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positive linear functional

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0.0.1 Definition

Let $\mathcal{A}$ be a http://planetmath.org/CAlgebraC^*-algebra and ϕ a linear functional on $\mathcal{A}$.

We say that ϕ is a **positive linear functional** on $\mathcal{A}$ if ϕ is such that $\phi(x) \geq 0$ for every $x \geq 0$, i.e. for every positive element $x \in \mathcal{A}$.

0.0.2 Properties

Let ϕ be a positive linear functional on $\mathcal{A}$. Then

- $\phi(x^*) = \overline{\phi(x)}$ for every $x \in \mathcal{A}$.

- $|\phi(x^*y)|^2 \leq \phi(x^*x)\phi(y^*y)$ for every $x, y \in \mathcal{A}$. This is an analog of the Cauchy-Schwartz inequality

Let ϕ be a linear functional on a C^*-algebra $\mathcal{A}$ with identity element e. Then

- ϕ is positive if and only if ϕ is http://planetmath.org/ContinuousLinearMappingbounded and $\|\phi\| = \phi(e)$.

0.0.3 Examples

- Let X be a locally compact Hausdorff space and $C_0(X)$ the C^*-algebra of continuous functions $X \longrightarrow \mathbb{C}$ that vanish at infinity. Let μ be a regular Radon measure on X. The linear functional ϕ defined by integration against μ,

$$\phi(f) := \int_X f \; d\mu \; , \qquad\qquad f \in C_0(x)$$

is a positive linear functional on $C_0(X)$. In fact, by the http://planetmath.org/RieszRepre representation theorem, all positive linear functionals of $C_0(X)$ are of this form.