adjoint

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Let $\mathscr{H}$ be a Hilbert space and let $A\colon \mathscr{D}(A) \subset \mathscr{H} \to \mathscr{H}$ be a densely defined linear operator. Suppose that for some $y \in \mathscr{H}$, there exists $z \in \mathscr{H}$ such that $(Ax, y) = (x, z)$ for all $x \in \mathscr{D}(A)$. Then such z is unique, for if z' is another element of $\mathscr{H}$ satisfying that condition, we have $(x, z-z') = 0$ for all $x \in \mathscr{D}(A)$, which implies $z-z' = 0$ since $\mathscr{D}(A)$ is http://planetmath.org/Densedense. Hence we may define a new operator $A^* : \mathscr{D}(A^*) \subset \mathscr{H} \to \mathscr{H}$ by

$$\begin{aligned} \mathscr{D}(A^*) =& \{y \in \mathscr{H} : \text{there is} z \in \mathscr{H} \text{such that} (Ax, y) = (x, z)\}, \\ A^*(y) =& z. \end{aligned}$$

It is easy to see that A^* is linear, and it is called the **adjoint** of A.

Remark. The requirement for A to be densely defined is essential, for otherwise we cannot guarantee A^* to be well defined.