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# special elements in a $C^*$-algebra and their spectral properties

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| Defines | spectrum of self-adjoint elements |
| Defines | spectrum of unitary elements |
| Defines | spectrum of projections |
| Defines | spectrum of positive elements |

**Definition -** Suppose $\mathcal{A}$ is a `http://planetmath.org/CAlgebra`$C^*$-algebra. An element $x \in \mathcal{A}$ is said to be:

- **normal** if $x^*x = xx^*$
- **self-adjoint** if $x^* = x$
- **unitary** if $\mathcal{A}$ has an identity element $e$ and $x^*x = xx^* = e$
- **positive** if $x = y^*y$ for some element $y \in \mathcal{A}$
- **projection** if $x^* = x$ and $x^2 = x$
- **partial isometry** if $x^*x$ and $xx^*$ are both projections

#### 0.0.1 Properties of the special elements in terms of their spectrum

In the following $\sigma(x)$ denotes the spectrum of an element $x$ and $R_\sigma(x)$ its spectral radius.

**Theorem 1 -** Suppose $\mathcal{A}$ is a $C^*$-algebra and $x \in \mathcal{A}$. If $x$ is normal then $\|x\| = R_\sigma(x)$

**Theorem 2 -** Suppose $\mathcal{A}$ is a $C^*$-algebra and $x \in \mathcal{A}$.

- If $x$ is self-adjoint, then $\sigma(x) \subset \mathbb{R}$.
- If $x$ is unitary, then $\sigma(x) \subset \partial D$, where $D \subset \mathbb{C}$ is the unit disk.
- If $x$ is positive, then $\sigma(x) \subset \mathbb{R}^+$
- If $x$ is a projection, then $\sigma(x) \subset \{0, 1\}$

**Theorem 3 -** Suppose $\mathcal{A}$ is a commutative $C^*$-algebra and $x \in \mathcal{A}$. Then

- $x$ is self-adjoint if and only if $\sigma(x) \subset \mathbb{R}$.
- $x$ is unitary if and only if $\sigma(x) \subset \partial D$, where $D \subset \mathbb{C}$ is the unit disk.
- $x$ is positive if and only if $\sigma(x) \subset \mathbb{R}^+$
- $x$ is a projection if and only if $\sigma(x) \subset \{0, 1\}$

**Theorem 4 -** Suppose $\mathcal{A}$ is a $C^*$-algebra and $x$ is normal in $\mathcal{A}$. Then

- $x$ is self-adjoint if and only if $\sigma(x) \subset \mathbb{R}$.
- $x$ is unitary if and only if $\sigma(x) \subset \partial D$, where $D \subset \mathbb{C}$ is the unit disk.
- $x$ is positive if and only if $\sigma(x) \subset \mathbb{R}^+$
- $x$ is a projection if and only if $\sigma(x) \subset \{0, 1\}$