derivative operator is unbounded in the sup norm

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Consider $C^\infty([-1,1])$ the vector space of functions with derivatives or arbitrary order on the set $[-1,1]$.

This space admits a norm called the supremum norm given by

$$|f| = \sup\{|f(x)| : x \in [-1,1]\}$$

This norm makes this vector space into a metric space.

We claim that the derivative operator $D : (Df)(x) = f'(x)$ is an unbounded operator.

All we need to prove is that there exists a succession of functions $f_n \in C^\infty([-1,1])$ such that $\frac{|D(f_n)|}{|f_n|}$ is divergent as $n \to \infty$

consider

$$f_n(x) = \exp(-n^4 x^2)$$

$$(Df_n)(x) = -2xn^4 \exp(-n^4 x^2)$$

Clearly $|f_n| = f_n(0) = 1$

To find $|Df_n|$ we need to find the extrema of the derivative of f_n, to do that calculate the second derivative and equal it to zero. However for the task at hand a crude estimate will be enough.

$$|Df_n| \geq |(Df_n)(\frac{1}{n^2})| = \frac{2n^2}{e}$$

So we finally get

$$\frac{|Df_n|}{|f_n|} \geq \frac{2n^2}{e}$$

showing that the derivative operator is indeed unbounded since $\frac{2n^2}{e}$ is divergent as $n \to \infty$.