polynomial functional calculus

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Let $\mathcal{A}$ be an unital associative algebra over $\mathbb{C}$ with identity element e and let $a \in \mathcal{A}$.

The **polynomial functional calculus** is the most basic form of a functional calculus. It allows the expression

$$p(a)$$

to make sense as an element of $\mathcal{A}$, for any polynomial $p : \mathbb{C} \longrightarrow \mathbb{C}$.

This is achieved in the following natural way: for any polynomial $p(\lambda) := \sum c_n \lambda^n$ we the element $p(a) := \sum c_n a^n \in \mathcal{A}$.

1 Definition

Recall that the set of polynomial functions in $\mathbb{C}$, denoted by $\mathbb{C}[\lambda]$, is an associative algebra over $\mathbb{C}$ under pointwise operations and is generated by the constant polynomial 1 and the variable λ (corresponding to the identity function in $\mathbb{C}$).

Moreover, any homomorphism from the algebra $\mathbb{C}[\lambda]$ is perfectly determined by the values of 1 and λ.

Definition - *Consider the algebra homomorphism $\pi : \mathbb{C}[\lambda] \longrightarrow \mathcal{A}$ such that $\pi(1) = e$ and $\pi(\lambda) = a$. This homomorphism is denoted by*

$$p \longmapsto p(a)$$

and it is called the **polynomial functional calculus** *for a.*

It is clear that for any polynomial $p(\lambda) := \sum c_n \lambda^n$ we have $p(a) = \sum c_n a^n$.

2 Spectral Properties

We will denote by $\sigma(x)$ the `http://planetmath.org/Spectrum`spectrum of an element $x \in \mathcal{A}$.

Theorem - (polynomial spectral mapping theorem) - *Let $\mathcal{A}$ be an unital associative algebra over $\mathbb{C}$ and a an element in $\mathcal{A}$. For any polynomial p we have that*

$$\sigma(p(a)) = p(\sigma(a))$$

: Let us first prove that $\sigma(p(a)) \subseteq p(\sigma(a))$. Suppose $\widetilde{\lambda} \in \sigma(p(a))$, which means that $p(a)-\widetilde{\lambda}e$ is not invertible. Now consider the polynomial in $\mathbb{C}$ given by $q := p-\widetilde{\lambda}$. It is clear that $q(a) = p(a)-\widetilde{\lambda}e$, and therefore $q(a)$ is not invertible. Since $\mathbb{C}$ is `http://planetmath.org/FundamentalTheoremOfAlgebra`algebraically closed, we have that

$$q(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k}$$

for some $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ and $n_1, \dots, n_k \in \mathbb{N}$. Thus, we can also write a similar product for $q(a)$ as

$$q(a) = (a - \lambda_1 e)^{n_1} \cdots (a - \lambda_k e)^{n_k}$$

Now, since $q(a)$ is not invertible we must have that at least one of the factors $(a - \lambda_i e)$ is not invertible, which means that for that particular λ_i we have $\lambda_i \in \sigma(a)$. But we also have that $q(\lambda_i) = 0$, i.e. $p(\lambda_i) = \widetilde{\lambda}$, and hence $\widetilde{\lambda} \in p(\sigma(a))$.

We now prove the inclusion $\sigma(p(a)) \supseteq p(\sigma(a))$. Suppose $\widetilde{\lambda} \in p(\sigma(a))$, which means that $\widetilde{\lambda} = p(\lambda_0)$ for some $\lambda_0 \in \sigma(a)$. The polynomial $p - \widetilde{\lambda}$ has a zero at λ_0, hence there is a polynomial d such that

$$p(\lambda) - \widetilde{\lambda} = d(\lambda)(\lambda - \lambda_0)\,, \qquad \lambda \in \mathbb{C}$$

Thus, we can also write a similar product for $q(a)$ as

$$p(a) - \widetilde{\lambda}e = d(a)(a - \lambda_0 e)$$

If $p(a)-\widetilde{\lambda}e$ was invertible, then we would see that $a-\lambda_0 e$ had a `http://planetmath.org/InversesI` and a `http://planetmath.org/InversesInRings`right inverse, thus being invertible. But we know that $\lambda_0 \in \sigma(a)$, hence we conclude that $p(a) - \widetilde{\lambda}e$ cannot be invertible, i.e. $\widetilde{\lambda} \in \sigma(p(a))$. $\square$