C^*-algebra homomorphisms have closed images

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Theorem - Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a *-homomorphism between the `http://planetmath.org/CAlg`
algebras $\mathcal{A}$ and $\mathcal{B}$. Then f has `http://planetmath.org/ClosedSet`closed `http://planetmath.org/Function`image, i.e. $f(\mathcal{A})$ is closed in $\mathcal{B}$.

Thus, the image $f(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}$.

Proof: The kernel of f, Ker$f$, is a closed two-sided ideal of $\mathcal{A}$, since f is continuous (see `http://planetmath.org/HomomorphismsOfCAlgebrasAreContinuous`this entry). Factoring threw the quotient C^*-algebra $\mathcal{A}/\mathrm{Ker}f$ we obtain an injective *-homomorphism $\widetilde{f} : \mathcal{A}/\mathrm{Ker}f \longrightarrow \mathcal{B}$.

Injective *-homomorphisms between C^*-algebras are known to be isometric (see `http://planetmath.org/InjectiveCAlgebraHomomorphismIsIsometric`this entry), hence the image $\widetilde{f}(\mathcal{A}/\mathrm{Ker}f)$ is closed in $\mathcal{B}$.

Since the images $\widetilde{f}(\mathcal{A}/\mathrm{Ker}f)$ and $f(\mathcal{A})$ coincide we conclude that $f(\mathcal{A})$ is closed in $\mathcal{B}$. $\square$