multiplicative linear functional

Canonical name	MultiplicativeLinearFunctional
Date of creation	2013-03-22 17:22:25
Last modified on	2013-03-22 17:22:25
Owner	asteroid (17536)
Last modified by	asteroid (17536)
Numerical id	29
Author	asteroid (17536)
Entry type	Definition
Classification	msc 46H05
Synonym	character (of an algebra)
Related topic	LinearFunctional
Related topic	GelfandTransform
Related topic	BanachAlgebra
Defines	character
Defines	maximal ideal space
Defines	character space

1 Definition

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$.

A **multiplicative linear functional** is an nontrivial algebra homomorphism $\phi : \mathcal{A} \longrightarrow \mathbb{C}$, i.e. ϕ is a non-zero linear functional such that $\phi(x \cdot y) = \phi(x) \cdot \phi(y), \quad \forall x, y \in \mathcal{A}$.

Multiplicative linear functionals are also called **characters** of $\mathcal{A}$.

2 Properties

- If ϕ is a multiplicative linear functional in a Banach algebra $\mathcal{A}$ over $\mathbb{C}$ then ϕ is continuous. Moreover, if $\mathcal{A}$ has an identity element then $\|\phi\| = 1$.

- Suppose $\mathcal{A}$ is a Banach algebra over $\mathbb{C}$. The set of multiplicative linear functionals in $\mathcal{A}$ is a locally compact Hausdorff space in the weak-* topology. Moreover, this set is compact if $\mathcal{A}$ has an identity element.

- Suppose $\mathcal{A}$ is a commutative Banach algebra over $\mathbb{C}$ with an identity element. There is a bijective correspondence between the set of maximal ideals in $\mathcal{A}$ and the set of multiplicative linear functionals in $\mathcal{A}$. This correspondence is given by

$$\phi \longmapsto Ker\ \phi$$

- Suppose $\mathcal{A}$ is a commutative `http://planetmath.org/CAlgebra`C^*-algebra. Multiplicative linear functionals in $\mathcal{A}$ are exactly the `http://planetmath.org/Banac` representations of $\mathcal{A}$.

3 Character space of a Banach algebra

As stated above, the set of all multiplicative linear functionals in a Banach algebra $\mathcal{A}$ is a locally compact Hausdorff space with the weak-* topology. It becomes a compact set if $\mathcal{A}$ has an identity element.

There are several designations for this space, such as: the of $\mathcal{A}$, the **maximal ideal space**, the **character space**.

4 Examples

- Let X be a topological space and $C(X)$ the algebra of continuous functions $X \longrightarrow \mathbb{C}$. Every point evaluation is a multiplicative linear functional of $C(X)$. In other words, for every point $x \in X$, the function

$$\begin{aligned} ev_x : C(X) &\longrightarrow \mathbb{C} \\ ev_x(f) &= f(x) \end{aligned}$$

  that gives the evaluation in x, is a multiplicative linear functional of $C(X)$.