vector norm

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Related topic	OperatorNorm
Related topic	BoundedOperator
Related topic	SemiNorm
Related topic	BanachSpace
Related topic	HilbertSpace
Related topic	UnitVector
Defines	normed vector space
Defines	Euclidean norm

A vector norm on the real vector space V is a function $f : V \to \mathbb{R}$ that satisfies the following properties:

$$\begin{aligned} f(x) = 0 \iff x = 0 & \\ f(x) \geq 0 \quad & x \in V \\ f(x+y) \leq f(x) + f(y) \quad & x, y \in V \\ f(\alpha x) = |\alpha| f(x) \quad & \alpha \in \mathbb{R}, x \in V \end{aligned}$$

Such a function is denoted as $||\, x \,||$. Particular norms are distinguished by subscripts, such as $||\, x \,||_V$, when referring to a norm in the space V. A *unit vector* with respect to the norm $||\cdot||$ is a vector x satisfying $||\, x \,|| = 1$.

A vector norm on a complex vector space is defined similarly.

A common (and useful) example of a real norm is the Euclidean norm given by $||x|| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ defined on $V = \mathbb{R}^n$. Note, however, that there exists vector spaces which are metric, but upon which it is not possible to define a norm. If it possible, the space is called a *normed vector space*. Given a metric on the vector space, a necessary and sufficient condition for this space to be a normed space, is

$$\begin{aligned} d(x+a, y+a) = & \quad d(x,y) \quad \forall x, y, a \in V \\ d(\alpha x, \alpha y) = & \quad |\alpha| d(x,y) \quad \forall x, y \in V, \alpha \in \mathbb{R} \end{aligned}$$

But given a norm, a metric can always be defined by the equation $d(x,y) = ||x - y||$. Hence every normed space is a metric space.