quotients of Banach algebras

Canonical name	QuotientsOfBanachAlgebras
Date of creation	2013-03-22 17:41:54
Last modified on	2013-03-22 17:41:54
Owner	asteroid (17536)
Last modified by	asteroid (17536)
Numerical id	6
Author	asteroid (17536)
Entry type	Theorem
Classification	msc 46H10
Classification	msc 46H05
Classification	msc 46B99

Theorem - Let $\mathcal{A}$ be a Banach algebra and $\mathcal{I} \subseteq \mathcal{A}$ a http://planetmath.org/ClosedSetclosed ideal. Then $\mathcal{A}/\mathcal{I}$ is Banach algebra under the quotient norm.

Proof: Denote the quotient norm by $\|\cdot\|_q$.

By the http://planetmath.org/QuotientsOfBanachSpacesByClosedSubspacesAreBanachSp entry we know that $\mathcal{A}/\mathcal{I}$ is a Banach space under the quotient norm. Thus, we only need to show the normed algebra inequality:

$$\|ab+\mathcal{I}\|_q \leq \|a+\mathcal{I}\|_q \|b+\mathcal{I}\|_q$$

for every $a, b \in \mathcal{A}$.

Using the fact that $\mathcal{A}$ is a Banach algebra and the definition of quotient norm we have that:

$$\begin{aligned}
\|ab+\mathcal{I}\|_q &= \inf_{z \in ab+\mathcal{I}} \|z\| = \inf_{\substack{u \in a+\mathcal{I} \\ v \in b+\mathcal{I}}} \|uv\| \\
&\leq \inf_{\substack{u \in a+\mathcal{I} \\ v \in b+\mathcal{I}}} \|u\|\|v\| \leq \inf_{u \in a+\mathcal{I}} \|u\| \inf_{v \in b+\mathcal{I}} \|v\| \\
&= \|a+\mathcal{I}\|_q \|b+\mathcal{I}\|_q
\end{aligned}$$

$\square$