spectral permanence theorem

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Author	asteroid (17536)
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Let $\mathcal{A}$ be a unital complex Banach algebra and $\mathcal{B} \subseteq \mathcal{A}$ a Banach subalgebra that contains the identity of $\mathcal{A}$.

For every element $x \in \mathcal{B}$ it makes sense to speak of the spectrum $\sigma_{\mathcal{B}}(x)$ of x relative to $\mathcal{B}$ as well as the spectrum $\sigma_{\mathcal{A}}(x)$ of x relative to $\mathcal{A}$.

We provide here three results of increasing sophistication which relate both these spectrums, $\sigma_{\mathcal{B}}(x)$ and $\sigma_{\mathcal{A}}(x)$. Any of the last two is usually refered to as the **spectral permanence theorem**.

- Let $\mathcal{B} \subseteq \mathcal{A}$ be as above. For every element $x \in \mathcal{B}$ we have

$$\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{B}}(x).$$

This first result is purely . It is a straightforward consequence of the fact that invertible elements in $\mathcal{B}$ are also invertible in $\mathcal{A}$.

The other inclusion, $\sigma_{\mathcal{B}}(x) \subseteq \sigma_{\mathcal{A}}(x)$, is not necessarily true. It is true, however, if one considers the boundary $\partial\sigma_{\mathcal{B}}(x)$ instead.

Theorem - Let $\mathcal{B} \subseteq \mathcal{A}$ be as above. For every element $x \in \mathcal{B}$ we have

$$\partial\sigma_{\mathcal{B}}(x) \subseteq \sigma_{\mathcal{A}}(x).$$

Since the spectrum is a non-empty compact set in $\mathbb{C}$, one can decompose $\mathbb{C}-\sigma_{\mathcal{A}}(x)$ into its connected components, obtaining an unbounded component Ω_∞ together with a sequence of bounded components $\Omega_1, \Omega_2, \dots$,

$$\mathbb{C} - \sigma_{\mathcal{A}}(x) = \Omega_\infty \cup \Omega_1 \cup \Omega_2 \cup \cdots$$

Of course there may be only a finite number of bounded components or none.

Theorem - Let $x \in \mathcal{B} \subseteq \mathcal{A}$ be as above. Then $\sigma_{\mathcal{B}}(x)$ is obtained from $\sigma_{\mathcal{A}}(x)$ by adjoining to it some (possibly none) bounded components of $\mathbb{C} - \sigma_{\mathcal{A}}(x)$.

As an example, if $\sigma_{\mathcal{A}}(x)$ is the unit circle, then $\sigma_{\mathcal{B}}(x)$ can only possibly be the unit circle or the closed unit disk.