quotients of Banach spaces by closed subspaces are Banach spaces under the quotient norm

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Theorem - Let X be a Banach space and M a closed subspace. Then X/M with the quotient norm is a Banach space.

Proof : In to prove that X/M is a Banach space it is enough to prove that every series in X/M that converges absolutely also converges in X/M.

Let $\sum_n X_n$ be an absolutely convergent series in X/M, i.e., $\sum_n \|X_n\|_{X/M} < \infty$. By definition of the quotient norm, there exists $x_n \in X_n$ such that

$$\|x_n\| \leq \|X_n\|_{X/M} + 2^{-n}$$

It is clear that $\sum_n \|x_n\| < \infty$ and so, as X is a Banach space, $\sum_n x_n$ is convergent.

Let $x = \sum_n x_n$ and $s_k = \sum_{n=1}^k x_n$. We have that

$$x - s_k + M = (x+M) - (s_k+M) = (x+M) - \sum_{n=1}^{k} (x_n+M) = (x+M) - \sum_{n=1}^{k} X_n$$

Since $\|x - s_k + M\|_{X/M} \leq \|x - s_k\| \longrightarrow 0$ we see that $\sum_n X_n$ converges in X/M to $x + M$. $\square$