polar decomposition

Canonical name	PolarDecomposition
Date of creation	2013-03-22 16:01:54
Last modified on	2013-03-22 16:01:54
Owner	aube (13953)
Last modified by	aube (13953)
Numerical id	10
Author	aube (13953)
Entry type	Definition
Classification	msc 47A05

The polar decomposition of an operator is a generalization of the familiar factorization of a complex number z in a radial part $|z|$ and an angular part $z/|z|$.

Let $\mathscr{H}$ be a Hilbert space, x a bounded operator on $\mathscr{H}$. Then there exist a pair (h, u), with h a bounded positive operator and u a partial isometry on $\mathscr{H}$, such that

$$x = uh.$$

If we impose the further conditions that $1 - u^*u$ is the projection to the kernel of x, and $\ker(h) = \ker(x)$, then (h, u) is unique, and is called the *polar decomposition* of x. The operator h will be $|x|$, the square root of x^*x, and u will be the partial isometry, determined by

- $u\xi = 0$ for $\xi \in \ker(x)$
- $u(|x|\xi) = x\xi$ for $\xi \in \mathscr{H}$.

If x is a closed, densely defined unbounded operator on $\mathscr{H}$, the polar decomposition (u, h) still exists, where now h will be the unbounded positive operator $|x|$ with the same domain $\mathscr{D}(x)$ as x, and u still the partial isometry determined by

- $u\xi = 0$ for $\xi \in \ker(x)$
- $u(|x|\xi) = x\xi$ for $\xi \in \mathscr{D}(x)$.

If x is affiliated with a von Neumann algebra M, both u and h will be affiliated with M.