operator norm of multiplication operator on L^2

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The operator norm of the multiplication operator M_ϕ is the essential supremum of the absolute value of ϕ. (This may be expressed as $\|M_\phi\|_{\text{op}} = \|\phi\|_{L^\infty}$.) In particular, if ϕ is essentially unbounded, the multiplication operator is unbounded.

For the time being, assume that ϕ is essentially bounded.

On the one hand, the operator norm is bounded by the essential supremum of the absolute value because, for any $\psi \in L^2$,

$$\begin{aligned}\|M_\phi\psi\|_{L^2} &= \sqrt{\int \psi(x)^2\phi(x)^2\, d\mu(x)} \\ &\leq \sqrt{(\operatorname{ess\,sup}\ \phi^2)\int \psi(x)^2\, d\mu(x)} \\ &= (\operatorname{ess\,sup}\ |\phi|)\, \|\psi\|_{L^2}\end{aligned}$$

and, hence

$$\|M_\phi\|_{\text{op}} = \sup \frac{\|M_\phi\psi\|_{L^2}}{\|\psi\|_{L^2}} \leq (\operatorname{ess\,sup}\ |\phi|).$$

On the other hand, the operator norm bounds by the essential supremum of the absolute value . For any $\epsilon > 0$, the measure of the set

$$A = \{x \mid\ |\phi(x)| \geq \operatorname{ess\,sup}\ |\phi| - \epsilon\}$$

is greater than zero. If $\mu(A) < \infty$, set $B = A$, otherwise let B be a subset of A whose measure is finite. Then, if χ_B is the characteristic function of B, we have

$$\begin{aligned}\|M_\phi\chi_B\|_{L^2} &= \sqrt{\int \phi(x)^2\chi_B(x)^2\, d\mu(x)} \\ &= \sqrt{\int_B \phi(x)^2\, d\mu(x)} \\ &\geq \mu(B)(\operatorname{ess\,sup}\ |\phi| - \epsilon)\end{aligned}$$

and, hence

$$\|M_\phi\|_{\text{op}} = \sup \frac{\|M_\phi\psi\|_{L^2}}{\|\psi\|_{L^2}} \geq \frac{\|M_\phi\chi_B\|_{L^2}}{\|\chi_B\|_{L^2}} = \operatorname{ess\,sup}\ |\phi| - \epsilon.$$

Since this is true for every $\epsilon > 0$, we must have

$$\|M_\phi\|_{\text{op}} \geq \operatorname{ess\,sup} |\phi|.$$

Combining with the inequality in the opposite direction,

$$\|M_\phi\|_{\text{op}} = \operatorname{ess\,sup} |\phi|.$$

It remains to consider the case where $|\phi|$ is essentially unbounded. This can be dealt with by a variation on the preceeding argument.

If ϕ is unbounded, then $\mu(\{x \mid |\phi(x)| \geq R\}) > 0$ for all $R > 0$. Furthermore, for any $R > 0$, we can find $N > R$ such that $\mu(A) > 0$, where

$$A = \{x \mid N + 1 \geq |\phi(x)| \geq N\}.$$

If $\mu(A) < \infty$, set $B = A$, otherwise let B be a subset of A whose measure is finite. Then, if χ_B is the characteristic function of B, we have

$$\begin{aligned} \|M_\phi \chi_B\|_{L^2} &= \sqrt{\int \phi(x)^2 \chi_B(x)^2 \, d\mu(x)} \\ &= \sqrt{\int_B \phi(x)^2 \, d\mu(x)} \\ &\geq \mu(B) N \end{aligned}$$

and, hence

$$\|M_\phi\|_{\text{op}} = \sup \frac{\|M_\phi \psi\|_{L^2}}{\|\psi\|_{L^2}} \geq \frac{\|M_\phi \chi_B\|_{L^2}}{\|\chi_B\|_{L^2}} = N \geq R.$$

Since this is true for every R, we see that the operator norm is infinite, i.e. the operator is unbounded.