closure of a vector subspace is a vector subspace

Canonical name	ClosureOfAVectorSubspaceIsAVectorSubspace
Date of creation	2013-03-22 15:00:19
Last modified on	2013-03-22 15:00:19
Owner	loner (106)
Last modified by	loner (106)
Numerical id	8
Author	loner (106)
Entry type	Theorem
Classification	msc 46B99
Classification	msc 15A03
Classification	msc 54A05
Related topic	ClosureOfAVectorSubspaceIsAVectorSubspace
Related topic	ClosureOfSetsClosedUnderAFinitaryOperation

Theorem 1. *In a topological vector space the* `http://planetmath.org/Closure`*closure of a vector subspace is a vector subspace.*

Proof. Let X be the topological vector space over $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, let V be a vector subspace in X, and let $\overline{V}$ be the closure of V. To prove that $\overline{V}$ is a vector subspace of X, it suffices to prove that $\overline{V}$ is non-empty, and

$$\lambda x + \mu y \in \overline{V}$$

whenever $\lambda, \mu \in \mathbb{F}$ and $x, y \in \overline{V}$.

First, as $V \subseteq \overline{V}$, $\overline{V}$ contains the zero vector, and $\overline{V}$ is non-empty. Suppose λ, μ, x, y are as above. Then there are nets $(x_i)_{i\in I}$, $(y_j)_{j\in J}$ in V converging to x, y, respectively. In a topological vector space, addition and multiplication are continuous operations. It follows that there is a net $(\lambda x_k + \mu y_k)_{k\in K}$ that converges to $\lambda x + \mu y$.

We have proven that $\lambda x + \mu y \in \overline{V}$, so $\overline{V}$ is a vector subspace. $\square$