resolvent function is analytic

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Theorem - Let $\mathcal{A}$ be a complex Banach algebra with identity element e. Let $x \in \mathcal{A}$ and $\sigma(x)$ denote its spectrum.

Then, the `http://planetmath.org/ResolventMatrix`resolvent function $R_x : \mathbb{C} - \sigma(x) \longrightarrow \mathcal{A}$ defined by $R_x(\lambda) = (x - \lambda e)^{-1}$ is `http://planetmath.org/BanachSpaceValuec`

Moreover, for each $\lambda_0 \in \mathbb{C} - \sigma(x)$ it has the power series

$$R_x(\lambda) = \sum_{n=0}^{\infty} R_x(\lambda_0)^{n+1}(\lambda - \lambda_0)^n \tag{1}$$

where the series converges absolutely for each λ in the open disk centered in λ_0 given by

$$|\lambda - \lambda_0| < \frac{1}{\|R_x(\lambda_0)\|} \tag{2}$$

Proof : Analyticity is defined for functions whose domain is open.

Thus, we start by proving that $\mathbb{C} - \sigma(x)$ is an open set in $\mathbb{C}$. To do so it is enough to prove that for every $\lambda_0 \in \mathbb{C} - \sigma(x)$ the open disk defined by (2) above is contained in $\mathbb{C} - \sigma(x)$.

Let $\lambda_0 \in \mathbb{C} - \sigma(x)$ and λ be such that

$$|\lambda - \lambda_0| < \frac{1}{\|R_x(\lambda_0)\|}$$

Then $\|(\lambda - \lambda_0)R_x(\lambda_0)\| < 1$ and by the `http://planetmath.org/NeumannSeriesInBanachAlgeb` series $e - (\lambda - \lambda_0)R_x(\lambda_0)$ is invertible.

Since $\lambda_0 \notin \sigma(x)$ it follows that $(x - \lambda_0 e)$ is invertible.

Hence, from the equality

$$x - \lambda e = x - \lambda_0 e - (\lambda - \lambda_0)e = (x - \lambda_0 e) \cdot [e - (\lambda - \lambda_0)R_x(\lambda_0)] \tag{3}$$

we conclude that $x - \lambda e$ is also invertible, i.e. $\lambda \in \mathbb{C} - \sigma(x)$. Thus $\mathbb{C} - \sigma(x)$ is open.

The above proof also pointed out that for every $\lambda_0 \in \mathbb{C}$, R_x is defined in the open disk of radius $\dfrac{1}{\|R_x(\lambda_0)\|}$ centered in λ_0.

We now prove the analyticity of the .

Taking inverses on the equality (3) above one obtains

$$R_x(\lambda) = (e - (\lambda - \lambda_0)R_x(\lambda_0))^{-1} \cdot R_x(\lambda_0)$$

Again, by the http://planetmath.org/NeumannSeriesInBanachAlgebrasNeumann series, one obtains

$$R_x(\lambda) = \left[\sum_{n=0}^{\infty} R_x(\lambda_0)^n (\lambda - \lambda_0)^n\right] \cdot R_x(\lambda_0) = \sum_{n=0}^{\infty} R_x(\lambda_0)^{n+1} (\lambda - \lambda_0)^n \quad \square$$