homology of the sphere

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Every loop on the sphere S^2 is contractible to a point, so its fundamental group, $\pi_1(S^2)$, is trivial.

Let $H_n(S^2, \mathbb{Z})$ denote the n-th homology group of S^2. We can compute all of these groups using the basic results from algebraic topology:

- S^2 is a compact orientable smooth manifold, so $H_2(S^2, \mathbb{Z}) = \mathbb{Z}$;
- S^2 is connected, so $H_0(S^2, \mathbb{Z}) = \mathbb{Z}$;
- $H_1(S^2, \mathbb{Z})$ is the abelianization of $\pi_1(S^2)$, so it is also trivial;
- S^2 is two-dimensional, so for $k > 2$, we have $H_k(S^2, \mathbb{Z}) = 0$

In fact, this pattern generalizes nicely to higher-dimensional spheres:

$$H_k(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{else} \end{cases}$$

This also provides the proof that the hyperspheres S^n and S^m are non-homotopic for $n \neq m$, for this would imply an isomorphism between their homologies.