Baker-Campbell-Hausdorff formula(e)

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Given a linear operator A, we define:

$$\exp A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k. \tag{1}$$

It follows that

$$\frac{\partial}{\partial \tau} e^{\tau A} = A e^{\tau A} = e^{\tau A} A. \tag{2}$$

Consider another linear operator B. Let $B(\tau) = e^{\tau A} B e^{-\tau A}$. Then one can prove the following series representation for $B(\tau)$:

$$B(\tau) = \sum_{m=0}^{\infty} \frac{\tau^m}{m!} B_m, \tag{3}$$

where $B_m = [A, B]_m := [A, [A, B]_{m-1}]$ and $B_0 := B$. A very important special case of eq. (**??**) is known as the *Baker-Campbell-Hausdorff (BCH)* formula. Namely, for $\tau = 1$ we get:

$$e^A \; B e^{-A} = \sum_{m=0}^{\infty} \frac{1}{m!} B_m. \tag{4}$$

Alternatively, this expression may be rewritten as

$$[B, e^{-A}] = e^{-A} \left([A, B] + \frac{1}{2}[A, [A, B]] + \cdots \right), \tag{5}$$

or

$$[e^A, B] = \left([A, B] + \frac{1}{2}[A, [A, B]] + \cdots \right) e^A. \tag{6}$$

There is a descendent of the BCH formula, which often is also referred to as BCH formula. It provides us with the multiplication law for two exponentials of linear operators: Suppose $[A, [A, B]] = [B, [B, A]] = 0$. Then,

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}. \tag{7}$$

Thus, if we want to commute two exponentials, we get an extra factor

$$e^A e^B = e^B e^A e^{[A,B]}. \tag{8}$$