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all orthonormal bases have the same cardinality

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Theorem. – All orthonormal bases of an Hilbert space H have the same cardinality. It follows that the concept of dimension of a Hilbert space is well-defined.

Proof: When H is finite-dimensional (as a vector space), every orthonormal basis is a Hamel basis of H. Thus, the result follows from the fact that all Hamel bases of a vector space have the same cardinality (see `http://planetmath.org/AllBasesForAVectorSpaceHaveTheSameCardinality`this entry).

We now consider the case where H is infinite-dimensional (as a vector space). Let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be two orthonormal basis of H, indexed by the sets I and J, respectively. Since H is infinite dimensional the sets I and J must be infinite.

We know, from Parseval's equality, that for every $x \in H$

$$\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2$$

We know that, in the above sum, $\langle x, e_i\rangle \neq 0$ for only a countable number of $i \in I$. Thus, considering x as f_j, the set $I_j := \{i \in I : \langle f_j, e_i\rangle \neq 0\}$ is countable. Since for each $i \in I$ we also have

$$\|e_i\|^2 = \sum_{j\in J} |\langle e_i, f_j\rangle|^2$$

there must be $j \in J$ such that $\langle f_j, e_i\rangle \neq 0$. We conclude that $I = \bigcup_{j\in J} I_j$.

Hence, since each I_j is countable, $I \leq J\times\mathbb{N} \cong J$ (because J is infinite).

An analogous proves that $J \leq I$. Hence, by the Schroeder-Bernstein theorem J and I have the same cardinality. $\square$