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classification of Hilbert spaces

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Defines	every Hilbert space is isometrically isomorphic to a $\ell^2(X)$ space

Hilbert spaces can be classified, up to isometric isomorphism, according to their dimension. Recall that an isometric isomorphism of Hilbert spaces is an unitary transformation, therefore it preserves the vector space structure along with the inner product structure (hence, preserving also the topological structure). Recall also that the dimension of a Hilbert space is a well defined concept, i.e. all orthonormal bases of an Hilbert space share the same cardinality.

The classification theorem we describe here states that two Hilbert spaces H_1 and H_2 are isometrically isomorphic if and only if they have the same dimension, i.e. if and only if an orthonormal basis of H_1 has the same cardinality of an orthonormal basis of H_2.

This will be achieved by proving that every Hilbert space is isometrically isomorphic to an `http://planetmath.org/EllpXSpace`$\ell^2(X)$ space, where X has the cardinality of any orthonormal basis of the Hilbert space in consideration.

Theorem 1 - Suppose H is an Hilbert space and let I be a set that indexes one (and hence, any) orthonormal basis of H. Then, H is isometrically isomorphic to $\ell^2(I)$.

Theorem [**Classification of Hilbert spaces**] **-** Two Hilbert spaces H_1 and H_2 are isometrically isomorphic if and only if they have the same dimension.

Proof of Theorem 1: Let $\{e_i\}_{i\in I}$ an orthonormal basis indexed by the set I. Let $U : H \longrightarrow \ell^2(I)$ be defined by

$$Ux\,(i) := \langle x, e_i\rangle$$

We claim that U is an isometric isomorphism. It is clear that U is linear. Using Parseval's equality and the definition of norm in $\ell^2(I)$ it follows that

$$\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2 = \sum_{i\in I} |Ux\,(i)|^2 = \|Ux\|^2_{\ell^2(I)}$$

We conclude that U is isometric. It remains to see that it is surjective (since injectivity follows from the isometric condition).

Let $f \in \ell^2(I)$. By definition of the space $\ell^2(I)$ we must have $\sum_{i \in I} |f(i)|^2 < \infty$, and therefore, using the Riesz-Fischer theorem, the series $\sum_{i \in I} f(i) e_i$ converges to an element $x_0 \in H$. We now see that

$$Ux_0\,(j) = \langle x_0, e_j \rangle = \sum_{i \in I} f(i) \langle e_i, e_j \rangle = f(j)$$

or in other , $Ux_0 = f$. Hence, U is surjective. $\square$

Proof of the classification theorem :

- ($\Longrightarrow$) Of course, if the Hilbert spaces H_1 and H_2 are isometrically isomorphic, with isometric isomorphism U, then if $\{e_i\}_{i \in I}$ is an orthonormal basis for H_1 than $\{Ue_i\}_{i \in I}$ is an orthonormal basis for H_2. Hence, H_1 and H_2 have the same dimension.

- ($\Longleftarrow$) If the Hilbert spaces H_1 and H_2 have the same dimension, then we can index any orthonormal basis of H_1 and any orthonormal basis of H_2 by the same set I. Using Theorem 1 we see that H_1 and H_2 are both isometrically isomorphic to $\ell^2(I)$. Hence H_1 and H_2 are isometrically isomorphic. $\square$