proof of Ascoli-Arzelà theorem

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Given $\epsilon > 0$ we aim at finding a 4ϵ-net in F i.e. a finite set of points F_ϵ such that

$$\bigcup_{f \in F_\epsilon} B_{4\epsilon}(f) \supset F$$

(see the definition of totally bounded). Let $\delta > 0$ be given with respect to ϵ in the definition of equi-continuity (see uniformly equicontinuous) of F. Let X_δ be a δ-lattice in X and Y_ϵ be a ϵ-lattice in Y. Let now $Y_\epsilon^{X_\delta}$ be the set of functions from X_δ to Y_ϵ and define $G_\epsilon \subset Y_\epsilon^{X_\delta}$ by

$$G_\epsilon = \{g \in Y_\epsilon^{X_\delta} \colon \exists f \in F \ \forall x \in X_\delta \quad d(f(x), g(x)) < \epsilon\}.$$

Since $Y_\epsilon^{X_\delta}$ is a finite set, G_ϵ is finite too: say $G_\epsilon = \{g_1, \ldots, g_N\}$. Then define $F_\epsilon \subset F$, $F_\epsilon = \{f_1, \ldots, f_N\}$ where $f_k \colon X \to Y$ is a function in F such that $d(f_k(x), g_k(x)) < \epsilon$ for all $x \in X_\delta$ (the existence of such a function is guaranteed by the definition of G_ϵ).

We now will prove that F_ϵ is a 4ϵ-lattice in F. Given $f \in F$ choose $g \in Y_\epsilon^{X_\delta}$ such that for all $x \in X_\delta$ it holds $d(f(x), g(x)) < \epsilon$ (this is possible as for all $x \in X_\delta$ there exists $y \in Y_\epsilon$ with $d(f(x), y) < \epsilon$). We conclude that $g \in G_\epsilon$ and hence $g = g_k$ for some $k \in \{1, \ldots, N\}$. Notice also that for all $x \in X_\delta$ we have $d(f(x), f_k(x)) \leq d(f(x), g_k(x)) + d(g_k(x), f_k(x)) < 2\epsilon$.

Given any $x \in X$ we know that there exists $x_\delta \in X_\delta$ such that $d(x, x_\delta) < \delta$. So, by equicontinuity of F,

$$d(f(x), f_k(x)) \leq d(f(x), f(x_\delta)) + d(f_k(x), f_k(x_\delta)) + d(f(x_\delta), f_k(x_\delta)) < 4\epsilon.$$