finite projective planes have q^2+q+1 points and q^2+q+1 lines

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Given a finite projective plane that contains a quadrangle OXYZ (i.e. no three of these four points are on one line). **To prove**: the plane has q^2+q+1 points and q^2+q+1 lines for some integer q, and there are $q+1$ points on each line and $q+1$ lines through each point.

Let x and y be the lines OX and OY, which must exist by the axioms. By the assumption OXYZ is a quadrangle these lines are distinct and Z is not on them. Let there be p points X_i on x other than O, for each of them one line ZX_i exists, and is distinct (one lines cannot pass through two X_i unless it is x but that's not a line through Z). Conversely every line through Z must intersect x in a unique point (two lines intersecting in Z cannot intersect at another point, and Z is not a point on x). So there are $p+1$ lines through Z (OZ is one of them). By the same reasoning, using y, there are $q+1$ lines through Z so $p=q$. We also found $q+1$ points (including O) on y and the same number on x. Intersecting the $q+1$ lines through Z with XY (on which Z does not lie, the quadrangle again) reveals at least $q+1$ distinct points there and at most $q+1$ because for each point there there is a line through it and Z.

The lines not through O intersect x in one of the q points X_i and y in one of the q points Y_j. There are q^2 possibilities and each of them is a distinct line, because there is only one line through a given X_i and Y_j. The lines that do pass through O intersect XY in one of the $q+1$ points there, again one line for each such point and vice versa. That's $q+1$ lines through O and q^2 not through O, q^2+q+1 in all.

There are $q+1$ lines through X (to each of the points of y) and $q+1$ lines through Y (to each of the points of x). Intersect the q lines through X other than XY with the q lines through Y other than XY, these q^2 intersections are all distinct because for any P there's only one line PX and one line PY. Note we did not use the line XY. Conversely for any P not on XY there must be some PX and some PY, so there are exactly q^2 points not on XY. Add the $q+1$ points on XY for a total of q^2+q+1.

The constructions above already showed $q+1$ lines through some points (X, Y and Z), by the same games as before that implies for each of them $q+1$ points on every line not through that point. We also saw $q+1$ points on some lines (x, y, XY) which implies for each of them $q+1$ lines through every point not on that line. Such reasoning covers q^2 items on first application and rapidly mops up stragglers on repeated application.

Some form of this proof is standard math lore; this version was half remembered and half reconstructed.