ternary ring

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Let R be a set containing at least two distinct elements 0 and 1, and $*$ a ternary operation on R. Write $a*b*c$ the image of (a,b,c) under $*$. We call $(R,*,0,1)$, or simply just R, a *ternary ring* if

1. $a*0*b=0*a*b=b$ for any $a,b\in R$,
2. $1*a*0=a*1*0=a$ for any $a\in R$,
3. given $a,b,c,d\in R$ with $a\neq c$, the equation $x*a*b=x*c*d$ has a unique solution for x,
4. given $a,b,c\in R$, the equation $a*b*x=c$ has a unique solution for x,
5. given $a,b,c,d\in R$, with $a\neq c$, the system of equations
$$\begin{cases} a*x*y=b \\ c*x*y=d \end{cases}$$
has a unique solution for (x,y).

Given a ternary ring R, we may form two binary operations on R, one called the *addition* $+$, and the other the *multiplication* $\cdot$ on R:

$$\begin{aligned} a+b &:= a*1*b, \\ a\cdot b &:= a*b*0. \end{aligned}$$

Proposition 1. *$(R,+,0)$ and $(R-\{0\},\cdot,1)$ are loops, and 0 is the zero element in R under the multiplication $\cdot$.*

Proof. We first show that $(R,+,0)$ is a loop. Given $a,b\in R$, there is a unique $c\in R$ such that $a*1*c=b$, but this is exactly $a+c=b$. In addition, there is a unique $d\in R$ such that $d*1*a=b=d*0*b$. But $d*1*a=b$ means $d+a=b$. This shows that $(R,+)$ is a quasigroup. Now, $a+0=a*1*0=a$ and $0+a=0*1*a=a$, so 0 is the identity with respect to $+$. Therefore, $(R,+,0)$ is a loop.

Next we show that $(S,\cdot,1)$ is a loop, where $S=R-\{0\}$. Given $a,b\in S$, there is a unique $c\in R$ such that $c*a*0=b=c*0*b=b$, since $a\neq 0$. From $c*a*0=b$, we get $c\cdot a=b$. Furthermore, $c\neq 0$, for otherwise $b=c*a*0=0*a*0=0$, contradicting the fact that $b\in S$. In addition, there is a unique $d\in R$ such that we have a system of equations $a*d*0=b$ and $0*d*0=0$. From $a*d*0=b$ we get $a\cdot d=b$. Furthermore $d\neq 0$, for otherwise $b = a*d*0 = a*0*0 = 0$, contradicting the fact that $b \in S$. Thus, $(S, \cdot, 1)$ is a quasigroup. Now, $a \cdot 1 = a * 1 * 0 = a$ and $1 \cdot a = 1 * a * 0 = a$, showing that $(S, \cdot, 1)$ is a loop.

Finally, for any $a \in R$, $a \cdot 0 = a * 0 * 0 = 0$ and $0 \cdot a = 0 * a * 0 = 0$. $\square$

Another property of a ternary ring is that, if the ternary ring is finite, then conditions 4 and 5 are equivalent in the presence of the first three.

Let R be a ternary ring, and a, b, c are arbitrary elements of R. R is said to be *linear* if $a * b * c = a \cdot b + c$ for all $a, b, c \in R$, *left distributive* if $a \cdot (b + c) = a \cdot b + a \cdot c$, *right distributive* if $(a + b) \cdot c = a \cdot c + b \cdot c$, and *distributive* if it is both left and right distributive.

For example, any division ring D, associative or not, is a linear ternary ring if we define the ternary operation $*$ on D by $a * b * c := a \cdot b + c$. Any associative division ring D is a distributive ternary ring. This easy verification is left to the reader. Semifields, near fields are also examples of ternary rings.

Remark. Ternary rings were invented by Marshall Hall in his studies of axiomatic projective and affine planes. Therefore, a ternary ring is also called a *Hall ternary ring*, or a *planar ternary ring*. It can be shown that in every affine plane, one can set up a coordinate system, and from this coordinate system, one can construct a ternary ring. Conversely, given any ternary ring, one can define an affine plane so that its coordinate system corresponds to this ternary ring.

References

[1] R. Artzy, *Linear Geometry*, Addison-Wesley (1965)