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state space is non-empty

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In this entry we prove the existence of states for every http://planetmath.org/CALgebraC^*-algebra.

Theorem - Let $\mathcal{A}$ be a C^*-algebra. For every http://planetmath.org/InvolutaryRingself-adjoint element $a \in \mathcal{A}$ there exists a state ψ on $\mathcal{A}$ such that $|\psi(a)| = \|a\|$.

Proof : We first consider the case where $\mathcal{A}$ is http://planetmath.org/Ringunital, with identity element e.

Let $\mathcal{B}$ be the C^*-subalgebra generated by a and e. Since a is self-adjoint, $\mathcal{B}$ is a comutative C^*-algebra with identity element.

Thus, by the Gelfand-Naimark theorem, $\mathcal{B}$ is isomorphic to $C(X)$, the space of continuous functions $X \longrightarrow \mathbb{C}$ for some compact set X.

Regarding a as an element of $C(X)$, a attains a maximum at a point $x_0 \in X$, since X is compact. Hence, $\|a\| = |a(x_0)|$.

The evaluation function at x_0,

$$ev_{x_0} : C(X) \longrightarrow \mathbb{C}$$
$$ev_{x_0}(f) := f(x_0)$$

is a multiplicative linear functional of $C(X)$. Hence, $\|ev_{x_0}\| = 1$ and also $|ev_{x_0}(a)| = |a(x_0)| = \|a\|$.

We can now extend ev_{x_0} to a linear functional ψ on $\mathcal{A}$ such that $\|\psi\| = \|ev_{x_0}\| = 1$, using the Hahn-Banach theorem.

Also, $\psi(e) = ev_{x_0}(e) = 1$ and so ψ is a norm one positive linear functional, i.e. ψ is a state on $\mathcal{A}$.

Of course, ψ is such that $|\psi(a)| = |ev_{x_0}(a)| = \|a\|$.

In case $\mathcal{A}$ does not have an identity element we can consider its minimal unitization $\widetilde{\mathcal{A}}$. By the preceding there is a state $\widetilde{\psi}$ on $\widetilde{\mathcal{A}}$ satisfying the required . Now, we just need to take the http://planetmath.org/RestrictionOfAFunctionrestriction of $\widetilde{\psi}$ to $\mathcal{A}$ and this restriction is a state in $\mathcal{A}$ satisfying the required . $\square$