partial isometry on Hilbert spaces

Canonical name	PartialIsometryOnHilbertSpaces
Date of creation	2013-03-22 18:35:00
Last modified on	2013-03-22 18:35:00
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Numerical id	7
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Entry type	Definition
Classification	msc 47C10

Definition 1. Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. An operator $W \in L(\mathscr{H}, \mathscr{K})$ is called a *partial isometry* if W is an isometry on $M = (\ker W)^{\perp}$. We then call $M = (\ker W)^{\perp}$ the *initial space* and $N = WM$ *final space* of W.

We need to show that the above definition is compatible with the general definition of partial isometry on rings. Indeed we have the following:

Proposition 1. *$W \in L(\mathscr{H}, \mathscr{K})$ is a partial isometry iff W^*W is a projection from $\mathscr{H}$ to M.*

Proof. We have:

$$\begin{aligned} & W \text{ partial isometry with initial space } M \\ \Leftrightarrow\ & \langle Wf, Wg\rangle = \langle f, g\rangle \ \forall\ f, g \in M \\ \Leftrightarrow\ & \langle W^*Wf, g\rangle = \langle f, g\rangle \ \forall\ f \in M, g \in \mathscr{H} \\ \Leftrightarrow\ & W^*Wf = f, f \in M \\ \text{and}\ & W^*Wf = 0, f \in M^{\perp} = \ker W \end{aligned}$$

$\square$

Remark 1. *If $W \in L(\mathscr{H}, \mathscr{K})$ is a partial isometry with initial space M and final space N we have:*

$$\begin{aligned} W^*(Wf) &= f \ \forall\ f \in M \\ \ker W^* &= (\text{ran} W)^{\perp} = N^{\perp} \end{aligned}$$

Thus N is the initial space and M the final space of W^*.