Fréchet derivative is unique

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Theorem The Fréchet derivative is unique.

Proof. Assume that both A and B in $L(\mathsf{V},\mathsf{W})$ satisfy the condition for the `http://planetmath.org/derivative2`Fréchet derivative at the point $\mathbf{x}$. To prove that they are equal we will show that for all $\varepsilon>0$ the operator norm $\|A-B\|$ is not greater than ε. By the definition of limit there exists a positive δ such that for all $\|\mathbf{h}\|\leq\delta$

$$\|f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})-A\mathbf{h}\|\leq\frac{\varepsilon}{2}\cdot\|\mathbf{h}\|\text{ and }\|f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})-B\mathbf{h}\|\leq\frac{\varepsilon}{2}\cdot\|\mathbf{h}\|$$

holds. This gives

$$\begin{aligned}\|(A-B)\mathbf{h}\| &= \|(f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})-A\mathbf{h})-(f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})-B\mathbf{h})\|\\ &\leq \|f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})-A\mathbf{h}\|+\|f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})-B\mathbf{h}\|\\ &< \varepsilon\cdot\|\mathbf{h}\|.\end{aligned}$$

Now we have

$$\delta\cdot\|A-B\|=\delta\cdot\sup_{\|\mathbf{g}\|\leq 1}\|(A-B)\mathbf{g}\|=\sup_{\|\mathbf{g}\|\leq\delta}\|(A-B)\mathbf{g}\|\leq\sup_{\|\mathbf{g}\|\leq\delta}\varepsilon\cdot\|\mathbf{g}\|\leq\varepsilon\cdot\delta,$$

thus $\|A-B\|\leq\varepsilon$ as we wanted to show.