proof of topologically irreducible representations are algebraically irreducible for C^*-algebras

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Denote by $\mathcal{H}$ an arbitrary Hilbert space. To fix notation let $\mathcal{U} \subset \mathcal{L}(\mathcal{H})$ be a C^* subalgebra of $\mathcal{L}(\mathcal{H})$. We then define the *commutator* of $\mathcal{U}$ by

$$\mathcal{U}' := \{T \in \mathcal{L}(\mathcal{H}) : TU = UT \ \forall U \in \mathcal{U}\}$$

Note that $\mathcal{U}'$ is closed with regard to the weak topology (see `http://planetmath.org/Commutant` entry). So $\mathcal{U}'$ is always a von Neumann algebra.

As an immediate consequence of Schur's Lemma for group representations on a Hilbert space we obtain the following result.

Lemma. Let $\mathcal{U}$ be a $*$-algebra and let π be a $*$-representation of $\mathcal{U}$ on the Hilbert space $\mathcal{H}$. Then π is topologically irreducible iff $\pi(\mathcal{U})' = \mathbb{C}I$.

We can now prove the result.

Theorem. Let $\mathcal{U}$ be a C^* algebra. Assume the $*$-representation π of $\mathcal{U}$ on the Hilbert space $\mathcal{H}$ is topologically irreducible. Then π is algebraically irreducible.

Proof. By the Lemma it follows that $\pi(\mathcal{U})' = \mathbb{C}I$. Hence $\pi(\mathcal{U})'' = \mathcal{L}(\mathcal{H})$. By the `http://planetmath.org/VonNeumannDoubleCommutantTheorem`double commutant theorem every operator in $\mathcal{L}(\mathcal{H})_1$ (the unit ball in the set of bounded operators $\mathcal{L}(\mathcal{H})$) belongs to the strong operator closure of $\pi(\mathcal{U})_1$ (the unit ball in $\pi(\mathcal{U})$).

To show the algebraical irreducibility of $\pi(\mathcal{U})$ it is enough to find for two given vectors $x, y \in \mathcal{H}, x \neq 0$ an element $T \in \mathcal{U}$ such that $\pi(T)x = y$ holds. Indeed, it is enough to consider the case $\|x\| = \|y\| = 1$.

Now construct the rank one approximation $\tilde{T}_1 := y \otimes x$ ($\Leftrightarrow \tilde{T}_1 z = \langle x, z\rangle y, z \in \mathcal{H} \Rightarrow \tilde{T}_1 x = \|x\| y = y$) with a corresponding $T_1 \in \mathcal{U}, \pi(T_1) \in \pi(\mathcal{U})_1$, so that $\|y - \pi(T_1)x\| = \|\tilde{T}_1 x - \pi(T_1)x\| \leq \frac{1}{2}$.

Approximate further $\tilde{T}_2 := (y - \pi(T_1)x) \otimes x \in \frac{1}{2}\mathcal{L}(\mathcal{H})_1$ and choose $\pi(T_2) \in \frac{1}{2}\pi(\mathcal{U})_1$ with $\|y - \pi(T_1)x - \pi(T_2)x\| = \|\tilde{T}_2 x - \pi(T_2)x\| \leq \frac{1}{2^2}$.

Proceed by induction with $\tilde{T}_n := (y - \sum_{j=1}^{n-1} \pi(T_j)x) \otimes x \in 2^{-j}\mathcal{L}(\mathcal{H})_1$. Choose $\pi(T_n) \in 2^{-n}\pi(\mathcal{U})_1$ with $\|y - \sum_{j=1}^{n} \pi(T_j)x\| = \|\tilde{T}_n x - \pi(T_n)x\| \leq 2^{-n}$. Then we have $\pi(T) := \sum_{j=1}^{n} \pi(T_n)$ in $\mathcal{U}$ and $\pi(T)x = y$ which completes the proof. $\square$