proof of closed graph theorem

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Let $T\colon X \to Y$ be a linear mapping. Denote its graph by $G(T)$, and let $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ be the projections onto X and Y, respectively. We remark that these projections are continuous, by definition of the product of Banach spaces.

If T is bounded, then given a sequence $\{(x_i, Tx_i)\}$ in $G(T)$ which converges to $(x, y) \in X \times Y$, we have that

$$x_i = p_1(x_i, Tx_i) \xrightarrow[i\to\infty]{} p_1(x, y) = x$$

and

$$Tx_i = p_2(x_i, Tx_i) \xrightarrow[i\to\infty]{} p_2(x, y) = y,$$

by continuity of the projections. But then, since T is continuous,

$$Tx = \lim_{i\to\infty} Tx_i = y.$$

Thus $(x, y) = (x, Tx) \in G(T)$, proving that $G(T)$ is closed.

Now suppose $G(T)$ is closed. We remark that $G(T)$ is a vector subspace of $X \times Y$, and being closed, it is a Banach space. Consider the operator $\tilde{T} : X \to G(T)$ defined by $\tilde{T}x = (x, Tx)$. It is clear that $\tilde{T}$ is a bijection, its inverse being $p_1|_{G(T)}$, the restriction of p_1 to $G(T)$. Since p_1 is continuous on $X \times Y$, the restriction is continuous as well; and since it is also surjective, the open mapping theorem implies that $p_1|_{G(T)}$ is an open mapping, so its inverse must be continuous. That is, $\tilde{T}$ is continuous, and consequently $T = p_2 \circ \tilde{T}$ is continuous.