polar decomposition in von Neumann algebras

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- Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H and $T \in \mathcal{M}$. If $T = VR$ is the polar decomposition for T with $KerV = KerR$, then both V and R belong to $\mathcal{M}$.

Proof :

- As $\mathcal{M}$ is a `http://planetmath.org/CAlgebra`C^*-algebra, it is known that $R = \sqrt{T^*T}$ belongs to $\mathcal{M}$. (proof will be added later)

- To see that V also belongs to $\mathcal{M}$, by the double commutant theorem, it suffices to show that V belongs to $\mathcal{M}''$ (the double commutant of $\mathcal{M}$).

  Suppose $S \in \mathcal{M}'$. We intend to prove that V commutes with S.

  For $x \in H$ we have that

  $$TSx = STx = SVRx$$

  and

  $$TSx = VRSx = VSRx$$

  So SV and VS agree on $\overline{Ran\ R}$.

  As R is self-adjoint, $\overline{Ran\ R}^{\perp} = KerR$, and so it remains to show that SV and VS agree on $KerR$. Recall that, by hypothesis, $KerR = KerV$.

  Let $x \in KerR$. We have that $RSx = SRx = 0$ and therefore

  $$S(KerR) \subseteq KerR = KerV$$

  and so we can conclude that VS is identically zero in $KerR$.

  Clearly SV is also identically zero on $KerR = KerV$.

  Thus VS and SV agree on $KerR$. Therefore $SV = VS$ and so $V \in \mathcal{M}'' = \mathcal{M}$ $\square$