proof of Neumann series in Banach algebras

Canonical name	ProofOfNeumannSeriesInBanachAlgebras
Date of creation	2013-03-22 17:32:40
Last modified on	2013-03-22 17:32:40
Owner	FunctorSalad (18100)
Last modified by	FunctorSalad (18100)
Numerical id	5
Author	FunctorSalad (18100)
Entry type	Proof
Classification	msc 46H05

Let x be an element of a Banach algebra with identity, $\|x\| < 1$. By applying the properties of the Norm in a Banach algebra, we see that the partial sums form a Cauchy sequence: $\|\sum_{n=l}^{m} x^n\| \leq \sum_{n=l}^{m} \|x\|^n \to 0$ for $l, m \to \infty$ (as is well known from real analysis), so by completeness of the Banach Algebra, the series converges to some element $y = \sum_{n=0}^{\infty} x^n$.

We observe that for any $m \in \mathbb{N}$,

$$(1-x)\sum_{n=0}^{m} x^n = \sum_{n=0}^{m} x^n - \sum_{n=1}^{m+1} x^n = 1 - x^{m+1} \tag{1}$$

Furthermore, $\|x^{m+1}\| \leq \|x\|^{m+1}$, so $\lim_m x^{m+1} = 0$.

Thus, by taking the limit $m \to \infty$ on both sides of (1), we get

$$(1-x)y = 1$$

(We can exchange the limit with the multiplication by $(1-x)$, since the multiplication in Banach algebras is continuous)

Since the Banach algebra generated by a single element is commutative and $(1-x)$ and y are both in the Banach algebra generated by x, we also get $y(1-x) = 1$. Hence, $y = (1-x)^{-1}$.

As in the first paragraph, the last claim $y \leq \frac{1}{1-\|y\|}$ again follows by applying the geometric series for real numbers.