# Wiener algebra

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### 0.0.1 Definition and classification of the Wiener algebra

Let $W$ be the space of all complex functions on $[0, 2\pi[$ whose Fourier series converges absolutely, that is, all functions $f : [0, 2\pi[\longrightarrow \mathbb{C}$ whose Fourier series

$$f(t) = \sum_{n=-\infty}^{+\infty} \hat{f}(n)e^{int}$$

is such that $\sum_n |\hat{f}(n)| < \infty$ .

Under pointwise operations and the norm $\|f\| = \sum_n |\hat{f}(n)|$ , $W$ is a commutative Banach algebra of continuous functions, with an identity element. $W$ is usually called the **Wiener algebra**.

**Theorem -** $W$ is isometrically isomorphic to the Banach algebra $\ell^1(\mathbb{Z})$ with the convolution product. The isomorphism is given by:

$$(a_k)_{k\in\mathbb{Z}} \longleftrightarrow f(t) = \sum_{n=-\infty}^{+\infty} a_k e^{int}$$

### 0.0.2 Wiener's Theorem

**Theorem (Wiener) -** If $f \in W$ has no zeros then $1/f \in W$, that is, $1/f$ has an absolutely convergent Fourier series.

**Proof :** We want to prove that $f$ is invertible in $W$. As $W$ is commutative, that is the same as proving that $f$ does not belong to any maximal ideal of $W$. Therefore we only need to show that $f$ is not in the kernel of any multiplicative linear functional of $W$.

Let $\phi$ be a multiplicative linear functional in $W$. We have that

$$\phi(f) = \phi\Big(\sum_{n=-\infty}^{+\infty} \hat{f}(n)e^{int}\Big) = \sum_{n=-\infty}^{+\infty} \hat{f}(n)\phi(e^{int}) = \sum_{n=-\infty}^{+\infty} \hat{f}(n)\phi^n(e^{it})$$

Since $\|\phi\| = 1$ we have that

$$|\phi(e^{it})| \leq \|\phi\|\|e^{it}\| = \|e^{it}\| = 1$$

and

$$|\phi(e^{-it})| \leq \|\phi\|\|e^{-it}\| = \|e^{-it}\| = 1$$

Since $1 = |\phi(e^{it}e^{-it})| = |\phi(e^{it})||\phi(e^{-it})|$ we deduce that

$$|\phi(e^{it})| = 1$$

We can conclude that

$$\phi(e^{it}) = e^{it_0} \quad \text{for some } t_0 \in [0, 2\pi[$$

Therefore we obtain

$$\phi(f) = \sum_{n=-\infty}^{+\infty} \hat{f}(n) e^{int_0} = f(t_0)$$

which is non-zero by definition of $f$.

We conclude that $f$ does not belong to the kernel of any multiplicative linear functional $\phi$. $\square$

### 0.0.3 Remark

The Wiener algebra is a Banach *-algebra with the involution given by $f^*(t) := \overline{f(-t)}$, but it is not a `http://planetmath.org/CAlgebra`$C^*$-algebra under this involution.