weak-* topology

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Let X be a locally convex topological vector space (over $\mathbb{C}$ or $\mathbb{R}$), and let X^* be the set of continuous linear functionals on X (the continuous dual of X). If $f \in X^*$ then let p_f denote the seminorm $p_f(x) = |f(x)|$, and let $p_x(f)$ denote the seminorm $p_x(f) = |f(x)|$. Obviously any normed space is a locally convex topological vector space so X could be a normed space.

Definition. The topology on X defined by the seminorms $\{p_f \mid f \in X^*\}$ is called the *weak topology* and the topology on X^* defined by the seminorms $\{p_x \mid x \in X\}$ is called the *weak-$*$ topology*.

The weak topology on X is usually denoted by $\sigma(X, X^*)$ and the weak-$*$ topology on X^* is usually denoted by $\sigma(X^*, X)$. Another common notation is (X, wk) and $(X^*, wk - *)$

Topology defined on a space Y by seminorms p_ι, $\iota \in I$ means that we take the sets $\{y \in Y \mid p_\iota(y) < \epsilon\}$ for all $\iota \in I$ and $\epsilon > 0$ as a subbase for the topology (that is finite intersections of such sets form the basis).

The most striking result about weak-$*$ topology is the Alaoglu's theorem which asserts that for X being a normed space, a closed ball (in the operator norm) of X^* is weak-$*$ compact. There is no similar result for the weak topology on X, unless X is a reflexive space.

Note that X^* is sometimes used for the algebraic dual of a space and X' is used for the continuous dual. In functional analysis X^* always means the continuous dual and hence the term *weak-$*$ topology*.

References

[1] John B. Conway. , Springer-Verlag, New York, New York, 1990.