Gelfand-Mazur theorem

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Theorem - Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$ that is also a division algebra (i.e. every non-zero element is invertible). Then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.

Proof : Let e denote the unit of $\mathcal{A}$.

Let $x \in \mathcal{A}$ and $\sigma(x)$ be its spectrum. It is known that the `http://planetmath.org/SpectrumIs` is a non-empty set in $\mathbb{C}$.

Let $\lambda \in \sigma(x)$. Since $x - \lambda e$ is not invertible and $\mathcal{A}$ is a division algebra, we must have $x - \lambda e = 0$ and so $x = \lambda e$

Let $\phi : \mathbb{C} \longrightarrow \mathcal{A}$ be defined by $\phi(\lambda) = \lambda e$.

It is clear that ϕ is an injective algebra homomorphism.

By the above discussion, ϕ is also surjective.

It is isometric because $\|\lambda e\| = |\lambda| \|e\| = |\lambda|$

Therefore, $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$. $\square$