proof of Brouwer fixed point theorem

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The n-dimensional simplex $\mathcal{S}_n$ is the following subset of $\mathbb{R}^{n+1}$

$$\left\{ (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \,\middle|\, \sum_{i=1}^{n+1} \alpha_i = 1, \quad \alpha_i \geq 0 \quad \forall i = 1, \ldots, n+1 \right\}$$

Given an element $x = \sum_i \alpha_i e_i \in \mathcal{S}_n$ we denote $[x]_i = \alpha_i$ (i.e., the i-th barycentric coordinate). We also denote $F(x) = \{\, i \,|\, [x]_i \neq 0\}$. An I-face of $\mathcal{S}_n$ is the subset $\{\, x \,|\, F(x) \subseteq I\}$.

As was noted in the statement of the theorem, the 'shape' is unimportant. Therefore, we will prove the following variant of the theorem using the KKM lemma.

Theorem 1 (Brouwer's Fixed Point Theorem)**.** *Let $f : \mathcal{S}_n \to \mathcal{S}_n$ be a continuous function. Then, f has a fixed point, namely, there is an $L \in \mathcal{S}_n$ such that $L = f(L)$.*

Proof. Clearly, $\sum_{i=1}^{n} [y]_i = 1$ for any $y \in \mathcal{S}_n$ and $L = f(L)$ if and only if $[L]_i = [f(L)]_i$ for all $i = 1, 2, \ldots, n+1$. For each $i = 1, 2, \ldots, n+1$ we define the following subset C_i of $\mathcal{S}_n$:

$$C_i = \left\{ x \in \mathcal{S}_n \,\middle|\, [x]_i \geq [f(x)]_i \right\}$$

We claim that if x is in some I-face of $\mathcal{S}_n$ ($I \subseteq \{1, 2, \ldots, n+1\}$) then there is an index $i \in I$ such that $x \in C_i$. Indeed, if x is in some I-face then $F(v) \subseteq I$. Thus, if $[x]_i \neq 0$ then $i \in I$. This shows that

$$\sum_{i \in I} [x]_i = 1$$

Assuming by contradiction that $x \notin C_i$ for all $i \in I$ implies that $[x]_i < [f(x)]_i$ for all $i \in I$. But this leads to a contradiction as the following inequality shows:

$$1 = \sum_{i \in I} [x]_i < \sum_{i \in I} [f(x)]_i \leq \sum_{i=1}^{n} [f(x)]_i = 1$$

This dicussion establishes that each I-face is contained in the union $\cup_{i \in I} C_i$. In addition, the subsets C_i are all closed. Therefore, we have shown that the hypothesis of the KKM Lemma holds.

By the KKM lemma there is a point L that is in every C_i for $i = 1, 2, \ldots, n+1$. We claim that L is a fixed point of f. Indeed, $[L]_i \geq [f(L)]_i \geq 0$ for all $i = 1, 2, \ldots, n+1$ and thus:

$$1 = [L]_1 + [L]_2 + \cdots + [L]_{n+1} \geq [f(L)]_1 + [f(L)]_2 + \cdots + [f(L)]_{n+1} = 1$$

Therefore, $[L]_i = [f(L)]_i$ for all $i = 1, 2, \ldots, n+1$ which implies that $L = f(L)$. $\square$