proof of Banach-Steinhaus theorem

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Let

$$E_n = \{x \in X : \|T(x)\| \leq n \text{ for all } T \in \mathcal{F}\}.$$

From the hypothesis, we have that

$$\bigcup_{n=1}^{\infty} E_n = X.$$

Also, each E_n is closed, since it can be written as

$$E_n = \bigcap_{T \in \mathcal{F}} T^{-1}(B(0,n)),$$

where $B(0,n)$ is the closed ball centered at 0 with radius n in Y, and each of the sets in the intersection is closed due to the continuity of the operators. Now since X is a Banach space, Baire's category theorem implies that there exists n such that E_n has nonempty interior. So there is $x_0 \in E_n$ and $r > 0$ such that $B(x_0, r) \subset E_n$. Thus if $\|x\| \leq r$, we have

$$\|T(x)\| - \|T(x_0)\| \leq \|T(x_0) + T(x)\| = \|T(x_0 + x)\| \leq n$$

for each $T \in \mathcal{F}$, and so

$$\|T(x)\| \leq n + \|T(x_0)\|$$

so if $\|x\| \leq 1$, we have

$$\|T(x)\| = \frac{1}{r}\|T(rx)\| \leq \frac{1}{r}\left(n + \|T(x_0)\|\right) = c,$$

and this means that

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\} \leq c$$

for all $T \in \mathcal{F}$.