Newton's method

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Let $\mathbf{f}$ be a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Newton's method consists of starting at an $\mathbf{a_0}$ for the equation $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Then the function is linearized at $\mathbf{a_0}$ by replacing the increment $\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a_0})$ by a linear function of the increment $[\mathbf{D}(\mathbf{a_0})](\mathbf{x} - \mathbf{a_0})$.

Now we can solve the linear equation $\mathbf{f}(\mathbf{a_0}) + [\mathbf{D}(\mathbf{a_0})](\mathbf{x} - \mathbf{a_0}) = \mathbf{0}$. Since this is a system of n linear equations in n unknowns, $[\mathbf{D}(\mathbf{a_0})](\mathbf{x} - \mathbf{a_0}) = -\mathbf{f}(\mathbf{a_0})$ can be likened to the general linear system $A\mathbf{x} = \mathbf{b}$.

Therefore, if $[\mathbf{D}(\mathbf{a_0})]$ is invertible, then $\mathbf{x} = \mathbf{a_0} - [\mathbf{D}(\mathbf{a_0})]^{-1}\mathbf{f}(\mathbf{a_0})$. By renaming $\mathbf{x}$ to $\mathbf{a_1}$, you can reiterate Newton's method to get an $\mathbf{a_2}$. Thus, Newton's method states

$$\mathbf{a}_{n+1} = \mathbf{a}_n - [\mathbf{Df}(\mathbf{a}_n)]^{-1}\mathbf{f}(\mathbf{a}_n)$$

Thus we get a series of $\mathbf{a}$'s that we hope will converge to $\mathbf{x}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. When we solve an equation of the form $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, we call the solution a *root* of the equation. Thus, Newton's method is used to find roots of nonlinear equations.

Unfortunately, Newton's method does not always converge. There are tests for neighborhoods of $\mathbf{a_0}$'s where Newton's method will converge however. One such test is Kantorovitch's theorem, which combines what is needed into a concise mathematical equation.

Corollary 1: Newton's Method in one dimension - The above equation is simplified in one dimension to the well-used

$$a_1 = a_0 - \frac{f(a_0)}{f'(a_0)}$$

This intuitively cute equation is pretty much **the** equation of first year calculus. :)

Corollary 2: Finding a square root - So now that you know the equation, you need to know how to *use* it, as it is an algorithm. The construction of the primary equation, of course is the important part. Let's see how you do it if you want to find a square root of a number b.

We want to find a number x (x for unknown), such that $x^2 = b$. You might think "why not find a number such that $x = \sqrt{b}$?" Well, the problem with that approach is that we don't have a value for $\sqrt{b}$, so we'd be right back where we started. However, squaring both sides of the equation to get $x^2 = b$ lets us work with the number we *do* know, b.) Back to $x^2 = b$. With some manipulation, we see this means that $x^2 - b = 0$! Thus we have our $f(x) = 0$ scenario.

We can see that $f'(x) = 2x$ thus, $f'(a_0) = 2a_0$ and $f(a_0) = a_0^2 - b$. Now we have all we need to carry out Newton's method. By renaming x to be a_1, we have

$$a_1 = a_0 - \frac{1}{2a_0}(a_0^2 - b) = \frac{1}{2}\left(a_0 + \frac{b}{a_0}\right)$$

.

The equation on the far right is also known as the *divide and average* method, for those who have not learned the full Newton's method, and just want a fast way to find square roots. Let's see how this works out to find the square root of a number like 2:

Let $x^2 = 2$

$$x^2 - 2 = 0 = f(x)$$

Thus, by Newton's method,...

$$a_1 = a_0 - \frac{a_0^2 - 2}{2a_0}$$

All we did was plug in the expressions $f(a_0)$ and $f'(a_0)$ where Newton's method asks for them. Now we have to pick an a_0. Hmm, since

$$\sqrt{1} < \sqrt{2} < \sqrt{4}$$
$$1 < \sqrt{2} < 2$$

let's pick a reasonable number between 1 and 2 like 1.5

$$a_1 = 1.5 - \frac{1.5^2 - 2}{2(1.5)}$$
$$a_1 = 1.41\bar{6}$$

Looks like our guess was too high. Let's see what the next iteration says

$$a_2 = 1.41\bar{6} - \frac{1.41\bar{6}^2 - 2}{2(1.41\bar{6})}$$
$$a_2 = 1.414215686\ldots$$

getting better =) You can use your calculator to find that

$$\sqrt{2} = 1.414213562\ldots$$

Not bad for only two iterations! Of course, the more you iterate, the more decimal places your a_n will be accurate to. By the way, this is also how your calculator/computer finds square roots!

Geometric Interpretation: Consider an arbitrary function $f\colon \mathbb{R} \to \mathbb{R}$ such as $f(x) = x^2 - b$. Say you wanted to find a root of this function. You know that in the neighborhood of $x = a_0$, there is a root (Maybe you used Kantorovitch's theorem or tested and saw that the function changed signs in this neighborhood). We want to use our knowledge a_0 to find an a_1 that is a better approximation to x_0 (in this case, closer to it on the x-axis).

So we know that $x_0 \leq a_1 \leq a_0$ or in another case $a_0 \leq a_1 \leq x_0$. What is an efficient algorithm to bridge the gap between a_0 and x_0 ? Let's look at a tangent line to the graph.

Note that the line intercepts the x-axis between a_0 and x_0, which is exactly what we want. The slope of this tangent line is $f'(a_0)$ by definition of the derivative at a_0, and we know one point on the line is $(a_1, 0)$, since that is the x-intercept. That is all we need to find the formula of the line, and solve for a_1.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
f(a_0) - 0 &= f'(a_0)(a_0 - a_1) && \text{Substituting} \\
\frac{f(a_0)}{f'(a_0)} &= a_0 - a_1 \\
-a_1 &= -a_0 + \frac{f(a_0)}{f'(a_0)} && \text{Aesthetic change. Flipped the equation around.} \\
a_1 &= a_0 - \frac{f(a_0)}{f'(a_0)} && \text{Newton's method!}
\end{aligned}$$