Pasch's theorem

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Theorem. *(Pasch) Let $\triangle abc$ be a triangle with non-collinear vertices a, b, c in a linear ordered geometry. Suppose a line ℓ intersects one side, say open line segment $\overline{ab}$, at a point strictly between a and b, then ℓ also intersects exactly one of the following:*

$$\overline{bc}, \qquad \overline{ac}, \qquad c.$$

Proof. First, note that vertices a and b are on opposite sides of line ℓ. Then either c lies on ℓ, or c does not. if c does not, then it must lie on either side (half plane) of ℓ. In other words, c and a must be on the opposite sides of ℓ, or c and b must be on the opposite sides of ℓ. If c and a are on the opposite sides, ℓ has a non-empty intersection with $\overline{ac}$. But if c and a are on the opposite sides, then c and b are on the same side, which means that $\overline{bc}$ does not intersect ℓ. □

Remark A companion property states that if line ℓ passes through one vertex a of a triangle $\triangle abc$ and at least one other point on $\triangle abc$, then it must intersect exactly one of the following:

$$b, \qquad c, \qquad \overline{bc}.$$

Of course, if ℓ passes through b, $\overline{ab}$ must lie on ℓ. Similarly, $\overline{ac}$ lies on ℓ if ℓ passes through c.