Gelfand transform

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Defines	commutative C^*-algebras classification
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The Gelfand Transform

Let $\mathcal{A}$ be a Banach algebra over $\mathbb{C}$. Let $\triangle$ be the space of all multiplicative linear functionals in $\mathcal{A}$, endowed with the weak-* topology. Let $C(\triangle)$ denote the algebra of complex valued continuous functions in $\triangle$.

The **Gelfand transform** is the mapping

$$\begin{aligned} \widehat{\ } : \mathcal{A} &\longrightarrow C(\triangle) \\ x &\longmapsto \widehat{x} \end{aligned}$$

where $\widehat{x} \in C(\triangle)$ is defined by $\quad \widehat{x}(\phi) := \phi(x), \quad \forall \phi \in \triangle$

The Gelfand transform is a continuous homomorphism from $\mathcal{A}$ to $C(\triangle)$.

Theorem - Let $C_0(\triangle)$ denote the algebra of complex valued continuous functions in $\triangle$, that vanish at infinity. The image of the Gelfand transform is contained in $C_0(\triangle)$.

The Gelfand transform is a very useful tool in the study of commutative Banach algebras and, particularly, commutative `http://planetmath.org/CAlgebra`C^*-algebras.

Classification of commutative C^*-algebras: Gelfand-Naimark theorems

The following results are called the Gelfand-Naimark theorems. They classify all commutative C^*-algebras and all commutative C^*-algebras with identity element.

Theorem 1 - Let $\mathcal{A}$ be a C^*-algebra over $\mathbb{C}$. Then $\mathcal{A}$ is *-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. Moreover, the Gelfand transform is a *-isomorphism between $\mathcal{A}$ and $C_0(\triangle)$.

Theorem 2 - Let $\mathcal{A}$ be a unital C^*-algebra over $\mathbb{C}$. Then $\mathcal{A}$ is *-isomorphic to $C(X)$ for some compact Hausdorff space X. Moreover, the Gelfand transform is a *-isomorphism between $\mathcal{A}$ and $C(\triangle)$.

The above theorems can be substantially improved. In fact, there is an `http://planetmath.org/EquivalenceOfCategories`equivalence between the category of commutative C^*-algebras and the category of locally compact Hausdorff spaces. For more and details about this, see the entry about the general `http://planetmath.org/GelfandNaimarkTheorem`Gelfand-Naimark theorem.