Hahn-Banach theorem

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The Hahn-Banach theorem is a foundational result in functional analysis. Roughly speaking, it asserts the existence of a great variety of bounded (and hence continuous) linear functionals on an normed vector space, even if that space happens to be infinite-dimensional. We first consider an abstract version of this theorem, and then give the more classical result as a corollary.

Let V be a real, or a complex vector space, with K denoting the corresponding field of scalars, and let

$$\mathrm{p} : V \to \mathbb{R}^+$$

be a seminorm on V.

Theorem 1 *Let $f : U \to K$ be a linear functional defined on a subspace $U \subset V$. If the restricted functional satisfies*

$$|f(\mathbf{u})| \leq \mathrm{p}(\mathbf{u}), \quad \mathbf{u} \in U,$$

then it can be extended to all of V without violating the above property. To be more precise, there exists a linear functional $F : V \to K$ such that

$$\begin{aligned} F(\mathbf{u}) &= f(\mathbf{u}), \quad \mathbf{u} \in U \\ |F(\mathbf{u})| &\leq \mathrm{p}(\mathbf{u}), \quad \mathbf{u} \in V. \end{aligned}$$

Definition 2 *We say that a linear functional $f : V \to K$ is* bounded *if there exists a bound $B \in \mathbb{R}^+$ such that*

$$|f(\mathbf{u})| \leq B\,\mathrm{p}(\mathbf{u}), \quad \mathbf{u} \in V. \tag{1}$$

If f is a bounded linear functional, we define $\|f\|$, the norm of f, according to

$$\|f\| = \sup\{|f(\mathbf{u})| : \mathrm{p}(\mathbf{u}) = 1\}.$$

One can show that $\|f\|$ is the infimum of all the possible B that satisfy (**??**)

Theorem 3 (Hahn-Banach) *Let $f : U \to K$ be a bounded linear functional defined on a subspace $U \subset V$. Let $\|f\|_U$ denote the norm of f relative to the restricted seminorm on U. Then there exists a bounded extension $F : V \to K$ with the same norm, i.e.*

$$\|F\|_V = \|f\|_U.$$