L^2-spaces are Hilbert spaces

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Defines	linear space of square integrable functions
Defines	sequilinearity

Let $(X, \mathfrak{B}, \mu)$ be a measure space. Let $L^2(X)$ denote the http://planetmath.org/LpSpaceL^2-space associated with this measure space, i.e. $L^2(X)$ consists of measurable functions $f : X \longrightarrow \mathbb{C}$ such that

$$\|f\|_2 := \left(\int_X |f|^2 d\mu\right)^{\frac{1}{2}} < \infty$$

identified up to equivalence almost everywhere.

It is known that all http://planetmath.org/LpSpaceL^p-spaces, with $1 \leq p \leq \infty$, are Banach spaces with respect to the http://planetmath.org/LpSpaceL^p-norm $\|\cdot\|_p$. For $L^2(X)$ we can say more:

Theorem - $L^2(X)$ is an Hilbert Space with respect to the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle = \int_X f\overline{g}\, d\mu$$

Proof:

Sesquilinearity follows from the http://planetmath.org/PropertiesOfTheLebesgueIntegral of the Lebesgue integral (that is, the inner product defined above is linear in the first argument and conjugate linear in the second one). The conjugate symmetry is evident.

Positive definiteness holds by construction: If $\int_X |f|^2 d\mu = 0$, then $|f|^2$ (and therefore f) is zero almost everywhere, thus the equivalence class of f is the equivalence class of the zero function (which is the additive neutral element of the space).

Completeness is proved for the general case of L^p-spaces in http://planetmath.org/ProofThat article.$\square$

0.0.1 Remarks

- The spaces $\mathbb{C}^n$ or $\mathbb{R}^n$ with the usual inner product are particular examples of $L^2(X)$, choosing $X = \{1, \dots, n\}$ with the counting measure.

- Choosing appropriate spaces X it can be shown that all Hilbert spaces are isometrically isomorphic to a L^2-space.