bound on area of right triangle

Canonical name	BoundOnAreaOfRightTriangle
Date of creation	2013-03-22 16:30:41
Last modified on	2013-03-22 16:30:41
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	17
Author	rspuzio (6075)
Entry type	Theorem
Classification	msc 51-00

We may bound the area of a right triangle in terms of its perimeter. The derivation of this bound is a good exercise in constrained optimization using Lagrange multipliers.

Theorem 1. *If a right triangle has perimeter P, then its area is bounded as*

$$A \leq \frac{3 - 2\sqrt{2}}{4} P^2$$

with equality when one has an isosceles right triangle.

Proof. Suppose a triangle has legs of length x and y. Then its hypotenuse has length $\sqrt{x^2 + y^2}$, so the perimeter is given as

$$P = x + y + \sqrt{x^2 + y^2}.$$

The area, of course, is

$$A = \frac{1}{2} xy.$$

We want to maximize A subject to the constraint that P be constant. This means that the gradient of A will be proportional to the gradient of P. That is to say, for some constant λ, we will have

$$\begin{aligned} \frac{\partial A}{\partial x} &= \lambda \frac{\partial P}{\partial x} \\ \frac{\partial A}{\partial y} &= \lambda \frac{\partial P}{\partial y} \end{aligned}$$

Together with the constraint, these form a system of three equations for the three quantities x, y, and λ. Writing them out explicitly,

$$\begin{aligned} \frac{1}{2} y &= \lambda \left(1 + \frac{x}{\sqrt{x^2 + y^2}} \right) \\ \frac{1}{2} x &= \lambda \left(1 + \frac{y}{\sqrt{x^2 + y^2}} \right) \\ P &= x + y + \sqrt{x^2 + y^2} \end{aligned}$$

Not that we cannot have $\lambda = 0$ because that would mean that all sides of our triangle would have zero length. Hence, we may eliminate λ between the first two equations to obtain

$$x \left(1 + \frac{x}{\sqrt{x^2 + y^2}} \right) = y \left(1 + \frac{y}{\sqrt{x^2 + y^2}} \right),$$

which may be manipulated to yield

$$(x-y)\left(1+\frac{x+y}{\sqrt{x^2+y^2}}\right)=0.$$

We have two case to consider — either the first factor or the second factor may equal zero. If the second factor equals zero,

$$1+\frac{x+y}{\sqrt{x^2+y^2}}=0,$$

move the "1" to the other side of the equation and cross-multiply to obtain

$$x+y=-\sqrt{x^2+y^2}.$$

Since we want $x \geq 0$ and $y \geq 0$ but the right-hand side is non-positive, the only option would be to have a trianagle of zero area. The other possibility was to have the second factor equal zero, which would give

$$x-y=0.$$

In this case, x equals y. Imposing this condition on the constraint, we see that

$$P=(2+\sqrt{2})x,$$

so we have the solution

$$\begin{aligned} x &= \frac{P}{2+\sqrt{2}}=\frac{2-\sqrt{2}}{2}P \\ y &= \frac{P}{2+\sqrt{2}}=\frac{2-\sqrt{2}}{2}P. \end{aligned}$$

□