existence of adjoints of bounded operators

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Let $\mathscr{H}$ be a Hilbert space and let $T : \mathscr{D}(T) \subset \mathscr{H} \longrightarrow \mathscr{H}$ be a densely defined linear operator.

Theorem - If T is `http://planetmath.org/ContinuousLinearMapping`bounded then its adjoint T^* is everywhere defined and is also bounded.

Proof : Since T is densely defined and bounded, it extends uniquely to a bounded (everywhere defined) linear operator on $\mathscr{H}$, which we denote by $\widetilde{T}$.

For each $z \in \mathscr{H}$, the function $f : \mathscr{H} \longrightarrow \mathbb{C}$ defined by $f(x) = \langle \widetilde{T}x, z \rangle$ defines a bounded linear functional on $\mathscr{H}$. By the Riesz representation theorem there exists $u \in \mathscr{H}$ such that

$$f(x) = \langle x, u \rangle$$

i.e.

$$\langle \widetilde{T}x, z \rangle = \langle x, u \rangle.$$

Since $\widetilde{T}$ extends T, we also have that for every $z \in \mathscr{H}$ there exists $u \in \mathscr{H}$ such that

$$\langle Tx, z \rangle = \langle x, u \rangle \text{ for every } x \in \mathscr{D}(T).$$

We conclude that T^* is everywhere defined. To see that it is bounded one just needs to check that

$$\sup_{z \neq 0} \frac{\|T^* z\|}{\|z\|} = \sup_{\substack{z \neq 0 \\ T^* z \neq 0}} \frac{|\langle T^* z, T^* z \rangle|}{\|T^* z\| \|z\|} \leq \sup_{\substack{z \neq 0 \\ x \neq 0}} \frac{|\langle x, T^* z \rangle|}{\|x\| \|z\|} = \sup_{\substack{z \neq 0 \\ x \neq 0}} \frac{|\langle Tx, z \rangle|}{\|x\| \|z\|} \leq \|T\|$$

where the last inequality comes from the Cauchy-Schwarz inequality and the fact that T is bounded. $\square$

Remark - This theorem shows in particular that bounded linear operators $T : \mathscr{H} \longrightarrow \mathscr{H}$ have bounded adjoints $T^* : \mathscr{H} \longrightarrow \mathscr{H}$.