proof of Taylor's formula for matrix functions

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Theorem. *Let p be a polynomial and suppose $\mathbf{A}$ and $\mathbf{B}$ are squared matrices of the same size, then* $p(\mathbf{A}+\mathbf{B}) = \sum_{k=0}^{n} \frac{1}{k!} p^{(k)}(\mathbf{A})\mathbf{B}^k$ *where* $n = \deg(p)$.

Proof. Since p is a polynomial, we can apply the Taylor expansion:

$$p(x) = \sum_{k=0}^{n} \frac{1}{k!} p^{(k)}(x_0)(x-x_0)^k$$

where $n = \deg(p)$. Now let $x = \mathbf{A}+\mathbf{B}$ and $x_0 = \mathbf{A}$.

The Taylor expansion can be checked as follows: let $p(x) = \sum_{k=0}^{n} a_k x^k$ for coefficients a_k (note that this coefficients can be taken from the space of square matrices defined over a field). We define the formal derivative of this polynomial as $p^{(1)}(x) = \frac{dp}{dx} = \sum_{k=1}^{n} a_k k x^{k-1}$ and we define $p^{(k)} = \frac{dp^{(k-1)}}{dx}$.

Then $p^{(k)}(x) = \sum_{i=k}^{n} a_i \frac{i!}{(i-k)!} x^{i-k}$ and we have $\frac{1}{k!} p^{(k)}(x_0) = \sum_{i=k}^{n} a_i \frac{i!}{(i-k)!k!}(x_0)^{i-k}$. Now consider

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{1}{k!} p^{(k)}(x_0)(x-x_0)^k = \sum_{k=0}^{n} \left(\sum_{i=k}^{n} a_i \frac{i!}{(i-k)!k!} (x_0)^{i-k} (x-x_0)^k \right) \\
= \;& \sum_{i=0}^{n} a_i (x_0)^i + \sum_{i=1}^{n} a_i i (x_0)^{i-1} (x-x_0) + \cdots + \sum_{i=j}^{n} a_i \frac{i!}{(i-j)!j!} (x_0)^{i-j} (x-x_0)^j + \cdots + a_n (x-x_0)^n \\
= \;& a_0 + a_1(x) + \cdots + a_i \left(\sum_{j=0}^{i} \frac{i!}{(i-j)!j!} (x_0)^{i-j} (x-x_0)^j \right) + \cdots + a_n \left(\sum_{j=0}^{n} \frac{n!}{(n-j)!j!} (x_0)^{n-j} (x-x_0)^j \right) \\
= \;& \sum_{k=0}^{n} a_k x^i = p(x)
\end{aligned}$$

since $\sum_{j=0}^{i} \frac{i!}{(i-j)!j!} (x_0)^{i-j} (x-x_0)^j = (x)^i$. □