invariant subspaces for self-adjoint *-algebras of operators

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In this entry we provide few results concerning invariant subspaces of *-algebras of bounded operators on Hilbert spaces.

Let H be a Hilbert space and $B(H)$ its algebra of bounded operators. Recall that, given an operator $T \in B(H)$, a subspace $V \subseteq H$ is said to be invariant for T if $Tx \in V$ whenever $x \in V$.

Similarly, given a subalgebra $\mathcal{A} \subseteq B(H)$, we will say that a subspace $V \subseteq H$ is *invariant* for $\mathcal{A}$ if $Tx \in V$ whenever $T \in \mathcal{A}$ and $x \in V$, i.e. if V is invariant for all operators in $\mathcal{A}$.

Invariant subspaces for a single operator

Proposition 1 - *Let $T \in B(H)$. If a subspace $V \subset H$ is invariant for T, then so is its closure $\overline{V}$.*

Proof: Let $x \in \overline{V}$. There is a sequence $\{x_n\}$ in V such that $x_n \to x$. Hence, $Tx_n \to Tx$. Since V is invariant for T, all Tx_n belong to V. Thus, their limit Tx must be in $\overline{V}$. We conclude that $\overline{V}$ is also invariant for T. $\square$

Proposition 2 - *Let $T \in B(H)$. If a subspace $V \subset H$ is invariant for T, then its orthogonal complement $V^{\perp}$ is invariant for T^*.*

Proof: Let $y \in V^{\perp}$. For all $x \in H$ we have that $\langle x, T^*y \rangle = \langle Tx, y \rangle = 0$, where the last equality comes from the fact that $Tx \in V$, since V is invariant for T. Therefore T^*y must belong to $V^{\perp}$, from which we conclude that $V^{\perp}$ is invariant for T^*. $\square$

Proposition 3 - *Let $T \in B(H)$, $V \subset H$ a closed subspace and $P \in B(H)$ the orthogonal projection onto V. The following are statements are equivalent:*

1. *V is invariant for T.*
2. *$V^{\perp}$ is invariant for T^*.*
3. *$TP = PTP$.*

Proof: (1) $\Longrightarrow$ (2) This part follows directly from Proposition 2.

(2) $\Longrightarrow$ (1) From Proposition 2 it follows that $(V^{\perp})^{\perp}$ is invariant for $(T^*)^* = T$. Since V is closed, $V = \overline{V} = (V^{\perp})^{\perp}$. We conclude that V is invariant for T.

(1) $\Longrightarrow$ (3) Let $x \in H$. From the orthogonal decomposition theorem we know that $H = V \oplus V^{\perp}$, hence $x = y + z$, where $y \in V$ and $z \in V^{\perp}$. We now see that $TPx = Ty$ and $PTPx = PTy = Ty$, where the last equality comes from the fact that $Ty \in V$. Hence, $TP = PTP$.

(3) $\Longrightarrow$ (1) Let $x \in V$. We have that $Tx = TPx = PTPx$. Since $PTPx$ is obviously on the image of P, it follows that $Tx \in V$, i.e. V is invariant for T. $\square$

Proposition 4 - *Let $T \in B(H)$, $V \subset H$ a closed subspace and $P \in B(H)$ the orhtogonal projection onto V. The subspaces V and $V^{\perp}$ are both invariant for T if and only if $TP = PT$.*

Proof: ($\Longrightarrow$) From Proposition 3 it follows that V is invariant for both T and T^*. Then, again from Proposition 3, we see that $PT = (T^*P)^* = (PT^*P)^* = PTP = TP$.

($\Longleftarrow$) Suppose $TP = PT$. Then $PTP = TPP = TP$, and from Proposition 3 we see that V is invariant for T.

We also have that $PT^* = T^*P$, and we can conclude in the same way that V is invariant for T^*. From Proposition 3 it follows that $V^{\perp}$ is also invariant for T. $\square$

Invariant subspaces for *-algebras of operators

We shall now generalize some of the above results to the case of self-adjoint subalgebras of $B(H)$.

Proposition 5 - *Let $\mathcal{A}$ be a *-subalgebra of $B(H)$ and V a subspace of H. If a subspace V is invariant for $\mathcal{A}$, then so are its closure $\overline{V}$ and its orthogonal complement $V^{\perp}$.*

Proof: From Proposition 1 it follows that $\overline{V}$ is invariant for all operators in $\mathcal{A}$, which means that V is invariant for $\mathcal{A}$.

Also, from Proposition 2 it follows that $V^{\perp}$ is invariant for the adjoint of each operator in $\mathcal{A}$. Since $\mathcal{A}$ is self-adjoint, it follows that $V^{\perp}$ is invariant for $\mathcal{A}$. $\square$

Theorem - *Let $\mathcal{A}$ be a *-subalgebra of $B(H)$, $V \subset H$ a closed subspace and P the orthogonal projection onto V. The following are equivalent:*

1. *V is invariant for $\mathcal{A}$.*

2. *$V^\perp$ is invariant for $\mathcal{A}$.*

3. *$P \in \mathcal{A}'$, i.e. P belongs to the commutant of $\mathcal{A}$.*

Proof: (1) $\Longleftrightarrow$ (2) This equivalence follows directly from Proposition 5 and the fact that V is closed.

(1) $\Longrightarrow$ (3) Suppose V is invariant for $\mathcal{A}$. We have already proved that $V^\perp$ is also invariant for $\mathcal{A}$. Thus, from Proposition 4 it follows that P commutes with all operators in $\mathcal{A}$, i.e. $P \in \mathcal{A}'$.

(3) $\Longrightarrow$ (1) Suppose $P \in \mathcal{A}'$. Then P commutes with all operators in $\mathcal{A}$. From Proposition 4 it follows that V is invariant for each operator in $\mathcal{A}$, i.e. V is invariant for $\mathcal{A}$. $\Box$