Euler-Lagrange differential equation (advanced)

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Defines	Euler-Lagrange differential equation

Let M and N be C^2 manifolds. Let $L\colon M \times TN \to \mathbb{R}$ be twice differentiable. Define a functional $F\colon D \subset C^2(M,N) \to \mathbb{R}$ as

$$F(q) = \int_M L\left(x, q(x), \mathbf{D}q(x)\right)\, d^m x$$

where D is the subset of `http://planetmath.org/node/5555`$C^2(M,N)$ for which this integral converges.

Note that if $f \in D$ and $g \in C^2(M,N)$ and the set $\{x \in M \mid f(x) \neq g(x)\}$ is compact, then $g \in D$. We may impose a topology on D as follows: Suppose that $f \in D$, that $K \subset M$ is compact, and that $U_0 \subset C^2(K,N)$ is open. Then we define an open set $U \subset D$ as the set of all functions $g \in D$ such that $f(x) = g(x)$ when $x \notin K$ and such that the restriction of g to K lies in U_0.

It is not hard to show that the functional F is continuous in this topology, and hence it makes sense to speak of local extrema of F. Suppose that $q_0 \in C^2(M,N)$ is a local extremum. Furthermore, suppose that $f\colon M \times [-1,+1] \to N$ is twice differentiable and $f(x,0) = q_0(x)$ for all $x \in q_0$ and $f(x,y) = q_0(x)$ for all $y \in [-1,+1]$ when x does not lie in a certain compact subset $K \subset M$. Then, viewed as a map from $[-1,+1]$ to D, f will be continuous. Therefore, since q_0 is a local extremum of F, 0 wil be a local extremum of the function $y \mapsto F(f(\cdot,y))$. Because the function $y \mapsto F(f(\cdot,y))$ is differentiable, it will be the case that

$$\frac{d}{d\lambda} F(f(\cdot,\lambda))\big|_{\lambda=0} = 0$$

It can be shown (see the addendum to this entry) that this condition will be satisfied if and only if q_0 is a solution of the following differential equation:

$$dL - d\left(\frac{\partial L}{\partial(dq)}\right) = 0. \tag{1}$$

This differential equation is known as the *Euler-Lagrange differential equation* (or Euler-Lagrange condition).

The Euler-Lagrange equation can only be used to investigate local extrema which are smooth functions. To a certain extent, this limitation can be ameliorated — one can study piecewise smooth functions by supplementing the Euler-Lagrange equation with auxiliary conditions at discontinuities and, in some cases, one can consider non-smooth solutions as weak solutions of the Euler-Lagrange equation.

In the special cases $dL = 0$, the Euler-Lagrange equation can be replaced by the Beltrami identity.