# Euler-Lagrange differential equation (elementary)

| | |
|---|---|
| Canonical name | EulerLagrangeDifferentialEquationelementary |
| Date of creation | 2013-03-22 12:21:49 |
| Last modified on | 2013-03-22 12:21:49 |
| Owner | rspuzio (6075) |
| Last modified by | rspuzio (6075) |
| Numerical id | 34 |
| Author | rspuzio (6075) |
| Entry type | Definition |
| Classification | msc 47A60 |
| Classification | msc 70H03 |
| Classification | msc 49K05 |
| Synonym | Euler-Lagrange condition |
| Related topic | CalculusOfVariations |
| Related topic | BeltramiIdentity |
| Related topic | VersionOfTheFundamentalLemmaOfCalculusOfVariations |
| Defines | Euler-Lagrange differential equation |
| Defines | Lagrangian |

Let $q(t)$ be a twice differentiable function from $\mathbb{R}$ to $\mathbb{R}$ and let $L$ be a twice differentiable function from $\mathbb{R}^3$ to $\mathbb{R}$. Let $\dot{q}$ denote $\frac{d}{dt}q$.

Define the functional $I$ as follows:

$$I(q) = \int_a^b L(t, q(t), \dot{q}(t))\, dt$$

Suppose we regard the function $L$ and the limits of integratiuon $a$ and $b$ as fixed and allow $q$ to vary. Then we could ask for which functions $q$ (if any) this integral attains an extremal (minimum or maximum) value. (Note: especially in Physics literature, the function $L$ is known as the *Lagrangian*.)

Suppose that a differentiable function $q_0\colon [a,b] \to \mathbb{R}$ is an extremum of $I$. Then, for every differentiable function $f\colon [-1,+1] \times [a,b] \to \mathbb{R}$ such that $f(0,x) = q_0(x)$, the function $g\colon [-1,+1] \to \mathbb{R}$, defined as

$$g(\lambda) = \int_a^b L\left(t, f(\lambda, t), \frac{\partial f}{\partial t}(\lambda, t)\right)\, dt$$

will have an extremum at $\lambda = 0$. If this function is differentiable, then $dg/d\lambda = 0$ when $\lambda = 0$.

By studying the condition $dg/d\lambda = 0$ (see the addendum to this entry for details), one sees that, if a function $q$ is to be an extremum of the integral $I$, then $q$ must satisfy the following equation:

$$\frac{\partial}{\partial q}L - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}L\right) = 0. \tag{1}$$

This equation is known as the *Euler–Lagrange differential equation* or the Euler-Lagrange condition. A few comments on notation might be in . The notations $\frac{\partial}{\partial q}L$ and $\frac{\partial}{\partial \dot{q}}L$ denote the partial derivatives of the function $L$ with respect to its second and third arguments, respectively. The notation $\frac{d}{dt}$ means that one is to first make the argument a function of $t$ by replacing the second argument with $q(t)$ and the third argument with $\dot{q}(t)$ and secondly, differentiate the resulting function with respect to $t$. Using the chain rule, the Euler-Lagrange equation can be written as follows:

$$\frac{\partial}{\partial q}L - \frac{\partial^2}{\partial t \partial \dot{q}}L - \dot{q}\frac{\partial^2}{\partial q \partial \dot{q}}L - \ddot{q}\frac{\partial^2}{\partial \dot{q}^2}L = 0 \tag{2}$$

This equation plays an important role in the calculus of variations. In using this equation, it must be remembered that it is only a necessary condition and, hence, given a solution of this equation, one cannot  to the conclusion that this solution is a local extremum of the functional $F$. More work is needed to determine whether the solution of the Euler-Lagrange equation is an extremum of the integral $I$ or not.

In the special case $\frac{\partial}{\partial t}L = 0$, the Euler-Lagrange equation can be replaced by the Beltrami identity.