operator norm

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Definition

Let $A\colon \mathsf{V} \to \mathsf{W}$ be a linear map between normed vector spaces V and W. To each such map (operator) A we can assign a non-negative number $\|A\|_{\mathrm{op}}$ defined by

$$\|A\|_{\mathrm{op}} := \sup_{\substack{\mathbf{v} \in \mathsf{V} \\ \mathbf{v} \neq \mathbf{0}}} \frac{\|A\mathbf{v}\|}{\|\mathbf{v}\|},$$

where the supremum $\|A\|_{\mathrm{op}}$ could be finite or infinite. Equivalently, the above definition can be written as

$$\|A\|_{\mathrm{op}} := \sup_{\substack{\mathbf{v} \in \mathsf{V} \\ \|\mathbf{v}\|=1}} \|A\mathbf{v}\| = \sup_{\substack{\mathbf{v} \in \mathsf{V} \\ 0<\|\mathbf{v}\|\leq 1}} \|A\mathbf{v}\|.$$

By convention, if V is the zero vector space, any operator from V to W must be the zero operator and is assigned zero norm.

$\|A\|_{\mathrm{op}}$ is called the the *operator norm* (or the *induced norm*) of A, for reasons that will be clear in the next .

Operator norm is in fact a norm

Definition - If $\|A\|_{\mathrm{op}}$ is finite, we say that A is a . Otherwise, we say that A is .

It turns out that, for bounded operators, $\|\cdot\|_{\mathrm{op}}$ satisfies all the properties of a norm (hence the name *operator norm*). The proof follows immediately from the definition:

Positivity: Since $\|A\mathbf{v}\| \geq 0$, by definition $\|A\|_{\mathrm{op}} \geq 0$. Also, $\|A\mathbf{v}\| = 0$ identically only if $A = 0$. Hence $\|A\|_{\mathrm{op}} = 0$ only if $A = 0$.

Absolute homogeneity: Since $\|\lambda A\mathbf{v}\| = |\lambda|\|A\mathbf{v}\|$, by definition $\|\lambda A\|_{\mathrm{op}} = |\lambda|\|A\|_{\mathrm{op}}$.

Triangle inequality: Since $\|(A+B)\mathbf{v}\| = \|A\mathbf{v}+B\mathbf{v}\| \leq \|A\mathbf{v}\|+\|B\mathbf{v}\|$, by definition $\|A+B\|_{\mathrm{op}} \leq \|A\|_{\mathrm{op}} + \|B\|_{\mathrm{op}}$.

The set $L(\mathsf{V},\mathsf{W})$ of bounded linear maps from V to W forms a vector space and $\|\cdot\|_{\mathrm{op}}$ defines a norm in it.

Example

Suppose that $\mathsf{V} = (\mathbb{R}^n, \|\cdot\|_p)$ and $\mathsf{W} = (\mathbb{R}^n, \|\cdot\|_p)$, where $\|\cdot\|_p$ is the vector p-norm. Then the operator norm $\|\cdot\|_{\mathrm{op}} = \|\cdot\|_p$ is the matrix p-norm.