proof of Brouwer fixed point theorem

Canonical name	ProofOfBrouwerFixedPointTheorem
Date of creation	2013-03-22 13:11:24
Last modified on	2013-03-22 13:11:24
Owner	bwebste (988)
Last modified by	bwebste (988)
Numerical id	6
Author	bwebste (988)
Entry type	Proof
Classification	msc 47H10
Classification	msc 54H25
Classification	msc 55M20

Proof of the Brouwer fixed point theorem:

Assume that there does exist a map from $f : B^n \to B^n$ with no fixed point. Then let $g(x)$ be the following map: Start at $f(x)$, draw the ray going through x and then let $g(x)$ be the first intersection of that line with the sphere. This map is continuous and well defined only because f fixes no point. Also, it is not hard to see that it must be the identity on the boundary sphere. Thus we have a map $g : B^n \to S^{n-1}$, which is the identity on $S^{n-1} = \partial B^n$, that is, a retraction. Now, if $i : S^{n-1} \to B^n$ is the inclusion map, $g \circ i = \mathrm{id}_{S^{n-1}}$. Applying the reduced homology functor, we find that $g_* \circ i_* = \mathrm{id}_{\tilde{H}_{n-1}(S^{n-1})}$, where $_*$ indicates the induced map on homology.

But, it is a well-known fact that $\tilde{H}_{n-1}(B^n) = 0$ (since B^n is contractible), and that $\tilde{H}_{n-1}(S^{n-1}) = \mathbb{Z}$. Thus we have an isomorphism of a non-zero group onto itself factoring through a trivial group, which is clearly impossible. Thus we have a contradiction, and no such map f exists.