Borel functional calculus

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Let $B(H)$ be the http://planetmath.org/Algebraalgebra of bounded operators over a complex Hilbert space H and $T \in B(H)$ a normal operator.

The **Borel functional calculus** is a functional calculus which enables the expression

$$f(T)$$

to make sense as a bounded operator in H, for a http://planetmath.org/Boundedbounded Borel function f.

In particular, it allows the definition of operators $\chi_S(T)$ for any characteristic function χ_S, which are of significant importance on the of the of T.

The Borel functional calculus will be constructed by extending the continuous functional calculus for arbitrary bounded Borel functions.

1 Preliminary Facts

Let us set some notation first:

- $\sigma(T)$ will denote the http://planetmath.org/Spectrumspectrum of T.
- $C(\sigma(T))$ will denote the http://planetmath.org/CAlgebraC^*-algebra of continuous functions $\sigma(T) \to \mathbb{C}$.
- $B(\sigma(T))$ will denote the C^*-algebra of bounded Borel functions $\sigma(T) \to \mathbb{C}$, endowed with the sup norm.

The continuous functional calculus for T allows the expression $f(T)$ to make sense for continuous functions $f \in C(\sigma(T))$, by the assignment of a unital *-homomorphism

$$\begin{aligned} \pi : C(\sigma(T)) &\longrightarrow B(H) \\ f &\longmapsto f(T) := \pi(f) \end{aligned}$$

that sends the identity function to T. This unital *-homomorphism is in fact uniquely determined by this property (see the entry on the http://planetmath.org/ContinuousFu functional calculus for more details).

The objective is to extend π to a unital *-homomorphism $\widetilde{\pi} : B(\sigma(T)) \longrightarrow B(H)$.

Since $B(\sigma(T))$ is a much larger C^*-algebra than $C(\sigma(T))$, there is no reson to presume that there is only one extension of π. Which extension would be the most natural then? It turns out that there is a unique extension that satisfies a good continuity property.

It is known that *-homomorphisms between C^*-algebras are continuous (see `http://planetmath.org/HomomorphismsOfCAlgebrasAreContinuous`this entry), so that whenever a net $f_i \in B(\sigma(T))$ converges in the sup norm to a function $f \in B(\sigma(T))$ we will have that $f_i(T) \to f(T)$ in the operator norm. All extensions of π will automaticaly satisfy this continuity property, but this can be improved in a satisfactory manner.

Notation - *Let X be a compact Hausdorff space, $M(X)$ the space of all finite* ***`http://planetmath.org/OuterRegular`****regular Borel measures in X and $B(X)$ the C^*-algebra of all bounded Borel functions in X. The weakest topology in $B(X)$ for which integration against any measure ν is continuous will be reffered to as the* μ-topology. *This means that $f_i \to f$ in the μ-topology if and only if $\int f_i d\nu \to \int f d\nu$ for all $\nu \in M(X)$.*

Notice that we can identify each function $f \in B(X)$ with a bounded `http://planetmath.org/Functional`linear functional ω_f in $M(X)$, given by

$$\omega_f(\nu) := \int_X f d\nu\,, \qquad\qquad \nu \in M(X)$$

and the μ-topology corresponds exactly to the weak-* topology under this identification.

We will see in the next that there is an unique extension of π that is continuous from the μ-topology to the weak operator topology.

Just like the `http://planetmath.org/StoneWeierstrassTheoremComplexVersion`Stone-Weierstrass theorem allowed the passage from the polynomial functional calculus to the continuous functional calculus, the `http://planetmath.org/RieszRepresentationTh` representation theorem will allow the passage from the latter to the Borel functional calculus.

2 Definition

The following result is the key for the definition of the Borel functional calculus.

Theorem 1 - *Let T be a normal operator in $B(H)$ and $\pi : C(\sigma(T)) \longrightarrow B(H)$ the unital *-homomorphism corresponding to the continuous functional calculus for T. Then, π extends uniquely to a *-homomorphism $\widetilde{\pi} : B(\sigma(T)) \longrightarrow B(H)$ that is continuous from the μ-topology to the weak operator topology. Moreover, each operator $\pi(f)$ lies in* ***http://planetmath.org/OperatorTopologies*** *operator* ***http://planetmath.org/Closure****closure of the unital *-algebra generated by T.*

: See `http://planetmath.org/ProofOfBorelFunctionalCalculus`this attached entry

We are now able to define the Borel functional calculus:

Definition - *Let T be a normal operator in $B(H)$. Let $\widetilde{\pi} : B(\sigma(T)) \longrightarrow B(H)$ be the unique *-homomorphism defined in Theorem 1. This *-homomorphism is denoted by*

$$f \longmapsto f(T)\,, \qquad\qquad f \in B(\sigma(T))$$

and it is called the **Borel functional calculus** *for T.*

Since this functional calculus extends the polynomial functional calculus, we have that for any polynomial $p(z) := \sum c_{n,m} z^n \overline{z}^m$,

$$p(T) = \sum c_{n,m} T^n (T^*)^m$$

Moreover, since $f(T)$ lies in the strong operator closure of the unital *-algebra generated by T, for any function $f \in B(\sigma(T))$, we see that $f(T)$ is the strong operator limit of polynomials $\sum c_{n,m} T^n (T^*)^m$.

3 Borel Calculus in von Neumann Algebras

The Borel functional calculus is in fact applicable for any normal operator T in any von Neumann algebra $\mathcal{M}$.

That is due to the fact, expressed in Theorem 1, that for every $f \in B(\sigma(T))$ the operator $f(T)$ belongs to the strong operator closure of the unital *-algebra generated by T. Being a von Neumann algebra, $\mathcal{M}$ is http://planetmath.org/ClosedSetclosed in the strong operator topology, and therefore all operators $f(T)$ belong to $\mathcal{M}$.

Thus, by restriction, we have in fact a *-homomorphism

$$\begin{aligned} \widetilde{\pi} : B(\sigma(T)) &\longrightarrow \mathcal{M} \\ f &\longmapsto f(T) \end{aligned}$$

satisfying the properties of Theorem 1, i.e. we have a Borel functional calculus for normal operators of a von Neumann algebra.

References

[1] W. Arveson, *A Short Course on Spectral Theory*, Graduate Texts in Mathematics, 209, Springer, New York, 2002

[2] N. Weaver, *Mathematical Quantization*, Studies in Advanced Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2001