generalization of Young inequality

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It's straightforward to extend `http://planetmath.org/YoungInequality`Young inequality to an arbitrary finite number of : provided that $a_i > 0$, $c_i > 0$ and $\sum_{i=1}^{n} \frac{1}{c_i} = \frac{1}{r}$,

$$\left(\prod_{i=1}^{n} a_i\right)^r \leq r \sum_{i=1}^{n} \frac{a_i^{c_i}}{c_i}$$

In fact,

$$\begin{aligned}
\left(\prod_{i=1}^{n} a_i\right)^r &= \exp\left[\log\left(\prod_{i=1}^{n} a_i\right)^r\right] \\
&= \exp\left[r\sum_{i=1}^{n} \log a_i\right] \\
&= \exp\left[r\sum_{i=1}^{n} \frac{1}{c_i}\log\left(a_i^{c_i}\right)\right] \\
&= \exp\left[\frac{\sum_{i=1}^{n} \frac{1}{c_i}\log\left(a_i^{c_i}\right)}{\frac{1}{r}}\right] \\
\text{(by Jensen's inequality and monotonicity of exp)} &\leq \exp\left[\log\left(\frac{\sum_{i=1}^{n} \frac{1}{c_i} a_i^{c_i}}{\frac{1}{r}}\right)\right] \\
&= r\sum_{i=1}^{n} \frac{a_i^{c_i}}{c_i}
\end{aligned}$$

Remark: in the case

$$\frac{1}{c_i} = 1 \qquad \forall i$$

one obtains:

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} a_i$$

that is, the usual arithmetic-geometric mean inequality, which suggests Young inequality could be regarded as a generalization of this classical result. Actually, let's consider the following restatement of Young inequality. Having defined: $w_i = \frac{1}{c_i}, \quad \sum_{i=1}^n w_i = W = \frac{1}{r}, \quad x_i = a_i^{\frac{1}{w_i}}$ we have:

$$\left(\prod_{i=1}^n x_i^{w_i}\right)^{\frac{1}{W}} \leq \frac{1}{W}\sum_{i=1}^n w_i x_i$$

This expression shows that Young inequality is nothing else than geometric-arithmentic *weighted* `http://planetmath.org/ArithmeticMean`mean inequality.