closed operator

Canonical name	ClosedOperator
Date of creation	2013-03-22 13:48:20
Last modified on	2013-03-22 13:48:20
Owner	Koro (127)
Last modified by	Koro (127)
Numerical id	9
Author	Koro (127)
Entry type	Definition
Classification	msc 47A05
Synonym	closed
Defines	closure
Defines	closable
Defines	core

Let B be a Banach space. A linear operator $A\colon \mathscr{D}(A) \subset B \to B$ is said to be if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathscr{D}(A)$ converging to $x \in B$ such that $Ax_n \xrightarrow[n\to\infty]{} y \in B$, it holds $x \in \mathscr{D}(A)$ and $Ax = y$. Equivalently, A is closed if its graph is closed in $B \oplus B$.

Given an operator A, not necessarily closed, if the closure of its graph in $B \oplus B$ happens to be the graph of some operator, we call that operator the **closure** of A, and we say that A is **closable**. We denote the closure of A by $\overline{A}$. It follows easily that A is the restriction of $\overline{A}$ to $\mathscr{D}(A)$.

A **core** of a closable operator is a subset $\mathscr{C}$ of $\mathscr{D}(A)$ such that the closure of the restriction of A to $\mathscr{C}$ is $\overline{A}$.

The following properties are easily checked:

1. Any bounded linear operator defined on the whole space B is closed;

2. If A is closed then $A - \lambda I$ is closed;

3. If A is closed and it has an inverse, then A^{-1} is also closed;

4. An operator A admits a closure if and only if for every pair of sequences $\{x_n\}$ and $\{y_n\}$ in $\mathscr{D}(A)$, both converging to $z \in B$, and such that both $\{Ax_n\}$ and $\{Ay_n\}$ converge, it holds $\lim_n Ax_n = \lim_n Ay_n$.