proof of Riesz representation theorem for separable Hilbert spaces

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Let $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots\}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Define

$$c_i = f(\mathbf{e}_i) \qquad \text{and} \qquad v = \sum_{k=0}^{n} \bar{c}_i \mathbf{e}_i.$$

The `http://planetmath.org/ContinuousLinearMapping`linear map f is continuous if and only if it is bounded, i.e. there exists a constant C such that $|f(v)| \leq C\|v\|$. Then

$$f(v) = \sum_{k=0}^{n} \bar{c}_k f(\mathbf{e}_k) = \sum_{k=0}^{n} |c_k|^2 \leq C \sqrt{\sum_{k=0}^{n} |c_k|^2}.$$

Simplifying, $\sum_{k=0}^{n} |c_k|^2 \leq C^2$. Hence $\sum_{k=0}^{\infty} c_k \mathbf{e}_k$ converges to an element u in H.

For every basis element, $f(\mathbf{e}_i) = c_k = \langle u, \mathbf{e}_i \rangle$. By linearity, it will also be true that

$$f(v) = \langle u, v \rangle \text{ if } v \text{ is a finite superposition of basis vectors.}$$

Any vector in the Hilbert space can be written as the limit of a sequence of finite superpositions of basis vectors hence, by continuity,

$$f(v) = \langle u, v \rangle \text{ for all } v \in \mathcal{H}$$

It is easy to see that u is unique. Suppose there existed two vectors u_1 and u_2 such that $f(v) = \langle u_1, v \rangle = \langle u_2, v \rangle$. Then $\langle u_1 - u_2, v \rangle = 0$ for all vectors $v \in \mathcal{H}$. But then, $\langle u_1 - u_2, u_1 - u_2 \rangle = 0$ which is only possible if $u_1 - u_2 = 0$, i.e. if $u_1 = u_2$.