continuous functional calculus

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Let $B(H)$ be the algebra of bounded operators over a complex Hilbert space H. Let $T \in B(H)$ be a normal operator.

The **continuous functional calculus** is a functional calculus which enables the expression

$$f(T)$$

to make sense as a bounded operator in H, for continuous functions f.

More generally, when $\mathcal{A}$ is a http://planetmath.org/CAlgebraC^*-algebra with identity element e, and x is a normal element of $\mathcal{A}$, the *continuous functional calculus* allows one to define $f(x) \in \mathcal{A}$ when f is a continuous function.

More precisely, if $\sigma(x)$ denotes the spectrum of x and $C(\sigma(x))$ denotes the C^*-algebra of complex valued continuous functions on $\sigma(x)$, we will define a continuous homomorphism

$$\begin{aligned} C(\sigma(x)) &\longrightarrow \mathcal{A} \\ f &\mapsto f(x) \end{aligned}$$

that the http://planetmath.org/FunctionalCalculusfunctional calculus .

There are several reasons to require the continuity of f on the spectrum $\sigma(x)$.

For example, suppose $\lambda_0 \in \sigma(x)$. The function $f(\lambda) = \frac{1}{\lambda - \lambda_0}$ is clearly not continuous in λ_0. By the functional calculus we would obtain

$$f(x) = \frac{1}{x - \lambda_0 e} = (x - \lambda_0 e)^{-1}$$

but $x - \lambda_0 e$ is not invertible since $\lambda_0 \in \sigma(x)$.

The abstraction towards C^*-algebras is almost . Indeed, C^*-algebras are the appropriate where to and prove the continuous functional calculus. The conclusions towards $B(H)$ then follow as a particular case.

1 Preliminary construction

Let $\mathcal{A}$ be a unital C^*-algebra and x a normal element in $\mathcal{A}$. Let $\mathcal{B} \subseteq \mathcal{A}$ be the C^*-subalgebra generated by x and the identity e of $\mathcal{A}$.

Thus, $\mathcal{B}$ is the norm closure of the algebra generated by x, x^* and e.

Moreover, since x is , $x^*x = xx^*$, it follows that $\mathcal{B}$ is commutative and $\mathcal{B}$ consists of those elements $y \in \mathcal{A}$ that can be approximated by polynomials $p(x, x^*)$ in x and x^*.

Recall the following facts:

- Since $\mathcal{B}$ is a commutative unital C^*-algebra, the set $\triangle$ of multiplicative linear functionals on $\mathcal{B}$ is a compact Hausdorff space.
- Let $C(\triangle)$ denote the C^*-algebra of complex valued continuous functions on $\triangle$. The Gelfand transform $G : \mathcal{B} \longrightarrow C(\triangle)$ is a C^*-isomorphism.

The following result is perhaps the for the definition of the continuous functional calculus.

Theorem 1 - *$\triangle$ and $\sigma(x)$ are homeomorphic topological spaces. Moreover, the mapping $S : \triangle \to \sigma(x)$ defined by*

$$S(\phi) := \phi(x)$$

is such an homeomorphism.

: We need to check that S is well defined, i.e. $\phi(x) \in \sigma(x)$ for all $\phi \in \triangle$. From the identity

$$\phi(x - \phi(x)e) = \phi(x) - \phi(x)\phi(e) = \phi(x) - \phi(x) = 0$$

follows that $x - \phi(x)e$ cannot be invertible in $\mathcal{B}$ (recall that ϕ is a multiplicative linear functional on $\mathcal{B}$).

Thus, $\phi(x) \in \sigma_{\mathcal{B}}(x)$. By the spectral invariance theorem, we see that $\phi(x) \in \sigma(x) = \sigma_{\mathcal{B}}(x)$, and so S is well defined.

- S **is continuous -** Suppose ϕ_α is a net in $\triangle$ such that $\phi_\alpha \longrightarrow \phi$. Recall that the topology in $\triangle$ is the weak-* topology, so $\phi_\alpha(x) \longrightarrow \phi(x)$.

  Thus, $S(\phi_\alpha) \longrightarrow S(\phi)$ and so S is continuous.

- S **is -** Suppose $S(\phi_1) = S(\phi_2)$. Then, $\phi_1(x) = \phi_2(x)$. Since

  $$\phi_i(x^*) = G(x^*)(\phi_i) = \overline{G(x)(\phi_i)} = \overline{\phi_i(x)}$$

  we must also have $\phi_1(x^*) = \phi_2(x^*)$.

  This clearly implies that

  $$\phi_1(p(x, x^*)) = \phi_2(p(x, x^*))$$

  for every polynomial in two variables p.

  Recall that the "polynomials" $p(x, x^*)$ are dense in $\mathcal{B}$. So we must have $\phi_1(y) = \phi_2(y)$ for every $y \in \mathcal{B}$, i.e. $\phi_1 = \phi_2$.

- S **is -** Let $\lambda \in \sigma(x) = \sigma_{\mathcal{B}}(x)$. Then $x - \lambda e$ is not invertible.

  Since $\mathcal{B}$ is commutative, $x - \lambda e$ is contained in a maximal ideal $\mathcal{M}$.

  As $\mathcal{M}$ is maximal ideal, the quotient $\mathcal{B}/\mathcal{M}$ is a division algebra, and so by the Gelfand-Mazur theorem, $\mathcal{B}/\mathcal{M}$ must ne isomorphic to $\mathbb{C}$.

  Therefore the quotient homomorphism

$$\phi : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{M} = \mathbb{C}$$

  is a multiplicative linear functional such that $\phi(x-\lambda e) = 0$, i.e. $\phi(x) = \lambda$, i.e. $S(\phi) = \lambda$.

  Therefore, S is surjective.

Since S is a continuous bijective function from the compact Hausdorff space $\triangle$ to $\sigma(x)$, it follows that it must be a homeomorphism. $\square$

2 Definition of the continuous functional calculus

If S is the homeomorphism between $\triangle$ and $\sigma(x)$ defined as above, then the mapping $f \mapsto f \circ S$ is a *-isomorphism between the algebras $C(\sigma(x))$ and $C(\triangle)$. Since the Gelfand transform $G : \mathcal{B} \longrightarrow C(\triangle)$ is a also a *-isomorphism, we obtain a *-isomorphism

$$\Gamma : C(\sigma(x)) \longrightarrow \mathcal{B}$$

by setting $\Gamma(f) := G^{-1}(f \circ S)$.

Definition - *Suppose x is a normal element in a unital C^*-algebra $\mathcal{A}$. For every $f \in C(\sigma(x))$ we define*

$$f(x) := \Gamma(f) \quad \in \mathcal{B} \subseteq \mathcal{A}$$

The mapping Γ, such that $f \mapsto f(x)$, is called the **continuous functional calculus** *for x.*

We now prove the `http://planetmath.org/FunctionalCalculus`functional calculus for the continuous functional calculus and show its uniqueness:

Theorem 2 - *Let $\mathcal{A}$ be a unital C^*-algebra, $x \in \mathcal{A}$ a normal element and* id *the identity function in $\mathbb{C}$. The continuous functional calculus for x is the unique unital *-homomorphism between $C(\sigma(x))$ and $\mathcal{A}$ which sends* id *to x. In particular, for every polynomial p in $\mathbb{C}$ of the form $p(\lambda) := \sum c_{n,m}\, \lambda^n \overline{\lambda}^m$, we have $p(x) = \sum c_{n,m}\, x^n (x^*)^m$.*

: We have seen that the continuous functional calculus Γ for x is a *-homomorphism between $C(\sigma(x))$ and $\mathcal{A}$. Recall that Γ was defined by $\Gamma(f) := G^{-1}(f \circ S)$. It is clear by the definition that Γ is unital. Also, $G \circ \Gamma(f) = f \circ S$ for every $f \in C(\triangle)$. Taking the identity function id we obtain that for every $\phi \in \triangle$

$$\begin{aligned} G \circ \Gamma(\mathrm{id})(\phi) &= \mathrm{id}(S(\phi)) \\ &= \phi(x) \\ &= G(x)(\phi) \end{aligned}$$

Since the Gelfand transform is a *-isomorphism, we must have $\Gamma(\mathrm{id}) = x$.

Now, let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial of the form $p(\lambda) := \sum c_{n,m}\, \lambda^n \overline{\lambda}^m$. Notice that $p = \sum c_{n,m}\, \mathrm{id}^n \overline{\mathrm{id}}^m$. If F is any unital *-homomorphism such that $F(\mathrm{id}) = x$, then one must have $F(p) = \sum c_{n,m}\, x^n (x^*)^m$. Thus all such unital *-homomorphisms coincide on the subspace of polynomials of the above form. By the `http://planetmath.org/StoneWeierstrassTheoremComplexVersion`Stone-Weierstrass theorem, this subspace is dense in $C(\sigma(x))$. Thus, all such unital *-homomorphisms coincide in $C(\sigma(x))$, and uniqueness is proven. $\square$

3 Properties

- The spectral mapping theorem assures that for $f \in C(\sigma(x))$

$$\sigma(f(x)) = f(\sigma(x))$$

- When $\sigma(x) \subset \mathbb{R}^+$ the continuous functional calculus assures the existence of a square root $\sqrt{x}$ of x, since $\sqrt{\lambda}$ is defined and continuous on $\lambda \in \sigma(x)$.