proof of Heine-Cantor theorem

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Owner	paolini (1187)
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We prove this theorem in the case when X and Y are metric spaces.

Suppose f is not uniformly continuous. Then

$$\exists \epsilon > 0 \; \forall \delta > 0 \; \exists x, y \in X \quad d(x,y) < \delta \text{ but } d(f(x), f(y)) \geq \epsilon.$$

In particular by letting $\delta = 1/k$ we can construct two sequences x_k and y_k such that

$$d(x_k, y_k) < 1/k \text{ and } d(f(x_k), f(y_k) \geq \epsilon.$$

Since X is compact the two sequence have convergent subsequences i.e.

$$x_{k_j} \to \bar{x} \in X, \quad y_{k_j} \to \bar{y} \in X.$$

Since $d(x_k, y_k) \to 0$ we have $\bar{x} = \bar{y}$. Being f continuous we hence conclude $d(f(x_{k_j}), f(y_{k_j})) \to 0$ which is a contradiction being $d(f(x_k), f(y_k)) \geq \epsilon$.