proof of existence and uniqueness of best approximations

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- Proof of the theorem on existence and uniqueness of - (http://planetmath.org/BestApproximationInInnerProductSpacesentry)

Existence : Without loss of generality we can suppose $x = 0$ (we could simply translate by $-x$ the set A).

Let $d = d(0, A) = \inf\{\|a\| : a \in A\}$ be the distance of A to the origin. By defintion of infimum there exists a sequence (a_n) in A such that

$$\|a_n\| \longrightarrow d$$

Let us see that (a_n) is a Cauchy sequence. By the parallelogram law we have

$$\left\|\frac{a_n - a_m}{2}\right\|^2 + \left\|\frac{a_n + a_m}{2}\right\|^2 = \frac{1}{2}\|a_n\|^2 + \frac{1}{2}\|a_m\|^2$$

i.e.

$$\left\|\frac{a_n - a_m}{2}\right\|^2 = \frac{1}{2}\|a_n\|^2 + \frac{1}{2}\|a_m\|^2 - \left\|\frac{a_n + a_m}{2}\right\|^2$$

As A is convex, $\frac{a_n + a_m}{2} \in A$, and therefore

$$\left\|\frac{a_n + a_m}{2}\right\| \geq d$$

So we see that

$$\left\|\frac{a_n - a_m}{2}\right\|^2 \leq \frac{1}{2}\|a_n\|^2 + \frac{1}{2}\|a_m\|^2 - d^2 \longrightarrow 0 \qquad \text{when } \ m, n \to \infty$$

which means that $\|a_n - a_m\| \longrightarrow 0$ when $m, n \to \infty$, i.e. (a_n) is a Cauchy sequence.

Since A is http://planetmath.org/Completecomplete, $a_n \longrightarrow a_0$ for some $a_0 \in A$.

As $a_0 \in A$ its norm must be $\|a_0\| \geq d$. But also

$$\|a_0\| \leq \|a_0 - a_n\| + \|a_n\| \longrightarrow d$$

which shows that $\|a_0\| = d$. We have thus proven the existence of http://planetmath.org/BestApproximationInInnerProductSpacesbest approximations.

Uniqueness : Suppose there were $a_0, b_0 \in A$ such that $\|a_0\| = \|b_0\| = d$. Then, by the parallelogram law

$$\left\| \frac{a_0 - b_0}{2} \right\|^2 + \left\| \frac{a_n + b_0}{2} \right\|^2 = \frac{1}{2}\|a_0\|^2 + \frac{1}{2}\|b_0\|^2 = d^2$$

If $a_0 - b_0 \neq 0$ then we would have $\left\| \frac{a_0 + b_0}{2} \right\|^2 < d^2$, which is contradiction since $\frac{a_0 + b_0}{2} \in A$ (A is convex).

Therefore $a_0 = b_0$, which proves the uniqueness of the . $\square$