proof of necessary and sufficient conditions for a normed vector space to be a Banach space

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Owner	willny (13192)
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Author	willny (13192)
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We prove here that in order for a normed space, say X, with the norm, say $\|\cdot\|$ to be Banach, it is necessary and sufficient that convergence of every absolutely convergent series in X implies convergence of the series in X.

Suppose that X is Banach. Let a sequence (x_n) be in X such that the series

$$\sum_n \|x_n\|$$

converges. Then for all $\epsilon > 0$ there exists N such that for all $m > n > N$ we have

$$\left\| \sum_{n+1}^{m} x_n \right\| \leq \sum_{n+1}^{m} \|x_n\| < \epsilon$$

Hence

$$s_k = \sum_{n=1}^{k} x_n$$

is a Cauchy sequence in X. Since X is Banach, s_k converges in X.

Conversely, suppose that absolute convergence implies convergence. Let (x_n) be a Cauchy sequence in X. Then for all $m \geq 1$ there exists N_m such that for all $k, k' \geq N_m$ we have $\|x_k - x_{k'}\| < 1/m^2$. We'll conveniently choose N_m so that N_m is an increasing sequence in m. Then in particular, $\|x_{N_m} - x_{N_{m+1}}\| < 1/m^2$. Hence we have,

$$\sum_{m=1}^{M} \|x_{N_m} - x_{N_{m+1}}\| < \sum_{m=1}^{M} \frac{1}{m^2}$$

The sum on the right converges, so must the sum on the left. Since absolute convergence implies convergence, we must have

$$\sum_{m=1}^{M} (x_{N_m} - x_{N_{m+1}})$$

converges as M tends to infinity. So there is an s in X which is the limit of the sum above. As a telescoping series, however, the sum above converges to $\lim_{M\to\infty}(x_{N_1} - x_{N_m}) = s$. Since s and x_{N_1} are both in X, so is the limit of x_{N_m}, which is a subsequence of the Cauchy sequence (x_n). Hence (x_n)

converges in X. So X is Banach.

This completes the proof.