proof of Hadwiger-Finsler inequality

Canonical name	ProofOfHadwigerFinslerInequality
Date of creation	2013-03-22 12:45:21
Last modified on	2013-03-22 12:45:21
Owner	mathwizard (128)
Last modified by	mathwizard (128)
Numerical id	5
Author	mathwizard (128)
Entry type	Proof
Classification	msc 51M16

From the cosines law we get:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha,$$

α being the angle between b and c. This can be transformed into:

$$a^2 = (b-c)^2 + 2bc(1-\cos\alpha).$$

Since $A = \frac{1}{2}bc\sin\alpha$ we have:

$$a^2 = (b-c)^2 + 4A\frac{1-\cos\alpha}{\sin\alpha}.$$

Now remember that

$$1 - \cos\alpha = 2\sin^2\frac{\alpha}{2}$$

and

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}.$$

Using this we get:

$$a^2 = (b-c)^2 + 4A\tan\frac{\alpha}{2}.$$

Doing this for all sides of the triangle and adding up we get:

$$a^2 + b^2 + c^2 = (a-b)^2 + (b-c)^2 + (c-a)^2 + 4A\left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\right).$$

β and γ being the other angles of the triangle. Now since the halves of the triangle's angles are less than $\frac{\pi}{2}$ the function tan is convex we have:

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} \geq 3\tan\frac{\alpha+\beta+\gamma}{6} = 3\tan\frac{\pi}{6} = \sqrt{3}.$$

Using this we get:

$$a^2 + b^2 + c^2 \geq (a-b)^2 + (b-c)^2 + (c-a)^2 + 4A\sqrt{3}.$$

This is the Hadwiger-Finsler inequality. $\square$