C^n norm

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One can define an extended norm on the space $C^n(I)$ where I is a subset of $\mathbb{R}$ as follows:

$$\|f\|_{C^n} = \sup_{x \in I} \sup_{k \leq n} \left| \frac{d^k f}{dx^k} \right|$$

If f is a function of more than one variable (i.e. lies in $C^n(D)$ for a subset $D \in \mathbb{R}^m$), then one needs to take the supremum over all partial derivatives of order up to n.

That

$$\| \cdot \|_{C^n}$$

satisfies the defining conditions for an extended norm follows trivially from the properties of the absolute value (positivity, homogeneity, and the triangle inequality) and the inequality

$$\sup(|f| + |g|) < \sup |f| + \sup |g|.$$

If we are considering functions defined on the whole of $\mathbb{R}^m$ or an unbounded subset of $\mathbb{R}^m$, the C^n norm may be infinite. For example,

$$\|e^x\|_{C^n} = \infty$$

for all n because the n-th derivative of e^x is again e^x, which blows up as x approaches infinity. If we are considering functions on a compact (closed and bounded) subset of $\mathbb{R}^m$ however, the C^n norm is always finite as a consequence of the fact that every continuous function on a compact set attains a maximum. This also means that we may replace the "sup" with a "max" in our definition in this case.

Having a sequence of functions converge under this norm is the same as having their n-th derivatives converge uniformly. Therefore, it follows from the fact that the uniform limit of continuous functions is continuous that C^n is complete under this norm. (In other words, it is a Banach space.)

In the case of C^∞, there is no natural way to impose a norm, so instead one uses all the C^n norms to define the topology in C^∞. One does this by declaring that a subset of C^∞ is closed if it is closed in all the C^n norms. A space like this whose topology is defined by an infinite collection of norms is known as a multi-normed space.