functional calculus

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1 Basic Idea

Let X be a normed vector space over a field $\mathbb{K}$. Let T be a linear operator in X and I the identity operator in X.

The **functional calculus** refers to a specific process which enables the expression

$$f(T)$$

to make sense as a linear operator in X, for certain scalar functions $f : \mathbb{K} \longrightarrow \mathbb{K}$.

At first sight, and for most functions f, there is no reason why the above expression should be associated with a particular linear operator.

But, for example, when f is a polynomial $f(x) = a_k x^k + \cdots + a_2 x^2 + a_1 x + a_0$, the expression

$$f(T) := a_k T^k + \cdots + a_2 T^2 + a_1 T + a_0 I$$

does indeed refer to a linear operator in X.

As another example, when T is a matrix in $\mathbb{R}^n$ or $\mathbb{C}^n$ one is sometimes led to the `http://planetmath.org/MatrixExponential`exponential of T

$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$$

Thus, we are applying the scalar exponential function to a matrix.

Note in this last example that e^T is approximated by polynomials (the partial sums of the series). This provides an idea of how to make sense of $f(T)$ if f can be approximated by polynomials:

If f can be approximated by polynomials p_n then one could try to define

$$f(T) := \lim_{n \to \infty} p_n(T)$$

But for that one needs to define what "approximated" means and to assure the above limit exists.

2 More abstractly

There is no reason why one should restrict to linear operators in a normed vector space. In this , we can consider instead a unital topological algebra $\mathcal{A}$ over a field $\mathbb{K}$.

There is no definition in mathematics of **functional calculus**, but the ideas above show that a *functional calculus for an element* $T \in \mathcal{A}$ should be something like an homomorphism $(\cdot)(T) : \mathcal{F} \longrightarrow \mathcal{A}$ from some topological algebra of scalar functions $\mathcal{F}$ to $\mathcal{A}$, that satisfied the following :

- $\mathcal{F}$ must contain the polynomial functions.
- $(\cdot)(T)$ is continuous.
- $p(T) = a_k T^k + \cdots + a_2 T^2 + a_1 T + a_0 I$ for each polynomial $p(x) = a_k x^k + \cdots + a_2 x^2 + a_1 x + a_0$, where I denotes the identity element of $\mathcal{A}$.

3 Functional Calculi

There are some functional calculi of . We give a very brief descprition of each one of them (please follows the links for entries with more detailed explanation).

- *polynomial functional calculus* -

  This is valid for any element T in any algebra $\mathcal{A}$. It associates polynomials to elements in the algebra generated by T, as discussed above.

- *functional calculus* -

  This is valid for any element T in a complex Banach algebra $\mathcal{A}$. It associates complex analytic functions defined on the `http://planetmath.org/Spectrum`spectrum of T to elements in the algebra generated by T.

- *continuous functional calculus* -

  This is valid for normal elements in `http://planetmath.org/CAlgebra`C^*-algebras. It associates continuous functions on the spectrum of T to elements in the C^*-algebra generated by T.

- *Borel functional calculus* -

  This is valid for normal operators T in a von Neumann algebra $\mathcal{A}$. It associates bounded Borel measurable functions on the spectrum of T to elements in the von Neumann algebra generated by T.

- *functional calculus for Hermitian matrices* -

  A case. It is valid for Hermitian matrices T. It associates real valued functions on the spectrum of T to elements in the algebra generated by T.

4 Applications

- Functional calculi provide an of constructing new linear operators having specified out of given ones.
- There are strong with spectral theory since one usually has $f(\sigma(T)) = \sigma(f(T))$, where $\sigma(\cdot)$ denotes the spectrum of its . This is called the spectral mapping theorem.
- As the with spectral theory can possibly show, functional calculi are an tool for studying equations. For example, they can give sufficient conditions for the existence of a square root $\sqrt{T}$ of an T.