example of non-separable Hilbert space

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As an exaple of a Hilbert space which is not separable, one may consider the following function space:

Consider real-valued functions on the real line but, instead of the usual L^2 norm, use the following inner product:

$$(f, g) = \lim_{R \to \infty} \frac{1}{R} \int_{-R}^{+R} f(x) g(x) \, dx$$

The first thing to note about this is that non-trivial functions have norm 0. For instance, any function of L^2 has zero norm according to this inner product.

Define the Hilbert space as the set of equivalence classes of functions for which this norm is finite modulo functions for which it is zero. Note that $\sin ax$ and $\sin bx$ are orthogonal under this norm if $a \neq b$. Hence, the set of functions $\sin ax$, where a is a real number, form an orthonormal set. Since the number of real numbers is uncountable, we have an uncountably infinite orthonormal set, so this Hilbert space is not separable.

It is important not to confuse what we are doing here with the Fourier integral. In that case, we are dealing with L^2, the functions $\sin ax$ have infinite L^2 norm (so they are not elements of that Hilbert space) and the expansion of a function in terms of them is a direct integral. By contrast, in the case propounded here, the expansion of a function of this space in terms of them would take the form of a direct sum, just as with the Fourier series of a function on a finite interval.