arclength as filtered limit

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The http://planetmath.org/Rectifiablelength of a rectifiable curve may be phrased as a filtered limit. To do this, we will define a filter of partitions of an interval $[a,b]$. Let $\mathbf{P}$ be the set of all ordered tuplets of distinct elements of $[a,b]$ whose entries are increasing:

$$\mathbf{P} = \{(t_1, \ldots t_n) \mid (a \leq t_1 < t_2 < \cdots < t_n \leq b) \wedge (n \in \mathbb{Z}) \wedge (n > 0)\}$$

We shall refer to elements of $\mathbf{P}$ as partitions of the interval $[a,b]$. We shall say that $(t_1, \ldots, t_n)$ is a refinement of a partition $(s_1, \ldots, s_m)$ if $\{t_1, \ldots, t_n\} \supset \{s_1, \ldots, s_m\}$. Let $\mathbf{F} \subset \mathcal{P}(\mathbf{P})$ be the set of all subsets of $\mathbf{P}$ such that, if a certain partition belongs to $\mathbf{F}$ then so do all refinements of that partition.

Let us see that $\mathbf{F}$ is a filter basis. Suppose that A and B are elements of $\mathbf{F}$. If a partition belongs to both A and B then every one of its refinements will also belong to both A and B, hence $A \cap B \in \mathbf{F}$.

Next, note that, if a partition of B is a refinement of a partition of A then, by the triangle inequality, the length of $\Pi(B)$ is greater than the length of $\Pi(A)$. By definition, for every $\epsilon > 0$, we can pick a partition A such that the length of $\Pi(A)$ differs from the length of the curve by at most ϵ. Since the length of $\Pi(B)$ for any partition B refining A lies between the length of $\Pi(A)$ and the length of the curve, we see that the length of $\Pi(B)$ will also differ by at most ϵ, so the length of the curve is the limit of the length of polygonal lines according to the filter generated by $\mathbf{F}$.