Fuglede-Putnam-Rosenblum theorem

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Let A be a C^*-algebra with unit e.

The Fuglede-Putnam-Rosenblum theorem makes the assertion that for a normal element $a \in A$ the kernel of the commutator mapping $[a, -]\colon A \to A$ is a $*$-closed set.

The general formulation of the result is as follows:

Theorem. Let A be a C^*-algebra with unit e. Let two normal elements $a, b \in A$ be given and $c \in A$ with $ac = cb$. Then it follows that $a^*c = cb^*$.

Lemma. For any $x \in A$ we have that $\exp(x - x^*)$ is a element of A.

Proof. We have for $x \in A$ that $\exp(x - x^*)^* \exp(x - x^*) = \exp(x^* - x + x - x^*) = \exp(0) = e$. And similarly $\exp(x - x^*)\exp(x - x^*)^* = e$. $\square$

With this we can now give a proof the Theorem.

Proof. The condition $ac = cb$ implies by induction that $a^k c = cb^k$ holds for each $k \in \mathbb{N}$. Expanding in power series on both sides yields $\exp(a)c = c\exp(b)$. This is equivalent to $c = \exp(-a)c\exp(b)$. Set $U_1 := \exp(a^* - a), U_2 := \exp(b - b^*)$. From the Lemma we obtain that $\|U_1\|_A = \|U_2\|_A = 1$. Since a commutes with a^* und b with b^* we obtain that

$$\exp(a^*)c\exp(-b^*) = \exp(a^*)\exp(-a)c\exp(b)\exp(b^*)$$

which equals $\exp(a^* - a)c\exp(b - b^*) = U_1 c U_2$.

Hence

$$\|\exp(a^*)c\exp(-b^*)\| \leq \|c\|.$$

Define $f\colon \mathbb{C} \to A$ by $f(\lambda) := \exp(\lambda a^*)c\exp(-\lambda b^*)$. If we substitute $a \mapsto \lambda a, b \mapsto \lambda b$ in the last estimate we obtain

$$\|f(\lambda)\| \leq \|c\|,\ \lambda \in \mathbb{C}.$$

But f is clearly an entire function and therefore Liouville's theorem implies that $f(\lambda) = f(0) = c$ for each λ.

This yields the equality

$$c\exp(\lambda b^*) = \exp(\lambda a^*)c.$$

Comparing the terms of first order for λ small finishes the proof. $\square$