spectrum is a non-empty compact set

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Theorem - Let $\mathcal{A}$ be a complex Banach algebra with identity element. The spectrum of each $a \in \mathcal{A}$ is a non-empty compact set in $\mathbb{C}$.

Remark: For Banach algebras over $\mathbb{R}$ the spectrum of an element is also a compact set, although it can be empty. To assure that it is not the empty set, proofs usually involve http://planetmath.org/LiouvillesTheorem2Liouville's theorem for of a complex with values in a Banach algebra.

Proof : Let e be the identity element of $\mathcal{A}$. Let $\sigma(a)$ denote the spectrum of the element $a \in \mathcal{A}$.

- **-** For each $\lambda \in \mathbb{C}$ such that $|\lambda| > \|a\|$ one has $\|\lambda^{-1}a\| < 1$, and so, by the http://planetmath.org/NeumannSeriesInBanachAlgebrasNeumann series, $e - \lambda^{-1}a$ is invertible. Since

  $$a - \lambda e = -\lambda(e - \lambda^{-1}a)$$

  we see that $a - \lambda e$ is also invertible.

  We conclude that $\sigma(a)$ is contained in a disk of radius $\|a\|$, and therefore it is bounded.

  Let $\phi : \mathbb{C} \longrightarrow \mathcal{A}$ be the function defined by

  $$\phi(\lambda) = a - \lambda e$$

  It is known that the set $\mathcal{G}$ of the invertible elements of $\mathcal{A}$ is open (see http://planetmath.org/InvertibleElementsInABanachAlgebraFormAnOpenSetthis entry).

  Since $\phi^{-1}(\mathcal{G}) = \mathbb{C} - \sigma(a)$ and ϕ is a continuous function we see that that $\sigma(a)$ is a closed set in $\mathbb{C}$.

  As $\sigma(a)$ is a bounded closed subset of $\mathbb{C}$, it is compact.

- **Non-emptiness -** Suppose that $\sigma(a)$ was empty. Then the resolvent R_a is defined in $\mathbb{C}$.

  We can see that R_a is bounded since it is continuous in the closed disk $|\lambda| < \|a\|$ and, for $\lambda > \|a\|$, we have (again, by the http://planetmath.org/NeumannSeriesI series)

$$\begin{aligned}
\|R_a(\lambda)\| &= \|(a-\lambda e)^{-1}\| \\
&= \|\lambda^{-1}(e-\lambda^{-1}a)^{-1}\| \\
&\leq \frac{|\lambda|^{-1}}{1-|\lambda|^{-1}\|a\|} \\
&= \frac{1}{|\lambda|-\|a\|}
\end{aligned}$$

and therefore $\lim_{|\lambda|\to\infty} R_a(\lambda) = 0$, which shows that R_a is bounded.

The resolvent function, R_a, is http://planetmath.org/BanachSpaceValuedAnalyticFuncti (see http://planetmath.org/ResolventFunctionIsAnalyticthis entry). As it is defined in $\mathbb{C}$, it is a bounded entire function. Applying http://planetmath.org/LiouvillesTheorem2Liouville's theorem we conclude that it must be constant (see this http://planetmath.org/BanachSpaceValuedAna entry for an idea of how holds for Banach space valued functions).

Since $R_a(\lambda)$ converges to 0 as $|\lambda| \to \infty$ we see that R_a must be identically zero.

Thus, we have arrived to a contradiction since 0 is not invertible.

Therefore $\sigma(a)$ is non-empty.$\square$