proof of classification of separable Hilbert spaces

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The strategy will be to show that any separable, infinite dimensional Hilbert space H is equivalent to ℓ^2, where ℓ^2 is the space of all square summable sequences. Then it will follow that any two separable, infinite dimensional Hilbert spaces, being equivalent to the same space, are equivalent to each other.

Since H is separable, there exists a countable dense subset S of H. Choose an enumeration of the elements of S as $s_0, s_1, s_2, \ldots$. By the Gram-Schmidt orthonormalization procedure, one can exhibit an orthonormal set $e_0, e_1, e_2, \ldots$ such that each e_i is a finite linear combination of the s_i's.

Next, we will demonstrate that Hilbert space spanned by the e_i's is in fact the whole space H. Let v be any element of H. Since S is dense in H, for every integer n, there exists an integer m_n such that

$$\|v - s_{m_n}\| \leq 2^{-n}$$

The sequence $(s_{m_0}, s_{m_1}, s_{m_2}, \ldots)$ is a Cauchy sequence because

$$\|s_{m_i} - s_{m_j}\| \leq \|s_{m_i} - v\| + \|v - s_{m_j}\| \leq 2^{-i} + 2^{-j}$$

Hence the limit of this sequence must lie in the Hilbert space spanned by $\{s_0, s_1, s_2, \ldots\}$, which is the same as the Hilbert space spanned by $\{e_0, e_1, e_2, \ldots\}$. Thus, $\{e_0, e_1, e_2, \ldots\}$ is an orthonormal basis for H.

To any $v \in H$ associate the sequence $U(v) = (\langle v, s_0 \rangle, \langle v, s_1 \rangle, \langle v, s_2 \rangle, \ldots)$. That this sequence lies in ℓ^2 follows from the generalized Parseval equality

$$\|v\|^2 = \sum_{k=0}^{\infty} \langle v, s_k \rangle$$

which also shows that $\|U(v)\|_{\ell^2} = \|v\|_H$. On the other hand, let $(w_0, w_1, w_2, \ldots)$ be an element of ℓ^2. Then, by definition, the sequence of partial sums $(w_0^2, w_0^2 + w_1^2, w_0^2 + w_1^2 + w_2^2, \ldots)$ is a Cauchy sequence. Since

$$\|\sum_{i=0}^{m} w_i e_i - \sum_{i=0}^{n} w_i e_i\|^2 = \sum_{i=0}^{m} w_i^2 - \sum_{i=0}^{n} w_i^2$$

if $m > n$, the sequence of partial sums of $\sum_{k=0}^{\infty} w_i e_i$ is also a Cauchy sequence, so $\sum_{k=0}^{\infty} w_i e_i$ converges and its limit lies in H. Hence the operator U is invertible and is an isometry between H and ℓ^2.