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ordering of self-adjoints

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Let $\mathcal{A}$ be a http://planetmath.org/CAlgebraC^*-algebra. Let $\mathcal{A}^+$ denote the set of positive elements of $\mathcal{A}$ and $\mathcal{A}_{sa}$ denote the set of self-adjoint elements of $\mathcal{A}$.

Since $\mathcal{A}^+$ is a http://planetmath.org/Cone5proper convex cone (see this http://planetmath.org/PositiveElement3entry), we can define a partial order $\leq$ on the set $\mathcal{A}_{sa}$, by setting

$$a \leq b \text{ if and only if } b - a \in \mathcal{A}^+, \text{ i.e. } b - a \text{ is positive.}$$

Theorem - The relation $\leq$ is a partial order relation on $\mathcal{A}_{sa}$. Moreover, $\leq$ turns $\mathcal{A}_{sa}$ into an ordered topological vector space.

0.0.1 Properties:

- $a \leq b \;\Rightarrow\; c^* a\, c \leq c^* b\, c$ for every $c \in \mathcal{A}$.
- If a and b are invertible and $a \leq b$, then $b^{-1} \leq a^{-1}$.
- If $\mathcal{A}$ has an identity element e, then $-\|a\| e \leq a \leq \|a\| e$ for every $a \in \mathcal{A}_{sa}$.
- $-b \leq a \leq b \;\Rightarrow\; \|a\| \leq \|b\|$.

0.0.2 Remark:

The proof that $\leq$ is partial order makes no use of the self-adjointness . In fact, $\mathcal{A}$ itself is an ordered topological vector space under the relation $\leq$.

However, it turns out that this ordering relation provides its most usefulness when restricted to self-adjoint elements. For example, some of the above would not hold if we did not restrict to $\mathcal{A}_{sa}$.