proof of Banach-Alaoglu theorem

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For any $x \in X$, let $D_x = \{z \in \mathbb{C} : |z| \leq \|x\|\}$ and $D = \Pi_{x \in X} D_x$. Since D_x is a compact subset of $\mathbb{C}$, D is compact in product topology by Tychonoff theorem.

We prove the theorem by finding a homeomorphism that maps the closed unit ball B_{X^*} of X^* onto a closed subset of D. Define $\Phi_x : B_{X^*} \to D_x$ by $\Phi_x(f) = f(x)$ and $\Phi : B_{X^*} \to D$ by $\Phi = \Pi_{x \in X} \Phi_x$, so that $\Phi(f) = (f(x))_{x \in X}$. Obviously, Φ is one-to-one, and a net (f_α) in B_{X^*} converges to f in weak-* topology of X^* iff $\Phi(f_\alpha)$ converges to $\Phi(f)$ in product topology, therefore Φ is continuous and so is its inverse $\Phi^{-1} : \Phi(B_{X^*}) \to B_{X^*}$.

It remains to show that $\Phi(B_{X^*})$ is closed. If $(\Phi(f_\alpha))$ is a net in $\Phi(B_{X^*})$, converging to a point $d = (d_x)_{x \in X} \in D$, we can define a function $f : X \to \mathbb{C}$ by $f(x) = d_x$. As $\lim_\alpha \Phi(f_\alpha(x)) = d_x$ for all $x \in X$ by definition of weak-* convergence, one can easily see that f is a linear functional in B_{X^*} and that $\Phi(f) = d$. This shows that d is actually in $\Phi(B_{X^*})$ and finishes the proof.