spectral mapping theorem

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Let $\mathcal{A}$ be a unital `http://planetmath.org/CAlgebra`C^*-algebra. Let x be a normal element in $\mathcal{A}$ and $\sigma(x)$ be its spectrum.

The continuous functional calculus provides a C^*-isomorphism

$$\begin{aligned} C(\sigma(x)) &\longrightarrow \mathcal{A}[x] \\ f &\mapsto f(x) \end{aligned}$$

between the C^*-algebra $C(\sigma(x))$ of complex valued continuous functions on $\sigma(x)$ and the C^*-subalgebra $\mathcal{A}[x] \subseteq \mathcal{A}$ generated by x and the identity of $\mathcal{A}$.

Spectral Mapping Theorem - Let $x \in \mathcal{A}$ be as above. Let $f \in C(\sigma(x))$. Then

$$\sigma(f(x)) = f(\sigma(x)).$$

Proof : Since $C(\sigma(x))$ and $\mathcal{A}[x]$ are isomorphic we must have

$$\sigma(f) = \sigma_{\mathcal{A}[x]}(f(x))$$

where $\sigma_{\mathcal{A}[x]}(f(x))$ denotes the spectrum of $f(x)$ relative to the subalgebra $\mathcal{A}[x]$.

By the spectral invariance theorem we have $\sigma_{\mathcal{A}[x]}(f(x)) = \sigma(f(x))$. Hence

$$\sigma(f) = \sigma(f(x))$$

Thus, we only have to prove that $f(\sigma(x)) = \sigma(f)$.

f is defined on $\sigma(x)$ so $f(\sigma(x))$ is precisely the image of f.

Let $\lambda \in \mathbb{C}$. The function $f - \lambda$ is invertible if and only if $f - \lambda$ has no zeros.

Equivalently, $f - \lambda$ is not invertible if and only if $f - \lambda$ has a zero, i.e. $f(\lambda_0) = \lambda$ for some λ_0.

The previous statement can be reformulated as: $\lambda \in \sigma(f)$ if and only if λ is in the image of f.

We conclude that $\sigma(f) = f(\sigma(x))$, and this proves the theorem. $\square$