# finite rank approximation on separable Hilbert spaces

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**Theorem** Let $\mathcal{H}$ be a separable Hilbert space and let $T \in L(\mathcal{H})$. Then $T$ is a compact operator iff there is a sequence $\{F_n\}$ of finite rank operators with $\|T - F_n\| \to 0$.

*Proof.* ($\Rightarrow$): Assume $T$ is compact on $\mathcal{H}$ and $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$. Define:

$$P_n f = \sum_{k=0}^{n} \langle f, e_k \rangle e_k$$

It is clear that the $P_n$ have finite rank and that we have $\|P_n f\| \leq \|f\|$ for all $n \in \mathbb{N}$, $f \in \mathcal{H}$.

Let $\mathcal{B}$ be the unit ball in $\mathcal{H}$. We have that $P_n \to I$ pointwise. Since the $P_n$ are contractive they are equicontinuous, hence $P_n$ converges uniformly to $I$ on compact sets, and in particular on $\overline{T(\mathcal{B})}$, which is compact by assumption. Therefore $P_n T \to T$ uniformly on $\mathcal{B}$, hence $\|P_n T - T\| \to 0$. Since $P_n T$ is bounded and of finite rank the first direction follows.

($\Leftarrow$): Now let $\{F_n\}$ be a sequence of bounded operators of finite rank with $\|T - F_n\| \to 0$. We have to show that $T(\mathcal{B})$ is relatively compact in $\mathcal{H}$. This is equivalent to $T(\mathcal{B})$ being totally bounded in $\mathcal{H}$. So we are left to show that for all $\epsilon > 0$ there is an $\epsilon$-net $x_1, \cdots, x_n \in \mathcal{H}$ so that:

$$T(\mathcal{B}) \subseteq \bigcup_{k=1}^{n} B_\epsilon(x_k)$$

So choose $\epsilon > 0$ and $n \in \mathbb{N}$ fixed so that:

$$\|F_n - T\| < \frac{\epsilon}{2}$$

Choose $x_1, \cdots, x_m \in \mathcal{H}$ with:

$$F_n(\mathcal{B}) \subseteq \bigcup_{k=1}^{m} B_{\frac{\epsilon}{2}}(x_k)$$

Hence (by the triangle inequality):

$$T(\mathcal{B}) \subseteq \bigcup_{k=1}^{m} B_\epsilon(x_k)$$

and we are done. $\square$