proper subspaces of a topological vector space have empty interior

Canonical name	ProperSubspacesOfATopologicalVectorSpaceHaveEmptyInterior
Date of creation	2013-03-22 17:34:23
Last modified on	2013-03-22 17:34:23
Owner	asteroid (17536)
Last modified by	asteroid (17536)
Numerical id	6
Author	asteroid (17536)
Entry type	Theorem
Classification	msc 46A99

Theorem - Let V be a topological vector space. Every proper subspace $S \subset V$ has empty interior.

Proof : Let S be a subspace of V. Suppose there is a non-empty open set $A \subseteq S$.

Fix a point $a_0 \in A \subseteq S$. Since the vector sum operation is continuous, translations of open sets are again open sets. In particular, the set $A - a_0 := \{x - a_0 : x \in A\}$ is an open set of V that contains the origin 0.

As S is a vector subspace and $A \subseteq S$, we see that the translation $A - a_0$ is still contained in S.

Since the scalar multiplication operation is continuous it follows easily that, for every $x \in V$, the function $f_x : \mathbb{K} \longrightarrow V$ given by

$$f_x(\lambda) = \lambda x$$

is also continuous.

Consider now any vector $v \in V$. The set $f_v^{-1}(A - a_0)$ is an open set that contains 0. Thus, taking a value $\lambda \in f_v^{-1}(A - a_0)$ we see that

$$\lambda v \in A - a_0,$$

i.e. we can multiply v by a sufficiently small λ such that λv belongs to the open set $A - a_0$.

Since the set $A - a_0$ is contained in S, we see that $\lambda v \in S$, and therefore $v \in S$.

This proves that $V = S$, i.e. S is not proper.

We conclude that if S is proper then S has empty interior. $\square$