normed vector space

Canonical name	NormedVectorSpace
Date of creation	2013-03-22 12:13:45
Last modified on	2013-03-22 12:13:45
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Last modified by	rspuzio (6075)
Numerical id	14
Author	rspuzio (6075)
Entry type	Definition
Classification	msc 46B99
Synonym	normed space
Synonym	normed linear space
Related topic	CauchySchwarzInequality
Related topic	VectorNorm
Related topic	PseudometricSpace
Related topic	MetricSpace
Related topic	UnitVector
Related topic	ProofOfGramSchmidtOrthogonalizationProcedure
Related topic	EveryNormedSpaceWithSchauderBasisIsSeparable
Related topic	EveryNormedSpaceWithSchauderBasisIsSeparable2
Related topic	FrobeniusProduct
Defines	norm
Defines	metric induced by a norm
Defines	metric induced by the norm
Defines	induced norm

Let $\mathbb{F}$ be a field which is either $\mathbb{R}$ or $\mathbb{C}$. A over $\mathbb{F}$ is a pair $(V, \|\cdot\|)$ where V is a vector space over $\mathbb{F}$ and $\|\cdot\|\colon V \to \mathbb{R}$ is a function such that

1. $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$ in V (*positive definiteness*)
2. $\|\lambda v\| = |\lambda|\|v\|$ for all $v \in V$ and all $\lambda \in \mathbb{F}$
3. $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$ (the *triangle inequality*)

The function $\|\cdot\|$ is called a *norm* on V.
Some properties of norms:

1. If W is a subspace of V then W can be made into a normed space by simply restricting the norm on V to W. This is called the induced norm on W.
2. Any normed vector space $(V, \|\cdot\|)$ is a metric space under the metric $d\colon V \times V \to \mathbb{R}$ given by $d(u, v) = \|u - v\|$. This is called the *metric induced by the norm* $\|\cdot\|$.
3. It follows that any normed space is a locally convex topological vector space, in the topology induced by the metric defined above.
4. In this metric, the norm defines a continuous map from V to $\mathbb{R}$ - this is an easy consequence of the triangle inequality.
5. If $(V, \langle , \rangle)$ is an inner product space, then there is a natural induced norm given by $\|v\| = \sqrt{\langle v, v\rangle}$ for all $v \in V$.
6. The norm is a convex function of its argument.