proof of Riesz representation theorem

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Existence - If $f = 0$ we can just take $u = 0$ and thereby have $f(x) = 0 = \langle x, 0\rangle$ for all $x \in \mathcal{H}$.

Suppose now $f \neq 0$, i.e. $Ker f \neq \mathcal{H}$.

Recall that, since f is `http://planetmath.org/ContinuousMap`continuous, $Ker f$ is a closed subspace of $\mathcal{H}$ (continuity of f implies that $f^{-1}(0)$ is closed in $\mathcal{H}$). It then follows from the orthogonal decomposition theorem that

$$\mathcal{H} = Ker f \oplus (Ker f)^{\perp}$$

and as $Ker f \neq \mathcal{H}$ we can find $z \in (Ker f)^{\perp}$ such that $\|z\| = 1$.

It follows easily from the linearity of f that for every $x \in \mathcal{H}$ we have

$$f(x)z - f(z)x \in Ker f$$

and since $z \in (Ker f)^{\perp}$

$$\begin{aligned} 0 &= \langle f(x)z - f(z)x, z\rangle \\ &= f(x)\langle z, z\rangle - f(z)\langle x, z\rangle \\ &= f(x)\|z\|^2 - \langle x, \overline{f(z)}z\rangle \\ &= f(x) - \langle x, \overline{f(z)}z\rangle \end{aligned}$$

which implies

$$f(x) = \langle x, \overline{f(z)}z\rangle \ .$$

The theorem then follows by taking $u = \overline{f(z)}z$.

Uniqueness - Suppose there were $u_1, u_2 \in \mathcal{H}$ such that for every $x \in \mathcal{H}$

$$f(x) = \langle x, u_1\rangle = \langle x, u_2\rangle.$$

Then $\langle x, u_1 - u_2\rangle = 0$ for every $x \in \mathcal{H}$. Taking $x = u_1 - u_2$ we obtain $\|u_1 - u_2\|^2 = 0$, which implies $u_1 = u_2$. $\square$