absolute value in a vector lattice

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Let V be a vector lattice over $\mathbb{R}$, and V^+ be its positive cone. We define three functions from V to V^+ as follows. For any $x \in V$,

- $x^+ := x \vee 0$,
- $x^- := (-x) \vee 0$,
- $|x| := (-x) \vee x$.

It is easy to see that these functions are well-defined. Below are some properties of the three functions:

1. $x^+ = (-x)^-$ and $x^- = (-x)^+$.
2. $x = x^+ - x^-$, since $x^+ - x^- = (x \vee 0) - (-x) \vee 0 = (x \vee 0) + (x \wedge 0) = x + 0 = x$.
3. $|x| = x^+ + x^-$, since $x^+ + x^- = x + 2x^- = x + (-2x) \vee 0 = (x - 2x) \vee (x + 0) = |x|$.
4. If $0 \leq x$, then $x^+ = x$, $x^- = 0$ and $|x| = x$. Also, $x \leq 0$ implies $x^+ = 0$, $x^- = -x$ and $|x| = -x$.
5. $|x| = 0$ iff $x = 0$. The "only if" part is obvious. For the "if" part, if $|x| = 0$, then $(-x) \vee x = 0$, so $x \leq 0$ and $-x \leq 0$. But then $0 \leq x$, so $x = 0$.
6. $|rx| = |r||x|$ for any $r \in \mathbb{R}$. If $0 \leq r$, then $|rx| = (-rx) \vee (rx) = r\big((-x) \vee x\big) = r|x| = |r||x|$. On the other hand, if $r \leq 0$, then $|rx| = (-rx) \vee (rx) = (-r)\big(x \vee (-x)\big) = -r|x| = |r||x|$.
7. $|x| + |y| = |x + y| \vee |x - y|$, since
$$LHS = (-x) \vee x + (-y) \vee y = (-x-y) \vee (-x+y) \vee (x-y) \vee (x+y) = RHS.$$
8. (triangle inequality). $|x+y| \leq |x| + |y|$, since $|x+y| \leq |x+y| \vee |x-y| = |x| + |y|$.

Properties 5, 6, and 8 satisfy the axioms of an absolute value, and therefore $|x|$ is called the *absolute value* of x. However, it is not the "norm" of a vector in the traditional sense, since it is not a real-valued function.