Veblen-Wedderburn system

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A *Veblen-Wedderburn system* is an algebraic system over a set R with two binary operations $+$ (called addition) and $\cdot$ (called multiplication) on R such that

1. there is a $0 \in R$, and that R is an abelian group under $+$, with 0 the additive identity
2. $R - \{0\}$, together with $\cdot$, is a loop (we denote 1 as its identity element)
3. $\cdot$ is right distributive over $+$; that is, $(a + b) \cdot c = a \cdot c + b \cdot c$
4. if $a \neq b$, then the equation $x \cdot a = x \cdot b + c$ has a unique solution in x

A Veblen-Wedderburn system is also called a *quasifield*.

Usually, we write ab instead of $a \cdot b$.

For any $a, b, c \in R$, by defining a ternary operation $*$ on R, given by

$$a * b * c := ab + c,$$

it is not hard to see that $(R, *, 0, 1)$ is a ternary ring. In fact, it is a linear ternary ring because $ab = a * b * 0$ and $a + c = a * 1 * c$.

For example, any field, or more generally, any division ring, associative or not, is Veblen-Wedderburn. An example of a Veblen-Wedderburn system that is not a division ring is the Hall quasifield.

A well-known fact about Veblen-Wedderburn systems is that, the projective plane of a Veblen-Wedderburn system is a translation plane, and, conversely, every translation plane can be coordinatized by a Veblen-Wedderburn system. This is the reason why a translation plane is also called a Veblen-Wedderburn plane.

Remark. Let R be a Veblen-Wedderburn system. If the multiplication $\cdot$, in addition to be right distributive over $+$, is also left distributive over $+$, then R is a semifield. If $\cdot$, on the other hand, is associative, then R is an abelian nearfield (a nearfield such that $+$ is commutative).

References

[1] R. Casse, *Projective Geometry, An Introduction*, Oxford University Press (2006)