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proof of Schauder fixed point theorem

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The idea of the proof is to reduce to the finite dimensional case where we can apply the Brouwer fixed point theorem.

Given $\epsilon > 0$ notice that the family of open sets $\{B_\epsilon(x) \colon x \in K\}$ is an open covering of K. Being K compact there exists a finite subcover, i.e. there exists n points $x_1, \dots, x_n$ of K such that the balls $B_\epsilon(x_i)$ cover the whole set K.

Define the functions $g_1, \dots, g_n$ by

$$g_i(x) := \begin{cases} \epsilon - \|x - x_i\|, & \text{if } \|x - x_i\| \leq \epsilon \\ 0, & \text{if } \|x - x_i\| \geq \epsilon \end{cases}$$

It is clear that each g_i is continuous, $g_i(x) \geq 0$ and $\sum_{i=1}^n g_i(x) > 0$ for every $x \in K$.

Thus we can define a function in K by

$$g(x) := \frac{\sum_{i=1}^n g_i(x) x_i}{\sum_{i=1}^n g_i(x)}$$

The above function g is a continuous function from K to the convex hull K_0 of $x_1, \dots, x_n$. Moreover one can easily prove the following

$$\|g(x) - x\| \leq \epsilon \qquad \forall_{x \in K}$$

Now, define the function $B := g \circ f$. The restriction $\tilde{B}$ of B to K_0 provides a continuous function $K_0 \longrightarrow K_0$.

Since K_0 is compact convex subset of a finite dimensional vector space, we can apply the Brouwer fixed point theorem to assure the existence of $z \in K_0$ such that

$$B(z) = \tilde{B}(z) = z$$

Therefore $g(f(z)) = z$ and we have the inequality

$$\|f(z) - z\| = \|f(z) - g(f(z))\| \leq \epsilon$$

Summarizing, for each $\epsilon > 0$ there exists $z = z(\epsilon) \in K$ such that $\|f(z) - z\| \leq \epsilon$. Then

$$\forall_{m \in \mathbb{N}} \qquad \exists_{z_m \in K} \qquad \|f(z_m) - z_m\| \leq \frac{1}{m}$$

As $f(z_m)$ is in the compact space K, there is a subsequence z_{m_k} such that $f(z_{m_k}) \longrightarrow x_0$, for some $x_0 \in K$.

We then have

$$\begin{aligned} \|z_{m_k} - x_0\| &= \|z_{m_k} - f(z_{m_k}) + f(z_{m_k}) - x_0\| \\ &\leq \|f(z_{m_k}) - z_{m_k}\| + \|f(z_{m_k}) - x_0\| \\ &\leq \frac{1}{m_k} + \|f(z_{m_k}) - x_0\| \longrightarrow 0 \end{aligned}$$

which means that $z_{m_k} \longrightarrow x_0$.

As f is continuous we have $f(z_{m_k}) \longrightarrow f(x_0)$. Both limits of $f(z_{m_k})$ must coincide, so we conclude that

$$f(x_0) = x_0$$

i.e. f has a fixed point. $\square$