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least surface of revolution

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The points $P_1 = (x_1,\, y_1)$ and $P_2 = (x_2,\, y_2)$ have to be by an arc c such that when it rotates around the x-axis, the http://planetmath.org/SurfaceOfRevolutionarea of the surface of revolution formed by it is as small as possible.

The area in question, expressed by the path integral

$$A \;=\; 2\pi\int_{P_1}^{P_2} y\,ds, \tag{1}$$

along c, is to be minimised; i.e. we must minimise

$$\int_{P_1}^{P_2} y\,ds \;=\; \int_{x_1}^{x_2}\sqrt{1+y'^2}\,|dx|. \tag{2}$$

Since the integrand in (2) does not explicitly depend on x, the http://planetmath.org/EulerLa Lagrange differential equation of the problem, the necessary condition for (2) to give an extremal c, reduces to the Beltrami identity

$$y\sqrt{1+y'^2} - y'\cdot\frac{yy'}{\sqrt{1+y'^2}} \;\equiv\; \frac{y}{\sqrt{1+y'^2}} \;=\; a,$$

where a is a constant of integration. After solving this equation for the derivative y' and separation of variables, we get

$$\pm\frac{dy}{\sqrt{y^2-a^2}} \;=\; \frac{dx}{a},$$

by integration of which we choose the new constant of integration b such that $x = b$ when $y = a$:

$$\pm\int_a^y \frac{dy}{\sqrt{y^2-a^2}} \;=\; \int_b^x \frac{dx}{a}$$

We can write two http://planetmath.org/Equivalent3equivalent results

$$\ln\frac{y+\sqrt{y^2-a^2}}{a} \;=\; +\frac{x-b}{a}, \qquad \ln\frac{y-\sqrt{{}^2-a^2}}{a} \;=\; -\frac{x-b}{a},$$

i.e.

$$\frac{y+\sqrt{y^2-a^2}}{a} \;=\; e^{+\frac{x-b}{a}}, \qquad \frac{y-\sqrt{y^2-a^2}}{a} \;=\; e^{-\frac{x-b}{a}}.$$

Adding these yields

$$y \;=\; \frac{a}{2}\left(e^{\frac{x-b}{a}} + e^{-\frac{x-b}{a}}\right) \;=\; a\cosh\frac{x-b}{a}. \tag{3}$$

From this we see that the extremals c of the problem are catenaries. It means that the least surface of revolution in the question is a catenoid.