Banach limit

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Consider the set c_0 of all convergent complex-valued sequences $\{x(n)\}_{n\in\mathbb{N}}$. The limit operation $x \mapsto \lim_{n\to\infty} x(n)$ is a linear functional on c_0, by the usual limit laws. A *Banach limit* is, loosely speaking, any linear functional that generalizes lim to apply to non-convergent sequences as well. The formal definition follows:

Let ℓ^∞ be the set of bounded complex-valued sequences $\{x(n)\}_{n\in\mathbb{N}}$, equipped with the sup norm. Then $c_0 \subset \ell^\infty$, and $\lim\colon c_0 \to \mathbb{C}$ is a linear functional. A *Banach limit* is any continuous linear functional $\phi \in (\ell^\infty)^*$ satisfying:

i $\phi(x) = \lim_n x(n)$ if $x \in c_0$ (That is, ϕ extends lim.)

ii $\|\phi\| = 1$.

iii $\phi(Sx) = \phi(x)$, where $S\colon \ell^\infty \to \ell^\infty$ is the shift operator defined by $Sx(n) = x(n+1)$. (Shift invariance)

iv If $x(n) \geq 0$ for all n, then $\phi(x) \geq 0$. (Positivity)

There is not necessarily a unique Banach limit. Indeed, Banach limits are often constructed by extending lim with the Hahn-Banach theorem (which in turn invokes the Axiom of Choice).

Like the limit superior and limit inferior, the Banach limit can be applied for situations where one wants to algebraically manipulate limit equations or inequalities, even when it is not assured beforehand that the limits in question exist (in the classical sense).

1 Some consequences of the definition

The positivity condition ensures that the Banach limit of a real-valued sequence is real-valued, and that limits can be compared: if $x \leq y$, then $\phi(x) \leq \phi(y)$. In particular, given a real-valued sequence x, by comparison with the sequences $y(n) = \inf_{k\geq n} x(k)$ and $z(n) = \sup_{k\geq n} x(k)$, it follows that $\liminf_n x(n) \leq \phi(x) \leq \limsup_n x(n)$.

The shift invariance allows any finite number of terms of the sequence to be neglected when taking the Banach limit, as is possible with the classical limit.

On the other hand, ϕ can never be multiplicative, meaning that $\phi(xy) = \phi(x)\phi(y)$ fails. For a counter-example, set $x = (0, 1, 0, 1, \dots)$; then we would have $\phi(0) = \phi(x \cdot Sx) = \phi(x)\phi(Sx) = \phi(x)^2$, so $\phi(x) = 0$, but $1 = \phi(1) = \phi(x + Sx) = \phi(x) + \phi(Sx) = 2\phi(x) = 0$.

That ϕ is continuous means it is compatible with limits in ℓ^∞. For example, suppose that $\{x_k\}_{k\in\mathbb{N}} \subset \ell^\infty$, and that $\sum_{k=0}^\infty x_k$ is absolutely convergent in ℓ^∞. (In other words, $\sum_{k=0}^\infty \|x_k\|_\infty < \infty$.) Then $\phi(\sum_{k=0}^\infty x_k) = \sum_{k=0}^\infty \phi(x_k)$ by continuity. Observe that this is just the dominated convergence theorem, specialized to the case of the counting measure on $\mathbb{N}$, in disguise.

2 Other definitions

In some definitions of the Banach limit, condition (i) is replaced by the seemingly weaker condition that $\phi(1) = 1$ — the Banach limit of a constant sequence is that constant. In fact, the latter condition together with shift invarance implies condition (i).

If we restrict to real-valued sequences, condition (ii) is clearly redundant, in view of the other conditions.