Drazin inverse

Canonical name	DrazinInverse
Date of creation	2013-03-22 13:58:05
Last modified on	2013-03-22 13:58:05
Owner	kronos (12218)
Last modified by	kronos (12218)
Numerical id	29
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Entry type	Definition
Classification	msc 47S99
Related topic	MoorePenroseGeneralizedInverse

A Drazin inverse of an operator A is an operator, B, such that

$$AB = BA,$$

$$BAB = B,$$

$$ABA = A - U,$$

where the spectral radius $r(U) = 0$. The Drazin inverse (B) is denoted by A^D. It exists, if 0 is not an accumulation point of $\sigma(A)$.

For example, a projection operator is its own Drazin inverse, $P^D = P$, as $PPP = PP = P$; for a Shift operator $S^D = 0$ holds.

The following are some other useful properties of the Drazin inverse:

1. $(A^D)^* = (A^*)^D$;

2. $A^D = (A + \alpha P^{(A)})^{-1}(I - P^{(A)})$, where $P^{(A)} := I - A^D A$ is the spectral projection of A at 0 and $\alpha \neq 0$;

3. $A^\dagger = (A^* A)^D A^* = A^*(AA^*)^D$, where $A^\dagger$ is the Moore-Penrose pseudoinverse of A;

4. $A^D = A^m (A^{2m+1})^\dagger A^m$ for $m \geq \operatorname{ind}(A)$, if $\operatorname{ind}(A) := \min\{k : \operatorname{Im} A^k = \operatorname{Im} A^{k+1}\}$ is finite;

5. If the matrix is represented explicitly by its Jordan canonical form, (Λ regular and N nilpotent), then

$$\left(E \begin{bmatrix} \Lambda & 0 \\ 0 & N \end{bmatrix} E^{-1} \right)^D = E \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & 0 \end{bmatrix} E^{-1};$$

6. Let e_λ^A denote an eigenvector of A to the eigenvalue λ. Then $e_\lambda^A + t(\lambda I - A)^D h e_\lambda^A + O(t^2)$ is an eigenvector of $A + th$.