cross ratio

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The *cross ratio* of the points a, b, c, and d in $\mathbb{C} \cup \{\infty\}$ is denoted by $[a, b, c, d\,]$ and is defined by

$$[a, b, c, d\,] = \frac{a-c}{a-d} \cdot \frac{b-d}{b-c}.$$

Some authors denote the cross ratio by (a, b, c, d).

Examples

Example 1. The cross ratio of 1, i, -1, and $-i$ is

$$\frac{1-(-1)}{1-(-i)} \cdot \frac{i-(-i)}{i-(-1)} = \frac{4i}{(1+i)^2} = 2.$$

Example 2. The cross ratio of 1, $2i$, 3, and $4i$ is

$$\frac{1-3}{1-4i} \cdot \frac{2i-4i}{2i-3} = \frac{4i}{5+14i} = \frac{56+20i}{221}.$$

Properties

1. The cross ratio is invariant under Möbius transformations and projective transformations. This fact can be used to determine distances between objects in a photograph when the distance between certain reference points is known.

2. The cross ratio $[a, b, c, d\,]$ is real if and only if a, b, c, and d lie on a single circle on the Riemann sphere.

3. The function $T : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ defined by

   $$T(z) = [z, b, c, d\,]$$

   is the unique Möbius transformation which sends b to 1, c to 0, and d to ∞.

References

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[2] Beardon, A. F., *The Geometry of Discrete Groups*. Springer-Verlag, 1983.

[3] Henle, M., *Modern Geometries: Non-Euclidean, Projective, and Discrete*. Prentice-Hall, 1997 [2001].