Beltrami identity

Canonical name	BeltramiIdentity
Date of creation	2013-03-22 12:21:08
Last modified on	2013-03-22 12:21:08
Owner	PrimeFan (13766)
Last modified by	PrimeFan (13766)
Numerical id	8
Author	PrimeFan (13766)
Entry type	Definition
Classification	msc 47A60
Related topic	CalculusOfVariations
Related topic	EulerLagrangeDifferentialEquation

Let $q(t)$ be a function $\mathbb{R} \to \mathbb{R}$, $\dot{q} = \frac{d}{dt}q$, and $L = L(q, \dot{q}, t)$. Begin with the time-relative Euler-Lagrange condition

$$\frac{\partial}{\partial q}L - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}L\right) = 0. \tag{1}$$

If $\frac{\partial}{\partial t}L = 0$, then the Euler-Lagrange condition reduces to

$$L - \dot{q}\frac{\partial}{\partial \dot{q}}L = C, \tag{2}$$

which is the *Beltrami identity*. In the calculus of variations, the ability to use the Beltrami identity can vastly simplify problems, and as it happens, many physical problems have $\frac{\partial}{\partial t}L = 0$.

In space-relative terms, with $q' := \frac{d}{dx}q$, we have

$$\frac{\partial}{\partial q}L - \frac{d}{dx}\frac{\partial}{\partial q'}L = 0. \tag{3}$$

If $\frac{\partial}{\partial x}L = 0$, then the Euler-Lagrange condition reduces to

$$L - q'\frac{\partial}{\partial q'}L = C. \tag{4}$$

To derive the Beltrami identity, note that

$$\frac{d}{dt}\left(\dot{q}\frac{\partial}{\partial \dot{q}}L\right) = \ddot{q}\frac{\partial}{\partial \dot{q}}L + \dot{q}\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}L\right) \tag{5}$$

Multiplying (1) by $\dot{q}$, we have

$$\dot{q}\frac{\partial}{\partial q}L - \dot{q}\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}L\right) = 0. \tag{6}$$

Now, rearranging (5) and substituting in for the rightmost term of (6), we obtain

$$\dot{q}\frac{\partial}{\partial q}L + \ddot{q}\frac{\partial}{\partial \dot{q}}L - \frac{d}{dt}\left(\dot{q}\frac{\partial}{\partial \dot{q}}L\right) = 0. \tag{7}$$

Now consider the total derivative

$$\frac{d}{dt}L(q, \dot{q}, t) = \dot{q}\frac{\partial}{\partial q}L + \ddot{q}\frac{\partial}{\partial \dot{q}}L + \frac{\partial}{\partial t}L. \tag{8}$$

If $\frac{\partial}{\partial t}L = 0$, then we can substitute in the left-hand side of (8) for the leading portion of (7) to get

$$\frac{d}{dt}L - \frac{d}{dt}\left(\dot{q}\frac{\partial}{\partial \dot{q}}L\right) = 0. \tag{9}$$

Integrating with respect to t, we arrive at

$$L - \dot{q}\frac{\partial}{\partial \dot{q}}L = C, \tag{10}$$

which is the Beltrami identity.