polyhedron

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Related topic	RegularPolygon
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Defines	vertex
Defines	corner point
Defines	finite polyhedron
Defines	locally finite
Defines	polyhedra
Defines	bounded polyhedron
Defines	normal polyhedron
Defines	regular polyhedron
Defines	Euler polyhedron
Defines	convex polyhedron
Defines	simple polyhedron

At least four definitions of a polyhedron are used.

Combinatorics

In combinatorics a *polyhedron* is the solution set of a finite system of linear inequalities. The solution set is in $\mathbb{R}^n$ for integer n. Hence, it is a convex set. Each extreme point of such a polyhedron is also called a *vertex* (or *corner point*) of the polyhedron. A solution set could be empty. If the solution set is bounded (that is, is contained in some sphere) the polyhedron is said to be *bounded*.

Elementary Geometry

In elementary geometry a polyhedron is a solid bounded by a finite number of plane faces, each of which is a polygon. This of course is not a precise definition as it relies on the undefined term "solid". Also, this definition allows a polyhedron to be non-convex.

Careful Treatments of Geometry

In treatments of geometry that are carefully done a definition due to Lennes is sometimes used [**?**]. The intent is to rule out certain objects that one does not want to consider and to simplify the theory of dissection. A polyhedron is a set of points consisting of a finite set of triangles T, not all coplanar, and their interiors such that

(i) every side of a triangle is common to an even number of triangles of the set, and

(ii) there is no subset T' of T such that (i) is true of a proper subset of T'.

Notice that condition (ii) excludes, for example, two cubes that are disjoint. But two tetrahedra having a common edge are allowed. The faces of the polyhedron are the insides of the triangles. Note that the condition that the faces be triangles is not important, since a polygon an be dissected into triangles. Also note since a triangle meets an *even* number of other triangles, it is possible to meet 4,6 or any other even number of triangles. So for example, a configuration of 6 tetrahedra all sharing a common edge is allowed.

By dissections of the triangles one can create a set of triangles in which no face intersects another face, edge or vertex. If this done the polyhedron is said to be .

A *convex polyhedron* is one such that all its inside points lie on one side of each of the planes of its faces.

An *Euler polyhedron* P is a set of points consisting of a finite set of *polygons*, not all coplanar, and their insides such that

(i) each edge is common to just *two* polygons,

(ii) there is a way using edges of P from a given vertex to each vertex, and

(iii) any simple polygon p made up of edges of P, divides the polygons of P into two sets A and B such that any way, whose points are on P from any point inside a polygon of A to a point inside a polygon of B, meets p.

A *regular polyhedron* is a convex Euler polyhedron whose faces are congruent regular polygons and whose dihedral angles are congruent.

It is a theorem, proved `http://planetmath.org/ClassificationOfPlatonicSolids`here, that for a regular polyhedron, the number of polygons with the same vertex is the same for each vertex and that there are 5 types of regular polyhedra.

Notice that a cone, and a cylinder are not polyhedra since they have "faces" that are not polygons.

A *simple polyhedron* is one that is homeomorphic to a sphere. For such a polyhedron one has $V - E + F = 2$, where V is the number of vertices, E is the number of edges and F is the number of faces. This is called Euler's formula.

Algebraic Topology

In algebraic topology another definition is used:

If K is a simplicial complex in $\mathbb{R}^n$, then $|K|$ denotes the union of the elements of K, with the subspace topology induced by the topology of $\mathbb{R}^n$. $|K|$ is called a *polyhedron*. If K is a finite complex, then $|K|$ is called a *finite polyhedron*.

It should be noted that we allow the complex to have an infinite number of simplexes. As a result, spaces such as $\mathbb{R}$ and $\mathbb{R}^n$ are polyhedra.

Some authors require the simplicial complex to be *locally finite*. That is, given $x \in \sigma \in K$ there is a neighborhood of x that meets only finitely many $\tau \in K$.

References

[1] Henry George Forder, *The Foundations of Euclidean Geometry*, Dover Publications, New York , 1958.

[2] N.J. Lennes, *On the simple finite polygon and polyhedron*, Amer. J. Math. **33**, (1911), p. 37