C^*-algebra homomorphisms preserve continuous functional calculus

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Let us setup some notation first: Let $\mathcal{A}$ be a unital http://planetmath.org/CAlgebraC^*-algebra and z a normal element of $\mathcal{A}$. Then

- $\sigma(z)$ denotes the spectrum of z.
- $C(\sigma(z))$ denotes the C^*-algebra of continuous functions $\sigma(z) \longrightarrow \mathbb{C}$.
- If $f \in C(\sigma(z))$ then $f(z)$ is the element of $\mathcal{A}$ given by the continuous functional calculus.

Theorem - Let $\mathcal{A}$, $\mathcal{B}$ be unital http://planetmath.org/CAlgebraC^*-algebras and $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ a *-homomorphism. Let x be a normal element in $\mathcal{A}$. If $f \in C(\sigma(x))$ then

$$\Phi(f(x)) = f(\Phi(x))$$

Proof: The identity elements of $\mathcal{A}$ and $\mathcal{B}$ will be both denoted by e and it will be clear from the context which one we are referring to.

First, we need to check that $f(\Phi(x))$ is a well-defined element of $\mathcal{B}$, i.e. that $\sigma(\Phi(x)) \subseteq \sigma(x)$. This is clear since, if $x - \lambda e$ is invertible for some $\lambda \in \mathbb{C}$, then $\Phi(x) - \lambda e = \Phi(x - \lambda e)$ is also invertible.

Let $\{p_n\}$ be sequence of polynomials in $C(\sigma(x))$ converging uniformly to f. Then we have that

- $\Phi(p_n(x)) \longrightarrow \Phi(f(x))$, by the continuity of Φ (see http://planetmath.org/HomomorphismsO entry) and the continuity of the continuous functional calculus mapping.
- $p_n(\Phi(x)) \longrightarrow f(\Phi(x))$, by the continuity of the continuous functional calculus mapping.

It is easily checked that $\Phi(p_n(x)) = p_n(\Phi(x))$ (since Φ is an homomorphism). Hence we conclude that $\Phi(f(x)) = f(\Phi(x))$ as intended. $\square$