canonical basis for symmetric bilinear forms

Canonical name	CanonicalBasisForSymmetricBilinearForms
Date of creation	2013-03-22 14:56:25
Last modified on	2013-03-22 14:56:25
Owner	Mathprof (13753)
Last modified by	Mathprof (13753)
Numerical id	7
Author	Mathprof (13753)
Entry type	Definition
Classification	msc 47A07
Classification	msc 11E39
Classification	msc 15A63
Defines	Sylvester's Law of Inertia

If $B : V \times V \to K$ is a symmetric bilinear form over a finite-dimensional vector space, where the characteristic of the field is not 2, then we may prove that there is an orthogonal basis such that B is represented by

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}$$

Recall that a bilinear form has a well-defined rank, and denote this by r.

If $K = \mathbb{R}$ we may choose a basis such that $a_1 = \cdots = a_t = 1$, $a_{t+1} = \cdots = a_{t+p} = -1$ and $a_{t+p+j} = 0$, for some integers p and t, where $1 \leq j \leq n-t-p$. Furthermore, these integers are *invariants* of the bilinear form. This is known as *Sylvester's Law of Inertia.* B is *positive definite* if and only if $t = n$, $p = 0$. Such a form constitutes a *real inner product space.*

If $K = \mathbb{C}$ we may go further and choose a basis such that $a_1 = \cdots = a_r = 1$ and $a_{r+j} = 0$, where $1 \leq j \leq n - r$.

If $K = F_p$ we may choose a basis such that $a_1 = \cdots = a_{r-1} = 1$,

$a_r = n$ or $a_r = 1$; and $a_{r+j} = 0$, where $1 \leq j \leq n - r$, and n is the least positive quadratic non-residue.