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proof of Krein-Milman theorem

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The proof is consist of three steps for good understanding. We will show initially that the set of extreme points of K,$Ex(K)$ is non-empty, $Ex(K) \neq \emptyset$. We consider that $\mathcal{A} = \{A \subset K \colon A \subset K, \text{extreme}\}$.

Step1

The family set $\mathcal{A}$ ordered by $\subset$ has a minimal element, in other words there exist $A \in \mathcal{A}$ such as $\forall B \in \mathcal{A}, B \subset A$ we have that $B = A$.

Proof1

We consider $A < B \Leftrightarrow B \subset A, \forall A, B \in \mathcal{A}$. The ordering relation $<$ is a partially relation on $\mathcal{A}$. We must show that A is maximal element for $\mathcal{A}$. We apply Zorn's lemma.We suppose that

$$\mathcal{C} = \{A_i \colon i \in I\}$$

is a chain of $\mathcal{A}$.Witout loss of generality we take $A = \bigcap_{i \in I} A_i$ and then $A \neq \emptyset$. $\mathcal{C}$ has the property of finite intersections and it is consist of closed sets. So we have that $\bigcap_{i \in I} A_i \neq \emptyset$. It is easy to see that $A \in \mathcal{A}$. Also $A \subset A_i$, for any $i \in I$, so we have that $A > A_i$, for any $i \in I$.

Step2

Every minimal element of $\mathcal{A}$ is a set which has only one point.

Proof2

We suppose that there exist a minimal element A of $\mathcal{A}$ which has at least two points, $x, y \in A$. There exist $x^* \in X^*$ such as $x^*(x) \neq x^*(y)$, witout loss of generality we have that $x^*(x) < x^*(y)$. A is compact set (closed subset of the compact K). Also there exist $\alpha \in \mathbb{R}$ such that $\alpha = \sup_{z \in A} x^*(z)$ and $B = \{z \in A \colon x^*(z) = \alpha\} \neq \emptyset$. It is obvious that B is an extreme subset of A, B is an extreme subset of K,$B \in \mathcal{A}$. $x \notin B$ since $B \in \mathcal{A}$ and $B \subsetneqq A$ that contradicts to the fact that A is minimal extreme subset of $\mathcal{A}$.

From the above two steps we have that $Ex(K) \neq \emptyset$.

Step3

$K = \bar{co}(Ex(K))$ where $\bar{co}(Ex(K))$ denotes the closed convex hull of extreme points of K.

Proof3

Let $L = \bar{co}(Ex(K))$. Then L is closed subset of K, therefore it is compact, and convex clearly by the definition. We suppose that $L \subsetneqq K$. Then there exist $x \in K - L$. Let use Hahn-Banach theorem(geometric form). There exist $x^* \in X^*$ such as $\sup_{w \in L} x^*(w) < x^*(x)$. Let $\alpha = \sup\{x^*(y) \colon y \in K\}$, $B = \{y \in K \colon x^*(y) = \alpha\}$. Similar to Step2 B is extreme subset of K. B is compact and from step1 and step2 we have that $Ex(B) \neq \emptyset$. It is true that $Ex(B) \subset Ex(K) \subset L$. Now let $y \in Ex(B)$ then $x^*(y) = \alpha$ and if $y \in Lx^*(y) < x^*(x) \leq \alpha$. That is a contradiction.