positive element

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Defines	positive operator
Defines	positive cone
Defines	square root of positive element

Let H be a complex Hilbert space. Let $T : H \longrightarrow H$ be a bounded operator in H.

Definition - T is said to be a **positive operator** if there exists a bounded operator $A : H \longrightarrow H$ such that

$$T = A^* A$$

where A^* denotes the adjoint of A.

Every positive operator T satisfies the very strong condition $\langle Tv, v \rangle \geq 0$ for every $v \in H$ since

$$\langle Tv, v \rangle = \langle A^* Av, v \rangle = \langle Av, Av \rangle = \|Av\|^2 \geq 0$$

The converse is also true, although it is not so to prove. Indeed,

Theorem - T is positive if and only if $\langle Tv, v \rangle \geq 0 \quad \forall_{v \in H}$

0.1 Generalization to C^*-algebras

The above notion can be generalized to elements in an arbitrary http://planetmath.org/CAlgebra algebra.

In what follows $\mathcal{A}$ denotes a C^*-algebra.

Definition - An element $x \in \mathcal{A}$ is said to be **positive** (and denoted $0 \leq x$) if

$$x = a^* a$$

for some element $a \in \mathcal{A}$.

Remark— Positive elements are clearly http://planetmath.org/InvolutaryRingself-adjoint.

0.2 Positive spectrum

It can be shown that the positive elements of $\mathcal{A}$ are precisely the normal elements of $\mathcal{A}$ with a positive spectrum. We it here as a theorem:

Theorem - Let $x \in \mathcal{A}$ and $\sigma(x)$ denote its spectrum. Then x is positive if and only if x is and $\sigma(x) \subset \mathbb{R}_0^+$.

0.3 Square roots

Positive elements admit a unique positive square root.

Theorem - Let x be a positive element in $\mathcal{A}$. There is a unique $b \in \mathcal{A}$ such that

- b is positive
- $x = b^2$.

The converse is also true (with assumptions): If x admits a square root then x is positive, since

$$x = b^2 = bb = b^* b$$

0.4 The positive cone

Another interesting fact about positive elements is that they form a http://planetmath.org/Cone5 convex cone in $\mathcal{A}$, usually called the **positive cone** of $\mathcal{A}$. That is stated in following theorem:

Theorem - Let a, b be positive elements in $\mathcal{A}$. Then

- $a + b$ is also positive
- λa is also positive for every $\lambda \geq 0$
- If both a and $-a$ are positive then $a = 0$.

0.5 Norm closure

Theorem - The set of positive elements in $\mathcal{A}$ is norm closed.