spectral measure

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1 Definition

In this entry by a *projection* we an orthogonal projection over some Hilbert space. Also, we say that two projections are *orthogonal* if their images are orthogonal subspaces.

Let H be an Hilbert space, $B(H)$ the algebra of bounded operators in H and $(X, \mathcal{B})$ a measurable space.

Definition - *A* **spectral measure** *in X is a function $P : \mathcal{B} \longrightarrow B(H)$ such that*

a) *$P(E)$ is a projection in $B(H)$ for every $E \in \mathcal{B}$.*

b) *$P(\emptyset) = 0$.*

c) *$P(X) = I$, where I denotes the identity operator in $B(H)$.*

d) *If E_1 and E_2 are disjoint subsets of $\mathcal{B}$, then $P(E_1)$ and $P(E_2)$ are orthogonal.*

e) *$P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$ for every sequence $E_1, E_2, \dots$ of disjoint sets in $\mathcal{B}$.*

The in the last condition is interpreted as the pointwise limit of the partial sums. Since from condition (d) the projections $P(E_1), P(E_2), \dots$ are orthogonal, we know that the pointwise limit exists and is a projection (see `http://planetmath.org/LatticeOfProjections`this entry, Theorem 5).

: In the following $\mathrm{Ran}(T)$ denotes the `http://planetmath.org/Function`range of an operator $T \in B(H)$.

- $E_1 \subseteq E_2 \implies \mathrm{Ran}(P(E_1)) \subseteq \mathrm{Ran}(P(E_2))$.
- $P(E_1 \cap E_2) = P(E_1)P(E_2)$ for every $E_1, E_2 \in \mathcal{B}$.

Thus, a spectral measure is a countably additive vector measure whose values are projections. For that, spectral measures are also called *projection valued measures*.

2 Examples

- Let $(X, \mathcal{B}, \mu)$ be a measure space. Consider the Hilbert space http://planetmath.org/L2Spa
  We regard a function f in http://planetmath.org/LpSpace$L^\infty(X, \mu)$ as the multiplication operator $M_f \in B(L^2(X, \mu))$ given by

  $$M_f(\xi) = f\xi\,, \qquad \xi \in L^2(X, \mu)$$

  In this setting, the characteristic functions are projections in $B(L^2(X, \mu))$ and we have a spectral measure given by

  $$\begin{aligned} P : X &\longrightarrow B(L^2(X, \mu)) \\ P(E) &:= \chi_E \end{aligned}$$

- Let H be a Hilbert space, $T \in B(H)$ a normal operator and $\sigma(T)$ the spectrum of T. For any measurable subset $E \subseteq \sigma(T)$ the operators $\chi_E(T)$, given by the Borel functional calculus, are projections in $B(H)$. Moreover, we have a spectral measure given by:

  $$\begin{aligned} P : X &\longrightarrow B(H) \\ P(E) &:= \chi_E(T) \end{aligned}$$

3 Equivalent Definition

The following result provides a very useful equivalent definition of a spectral measure.

Theorem 1 - *A function $P : \mathcal{B} \longrightarrow B(H)$ whose values are projections is a spectral measure in X if and only if $P(X) = I$ and for every $\xi, \eta \in H$ the function $\mu_{\xi,\eta} : X \longrightarrow \mathbb{C}$ given by*

$$\mu_{\xi,\eta}(E) := \langle P(E)\xi, \eta \rangle$$

is a complex measure in X.

4 Integration against spectral measures

Let $f : X \longrightarrow \mathbb{C}$ be a http://planetmath.org/Boundedbounded measurable function and P a spectral measure in X. We are interested to give meaning to the integral

$$\int_X f dP$$

Since we are dealing with "measures" whose values are linear operators it is reasonable to expect that this integral is itself a linear operator.

There are two natural ways to define it that turn out to be equivalent. The first approach is a construction that resembles the approximation of f by simple functions in Lebesgue integral theory. Here the role of simple functions will be played by the operators of the form

$$\sum_i \lambda_i P(E_i)\,, \qquad \lambda_i \in \mathbb{C}$$

Theorem 2 - *There exists a unique operator $S \in B(H)$ with the following property: for any given $\epsilon > 0$ and for every measurable partition $\{E_1, \cdots, E_n\}$ of X that satisfies $|f(x) - f(x')| < \epsilon$ for all $x, x' \in E_i$, we have*

$$\|S - \sum_{i=1}^{n} f(x_i) P(E_i)\| < \epsilon$$

for any choice of points $x_i \in E_i$.

We can then define $\int_X f dP$ as the unique operator S described by Theorem 2.

The other approach to define this integral is by specifying an appropriate bounded sesquilinear form. Recall that from `http://planetmath.org/RieszRepresentationTheo` representation theorem, to every bounded sesquilinear form corresponds a unique bounded operator. The construction is as follows:

First we notice that, from the alternative defintion of spectral measure (Theorem 1), for every vectors $\xi, \eta \in H$ we can define a complex measure $\mu_{\xi,\eta}$ by

$$\mu_{\xi,\eta}(E) = \langle P(E)\,\xi, \eta\rangle,$$

whose total variation is estimated by $\|\mu_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$.

Then we notice that the function $[\cdot,\cdot] : H \times H \longrightarrow \mathbb{C}$ defined by

$$[\xi, \eta] := \int_X f \; d\mu_{\xi,\eta}$$

is a sesquilinear form.

Then, by the `http://planetmath.org/RieszRepresentationTheoremOfBoundedSesquilinea` representation theorem, there exists a unique operator $S \in B(H)$ such that

$$\langle S\xi, \eta \rangle = \int_X f \; d\mu_{\xi,\eta}, \qquad \xi, \eta \in H \tag{1}$$

We can then define $\displaystyle\int_X f \;\; dP$ as this operator S. Of course, the two definitions are equivalent. We summarize this in the following result

Theorem 3 - *Given a spectral measure P and a bounded Borel function f, an operator S that satisfies condition (1) also satisfies the conditions of Theorem 2. Therefore, both definitions of the integral of f with respect to P coincide and we have that:*

- $\displaystyle\Big\langle \int_X f \; dP \, \xi, \eta \Big\rangle = \int_X f \; d\mu_{\xi,\eta}$
- $\displaystyle\int_X f \;\; dP$ *can be arbitrarilly approximated in norm by operators of the form* $\displaystyle\sum_{i=1}^{n} f(x_i) P(E_i)$.

5 Remarks

The second example we gave above, of a spectral measure associated with a normal operator, is in some sense the general case: all spectral projections in $\mathbb{C}$ supported in a compact set arise from a normal operator. Thus, to any such spectral projection we can associate a normal operator and vice-versa. This interplay between spectral projections and normal operators is deeply explored in some versions of the spectral theorem.

References

[1] W. Arveson, *A Short Course on Spectral Theory*, Graduate Texts in Mathematics, 209, Springer, New York, 2002

[2] J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Graduate Texts in Mathematics, 96, Springer-Verlag, New York, Berlin, 1990.