proof of L^p-norm is dual to L^q

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Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space and p, q be Hölder conjugates. Then, we show that a measurable function $f: X \to \mathbb{R}$ has L^p-norm

$$\|f\|_p = \sup\left\{\|fg\|_1 : g \in L^q, \|g\|_q = 1\right\}. \tag{1}$$

Furthermore, if either $p < \infty$ and $\|f\|_p < \infty$ or $p = 1$ then μ is not required to be σ-finite.

If $\|f\|_p = 0$ then f is zero almost everywhere, and both sides of equality (**??**) are zero. So, we only need to consider the case where $\|f\|_p > 0$.

Let K be the right hand side of equality (**??**). For any $g \in L^q$ with $\|g\|_q = 1$, the Hölder inequality gives $\|fg\|_1 \le \|f\|_p$, so $K \le \|f\|_p$. Only the reverse inequality remains to be shown.

If $1 < p < \infty$ and $\|f\|_p < \infty$ then, setting $g = |f|^{p-1}$ gives

$$\|g\|_q = \left(\int |f|^p \, d\mu\right)^{\frac{1}{q}} = \|f\|_p^{p-1} < \infty.$$

Therefore, $g \in L^q$ and,

$$K \ge \|f(g/\|g\|_q)\|_1 = \||f|^p\|_1/\|g\|_q = \|f\|_p^p/\|f\|_p^{p-1} = \|f\|_p.$$

On the other hand, if $p = 1$ so that $q = \infty$, then setting $g = 1$ gives $\|g\|_q = 1$ and

$$K \ge \|fg\|_1 = \|f\|_1.$$

So, we have shown that $K = \|f\|_p$ when $p < \infty$ and $\|f\|_p < \infty$, and when $p = 1$. From now on, it is assumed that the measure is σ-finite. Then there is a sequence $A_n \in \mathfrak{M}$ increasing to the whole of X and such that $\mu(A_n) < \infty$.

Now consider the case where $1 < p < \infty$ and $\|f\|_p = \infty$. Let f_n be the sequence of functions

$$f_n = 1_{A_n} 1_{|f| \le n} f$$

then, $|f_n| \le |f|$ and monotone convergence gives $\|f_n\|_p \to \|f\|_p = \infty$. Therefore,

$$K \ge \sup\left\{\|f_n g\|_1 : g \in L^q, \|g\|_q = 1\right\} = \|f_n\|_p.$$

and letting n go to infinity gives $K = \infty$.

We finally consider $p = \infty$. Then, for any $L < \|f\|_p$ there exists a set $A \in \mathfrak{M}$ with $\mu(A) > 0$ such that $|f| \ge L$ on A. Also, monotone convergence gives $\mu(A \cap A_n) \to \mu(A)$ and, therefore, $\mu(A \cap A_n) > 0$ eventually. Replacing

A by $A \cap A_n$ if necessary, we may suppose that $\mu(A) < \infty$. So, setting $g = 1_A/\mu(A)$ gives $\|g\|_1 = 1$ and,

$$K \geq \|fg\|_1 = \int_A |f| \, d\mu/\mu(A) \geq L.$$

Letting L increase to $\|f\|_p$ gives $K \geq \|f\|_p$ as required.