orthonormal basis

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Definition

An *orthonormal basis* (or *Hilbert basis*) of an inner product space V is a subset B of V satisfying the following two properties:

- B is an orthonormal set.
- The linear span of B is dense in V.

The first condition means that all elements of B have norm 1 and every element of B is `http://planetmath.org/OrthogonalVectors`orthogonal to every other element of B. The second condition says that every element of V can be approximated arbitrarily closely by (finite) linear combinations of elements of B.

Orthonormal bases of Hilbert spaces

Every Hilbert space has an orthonormal basis. The cardinality of this orthonormal basis is called the *dimension* of the Hilbert space. (This is well-defined, as the cardinality does not depend on the choice of orthonormal basis. This dimension is not in general the same as `http://planetmath.org/Dimension2`the usual concept of dimension for vector spaces.)

If B is an orthonormal basis of a Hilbert space H, then for every $x \in H$ we have

$$x = \sum_{b \in B} \langle x, b \rangle b.$$

Thus x is expressed as a (possibly infinite) "linear combination" of elements of B. The expression is well-defined, because only countably many of the terms $\langle x, b \rangle b$ are non-zero (even if B itself is uncountable), and if there are infinitely many non-zero terms the series is unconditionally convergent. For any $x, y \in H$ we also have

$$\langle x, y \rangle = \sum_{b \in B} \langle x, b \rangle \langle b, y \rangle.$$