proof of Borel functional calculus

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In this entry we give a proof of the main result about the Borel functional calculus (1 on the parent entry). We will restate here the result for convenience. Please, check the parent entry for the details on notation.

Theorem - *Let T be a normal operator in $B(H)$ and $\pi : C(\sigma(T)) \longrightarrow B(H)$ the unital *-homomorphism corresponding to the continuous functional calculus for T. Then, π extends uniquely to a *-homomorphism $\widetilde{\pi} : B(\sigma(T)) \longrightarrow B(H)$ that is continuous from the μ-topology to the weak operator topology. Moreover, each operator $\pi(f)$ lies in* `http://planetmath.org/OperatorTopologies` *operator closure of the unital *-algebra generated by T.*

Proof : First we shall prove the existence of the `http://planetmath.org/ExtensionOfAFunct` $\widetilde{\pi}$ with the described continuity property, then we shall prove its uniqueness and for last we shall prove the last assertion of the theorem about the image of $\widetilde{\pi}$. For simplicity the proofs of some auxiliary results are given at the end of the entry

0.0.1 Existence

For each pair of vectors $\xi, \eta \in H$ consider the linear functional $\phi_{\xi,\eta} : C(\sigma(T)) \to \mathbb{C}$ given by

$$\phi_{\xi,\eta}(f) := \langle \pi(f)\xi, \eta \rangle$$

This linear functional is bounded with norm at most $\|\xi\|\|\eta\|$ because

$$\|\phi_{\xi,\eta}(f)\| \leq \|\pi\| \, \|f\| \, \|\xi\|\|\eta\| = \|f\| \, \|\xi\|\|\eta\|$$

where the last equality comes from the fact that π is a *-isomorphism between `http://planetmath.org/CAlgebras`C^*-algebras, therefore having norm 1 (see `http://planetmath.org/HomomorphismsOfCAlgebrasAreContinuous`this entry).

By the `http://planetmath.org/RieszRepresentationTheoremOfLinearFunctionalsOnFunc` representation theorem there is a unique complex Radon measure $\mu_{\xi,\eta}$ in $\sigma(T)$ such that

$$\langle \pi(f)\xi, \eta \rangle = \int_{\sigma(T)} f \, d\mu_{\xi,\eta} \,, \qquad \forall f \in C(\sigma(T)) \qquad (1)$$

Consider now the mapping $(\xi, \eta) \mapsto \mu_{\xi,\eta}$. This mapping has the following properties, whoose proofs are given at the end of the entry (section Auxiliary Results):

a) $\mu_{\lambda_1\xi_1+\lambda_2\xi_2,\eta} = \lambda_1\mu_{\xi_1,\eta} + \lambda_2\mu_{\xi_2,\eta}$, for all $\lambda_1, \lambda_2 \in \mathbb{C}$.

b) $\mu_{\xi,\lambda_1\eta_1+\lambda_2\eta_2} = \overline{\lambda_1}\mu_{\xi,\eta_1} + \overline{\lambda_2}\mu_{\xi,\eta_2}$, for all $\lambda_1, \lambda_2 \in \mathbb{C}$.

c) $\mu_{\xi,\eta} = \overline{\mu_{\eta,\xi}}$.

d) $g \cdot \mu_{\xi,\eta} = \mu_{\pi(g)\xi,\eta}$, for all $g \in C(\sigma(T)$.

Therefore, for each function $f \in B(\sigma(T))$ we have a sesquilinear form in H given by

$$[\xi, \eta] := \int_{\sigma(T)} f \, d\mu_{\xi,\eta}$$

Moreover, this sesquilinear form is `http://planetmath.org/BoundedSesquilinearForm`bound with norm at most $\|f\| \, \|\xi\|\|\eta\|$. By the Riesz lemma on bounded sesquilinear forms, there is a unique operator $\widetilde{\pi}(f) \in B(H)$ such that

$$\langle \widetilde{\pi}(f)\xi, \eta\rangle := \int_{\sigma(T)} f \, d\mu_{\xi,\eta}\,, \qquad\qquad \xi, \eta \in H$$

We will now see that the mapping $\widetilde{\pi} : B(\sigma(T) \to B(H)$, such that $f \mapsto \widetilde{\pi}(f)$, has the desired properties stated in the theorem.

First, it is clear that $\widetilde{\pi}$ is linear. Also clear is the fact that $\widetilde{\pi}$ coincides with π in $C(\sigma(T))$, because of equality (1) and the uniqueness part of Riesz lemma on sesquilinear forms. Now, for any real valued function $f \in B(\sigma(T))$ we have that

$$\begin{aligned} \langle \widetilde{\pi}(f)^*\xi, \eta\rangle &= \overline{\langle \widetilde{\pi}(f)\eta, \xi\rangle} \\ &= \overline{\int_{\sigma(T)} f \, d\mu_{\eta,\xi}} \\ &= \int_{\sigma(T)} f \, d\mu_{\xi,\eta} \\ &= \langle \pi(f)\xi, \eta\rangle \end{aligned}$$

which means that $\widetilde{\pi}(f)^* = \widetilde{\pi}(f)$, i.e. $\widetilde{\pi}(f)$ is self-adjoint. Decomposing an arbitrary function $f \in B(\sigma(T))$ in its real and imaginary parts we see that $\widetilde{\pi}(f)^* = \widetilde{\pi}(\overline{f})$.

We now show that $\widetilde{\pi}$ is multiplicative, i.e. $\widetilde{\pi}(fg) = \widetilde{\pi}(f)\widetilde{\pi}(g)$ for all $f, g \in B(\sigma(T))$. For that we need an additional property of the measures $\mu_{\xi,\eta}$, whoose proof is also at the end of the entry:

e) $f \cdot \mu_{\xi,\eta} = \mu_{\xi,\widetilde{\pi}(f)^*\eta}$, for all $f \in B(\sigma(T))$.

Given $f, g \in B(\sigma(T))$ we have, for every $\xi, \eta \in H$,

$$\begin{aligned} \langle\widetilde{\pi}(fg)\xi,\eta\rangle &= \int_{\sigma(T)} fg\, d\mu_{\xi,\eta} = \int_{\sigma(T)} g\, d\mu_{\xi,\widetilde{\pi}(f)*\eta} \\ &= \langle\widetilde{\pi}(g)\xi,\widetilde{\pi}(f)^*\eta\rangle = \langle\widetilde{\pi}(f)\widetilde{\pi}(g)\xi,\eta\rangle \end{aligned}$$

and therefore $\widetilde{\pi}(fg) = \widetilde{\pi}(f)\widetilde{\pi}(g)$. Thus, $\widetilde{\pi} : B(\sigma(T)) \to B(H)$ is a *-homomorphism that extends π.

0.0.2 Continuity Property

We now prove that the above defined $\widetilde{\pi}$ is continuous from the μ-topology to the weak operator topology.

Let $\{f_i\}$ be a net of functions in $B(\sigma(T))$ that converge in the μ-topology to a function $f \in B(\sigma(T))$. This means that for all Radon measures ν in $\sigma(T)$ we have $\int f_i\, d\nu \to \int f\, d\nu$.

Now for all $\xi, \eta \in H$ we have

$$|\langle\widetilde{\pi}(f_i - f)\xi,\eta\rangle| = \Big|\int_{\sigma(T)} f_i - f\, d\mu_{\xi,\eta}\Big| \to 0$$

Hence, $\widetilde{\pi}(f_i)$ converges to $\widetilde{\pi}(f)$ in the weak operator topology.

0.0.3 Uniqueness

Let $\pi' : B(\sigma(T)) \to B(H)$ be another *-homomorphism that extends π and is continuous from the μ-topology to the weak operator topology. For any measurable subset $S \subset \sigma(T)$ consider the set

$$W := \{(U,K) : U \supset S \text{is open and} K \subset S \text{is compact}\}$$

We give this set the partial order $\leq$ such that $(U_1,K_1) \leq (U_2,K_2)$ whenever $U_2 \subset U_1$ and $K_1 \subset K_2$. For any pair $(U,K) \in W$ there is a continuous function $f_{U,K} \in C(\sigma(T))$ such that f takes values on the interval $[0,1]$, $f|_K = 1$

and supp$f \subset U$ (see http://planetmath.org/ApplicationsOfUrysohnsLemmaToLocallyCompact entry).

We claim that $f_{U,K}$ converges to χ_S in the μ-topology. In fact, given a complex Radon measure ν in $\sigma(T)$, there is for every $\epsilon > 0$ a pair $(U_0, K_0) \in W$ such that $|\nu|(U_0 \setminus K_0) < \epsilon$. Of course, for all pairs (U, K) such that $(U_0, K_0) \leq (U, K)$ we also have $|\nu|(U \setminus K) < \epsilon$. Hence, we have

$$|\int_{\sigma(T)} f_{U,K} - \chi_S \, d\nu| \leq \int_{U \setminus K} |f_{U,K} - \chi_S| \, d|\nu| \leq \epsilon$$

We conclude that $f_{U,K}$ converges to χ_S in the μ-topology.

Since π' is continuous from the μ-topology to the weak operator topology we must have

$$\langle \pi'(f_{U,V})\xi, \eta \rangle \longrightarrow \langle \pi'(\chi_S)\xi, \eta \rangle$$

But since π' and $\widetilde{\pi}$ coincide with π on $C(\sigma(T))$ we also have

$$\langle \pi'(f_{U,V})\xi, \eta \rangle = \langle \widetilde{\pi}(f_{U,V})\xi, \eta \rangle \longrightarrow \langle \widetilde{\pi}(\chi_S)\xi, \eta \rangle$$

Hence, for any characteristic function χ_S we have $\pi'(\chi_S) = \widetilde{\pi}(\chi_S)$. Since any function $f \in B(\sigma(T))$ can be uniformly approximated by simple functions it follows that $\pi'(f) = \widetilde{\pi}(f)$, and we have proved the uniqueness of $\widetilde{\pi}$.

0.0.4 Image of $\widetilde{\pi}$

Let $\mathcal{A}$ be the unital *-algebra generated by T. We now prove that for any $f \in B(\sigma(T))$, the operator $\widetilde{\pi}(f)$ lies in the strong operator closure of $\mathcal{A}$, i.e. lies in the von Neumann algebra generated by T. For that it is enough to prove that $\widetilde{\pi}(f)$ is in the double commutant $\mathcal{A}''$ of $\mathcal{A}$.

Recall from the continuous functional calculus that $\pi(f)$ is in the norm closure of $\mathcal{A}$, and hence in $\mathcal{A}''$, for every $f \in C(\sigma(T))$.

We have seen above that for each characteristic function χ_S there is a net $f_{U,K}$ of functions in $C(\sigma(T))$ such that $\widetilde{\pi}(f_{U,K}) \to \widetilde{\pi}(\chi_S)$ in the weak operator topology. Given an element R in the commutant of $\mathcal{A}$ we have

$$\langle \widetilde{\pi}(f_{U,K})R\,\xi, \eta \rangle = \langle R\,\widetilde{\pi}(f_{U,K})\,\xi, \eta \rangle\,, \qquad \forall \xi, \eta \in H$$

The first term converges to $\langle \widetilde{\pi}(\chi_S)R\,\xi, \eta \rangle$, whereas the second to $\langle R\,\widetilde{\pi}(\chi_S)\,\xi, \eta \rangle$. Thus, $\widetilde{\pi}(\chi_S)R = R\,\widetilde{\pi}(\chi_S)$, and therefore $\widetilde{\pi}(\chi_S) \in \mathcal{A}''$.

Since every function $f \in B(\sigma(T))$ can be uniformly approximated by simple functions, it follows that $\widetilde{\pi}(f) \in \mathcal{A}''$.

0.0.5 Auxiliary Results

In this section we prove the properties of the measures $\mu_{\xi,\eta}$ stated and used above.

a) For all functions $f \in C(\sigma(T))$ we have $\langle \pi(f)(\lambda_1\xi_1 + \lambda_2\xi_2), \eta\rangle = \lambda_1\langle\pi(f)\xi_1,\eta\rangle + \lambda_2\langle\pi(f)\xi_2,\eta\rangle$. Hence,

$$\int_{\sigma(T)} f\,d\mu_{\lambda_1\xi_1+\lambda_2\xi_2,\eta} = \lambda_1\int_{\sigma(T)} f\,d\mu_{\xi_1,\eta} + \lambda_2\int_{\sigma(T)} f\,d\mu_{\xi_2,\eta}$$

Since this holds for every $f \in C(\sigma(T))$, the uniqueness part of tells us that

$$\mu_{\lambda_1\xi_1+\lambda_2\xi_2,\eta} = \lambda_1\mu_{\xi_1,\eta} + \lambda_2\mu_{\xi_2,\eta}$$

b) The proof is similar to a).

c) For every $f \in C(\sigma(T))$ we have

$$\begin{aligned}\int_{\sigma(T)} f\,d\mu_{\xi,\eta} &= \langle\pi(f)\xi,\eta\rangle = \overline{\langle\pi(\overline{f})\eta,\xi\rangle} \\ &= \overline{\int_{\sigma(T)} \overline{f}\,d\mu_{\eta,\xi}} = \int_{\sigma(T)} f\,d\overline{\mu_{\eta,\xi}}\end{aligned}$$

Hence we conclude that $\mu_{xi,\eta} = \overline{\mu_{\eta,\xi}}$.

d) For every $f \in C(\sigma(T))$ we have

$$\begin{aligned}\int_{\sigma(T)} f\,dg\cdot\mu_{\xi,\eta} &= \int_{\sigma(T)} fg\,d\mu_{\xi,\eta} = \langle\pi(fg)\xi,\eta\rangle \\ &= \langle\pi(f)\pi(g)\xi,\eta\rangle = \int_{\sigma(T)} f\,d\mu_{\pi(g)\xi,\eta}\end{aligned}$$

Hence, $g\cdot\mu_{\xi,\eta} = \mu_{\pi(g)\xi,\eta}$.

e) For all $h \in C(\sigma(T))$ we have

$$\begin{aligned}\int_{\sigma(T)} h\,df\cdot\mu_{\xi,\eta} &= \int_{\sigma(T)} hf\,d\mu_{\xi,\eta} = \int_{\sigma(T)} f\,dh\cdot\mu_{\xi,\eta} \\ &= \int_{\sigma(T)} f\,d\mu_{\pi(f)\xi,\eta} = \langle\widetilde{\pi}(f)\pi(g)\xi,\eta\rangle \\ &= \langle\pi(h)\xi,\widetilde{\pi}(f)^*\eta\rangle = \int_{\sigma(T)} h\,d\mu_{\xi,\widetilde{\pi}(f)^*\eta}\end{aligned}$$

Hence, $f \cdot \mu_{\xi,\eta} = \mu_{\xi,\widetilde{\pi}(f)^*\eta}$.