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projections and closed subspaces

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Theorem 1 - Let X be a Banach space and M a closed subspace. Then,

- M is topologically complemented in X if and only if there exists a continuous projection onto M.

- Given a topological complement N of M, there exists a unique continuous projection P onto M such that $P(x+y)=x$ for all $x \in M$ and $y \in N$.

The projection P in the second part of the above theorem is sometimes called the *projection onto* M *along* N.

The above result can be further improved for Hilbert spaces.

Theorem 2 - Let X be a Hilbert space and M a closed subspace. Then, M is topologically complemented in X if and only if there exists an orthogonal projection onto M (which is unique).

Since, by the orthogonal decomposition theorem, a closed subspace of a Hilbert space is always topologically complemented by its orthogonal complement ($X = M \oplus M^{\perp}$), it follows that

Corollary - Let X be a Hilbert space and M a closed subspace. Then, there exists a unique orthogonal projection onto M. This establishes a bijective correspondence between orthogonal projections and closed subspaces.