proof of Desargues' theorem

Canonical name	ProofOfDesarguesTheorem
Date of creation	2013-03-22 13:47:51
Last modified on	2013-03-22 13:47:51
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Last modified by	drini (3)
Numerical id	5
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Entry type	Proof
Classification	msc 51A30

The claim is that if triangles ABC and XYZ are perspective from a point P, then they are perspective from a line (meaning that the three points

$$AB \cdot XY \qquad BC \cdot YZ \qquad CA \cdot ZX$$

are collinear) and conversely.

Since no three of A, B, C, P are collinear, we can lay down homogeneous coordinates such that

$$A = (1, 0, 0) \qquad B = (0, 1, 0) \qquad C = (0, 0, 1) \qquad P = (1, 1, 1)$$

By hypothesis, there are scalars p, q, r such that

$$X = (1, p, p) \qquad Y = (q, 1, q) \qquad Z = (r, r, 1)$$

The equation for a line through (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$(y_1 z_2 - z_1 y_2)x + (z_1 x_2 - x_1 z_2)y + (x_1 y_2 - y_1 x_2)z = 0 \ ,$$

giving us equations for six lines:

$$\begin{array}{rcl} AB & : & z = 0 \\ BC & : & x = 0 \\ CA & : & y = 0 \\ XY & : & (pq - p)x + (pq - q)y + (1 - pq)z = 0 \\ YZ & : & (1 - qr)x + (qr - q)y + (qr - r)z = 0 \\ ZX & : & (rp - p)x + (1 - rp)y + (rp - r)z = 0 \end{array}$$

whence

$$\begin{array}{rcl} AB \cdot XY & = & (pq - q, -pq + p, 0) \\ BC \cdot YZ & = & (0, qr - r, -qr + q) \\ CA \cdot ZX & = & (-rp + r, 0, rp - p). \end{array}$$

As claimed, these three points are collinear, since the determinant

$$\begin{vmatrix} pq - q & -pq + p & 0 \\ 0 & qr - r & -qr + q \\ -rp + r & 0 & rp - p \end{vmatrix}$$

is zero. (More precisely, all three points are on the line

$$p(q - 1)(r - 1)x + (p - 1)q(r - 1)y + (p - 1)(q - 1)rz = 0 \ .)$$

Since the hypotheses are self-dual, the converse is true also, by the principle of duality.