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Stone-Weierstrass theorem (complex version)

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Theorem - Let X be a compact space and $C(X)$ the algebra of continuous functions $X \longrightarrow \mathbb{C}$ endowed with the sup norm $\| \cdot \|_\infty$. Let $\mathcal{A}$ be a subalgebra of $C(X)$ for which the following conditions hold:

1. $\forall x, y \in X, x \neq y, \exists f \in \mathcal{A} : f(x) \neq f(y)$, i.e. $\mathcal{A}$ separates points
2. $1 \in \mathcal{A}$, i.e. $\mathcal{A}$ contains all constant functions
3. If $f \in \mathcal{A}$ then $\overline{f} \in \mathcal{A}$, i.e. $\mathcal{A}$ is a `http://planetmath.org/InvolutaryRing`self-adjoint subalgebra of $C(X)$

Then $\mathcal{A}$ is dense in $C(X)$.

Proof: The proof follows easily from the real version of this theorem (see the `http://planetmath.org/StoneWeierstrassTheorem`parent entry).

Let $\mathcal{R}$ be the set of the real parts of elements $f \in \mathcal{A}$, i.e.

$$\mathcal{R} := \{\mathrm{Re}(f) : f \in \mathcal{A}\}$$

It is clear that $\mathcal{R}$ contains (it is in fact equal) to the set of the imaginary parts of elements of $\mathcal{A}$. This can be seen just by multiplying any function $f \in \mathcal{A}$ by $-i$.

We can see that $\mathcal{R} \subseteq \mathcal{A}$. In fact, $\mathrm{Re}(f) = \frac{f+\overline{f}}{2}$ and by condition 3 this element belongs to $\mathcal{A}$.

Moreover, $\mathcal{R}$ is a subalgebra of $\mathcal{A}$. In fact, since $\mathcal{A}$ is an algebra, the product of two elements $\mathrm{Re}(f)$, $\mathrm{Re}(g)$ of $\mathcal{R}$ gives an element of $\mathcal{A}$. But since $\mathrm{Re}(f).\mathrm{Re}(g)$ is a real valued function, it must belong to $\mathcal{R}$. The same can be said about sums and products by real scalars.

Let us now see that $\mathcal{R}$ separates points. Since $\mathcal{A}$ separates points, for every $x \neq y$ in X there is a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. But this implies that $\mathrm{Re}(f(x)) \neq \mathrm{Re}(f(y))$ or $\mathrm{Im}(f(x)) \neq \mathrm{Im}(f(y))$, hence there is a function in $\mathcal{R}$ that separates x and y.

Of course, $\mathcal{R}$ contains the constant function 1.

Hence, we can apply the real version of the Stone-Weierstrass theorem to conclude that every real valued function in X can be uniformly approximated by elements of $\mathcal{R}$.

Let us now see that $\mathcal{A}$ is dense in $C(X)$. Let $f \in C(X)$. By the previous observation, both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are the uniform limits of sequences $\{g_n\}$ and $\{h_n\}$ in $\mathcal{R}$. Hence,

$$\|f - (g_n + ih_n)\|_\infty \leq \|\mathrm{Re}(f) - g_n\|_\infty + \|\mathrm{Im}(f) - h_n\|_\infty \longrightarrow 0$$

Of course, the sequence $\{g_n + ih_n\}$ is in $\mathcal{A}$. Hence, $\mathcal{A}$ is dense in $C(X)$.

$\square$