# proof of Sobolev inequality for $\Omega = \mathbf{R}^n$


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**Step 1: $u$ is smooth and $p = 1$** First suppose $u$ is a compactly supported smooth function, and let $(e_k)_{1\le k\le n}$ denote a basis of $\mathbf{R}^n$. For every $1 \le k \le n$,

$$u(x) = \int_{-\infty}^{0} \frac{\partial u}{\partial x_k}(x + se_k)\, ds.$$

Therefore,

$$|u(x)| \le S_k(x) := \int_{\mathbf{R}} \Big|\frac{\partial u}{\partial x_k}(x_1, \dots, x_{k-1}, s, x_{k+1}, \dots, x_n)\Big|\, ds.$$

Note that $S_k$ does not depend on $x_k$. One also has

$$|u(x)|^{n/(n-1)} \le \prod_{k=1}^{n} |S_k(x)|^{1/(n-1)}.$$

The integration of this inequality yields,

$$\int_{\mathbf{R}^n} |u(x)|^{n/(n-1)}\, dx \le \int_{\mathbf{R}^n} \prod_{k=1}^{n} |S_k(x)|^{1/(n-1)}\, dx.$$

Since $S_1$ does not depend on $x_k$, we can apply the generalized Hölder inequality with $n-1$ for the integration with respect to $x_1$ in order to obtain:

$$\int_{\mathbf{R}^n} |u(x)|^{n/(n-1)}\, dx \le \int_{\mathbf{R}^{n-1}} S_1(x) \prod_{k=2}^{n} \Big(\int_{\mathbf{R}} S_k(x)\, dx_1\Big)^{1/(n-1)} dx_1 \dots dx_n.$$

The repetition of this process for the variables $x_2, \dots, x_n$ gives

$$\int_{\mathbf{R}^n} |u(x)|^{n/(n-1)}\, dx \le \prod_{k=1}^{n} \Big(\int_{\mathbf{R}^n} \Big|\frac{\partial u}{\partial x_k}\Big|\, dx\Big)^{1/(n-1)}.$$

By the arithmetic-geometric means inequality, one obtains

$$\int_{\mathbf{R}^n} |u(x)|^{n/(n-1)}\, dx \le n^{-n/(n-1)} \Big(\sum_{k=1}^{n} \Big(\int_{\mathbf{R}^n} \Big|\frac{\partial u}{\partial x_k}\Big|\, dx\Big)\Big)^{n/(n-1)}.$$

One finally concludes

$$\|u\|_{L^{n/(n-1)}} \le n^{1/2 - n/(n-1)} \|\nabla u\|_{L^{n/(n-1)}}.$$

**Step 2: general $u$ and $p = 1$** In general if $u \in W^{1,1}(\mathbf{R}^n)$. It can be approximated by a sequence of compactly supported smooth functions $(u_m)$. By step 1, one has

$$\|u_m - u_\ell\|_{L^{n/(n-1)}} \leq n^{1/2-n/(n-1)} \|\nabla u_m - \nabla u_\ell\|_{L^1}.$$

therefore $(u_m)$ is a Cauchy sequence in $L^{n/(n-1)}(\mathbf{R}^n)$. Since it converges to $u$ in $L^1(\mathbf{R}^n)$, the limit of $(u_m)$ is $u$ in $L^{n/(n-1)}(\mathbf{R}^n)$ and one has

$$\|u\|_{L^{n/(n-1)}} \leq n^{1/2-n/(n-1)} \|\nabla u\|_{L^{n/(n-1)}}.$$

**Step 3: $1 < p < n$ and $u$ is smooth** Suppose $1 < p < n$ and $u$ is a smooth compactly supported function. Let

$$r = \frac{p(n-1)}{n-p}$$

and

$$v = |u|^r.$$

Since $u$ is smooth, $v \in W^{1,1}$ (It is however not necessarily smooth), and its weak derivative is

$$\nabla v = r u |u|^{r-2} \nabla u.$$

One has, by the Hölder inequality,

$$\|\nabla v\|_{L^1(\mathbf{R}^N)} \leq r \||u|^r\|_{L^{p/p-1}(\mathbf{R}^N)} \|\nabla u\|_{L^p(\mathbf{R}^N)} = r \|u\|^{r-1}_{L^{np/(n-p)}(\mathbf{R}^N)} \|\nabla u\|_{L^p(\mathbf{R}^N)}$$

Therefore, the Sobolev inequality yields

$$\|u\|^r_{L^{np/(n-p)}(\mathbf{R}^N)} = \|v\|_{L^{n/(n-1)}(\mathbf{R}^N)} \leq r n^{1/2-n/(n-1)} \|u\|^{r-1}_{L^{np/(n-p)}(\mathbf{R}^N)} \|\nabla u\|_{L^p(\mathbf{R}^N)}.$$

This yields

$$\|u\|_{L^{np/(n-p)}(\mathbf{R}^N)} \leq n^{1/2-n/(n-1)} \frac{p(n-1)}{n-p} \|\nabla u\|_{L^p(\mathbf{R}^N)}.$$

**Step 4: $1 < p < n$ and $u \in W^{1,p}$** This is done as step 2.

This proof is due to Gagliardo and Nirenberg, who were the first to prove the inequality for $p = 1$. This proof can be also found in [**?**, **?**, **?**].

# References

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[3] Michel Willem, *Analyse fonctinnelle lmentaire*, Cassini, Paris, 2003.