Wielandt theorem for unital normed algebras

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Theorem. (Wielandt (1949)) Let A be a normed unital algebra (with unit e). If $x, y \in A$ then $xy - yx \neq e$.

Proof. Assume there are $x, y \in A$ such that $xy - yx = e$. Then for all $n \in \mathbb{N}$ we have

$$x^n y - y x^n = n x^{n-1} \neq 0$$

We prove this by induction over $n \in \mathbb{N}$. It holds for $n = 1$ by assumption. Assume it is valid for $n \in \mathbb{N}$. Then $x^n \neq 0$ and

$$\begin{aligned} x^{n+1} y - y x^{n+1} &= x^n (xy - yx) + (x^n y - y x^n) x \\ &= x^n e + n x^{n-1} x = x^n e + n x^n = (n + e) x^n \end{aligned}$$

From this identity it follows that

$$n \|x^{n-1}\| = \|x^n y - y x^n\| \leq 2 \|x^n\| \|y\| \leq 2 \|x^{n-1}\| \|x\| \|y\|$$

It follows that $n \leq 2||x||||y||$ for all $n \in \mathbb{N}$ which is impossible. $\square$

Corollary. The identity operator on a Hilbert space $\mathcal{H}$ cannot be expressed as a commutator of two bounded linear operators in $\mathcal{L}(\mathcal{H})$.

Remark. The above can be understood as a version of the uncertainty principle in one dimension. Let $H = L^2(\mathbb{R})$. Let $q \colon H \to H$ be $q(f)(x) := xf(x)$ with $D(q) = \{f \in L^2(\mathbb{R}) : x \mapsto xf(x) \in L^2(\mathbb{R})\}$, the coordinate operator and $p \colon H \to H, p(f)(x) = -if'(x)$ the momentum operator with $D(p) := \{f \in L^2(\mathbb{R}) : f \text{ absolutely continuous}, f' \in L^2(\mathbb{R})\}$. It follows that

$$pq - qp = -i \mathrm{id}_D \text{ on } D = D(q) \cap D(p)$$

According to the corollary $D = L^2(\mathbb{R})$ can never be the case.