example of Dirac sequence

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We can construct a Dirac sequence $\{\delta_n\}_{n\in\mathbb{N}_+}$ by choosing

$$\delta_n(x) = \frac{n}{\pi(1+n^2x^2)}.$$

To show that conditions 1 and 3 in the definition of a Dirac sequence are satisfied is trivial and condition 2 is also fulfilled since

$$\int_{-\infty}^{\infty} \delta_n(x)dx = \frac{1}{\pi}\cdot\int_{-\infty}^{\infty} \frac{n}{1+n^2x^2}dx = \left[\begin{array}{l} y = nx \\ dy = n\cdot dx \end{array}\right] = \frac{1}{\pi}\cdot\int_{-\infty}^{\infty}\frac{1}{1+y^2}dy = \frac{1}{\pi}\cdot\arctan y\Big|_{y=-\infty}^{\infty} = \frac{1}{\pi}\cdot$$

for all $n \in \mathbb{N}_+$, hence $\{\delta_n\}_{n\in\mathbb{N}_+}$ is a Dirac sequence.

To prove that it actually converges in $\mathcal{D}'(\mathbb{R})$ (the space of all distributions on $\mathcal{D}(\mathbb{R})$) to the Dirac delta distribution δ, we must show that

$$\lim_{n\to\infty}\int_{\mathbb{R}} \delta_n(x)\varphi(x)dx = \varphi(0)$$

for any test function $\varphi \in \mathcal{D}(\mathbb{R})$ (a topological vector space of smooth functions with compact support). Let us take an arbitrary test function $\varphi \in \mathcal{D}(\mathbb{R})$ and assume that the closed and compact set $\text{supp}(\varphi)$ is contained in some open interval $(a,b) \subset \mathbb{R}$ ($a < 0$ and $b > 0$). Using the triangle inequality and the fact that $\int_{\mathbb{R}} \delta_n(x)dx = 1$ for all $n \in \mathbb{N}_+$ we can write

$$\left|\int_{-\infty}^{\infty}\delta_n(x)\varphi(x)dx - \varphi(0)\right| = \left|\int_{-\infty}^{\infty}\delta_n(x)(\varphi(x)-\varphi(0))dx\right| \leq$$

$$\leq \underbrace{\varphi(0)\int_{-\infty}^{a}|\delta_n(x)|\,dx}_{I_1} + \underbrace{\int_a^b |\delta_n(x)(\varphi(x)-\varphi(0))|\,dx}_{I_2} + \underbrace{\varphi(0)\int_b^{\infty}|\delta_n(x)|\,dx}_{I_3}$$

It is easy to see that $\lim_{n\to\infty}\delta_n(x) = 0$, $\forall x \in (-\infty, a] \cup [b, \infty)$ and therefore $\lim_{n\to\infty} I_1 = 0$ and $\lim_{n\to\infty} I_3 = 0$. Finally we want to estimate I_2 when $n \to \infty$.

$$I_2 = \int_a^b |\delta_n(x)|\underbrace{|(\varphi(x)-\varphi(0))|}_{\leq |x|\cdot\sup|\varphi'(x)|}dx \leq \sup|\varphi'(x)|\cdot\int_a^b|\delta_n(x)x|\,dx =$$

$$= \sup|\varphi'(x)|\cdot\frac{1}{\pi}\int_a^b\left|\frac{nx}{1+(nx)^2}\right|dx = \sup|\varphi'(x)|\cdot\frac{1}{\pi}\left(-\int_a^0\frac{nx}{1+(nx)^2}dx + \int_0^b\frac{nx}{1+(nx)^2}dx\right) =$$

$$= \sup|\varphi'(x)| \cdot \frac{1}{\pi}\left(-\left(\frac{1}{2n}\cdot \ln|1+(nx)^2|\Big|_{x=a}^{0}\right)+\left(\frac{1}{2n}\cdot \ln|1+(nx)^2|\Big|_{x=0}^{b}\right)\right) =$$

$$= \sup|\varphi'(x)| \cdot \frac{1}{\pi}\left(\frac{1}{2n}\cdot \ln|1+(na)^2| + \frac{1}{2n}\cdot \ln|1+(nb)^2|\right)$$

We now conclude that $\lim_{n\to\infty} I_2 = 0$. This means that $\lim_{n\to\infty} I_1+I_2+I_3 = 0$ which shows that $\{\delta_n\}_{n\in\mathbb{N}_+}$ converges to the Dirac delta distribution δ.