lattice of projections

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Let H be a Hilbert space and $B(H)$ the algebra of bounded operators in H. By a projection in $B(H)$ we always an orthogonal projection.

Recall that a projection P in $B(H)$ is a http://planetmath.org/BoundedOperatorbounded self-adjoint operator satisfying $P^2 = P$.

The set of projections in $B(H)$, although not forming a vector space, has a very rich structure. In this entry we are going to endow this set with a partial ordering in a that it becomes a complete lattice. The lattice structure of the set of projections has profound consequences on the structure of von Neumann algebras.

1 The Lattice of Projections

In Hilbert spaces there is a bijective correspondence between closed subspaces and projections (see http://planetmath.org/ProjectionsAndClosedSubspacesthis entry). This correspondence is given by

$$P \longleftrightarrow \mathrm{Ran}(P)$$

where P is a projection and $\mathrm{Ran}(P)$ denotes the range of P.

Since the set of closed subspaces can be partially ordered by inclusion, we can define a partial order $\leq$ in the set of projections using the above correspondence:

$$P \leq Q \iff \mathrm{Ran}(P) \subseteq \mathrm{Ran}(Q)$$

But since projections are self-adjoint operators (in fact they are positive operators, as $P = P^*P$), they inherit the natural http://planetmath.org/OrderingOfSelfAdjoin ordering of self-adjoint operators, which we denote by $\leq_{sa}$, and whose definition we recall now

$$P \leq_{sa} Q \iff Q - P \text{is a positive operator}$$

As the following theorem shows, these two orderings coincide. Thus, we shall not make any more distinctions of notation between them.

Theorem 1 - Let P, Q be projections in $B(H)$. The following conditions are equivalent:

- $\mathrm{Ran}(P) \subseteq \mathrm{Ran}(Q)$ (i.e. $P \leq Q$)
- $QP = P$

- $PQ = P$
- $\|Px\| \leq \|Qx\|$ for all $x \in H$
- $P \leq_{sa} Q$

Two closed subspaces Y, Z in H have a greatest lower bound $Y \wedge Z$ and a least upper bound $Y \vee Z$. Specifically, $Y \wedge Z$ is precisely the intersection $Y \cap Z$ and $Y \vee Z$ is precisely the closure of the subspace generated by Y and Z. Hence, if P, Q are projections in $B(H)$ then $P \wedge Q$ is the projection onto $\mathrm{Ran}(P) \cap \mathrm{Ran}(Q)$ and $P \vee Q$ is the projection onto the closure of $\mathrm{Ran}(P) + \mathrm{Ran}(Q)$.

The above discussion clarifies that the set of projections in $B(H)$ has a lattice structure. In fact, the set of projections forms a complete lattice, by somewhat as above:

Every family $\{Y_\alpha\}$ of closed subspaces in H possesses an infimum $\bigwedge Y_\alpha$ and a supremum $\bigvee Y_\alpha$, which are, respectively, the intersection of all Y_α and the closure of the subspace generated by all Y_α. There is, of course, a correspondent in terms of projections: every family $\{P_\alpha\}$ of projections has an infimum $\bigwedge P_\alpha$ and a supremum $\bigvee P_\alpha$, which are, respectively, the projection onto the intersection of all $\mathrm{Ran}(P_\alpha)$ and the projection onto the closure of the subspace generated by all $\mathrm{Ran}(P_\alpha)$.

2 Additional Lattice Features

- The lattice of projections in $B(H)$ is never `http://planetmath.org/DistributiveLatticed` (unless H is one-dimensional).
- Also, it is modular if and only if H is finite dimensional. Nevertheless, there are important of von Neumann algebras (a particular type of subalgebras of $B(H)$ that are "rich" in projections) over an infinite-dimensional H, whose lattices of projections are in fact modular.
- Projections on one-dimensional subspaces are usually called **minimal projections** and they are in fact minimal in the sense that: there are no closed subspaces strictly between $\{0\}$ and a one-dimensional subspace, and every closed subspace other than $\{0\}$ contains a one-dimensional subspace. This means that the lattice of projections in

$B(H)$ is an atomic lattice and its atoms are precisely the projections on one-dimensional subspaces.

Moreover, every closed subspace of H is the closure of the span of its one-dimensional subspaces. Thus, the lattice of projections in $B(H)$ is an atomistic lattice.

- In Hilbert spaces every closed subspace Z is topologically complemented by its orthogonal complement ($H = Z \oplus Z^{\perp}$), and this fact is reflected in the structure of projections. The lattice of projections is then an orthocomplemented lattice, where the orthocomplement of each projection P is the projection $I - P$ (onto $\mathrm{Ran}(P)^{\perp}$).
- We shall see further ahead in this entry, when we discuss orthogonal projections, that the lattice of projections in $B(H)$ is an orthomodular lattice.

3 Commuting and Orthogonal Projections

When two projections P, Q commute, the projections $P \wedge Q$ and $P \vee Q$ can be described algebraically in a very . We shall see at the end of this section that P and Q commute precisely when its corresponding subspaces $\mathrm{Ran}(P)$ and $\mathrm{Ran}(Q)$ are "perpendicular".

Theorem 2 - Let P, Q be commuting projections (i.e. $PQ = QP$), then

- $P \wedge Q = PQ$
- $P \vee Q = P + Q - PQ$
- $\mathrm{Ran}(P) \vee \mathrm{Ran}(Q) = \mathrm{Ran}(P) + \mathrm{Ran}(Q)$. In particular, $\mathrm{Ran}(P) + \mathrm{Ran}(Q)$ is closed.

Two projections P, Q are said to be **orthogonal** if $P \leq Q^{\perp}$. This is equivalent to say that its corresponding subspaces are orthogonal ($\mathrm{Ran}(P)$ lies in the orthogonal complement of $\mathrm{Ran}(Q)$).

Corollary 1 - Two projections P, Q are orthogonal if and only if $PQ = 0$. When this is so, then $P \vee Q = P + Q$.

Corollary 2 - Let P, Q be projections in $B(H)$ such that $P \leq Q$. Then $Q - P$ is the projection onto $\mathrm{Ran}(Q) \cap \mathrm{Ran}(P)^{\perp}$.

We can now see that P, Q commute if and only if $\mathrm{Ran}(P)$ and $\mathrm{Ran}(Q)$ are "perpendicular". A somewhat informal and intuitive definition of "perpendicular" is that of requiring the two subspaces to be orthogonal outside their intersection (this is different of , since orthogonal subspaces do not intersect each other). More rigorously, P and Q commute if and only if the subspaces $\mathrm{Ran}(P) \cap (\mathrm{Ran}(P) \cap \mathrm{Ran}(Q))^{\perp}$ and $\mathrm{Ran}(Q) \cap (\mathrm{Ran}(P) \cap \mathrm{Ran}(Q))^{\perp}$ are orthogonal.

This can be proved using all the above results: The two subspaces are orthogonal iff

$$0 = (P - P \wedge Q)(Q - P \wedge Q) = PQ - P \wedge Q$$

and $PQ = P \wedge Q$ iff

$$PQ = P \wedge Q = (P \wedge Q)^{*} = (PQ)^{*} = QP$$

We can now also see that the lattice of projections is orthomodular: Suppose $P \leq Q$. Then, using the above results,

$$P \vee (Q \wedge P^{\perp}) = P \vee (Q - P) = P + (Q - P) - P(Q - P) = Q$$

4 Nets of Projections

In the following we discuss some useful and interesting results about convergence and limits of projections.

Let Λ be a poset. A net of projections $\{P_{\alpha}\}_{\alpha \in \Lambda}$ is said to be increasing if $\alpha \leq \beta \Longrightarrow P_{\alpha} \leq P_{\beta}$. Decreasing nets are defined similarly.

Theorem 3 - Let $\{P_{\alpha}\}$ be an increasing net of projections. Then $\lim_{\alpha} P_{\alpha} x = \bigvee_{\alpha} P_{\alpha}\, x$ for every $x \in H$.

In other words, P_{α} converges to $\bigvee_{\alpha} P_{\alpha}$ in the strong operator topology.

Similarly for decreasing nets of projections,

Theorem 4 - Let $\{P_\alpha\}$ be a decreasing net of projections. Then $\lim_\alpha P_\alpha x = \bigwedge_\alpha P_\alpha \, x$ for every $x \in H$.

In other words, P_α converges to $\bigwedge_\alpha P_\alpha$ in the strong operator topology.

Theorem 5 - Let Λ be a set and $\{P_\alpha\}_{\alpha\in\Lambda}$ be a family of pairwise orthogonal projections. Then $\sum P_\alpha$ is summable and $\sum P_\alpha \, x = \bigvee_\alpha P_\alpha \, x$ for all $x \in H$.