proof of basic criterion for self-adjointness

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1. ($1 \implies 2$) If A is self-adjoint and $Ax = ix$, then

$$i\|x\|^2 = (ix, x) = (Ax, x) = (x, A^*x) = (x, Ax) = (x, ix) = \overline{(ix, x)} = -i\|x\|^2,$$

so $x = 0$. Similarly we prove that $Ax = -ix$ implies $x = 0$. That A is closed follows from the fact that the adjoint of an operator is always closed.

2. ($2 \implies 3$) If 2 holds, then $\{0\} = \operatorname{Ker}(A^* \pm i)^* = \operatorname{Ker}(A \mp i)^* = \operatorname{Ran}(A \mp i)^\perp$, so that $\operatorname{Ran} A \mp i$ is dense in $\mathscr{H}$. Also, since A is symmetric, for $x \in D(A)$,

$$\|(A+i)x\|^2 = \|Ax\|^2 + \|x\|^2 + (Ax, ix) + (ix, Ax) = \|Ax\|^2 + \|x\|^2$$

because $(Ax, ix) = (x, iA^*x) = (x, iAx) = -(ix, Ax)$. Hence $\|x\| \leq \|(A+i)x\|$, so that given a sequence $x_n \in D(A)$ such that $(A+i)x_n \to y$, we have that $\{(A+i)x_n\}$ is a Cauchy sequence and thus $\{x_n\}$ itself is a Cauchy sequence. Hence $\{x_n\}$ converges to some $x \in \mathscr{H}$ and since A is closed it follows that $x \in D(A)$ and $(A+i)x = y$. This proves that $y \in \operatorname{Ran}(A+i)$, so that $\operatorname{Ran}(A+i)$ is closed (and similarly, $\operatorname{Ran}(A-i)$ is closed. Thus $\operatorname{Ran}(A \pm i) = \mathscr{H}$.

3. ($3 \implies 1$) Suppose 3. If $y \in D(A^*)$, then there is $x \in D(A)$ such that $(A+i)x = (A^* - i)y$. Since A is symmetric, $(A+i)x = (A^*+i)x = (A-i)^*x$, so that $(A^* - i)(x - y) = 0$. But since $\operatorname{Ker}(A^* - i) = \operatorname{Ran}(A+i)^\perp = \{0\}$, it follows that $x = y$, so that $y \in D(A)$. Hence $D(A) = D(A^*)$, and therefore A is self-adjoint.