example of Banach algebra which is not a C^*-algebra for any involution

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Defines	finite dimensional C^*-algebras are semi-simple

Consider the Banach algebra $\mathcal{A} = \left\{ \begin{bmatrix} \lambda I_n & A \\ 0 & \lambda I_n \end{bmatrix} : \ \lambda \in \mathbb{C}, \ \ A \in Mat_{n \times n}(\mathbb{C}) \right\}$ with the usual matrix operations and matrix norm, where I_n denotes the identity matrix in $Mat_{n\times n}(\mathbb{C})$.

Claim - $\mathcal{A}$ is not a `http://planetmath.org/CAlgebra`C^*-algebra for any involution $*$.

To prove the above claim we will give a proof of a more general fact about finite dimensional C^*-algebras, which clearly shows the for a Banach algebra to be a C^*-algebra for some involution.

Theorem - Every finite dimensional C^*-algebra is semi-simple, i.e. its Jacobson radical is $\{0\}$.

Proof : Let $\mathcal{B}$ be a finite dimensional C^*-algebra. Let a be an element of $J(\mathcal{B})$, the Jacobson radical of $\mathcal{B}$.

$J(\mathcal{B})$ is an ideal of $\mathcal{B}$, so $a^*a \in J(\mathcal{B})$.

The Jacobson radical of a finite dimensional algebra is nilpotent, therefore there exists $n \in \mathbb{N}$ such that $(a^*a)^n = 0$. Then, by the C^* condition and the fact that a^*a is selfadjoint,

$$0 = \|(a^*a)^{2^n}\| = \|a^*a\|^{2^n} = \|a\|^{2^{n+1}}$$

so $a = 0$ and $J(\mathcal{B})$ is trivial. $\square$

We now prove the above claim.

Proof of the claim: It is easy to see that $\left\{ \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} : \ A \in Mat_{n\times n}(\mathbb{C}) \right\}$ is the only maximal ideal of $\mathcal{A}$. Therefore the Jacobson radical of $\mathcal{A}$ is not trivial.

By the theorem we conclude that there is no involution $*$ that makes $\mathcal{A}$ into a C^*-algebra.$\square$

Remark - It could happen that there were no involutions in $\mathcal{A}$ and so the above claim would be uninteresting. That's not the case here. For example, one can see that $[a_{i,j}] \longrightarrow [\overline{a}_{2n+1-j,2n+1-i}]$ defines an involution in $\mathcal{A}$ (this is just the taken over the other diagonal of the matrix).