seminorm

Canonical name	Seminorm
Date of creation	2013-03-22 12:24:57
Last modified on	2013-03-22 12:24:57
Owner	rmilson (146)
Last modified by	rmilson (146)
Numerical id	20
Author	rmilson (146)
Entry type	Definition
Classification	msc 46B20
Synonym	semi-norm
Defines	homogeneous

Let V be a real, or a complex vector space, with K denoting the corresponding field of scalars. A *seminorm* is a function

$$\mathrm{p} : V \to \mathbb{R}^+,$$

from V to the set of non-negative real numbers, that satisfies the following two properties.

$$\begin{aligned} \mathrm{p}(k\,\mathbf{u}) &= |k|\,\mathrm{p}(\mathbf{u}), \quad k \in K,\ \mathbf{u} \in V && \textbf{Homogeneity} \\ \mathrm{p}(\mathbf{u}+\mathbf{v}) &\leq \mathrm{p}(\mathbf{u}) + \mathrm{p}(\mathbf{v}), \quad \mathbf{u},\mathbf{v} \in V, && \textbf{Sublinearity} \end{aligned}$$

A seminorm differs from a norm in that it is permitted that $\mathrm{p}(\mathbf{u}) = 0$ for some non-zero $\mathbf{u} \in V$.

It is possible to characterize the seminorms properties geometrically. For $k > 0$, let

$$B_k = \{\mathbf{u} \in V : \mathrm{p}(\mathbf{u}) \leq k\}$$

denote the ball of radius k. The homogeneity property is equivalent to the assertion that

$$B_k = kB_1,$$

in the sense that $\mathbf{u} \in B_1$ if and only if $k\mathbf{u} \in B_k$. Thus, we see that a seminorm is fully determined by its unit ball. Indeed, given $B \subset V$ we may define a function $\mathrm{p}_B : V \to \mathbb{R}^+$ by

$$\mathrm{p}_B(\mathbf{u}) = \inf\{\lambda \in \mathbb{R}^+ : \lambda^{-1}\mathbf{u} \in B\}.$$

The geometric nature of the unit ball is described by the following.

Proposition 1 *The function p_B satisfies the homegeneity property if and only if for every $\mathbf{u} \in V$, there exists a $k \in \mathbb{R}^+ \cup \{\infty\}$ such that*

$$\lambda\,\mathbf{u} \in B \quad \textit{if and only if} \quad \|\lambda\| \leq k.$$

Proposition 2 *Suppose that p is homogeneous. Then, it is sublinear if and only if its unit ball, B_1, is a convex subset of V.*

Proof. First, let us suppose that the seminorm is both sublinear and homogeneous, and prove that B_1 is necessarily convex. Let $\mathbf{u},\mathbf{v} \in B_1$, and let k be a real number between 0 and 1. We must show that the weighted average $k\mathbf{u} + (1-k)\mathbf{v}$ is in B_1 as well. By assumption,

$$\mathrm{p}(k\,\mathbf{u} + (1-k)\mathbf{v}) \leq k\,\mathrm{p}(\mathbf{u}) + (1-k)\,\mathrm{p}(\mathbf{v}).$$

The right side is a weighted average of two numbers between 0 and 1, and is therefore between 0 and 1 itself. Therefore

$$k\,\mathbf{u} + (1-k)\mathbf{v} \in B_1,$$

as desired.

Conversely, suppose that the seminorm function is homogeneous, and that the unit ball is convex. Let $\mathbf{u}, \mathbf{v} \in V$ be given, and let us show that

$$\mathrm{p}(\mathbf{u}+\mathbf{v}) \leq \mathrm{p}(\mathbf{u}) + \mathrm{p}(\mathbf{v}).$$

The essential complication here is that we do not exclude the possibility that $\mathrm{p}(\mathbf{u}) = 0$, but that $\mathbf{u} \neq 0$. First, let us consider the case where

$$\mathrm{p}(\mathbf{u}) = \mathrm{p}(\mathbf{v}) = 0.$$

By homogeneity, for every $k > 0$ we have

$$k\mathbf{u},\ k\mathbf{v} \in B_1,$$

and hence

$$\frac{k}{2}\,\mathbf{u} + \frac{k}{2}\,\mathbf{v} \in B_1,$$

as well. By homogeneity, again,

$$\mathrm{p}(\mathbf{u}+\mathbf{v}) \leq \frac{2}{k}.$$

Since the above is true for all positive k, we infer that

$$\mathrm{p}(\mathbf{u}+\mathbf{v}) = 0,$$

as desired.

Next suppose that $\mathrm{p}(\mathbf{u}) = 0$, but that $\mathrm{p}(\mathbf{v}) \neq 0$. We will show that in this case, necessarily,

$$\mathrm{p}(\mathbf{u}+\mathbf{v}) = \mathrm{p}(\mathbf{v}).$$

Owing to the homogeneity assumption, we may without loss of generality assume that

$$\mathrm{p}(\mathbf{v}) = 1.$$

For every k such that $0 \leq k < 1$ we have

$$k\,\mathbf{u} + k\,\mathbf{v} = (1-k)\frac{k\,\mathbf{u}}{1-k} + k\,\mathbf{v}.$$

The right-side expression is an element of B_1 because

$$\frac{k\,\mathbf{u}}{1-k},\ \mathbf{v} \in B_1.$$

Hence

$$k\,\mathrm{p}(\mathbf{u}+\mathbf{v}) \leq 1,$$

and since this holds for k arbitrarily close to 1 we conclude that

$$\mathrm{p}(\mathbf{u}+\mathbf{v}) \leq \mathrm{p}(\mathbf{v}).$$

The same argument also shows that

$$\mathrm{p}(\mathbf{v}) = \mathrm{p}(-\mathbf{u} + (\mathbf{u}+\mathbf{v})) \leq \mathrm{p}(\mathbf{u}+\mathbf{v}),$$

and hence

$$\mathrm{p}(\mathbf{u}+\mathbf{v}) = \mathrm{p}(\mathbf{v}),$$

as desired.

Finally, suppose that neither $\mathrm{p}(\mathbf{u})$ nor $\mathrm{p}(\mathbf{v})$ is zero. Hence,

$$\frac{\mathbf{u}}{\mathrm{p}(u)},\ \frac{\mathbf{v}}{\mathrm{p}(v)}$$

are both in B_1, and hence

$$\frac{\mathrm{p}(u)}{\mathrm{p}(u)+\mathrm{p}(v)}\frac{\mathbf{u}}{\mathrm{p}(u)} + \frac{\mathrm{p}(v)}{\mathrm{p}(u)+\mathrm{p}(v)}\frac{\mathbf{v}}{\mathrm{p}(v)} = \frac{\mathbf{u}+\mathbf{v}}{\mathrm{p}(u)+\mathrm{p}(v)}$$

is in B_1 also. Using homogeneity, we conclude that

$$\mathrm{p}(u+v) \leq \mathrm{p}(u) + \mathrm{p}(v),$$

as desired.