scaling of the open ball in a normed vector space

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Let V be a vector space over a field F (real or complex), and let $\|\cdot\|$ be a norm on V. Further, for $r > 0$, $v \in V$, let

$$B_r(v) = \{w \in V : \|w - v\| < r\}.$$

Then for any non-zero $\lambda \in F$, we have

$$\lambda B_r(v) = B_{|\lambda| r}(\lambda v).$$

The claim is clear for $\lambda = 0$, so we can assume that $\lambda \neq 0$. Then

$$\begin{aligned}
\lambda B_r(v) &= \{z \in V : \|w - v\| < r \text{ and } z = \lambda w\} \\
&= \{z \in V : \|\frac{z}{\lambda} - v\| < r\} \\
&= \{z \in V : \|z - \lambda v\| < |\lambda| r\} \\
&= B_{|\lambda| r}(\lambda v).
\end{aligned}$$