commutant is a weak operator closed subalgebra

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Let H be a Hilbert space and $B(H)$ the algebra of bounded operators in H. Recall that the commutant of a subset $\mathcal{F} \subset B(H)$ is the set of all bounded operators that commute with those of $\mathcal{F}$, i.e.

$$\mathcal{F}' := \{T \in B(H) : \ TS = ST\,, \quad \forall S \in \mathcal{F}\}.$$

- If $\mathcal{F} \subset B(H)$, then $\mathcal{F}'$ is a subalgebra of $B(H)$ that contains the identity operator and is closed in the weak operator topology.

: It is clear that $\mathcal{F}'$ contains the identity operator, since it commutes with all operators in $B(H)$ and in particular with those of $\mathcal{F}$.

Let us now see that $\mathcal{F}'$ is a subalgebra of $B(H)$. Let $T_1, T_2 \in \mathcal{F}'$ and $\lambda \in \mathbb{C}$. We have that, for all $S \in \mathcal{F}$,

$$\begin{aligned} S(T_1 + T_2) = ST_1 + ST_2 = T_1S + T_2S = (T_1 + T_2)S \\ S(\lambda T_1) = \lambda ST_1 = \lambda T_1 S \\ S(T_1T_2) = T_1ST_2 = T_1T_2S \end{aligned}$$

thus, $T_1 + T_2$, λT_1 and T_1T_2 all belong to $\mathcal{F}'$, and therefore $\mathcal{F}'$ is a subalgebra of $B(H)$.

It remains to see that $\mathcal{F}'$ is weak operator closed. Suppose (T_i) is a net in $\mathcal{F}'$ that converges to T in the weak operator topology. Then, for all $x, y \in H$ we have that $\langle T_i x, y\rangle \to \langle Tx, y\rangle$. Thus, for all $S \in \mathcal{F}$, we have

$$\begin{aligned} \langle (TS - ST)x, y\rangle &= \langle TSx, y\rangle - \langle Tx, S^*y\rangle \\ &= \lim \big(\langle T_iSx, y\rangle - \langle T_ix, S^*y\rangle\big) \\ &= \lim \langle (T_iS - ST_i)x, y\rangle \\ &= \lim \langle (T_iS - T_iS)x, y\rangle \\ &= 0 \end{aligned}$$

Hence, $TS - ST = 0$, so that $T \in \mathcal{F}'$. We conclude that $\mathcal{F}'$ is closed in the weak operator topology. $\square$