bilinear form

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Defines	left map
Defines	right map

Definition. Let U, V, W be vector spaces over a field K. A *bilinear map* is a function $B : U \times V \to W$ such that

1. the map $x \mapsto B(x, y)$ from U to W is linear for each $y \in V$
2. the map $y \mapsto B(x, y)$ from V to W is linear for each $x \in U$.

That is, B is *bilinear* if it is linear in each parameter, taken separately.

Bilinear forms. A *bilinear form* is a bilinear map $B : V \times V \to K$. A W-valued bilinear form is a bilinear map $B : V \times V \to W$. One often encounters bilinear forms with additional assumptions. A bilinear form is called

- *symmetric* if $B(x, y) = B(y, x)$, $x, y \in V$;
- *skew-symmetric* if $B(x, y) = -B(y, x)$, $x, y \in V$;
- *alternating* if $B(x, x) = 0$, $x \in V$.

By expanding $B(x + y, x + y) = 0$, we can show alternating implies skew-symmetric. Further if K is not of characteristic 2, then skew-symmetric implies alternating.

Left and Right Maps. Let $B : U \times V \to W$ be a bilinear map. We may identify B with the linear map $B_\otimes : U \otimes V \to W$ (see tensor product). We may also identify B with the linear maps

$$\begin{aligned} B_L &: U \to L(V, W), \qquad B_L(x)(y) = B(x, y),\ x \in U,\ y \in V; \\ B_R &: V \to L(U, W), \qquad B_R(y)(x) = B(x, y),\ x \in U,\ y \in V. \end{aligned}$$

called the left and right map, respectively.

Next, suppose that $B : V \times V \to K$ is a bilinear form. Then both B_L and B_R are linear maps from V to V^*, the dual vector space of V. We can therefore say that B is symmetric if and only if $B_L = B_R$ and that B is anti-symmetric if and only if $B_L = -B_R$. If V is finite-dimensional, we can identify V and V^{**}, and assert that $B_L = (B_R)^*$; the left and right maps are, in fact, dual homomorphisms.

Rank. Let $B : U \times V \to K$ be a bilinear form, and suppose that U, V are finite dimensional. One can show that $\operatorname{rank} B_L = \operatorname{rank} B_R$. We call this integer $\operatorname{rank} B$, the of B. Applying the rank-nullity theorem to both the left and right maps gives the following results:

$$\begin{aligned}\dim U &= \dim \ker B_L + \operatorname{rank} B \\ \dim V &= \dim \ker B_R + \operatorname{rank} B\end{aligned}$$

We say that B is *non-degenerate* if both the left and right map are non-degenerate. Note that in for B to be non-degenerate it is necessary that $\dim U = \dim V$. If this holds, then B is non-degenerate if and only if $\operatorname{rank} B$ is equal to $\dim U, \dim V$.

Orthogonal complements. Let $B : V \times V \to K$ be a bilinear form, and let $S \subset V$ be a subspace. The left and right orthogonal complements of S are subspaces ${}^{\perp}S, S^{\perp} \subset V$ defined as follows:

$$\begin{aligned}{}^{\perp}S &= \{u \in V \mid B(u, v) = 0 \text{ for all } v \in S\}, \\ S^{\perp} &= \{v \in V \mid B(u, v) = 0 \text{ for all } u \in S\}.\end{aligned}$$

We may also realize $S^{\perp}$ by considering the linear map $B'_R : V \to S^*$ obtained as the composition of $B_R : V \to V^*$ and the dual homomorphism $V^* \to S^*$. Indeed, $S^{\perp} = \ker B'_R$. An analogous statement can be made for ${}^{\perp}S$.

Next, suppose that B is non-degenerate. By the rank-nullity theorem we have that

$$\begin{aligned}\dim V &= \dim S + \dim S^{\perp} \\ &= \dim S + \dim {}^{\perp}S.\end{aligned}$$

Therefore, if B is non-degenerate, then

$$\dim S^{\perp} = \dim {}^{\perp}S.$$

Indeed, more can be said if B is either symmetric or skew-symmetric. In this case, we actually have

$${}^{\perp}S = S^{\perp}.$$

We say that $S \subset V$ is a non-degenerate subspace relative to B if the restriction of B to $S \times S$ is non-degenerate. Thus, S is a non-degenerate subspace if and only if $S \cap S^\perp = \{0\}$, and also $S \cap {}^\perp S = \{0\}$. Hence, if B is non-degenerate and if S is a non-degenerate subspace, we have

$$V = S \oplus S^\perp = S \oplus {}^\perp S.$$

Finally, note that if B is positive-definite, then B is necessarily non-degenerate and that every subspace is non-degenerate. In this way we arrive at the following well-known result: if V is positive-definite inner product space, then

$$V = S \oplus S^\perp$$

for every subspace $S \subset V$.

Adjoints. Let $B : V \times V \to K$ be a non-degenerate bilinear form, and let $T \in L(V, V)$ be a linear endomorphism. We define the right adjoint $T^\star \in L(V, V)$ to be the unique linear map such that

$$B(Tu, v) = B(u, T^\star v), \quad u, v \in V.$$

Letting $T^* : V^* \to V^*$ denote the dual homomorphism, we also have

$$T^\star = {B_R}^{-1} \circ T^* \circ B_R.$$

Similarly, we define the left adjoint ${}^\star T \in L(V, V)$ by

$${}^\star T = {B_L}^{-1} \circ T^* \circ B_L.$$

We then have

$$B(u, Tv) = B({}^\star Tu, v), \quad u, v \in V.$$

If B is either symmetric or skew-symmetric, then ${}^\star T = T^\star$, and we simply use $T^\star$ to refer to the adjoint homomorphism.

Additional remarks.

1. if B is a symmetric, non-degenerate bilinear form, then the adjoint operation is represented, relative to an orthogonal basis (if one exists), by the matrix transpose.

2. If B is a symmetric, non-degenerate bilinear form then $T \in L(V, V)$ is then said to be a *normal operator* (with respect to B) if T commutes with its adjoint $T^\star$.

3. An $n \times m$ matrix may be regarded as a bilinear form over $K^n \times K^m$. Two such matrices, B and C, are said to be congruent if there exists an invertible P such that $B = P^T C P$.

4. The identity matrix, I_n on $\mathbb{R}^n \times \mathbb{R}^n$ gives the standard Euclidean inner product on $\mathbb{R}^n$.