Hilbert module

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Defines	pre-Hilbert module

Definition 1. A **(right) pre-Hilbert module** over a C^*-algebra A is a right A-module $\mathcal{E}$ equipped with an A-valued inner product $\langle -, - \rangle \colon \mathcal{E} \times \mathcal{E} \to A$, i.e. a sesquilinear pairing satisfying

$$\begin{aligned} \langle u, va \rangle &= \langle u, v \rangle a & (1) \\ \langle u, v \rangle &= \langle v, u \rangle^* & (2) \\ \langle v, v \rangle &\geq 0, \text{ with } \langle v, v \rangle = 0 \text{ iff } v = 0, & (3) \end{aligned}$$

for all $u, v \in \mathcal{E}$ and $a \in A$. Note, positive definiteness is well-defined due to the notion of positivity for C^*-algebras. The norm of an element $v \in \mathcal{E}$ is defined by $\|v\| = \sqrt{\|\langle v, v \rangle\|}$.

Definition 2. A **(right) Hilbert module** over a C^*-algebra A is a right pre-Hilbert module over A which is complete with respect to the norm.

Example 1 (Hilbert spaces)
A complex Hilbert space is a Hilbert $\mathbb{C}$*-module.*

Example 2 (C^*-algebras)
A C^**-algebra* A *is a Hilbert* A*-module with inner product* $\langle a, b \rangle = a^* b$*.*

Definition 3. A **Hilbert A-B-bimodule** is a (right) Hilbert module $\mathcal{E}$ over a C^*-algebra B together with a *-homomorphism π from a C^*-algebra A to $\mathrm{End}(\mathcal{E})$.