Brouwer fixed point in one dimension

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Theorem 1 [?, ?] Suppose f is a continuous function $f : [-1,1] \to [-1,1]$. Then f has a fixed point, i.e., there is a x such that $f(x) = x$.

Proof (Following [?]) We can assume that $f(-1) > -1$ and $f(+1) < 1$, since otherwise there is nothing to prove. Then, consider the function $g : [-1,1] \to \mathbb{R}$ defined by $g(x) = f(x) - x$. It satisfies

$$\begin{aligned} g(+1) &< 0, \\ g(-1) &> 0, \end{aligned}$$

so by the intermediate value theorem, there is a point x such that $g(x) = 0$, i.e., $f(x) = x$. $\Box$

Assuming slightly more of the function f yields the Banach fixed point theorem. In one dimension it states the following:

Theorem 2 Suppose $f : [-1,1] \to [-1,1]$ is a function that satisfies the following condition:

for some constant $C \in [0,1)$, we have for each $a, b \in [-1,1]$,

$$|f(b) - f(a)| \leq C|b - a|.$$

Then f has a unique fixed point in $[-1,1]$. In other words, there exists one and only one point $x \in [-1,1]$ such that $f(x) = x$.

Remarks The fixed point in Theorem 2 can be found by iteration from any $s \in [-1,1]$ as follows: first choose some $s \in [-1,1]$. Then form $s_1 = f(s)$, then $s_2 = f(s_1)$, and generally $s_n = f(s_{n-1})$. As $n \to \infty$, s_n approaches the fixed point for f. More details are given on the entry for the Banach fixed point theorem. A function that satisfies the condition in Theorem 2 is called a contraction mapping. Such mappings also satisfy the `http://planetmath.org/LipschitzCondition`Lipschitz condition.

References

[1] A. Mukherjea, K. Pothoven, *Real and Functional analysis*, Plenum press, 1978.