Riesz representation theorem of bounded sesquilinear forms

Canonical name	RieszRepresentationTheoremOfBoundedSesquilinearForms
Date of creation	2013-03-22 18:41:38
Last modified on	2013-03-22 18:41:38
Owner	asteroid (17536)
Last modified by	asteroid (17536)
Numerical id	10
Author	asteroid (17536)
Entry type	Theorem
Classification	msc 47A07
Classification	msc 46C05
Synonym	Riesz lemma on bounded sesquilinear forms
Synonym	correspondence between bounded operators and bounded sesquilinear forms
Defines	bounded sesquilinear form

Bounded sesquilinear forms

Let H_1, H_2 be two Hilbert spaces.

Definition - A sesquilinear form $[\cdot,\cdot] : H_1 \times H_2 \to \mathbb{C}$ is said to be *bounded* if there is a constant $C \geq 0$ such that

$$[\xi,\eta] \leq C\|\xi\|\|\eta\|$$

for all $\xi \in H_1$ and $\eta \in H_2$.

Bounded sesquilinear forms are precisely those which are continuous from $H_1 \times H_2$ to $\mathbb{C}$.

Examples :

- When H_1 and H_2 are the same Hilbert space, denoted by H, the inner product $\langle\cdot,\cdot\rangle$ in H is itself a bounded sesquilinear form. The boundedness condition follows from the Cauchy-Schwarz inequality.

- Let $T : H_1 \to H_2$ be a bounded linear operator and denote by $\langle\cdot,\cdot\rangle$ the inner product in H_2. The function $[\cdot,\cdot] : H_1 \times H_2 \to \mathbb{C}$ defined by

  $$[\xi,\eta] := \langle T\xi,\eta\rangle$$

  is a bounded sesquilinear form. The boundedness condition follows from the Cauchy-Schwarz inequality and the fact that T is bounded.

Riesz representation of bounded sesquilinear forms

The second example above is in fact the general case. To every bounded sesquilinear form one can associate to it a unique bounded operator. That is content of the following result:

Theorem - Riesz - Let H_1, H_2 be two Hilbert spaces and denote by $\langle\cdot,\cdot\rangle$ the inner product in H_2. For every bounded sesquilinear form $[\cdot,\cdot] : H_1 \times H_2 \to \mathbb{C}$ there is a unique bounded linear operator $T : H_1 \to H_2$ such that

$$[\xi,\eta] = \langle T\xi,\eta\rangle\,, \qquad \xi \in H_1, \eta \in H_2.$$

Thus, there is a correspondence between bounded linear operators and bounded sesquilinear forms. Actually, in the early twentieth century, spectral theory was formulated solely in terms of sesquilinear forms on Hilbert spaces. Only later it was realized that this could be achieved, perhaps in a more intuitive manner, by considering linear operators instead. The linear operator approach has its advantages, as for example one can define the composition of linear operators but not of sesquilinear forms. Nevertheless it is many times useful to define a linear operator by specifying its sesquilinear form.