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Dunkl-Williams inequality

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Let V be an inner product space and $a, b \in V$. If $a \neq 0$ and $b \neq 0$, then

$$\|a-b\| \geq \frac{1}{2}(\|a\|+\|b\|)\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\|. \tag{1}$$

Equality holds if and only if $a=0$, $b=0$, $\|a\|=\|b\|$ or $a\|b\|=b\|a\|$. In fact, if (1) holds and V is a normed linear space, then V is an inner product space.

If X is a normed linear space and $a \neq 0$ and $b \neq 0$ then

$$\|a-b\| \geq \frac{1}{4}(\|a\|+\|b\|)\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\|. \tag{2}$$

Equality holds if and only if $a=0$, $b=0$ or $a=b$. The constant $\frac{1}{4}$ is best possible. For example, let X be the set of ordered pairs of real numbers, with norm of (x_1, x_2) equal to $|x_1|+|x_2|$. Let $a=(1,\epsilon)$ and $b=(1,0)$ where ϵ is a small positive number. After a bit of routine calculation, it is easily seen that the best possible constant is $\frac{1}{4}$.

The inequality (2) has been generalized in the case where X is a normed linear space over the reals. In that case one can show:

$$\|a-b\| \geq c_p(\|a\|^p+\|b\|^p)^{1/p}\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\| \tag{3}$$

where $c_p=2^{-1-1/p}$ if $0<p\leq 1$ and $c_p=1/4$ if $p \geq 1$. The case $p=1$ is the Dunkl and Williams inequality.

If X is a normed linear space and $0<p\leq 1$ then (3) holds with $c_p=2^{-1/p}$ if and only if X is an inner product space.

The inequality (2) can be improved slightly to get:

$$\|a-b\| \geq \frac{1}{2}\max(\|a\|,\|b\|)\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\|. \tag{4}$$

Equality holds in (4) if and only if a and b span an ${\ell_2}^1$ in the underlying real vector space with $\pm\|b-a\|^{-1}(b-a)$ and $\pm\|a\|^{-1}a$ (or $\pm\|b\|^{-1}b$) as the vertices of the unit parallelogram.

References

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