closed ideals in C^*-algebras are self-adjoint

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Owner	asteroid (17536)
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Theorem - Every http://planetmath.org/ClosedSetclosed http://planetmath.org/Ideal
sided ideal $\mathcal{I}$ of a http://planetmath.org/CAlgebraC^*-algebra $\mathcal{A}$ is http://planetmath.org/Inv
adjoint, i.e.

$$\text{if } x \in \mathcal{I} \text{ then } x^* \in \mathcal{I}.$$

Proof : Let $\mathcal{I}^* := \{a^* : a \in \mathcal{I}\}$.

Since $\mathcal{I}$ is closed and the involution mapping is continuous, it follows that $\mathcal{I}^*$ is also closed.

We claim that $\mathcal{I}^*$ is also a of $\mathcal{A}$. To see this let $a, b \in \mathcal{I}$, $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Then

- $a^* + \lambda b^* = (a + \overline{\lambda} b)^* \in \mathcal{I}^*$ since $a + \overline{\lambda} b \in \mathcal{I}$
- $xa^* = (ax^*)^* \in \mathcal{I}^*$ since $ax^* \in \mathcal{I}$.
- $a^*x = (x^*a)^* \in \mathcal{I}^*$ since $x^*a \in \mathcal{I}$

Let $\mathcal{B} := \mathcal{I} \cap \mathcal{I}^*$.

$\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$ (it is a norm-closed, involution-closed, subalgebra of $\mathcal{A}$).

It is known that every C^*-algebra has an approximate identity consisting of positive elements with norm less than 1 (see this http://planetmath.org/CAlgebrasHaveAppro

Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $\mathcal{B}$ with the above :

1. each e_λ is positive (hence self-adjoint) and
2. $\|e_\lambda\| \leq 1 \quad \forall_{\lambda \in \Lambda}$

We now prove $\mathcal{I}$ is self-adjoint:

Let $a \in \mathcal{I}$. We have that

$$\begin{aligned}
\|a^* - a^*e_\lambda\|^2 &= \|(a^* - a^*e_\lambda)^* \cdot (a^* - a^*e_\lambda)\| \\
&= \|(a - e_\lambda a) \cdot (a^* - a^*e_\lambda)\| \\
&= \|(aa^* - aa^*e_\lambda) - e_\lambda(aa^* - aa^*e_\lambda)\| \\
&\leq \|aa^* - aa^*e_\lambda\| + \|e_\lambda\| \cdot \|aa^* - aa^*e_\lambda\| \\
&\leq \|aa^* - aa^*e_\lambda\| + \|aa^* - aa^*e_\lambda\| \\
&= 2\|aa^* - aa^*e_\lambda\|
\end{aligned}$$

Taking limits in both we obtain

$$\lim_\lambda \|a^* - a^*e_\lambda\|^2 \leq \lim_\lambda \ 2\|aa^* - aa^*e_\lambda\| = 0$$

since $aa^* \in \mathcal{I} \cap \mathcal{I}^* = \mathcal{B}$ and $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate identity for $\mathcal{B}$.

As $e_\lambda \in \mathcal{I}$ we see that $a^* e_\lambda \in \mathcal{I}$.

We conclude from the limit above that a^* is in the closure of $\mathcal{I}$. Therefore $a^* \in \mathcal{I}$.

Hence, $\mathcal{I}$ is self-adjoint. $\square$