Banach spaces of infinite dimension do not have a countable Hamel basis

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A Banach space of infinite dimension does not have a countable Hamel basis.

Proof

Let E be such space, and suppose it does have a countable Hamel basis, say $B = (v_k)_{k\in\mathbb{N}}$.

Then, by definition of Hamel basis and linear combination, we have that $x \in E$ if and only if $x = \lambda_1 \cdot v_1 + \cdots + \lambda_n \cdot v_n$ for some $n \in \mathbb{N}$. Consequently,

$$E = \bigcup_{i=1}^{\infty} (\mathrm{span}(v_j)_{j=1}^{i}).$$

This would mean that E is a countable union of proper subspaces of finite dimension (they are proper because E has infinite dimension), but every finite dimensional proper subspace of a normed space is nowhere dense, and then E would be first category. This is absurd, by the Baire Category Theorem.

Note

In fact, the Hamel dimension of an infinite-dimensional Banach space is always at least the cardinality of the continuum (even if the Continuum Hypothesis fails). A one-page proof of this has been given by H. Elton Lacey[**?**].

Examples

Consider the set of all real-valued infinite sequences (x_n) such that $x_n = 0$ for all but finitely many n.

This is a vector space, with the known operations. Morover, it has infinite dimension: a possible basis is $(e_k)_{k\in\mathbb{N}}$, where

$$e_i(n) = \begin{cases} 1, & \text{if } n = i \\ 0, & \text{otherwise.} \end{cases}$$

So, it has infinite dimension and a countable Hamel basis. Using our result, it follows directly that there is no way to define a norm in this vector space such that it is a complete metric space under the induced metric.

References

[1] H. Elton Lacey, *The Hamel Dimension of any Infinite Dimensional Separable Banach Space is c*, Amer. Math. Mon. 80 (1973), 298.