bounded inverse theorem

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The next result is a corollary of the open mapping theorem. It is often called the **bounded inverse theorem** or the **inverse mapping theorem**.

Theorem - Let X, Y be Banach spaces. Let $T : X \longrightarrow Y$ be an invertible bounded operator. Then T^{-1} is also .

Proof : T is a surjective continuous operator between the Banach spaces X and Y. Therefore, by the open mapping theorem, T takes open sets to open sets.

So, for every open set $U \subseteq X$, $T(U)$ is open in Y.

Hence $(T^{-1})^{-1}(U)$ is open in Y, which proves that T^{-1} is continuous, i.e. bounded. $\square$

0.0.1 Remark

It is usually of great importance to know if a bounded operator $T : X \longrightarrow Y$ has a bounded inverse. For example, suppose the equation

$$Tx = y$$

has unique solutions x for every given $y \in Y$. Suppose also that the above equation is very difficult to solve (numerically) for a given y_0, but easy to solve for a value $\tilde{y}$ "near" y_0. Then, if T^{-1} is continuous, the correspondent solutions x_0 and $\tilde{x}$ are also "near" since

$$\|x_0 - \tilde{x}\| = \|T^{-1}y_0 - T^{-1}\tilde{y}\| \leq \|T^{-1}\|\|y_0 - \tilde{y}\|$$

Therefore we can solve the equation for a "near" value $\tilde{y}$ instead, without obtaining a significant error.