any topological space with the fixed point property is connected

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Theorem Any topological space with the `http://planetmath.org/FixedPointProperty`fixed-point property is connected.

Proof. We will prove the contrapositive. Suppose X is a topological space which is not connected. So there are non-empty disjoint open sets $A, B \subseteq X$ such that $X = A \cup B$. Then there are elements $a \in A$ and $b \in B$, and we can define a function $f \colon X \to X$ by

$$f(x) = \begin{cases} a, & \text{when } x \in B, \\ b, & \text{when } x \in A. \end{cases}$$

Since $A \cap B = \varnothing$ and $A \cup B = X$, the function f is well-defined. Also, $a \notin B$ and $b \notin A$, so f has no fixed point. Furthermore, if V is an open set in X, a short calculation shows that $f^{-1}(V)$ is $\varnothing, A, B$ or X, all of which are open sets. So f is continuous, and therefore X does not have the fixed-point property. $\square$

References

[1] G.J. Jameson, *Topology and Normed Spaces*, Chapman and Hall, 1974.

[2] L.E. Ward, *Topology, An Outline for a First Course*, Marcel Dekker, Inc., 1972.