proof of convergence theorem

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Let us show the equivalence of (2) and (3). First, the proof that (3) implies (2) is a direct calculation. Next, let us show that (2) implies (3): Suppose $Tu_i \to 0$ in $\mathbb{C}$, and if K is a compact set in U, and $\{u_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{D}_K$ such that for any multi-index α, we have

$$D^\alpha u_i \to 0$$

in the supremum norm $\|\cdot\|_\infty$ as $i \to \infty$. For a contradiction, suppose there is a compact set K in U such that for all constants $C > 0$ and $k \in \{0, 1, 2, \dots\}$ there exists a function $u \in \mathcal{D}_K$ such that

$$|T(u)| > C \sum_{|\alpha| \leq k} ||D^\alpha u||_\infty.$$

Then, for $C = k = 1, 2, \dots$ we obtain functions $u_1, u_2, \dots$ in $\mathcal{D}(K)$ such that $|T(u_i)| > i \sum_{|\alpha| \leq i} ||D^\alpha u_i||_\infty$. Thus $|T(u_i)| > 0$ for all i, so for $v_i = u_i / |T(u_i)|$, we have

$$1 > i \sum_{|\alpha| \leq i} ||D^\alpha v_i||_\infty.$$

It follows that $||D^\alpha u_i||_\infty < 1/i$ for any multi-index α with $|\alpha| \leq i$. Thus $\{v_i\}_{i=1}^{\infty}$ satisfies our assumption, whence $T(v_i)$ should tend to 0. However, for all i, we have $T(v_i) = 1$. This contradiction completes the proof.