equation of catenary via calculus of variations

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Using the mechanical principle that the centre of mass itself as low as possible, determine the equation of the curve formed by a l when supported at its ends in the points $P_1 = (x_1,\, y_1)$ and $P_2 = (x_2,\, y_2)$.

We have an isoperimetric problem

$$\text{to minimise} \quad \int_{P_1}^{P_2} y\,ds \tag{1}$$

under the constraint

$$\int_{P_1}^{P_2} ds \;=\; l, \tag{2}$$

where both the path integrals are taken along some curve c. Using a Lagrange multiplier λ, the task changes to a free problem

$$\int_{P_1}^{P_2} (y-\lambda)\,ds \;=\; \int_{x_1}^{x_2} (y-\lambda)\sqrt{1+y'^2}\,|dx| \;=\; \text{min!} \tag{3}$$

(cf. example of calculus of variations).

The `http://planetmath.org/EulerLagrangeDifferentialEquation`Euler–Lagrange differential equation, the necessary condition for (3) to give an extremal c, reduces to the Beltrami identity

$$(y-\lambda)\sqrt{1+y'^2} - y'\cdot(y-\lambda)\cdot\frac{y'}{\sqrt{1+y'^2}} \;\equiv\; \frac{y-\lambda}{\sqrt{1+y'^2}} \;=\; a,$$

where a is a constant of integration. After solving this equation for the derivative y' and separation of variables, we get

$$\pm\frac{dy}{\sqrt{(y-\lambda)^2-a^2}} \;=\; \frac{dx}{a}$$

which may become clearer by notating $y-\lambda := u$; then by integrating

$$\pm\frac{du}{\sqrt{u^2-a^2}} \;=\; \frac{dx}{a}$$

we choose the new constant of integration b such that $x = b$ when $u = a$:

$$\pm\int_a^u \frac{du}{\sqrt{u^2-a^2}} \;=\; \int_b^x \frac{dx}{a}$$

We can write two http://planetmath.org/Equivalent3equivalent results

$$\ln\frac{u+\sqrt{u^2-a^2}}{a} \;=\; +\frac{x-b}{a}, \qquad \ln\frac{u-\sqrt{u^2-a^2}}{a} \;=\; -\frac{x-b}{a},$$

i.e.

$$\frac{u+\sqrt{u^2-a^2}}{a} \;=\; e^{+\frac{x-b}{a}}, \qquad \frac{u-\sqrt{u^2-a^2}}{a} \;=\; e^{-\frac{x-b}{a}}.$$

Adding these allows to eliminate the square roots and to obtain

$$u \;=\; \frac{a}{2}\left(e^{\frac{x-b}{a}}+e^{-\frac{x-b}{a}}\right),$$

or

$$y-\lambda \;=\; a\cosh\frac{x-b}{a}. \tag{4}$$

This is the sought form of the equation of the chain curve. The constants λ, a, b can then be determined for putting the curve to pass through the given points P_1 and P_2.

References

[1] E. Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset IV. Johdatus variatiolaskuun.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1946).