example of isogonal trajectory

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Determine the curves which intersect the origin-centered circles at an angle of 45°.

The differential equation of the circles $x^2+y^2 = R^2$ is $2x\,dx+2y\,dy = 0$, i.e.

$$\frac{x}{y}+\frac{dy}{dx} \;=\; 0.$$

Thus, by the model (2) of the `http://planetmath.org/IsogonalTrajectory`parent entry, the differential equation of the isogonal trajectory reads

$$\frac{x}{y}+\frac{y'-\tan\frac{\pi}{4}}{1+y'\tan\frac{\pi}{4}} \;=\; 0, \tag{1}$$

which can be rewritten as

$$y' \;=\; \frac{y\!-\!x}{y\!+\!x} \;=\; \frac{\frac{y}{x}\!-\!1}{\frac{y}{x}\!+\!1}.$$

Here, one may take $\frac{y}{x} := t$ as a new variable (see ODE types reductible to the variables separable case), when

$$y \;=\; xt, \quad y' \;=\; \frac{dy}{dx} \;=\; t+x\frac{dt}{dx},$$

and in the resulting equation

$$t+x\frac{dt}{dx} \;=\; \frac{t\!-\!1}{t\!+\!1}$$

one can `http://planetmath.org/SeparationOfVariables`separate the variables:

$$\frac{1\!+\!t}{1\!+\!t^2}\,dt \;=\; -\frac{dx}{x}$$

Multiplying here by 2 and integrating then give

$$2\arctan t+\ln(1\!+\!t^2) \;=\; -2\ln x+\ln C^2 \;\equiv\; -\ln\frac{x^2}{C^2},$$

or equivalently

$$\ln\frac{x^2\!+\!x^2t^2}{C^2} \;=\; -2\arctan t.$$

This is

$$\ln \frac{\sqrt{x^2+y^2}}{C} \;=\; -\arctan \frac{y}{x},$$

i.e.

$$\sqrt{x^2+y^2} \;=\; Ce^{-\arctan \frac{y}{x}}.$$

Expressing this in the polar coordinates r, φ gives the family of the integral curves of the equation (1) in the form

$$r \;=\; Ce^{-\varphi}.$$

Consequently, the family of the isogonal trajectories consists of logarithmic spirals.