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criteria for existence of antidervatives

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Let X be a normed space, Y a Banach space, $U \subset X$ a connected open set, $f\colon U \to L(X;Y)$ a continuous function, where $L(X;Y)$ is the space of continuous linear operators. In this article a path is a curve that has bounded variation. The following theorems give necessary and sufficient conditions for f to have an antiderivatives.

Theorem 1. *The following conditions are equivalent:*

1. *f has an antiderivative on U,*
2. *for any γ closed path in U $\int_\gamma f = 0$,*
3. *for any γ, δ paths in U that have the same starting and endpoints $\int_\gamma f = \int_\delta f$.*

The next theorem states criteria for the existence of local antiderivatives.

Theorem 2. *The following conditions are equivalent:*

1. *f has an antiderivative locally,*
2. *for γ, δ homotopic closed paths in U $\int_\gamma f = \int_\delta f$,*
3. *if γ is a triangular path such that its convex hull is in U, then $\int_\gamma f = 0$.*

With the stronger assumption that f is differenciable we can obtain a more easily applicable condition. We introduce the canonical isometric isomorphism

$$\pi_{1,1}\colon L(X;L(X;Y)) \to L_2(X;Y), \quad u \mapsto ((x_1,x_2) \mapsto u(x_1)(x_2))$$

where $L_2(X;Y)$ is the space of bilinear operators from X to Y. If F is an antiderivative of f, then $\pi_{1,1}(Df(x)) = D^2F(x)$ and by Clairaut's theorem the second derivative is symmetric. The following theorems assert that the reverse is also true.

Theorem 3. *If f is differentiable, then it has an antiderivative locally if and only if $\pi_{1,1}(Df(x))$ is symmetric for all $x \in U$.*

Combining these three theorems immediately gives the following.

Corollary 1. *If U is simply connected and f is differentiable, then it has an antiderivative on U if and only if $\pi_{1,1}(Df(x))$ is symmetric for all $x \in U$.*