hyperplane
Related topic	HahnBanachgeometricFormTheorem

Let X be a vector space, and Φ be any subspace of linear functionals on X. Impose on X the weak topology generated by Φ.

Theorem 1 (Hyperplane Separation Theorem I)**.** *Given a weakly closed convex subset $S \subset X$, and $a \in X \setminus S$. there is $\phi \in \Phi$ such that*

$$\phi(a) < \inf_{x \in S} \phi(x) .$$

Proof. The weak topology on X can be generated by the semi-norms $x \mapsto |p(x)|$ for $p \in \Phi$. A subbasis for the weak topology consists of neigborhoods of the form $\{x \in X : |p(x-y)| < \epsilon\}$ for $y \in X$, $p \in \Phi$ and $\epsilon > 0$. Since $X \setminus S$ is weakly open, there exist $f_1, \ldots, f_n \in \Phi$ and $\epsilon > 0$ such that

$$|f_i(x) - f_i(a)| = |f_i(x-a)| < \epsilon \,, \text{ for all } i = 1, \ldots, n \quad \text{ implies } x \in X \setminus S \,.$$

In other words, if $x \in S$ then at least one of $|f_i(x) - f_i(a)|$ is $\geq \epsilon$.

Define a map $F \colon X \to \mathbb{R}^n$ by $F(x) = (f_1(x), \ldots, f_n(x))$. The set $\overline{F(S)}$ is evidently closed and convex in $\mathbb{R}^n$, a Hilbert space under the standard inner product. So there is a point $b \in \overline{F(S)}$ that minimizes the norm $\|b - F(a)\|$.

It follows that $\langle y - b, b - F(a)\rangle \geq 0$ for all $y \in \overline{F(S)}$; for otherwise we can attain a smaller value of the norm by moving from the point b along a line towards y. (Formally, we have $0 \leq \frac{d}{dt}\big|_{t=0} \|ty + (1-t)b - F(a)\|^2 = 2\langle y - b, b - F(a)\rangle$.)

Take $\phi = \sum_{i=1}^n \lambda_i f_i$ where $\lambda = b - F(a)$. Then we find, for all $x \in S$,

$$\begin{aligned}
\phi(x-a) &= \langle b - F(a), F(x-a)\rangle \\
&= \langle b - F(a), b - F(a)\rangle + \langle b - F(a), y - b\rangle \,, \quad y = F(x) \in \overline{F(S)} \\
&\geq \|b - F(a)\|^2 + 0 \geq \epsilon^2 \,.
\end{aligned}$$

□

Theorem 2 (Hyperplane Separation Theorem II)**.** *Let $S \subset X$ be a weakly closed convex subset, and $K \subset X$ a compact convex subset, that do not intersect each other. Then there exists $\phi \in \Phi$ such that*

$$\sup_{y \in K} \phi(y) < \inf_{x \in S} \phi(x) .$$

Proof. We show that $S - K = \{x - y : x \in S \,, y \in K\}$ is weakly closed in X. Let $\{z_\alpha = x_\alpha - y_\alpha\} \subseteq A$ be a net convergent to z. Since K is compact, $\{y_\alpha\}$ has a subnet $\{y_{\alpha(\beta)}\}$ convergent to $y \in K$. Then the subnet

$x_{\alpha(\beta)} = z_{\alpha(\beta)} + y_{\alpha(\beta)}$ is convergent to $x = z + y$. The point x is in S since S is closed; therefore $z = x - y$ is in $S - K$.

Also, $S-K$ is convex since S and K are. Noting that $0 \notin S-K$ (otherwise S and K would have a common point), we apply the previous theorem to obtain a $\phi \in \Phi$ such that

$$0 = \phi(0) < \inf_{z \in S-K} \phi(z) \leq \phi(x-y)\,, \text{ for all } x \in S \text{ and } y \in K.$$

The desired conclusion follows at once. $\square$