planetmath.org

Math for the people, by the people.

proof of Bessel inequality

Canonical name	ProofOfBesselInequality
Date of creation	2013-03-22 12:46:41
Last modified on	2013-03-22 12:46:41
Owner	ariels (338)
Last modified by	ariels (338)
Numerical id	4
Author	ariels (338)
Entry type	Proof
Classification	msc 46C05

Let

$$r_n = x - \sum_{k=1}^{n} \langle x, e_k \rangle \cdot e_k.$$

Then for $j = 1, \ldots, n$,

$$\begin{aligned} \langle r_n, e_j \rangle &= \langle x, e_j \rangle - \textstyle\sum_{k=1}^{n} \langle \langle x, e_k \rangle \cdot e_k, e_j \rangle & (1) \\ &= \langle x, e_j \rangle - \langle x, e_j \rangle \langle e_j, e_j \rangle = 0 & (2) \end{aligned}$$

so $e_1, \ldots, e_n, r_n$ is an orthogonal series.

Computing norms, we see that

$$\|x\|^2 = \left\| r_n + \sum_{k=1}^{n} \langle x, e_k \rangle \cdot e_k \right\|^2 = \|r_n\|^2 + \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \geq \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 .$$

So the series

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$$

converges and is bounded by $\|x\|^2$, as required.