proof of Banach-Tarski paradox

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We deal with some technicalities first, mainly concerning the properties of equi-decomposability. We can then prove the paradox in a clear and unencumbered line of argument: we show that, given two unit balls U and U' with arbitrary origin, U and $U \cup U'$ are equi-decomposable, regardless whether U and U' are disjoint or not. The original proof can be found in [**?**].

Technicalities

Theorem 1. *Equi-decomposability gives rise to an equivalence relation on the subsets of Euclidean space.*

Proof. The only non-obvious part is transitivity. So let A, B, C be sets such that A and B as well as B and C are equi-decomposable. Then there exist disjoint decompositions $A_1, \dots A_n$ of A and $B_1, \dots B_n$ of B such that A_k and B_k are congruent for $1 \leq k \leq n$. Furthermore, there exist disjoint decompositions $B'_1, \dots, B'_m$ of B and $C_1, \dots C_m$ of C such that B'_l and C_l are congruent for $1 \leq l \leq m$. Define

$$B_{k,l} := B_k \cap B'_l \text{ for } 1 \leq k \leq n, 1 \leq l \leq m.$$

Now if A_k and B_k are congruent via some isometry $\theta\colon A_k \to B_k$, we obtain a disjoint decomposition of A_k by setting $A_{k,l} := \theta^{-1}(B_{k,l})$. Likewise, if B'_l and C_l are congruent via some isometry $\theta'\colon B'_l \to C_l$, we obtain a disjoint decomposition of C_l by setting $C_{k,l} := \theta'(B_{k,l})$. Clearly, the $A_{k,l}$ and $C_{k,l}$ define a disjoint decomposition of A and C, respectively, into nm parts. By transitivity of congruence, $A_{k,l}$ and $C_{k,l}$ are congruent for $1 \leq k \leq n$ and $1 \leq l \leq m$. Therefore, A and C are equi-decomposable. □

Theorem 2. *Given disjoint sets $A_1, , \dots A_n$, $B_1, \dots B_n$ such that A_k and B_k are equi-decomposable for $1 \leq k \leq n$, their unions $A = \bigcup_{k=1}^{n} A_k$ and $B = \bigcup_{k=1}^{n} B_k$ are equi-decomposable as well.*

Proof. By definition, there exists for every k, $1 \leq k \leq n$ an integer l_k such that there are disjoint decompositions

$$A_k = \bigcup_{i=1}^{l_k} A_{k,i}, \qquad B_k = \bigcup_{i=1}^{l_k} B_{k,i}$$

such that $A_{k,i}$ and $B_{k,i}$ are congruent for $1 \leq i \leq l_k$. Rewriting A and B in the form

$$A = \bigcup_{k=1}^{n} \bigcup_{i=1}^{l_k} A_{k,i}, \qquad B = \bigcup_{k=1}^{n} \bigcup_{i=1}^{l_k} B_{k,i}$$

gives the result. □

Theorem 3. *Let A, B, C be sets such that A and B are equi-decomposable and $C \subsetneq A$, then there exists $D \subsetneq B$ such that C and D are equi-decomposable.*

Proof. Let $A_1, \dots, A_n$ and $B_1, \dots B_n$ be disjoint decompositions of A and B, respectively, such that A_k and B_k are congruent via an isometry $\theta_k \colon A_k \to B_k$ for all $1 \leq k \leq n$. Let $\theta \colon A \to B$ a map such that $\theta(x) = \theta_k(x)$ if $x \in A_k$. Since the A_k are disjoint, θ is well-defined everywhere. Furthermore, θ is obviously bijective. Now set $C_k := C \cap A_k$ and define $D_k := \theta_k(C_k)$, so that C_k and D_k are congruent for $1 \leq k \leq n$, so the disjoint union $D := D_1 \cup \dots \cup D_n$ and C are equi-decomposable. By construction, $\theta(C) = D$. Since C is a proper subset of A and θ is bijective, D is a proper subset of B. □

Theorem 4. *Let A, B and C be sets such that A and C are equi-decomposable and $A \subseteq B \subseteq C$. Then B and C are equi-decomposable.*

Proof. Let $A_1, \dots, A_n$ and $C_1, \dots C_n$ disjoint decompositions of A and C, respectively, such that A_k and C_k are congruent via an isometry θ_k for $1 \leq k \leq n$. Like in the proof of theorem **??**, let $\theta \colon A \to C$ be the well-defined, bijective map such that $\theta(x) = \theta_k(x)$ if $x \in A_k$. Now, for every $b \in B$, let $\mathcal{C}(b)$ be the intersection of all sets $X \subseteq B$ satisfying

- $b \in X$,
- for all $x \in X$, the preimage $\theta^{-1}(x)$ lies in X,

- for all $x \in X \cap A$, the image $\theta(x)$ lies in X.

Let $b_1, b_2 \in B$ such that $\mathcal{C}(b_1)$ and $\mathcal{C}(b_2)$ are not disjoint. Then there is a $b \in \mathcal{C}(b_1) \cap \mathcal{C}(b_2)$ such that $b = \theta^r(b_1) = \theta^s(b_2)$ for suitable integers s and r. Given $b' \in \mathcal{C}(b_1)$, we have $b' = \theta^t(b_1) = \theta^{t+s-r}(b_2)$ for a suitable integer t, that is $b' \in \mathcal{C}(b_2)$, so that $\mathcal{C}(b_1) \subseteq \mathcal{C}(b_2)$. The reverse inclusion follows likewise, and we see that for arbitrary $b_1, b_2 \in B$ either $\mathcal{C}(b_1) = \mathcal{C}(b_2)$ or $\mathcal{C}(b_1)$ and $\mathcal{C}(b_2)$ are disjoint. Now set

$$D := \{b \in B \mid \mathcal{C}(b) \subseteq A\},$$

then obviously $D \subseteq A$. If for $b \in B$, $\mathcal{C}(b)$ consists of the sequence of elements $\ldots, \theta^{-1}(b), b, \theta(b), \ldots$ which is infinite in both directions, then $b \in D$. If the sequence is infinite in only one direction, but the final element lies in A, then $b \in D$ as well. Let $E := \theta(D)$ and $F := B \setminus D$, then clearly $E \cup F \subseteq C$.

Now let $c \in C$. If $c \notin E$, then $\theta^{-1}(c) \notin D$, so $\mathcal{C}(\theta^{-1}(c))$ consists of a sequence $\ldots, \theta^{-2}(c), \theta^{-1}(c), \ldots$ which is infinite in only one direction and the final element does not lie in A. Now $\theta^{-1}(c) \in \mathcal{C}(\theta^{-1}(c))$, but since $\theta^{-1}(c)$ *does* lie in A, it is not the final element. Therefore the subsequent element c lies in $\mathcal{C}(\theta^{-1}(c))$, in particular $c \in B$ and $\mathcal{C}(c) = \mathcal{C}(\theta^{-1}(c)) \nsubseteq A$, so $c \in B \setminus D = F$. It follows that $C = E \cup F$, and furthermore E and F are disjoint.

It now follows similarly as in the preceding proofs that D and $E = \theta(D)$ are equi-decomposable. By theorem **??**, $B = D \cup F$ and $C = E \cup F$ are equi-decomposable. $\square$

The proof

We may assume that the unit ball U is centered at the origin, that is $U = \mathbb{B}_3$, while the other unit ball U' has an arbitrary origin. Let S be the surface of U, that is, the unit sphere. By the Hausdorff paradox, there exists a disjoint decomposition

$$S = B' \cup C' \cup D' \cup E'$$

such that B', C', D' and $C' \cup D'$ are congruent, and E' is countable. For $r > 0$ and $A \subseteq \mathbb{R}^3$, let rA be the set of all vectors of A multiplied by r. Set

$$B := \bigcup_{0<r<1} rB', \quad C := \bigcup_{0<r<1} rC', \quad D := \bigcup_{0<r<1} rD', \quad E := \bigcup_{0<r<1} rE'.$$

These sets give a disjoint decomposition of the unit ball with the origin deleted, and obviously B, C, D and $C \cup D$ are congruent (but E is of course *not* countable). Set

$$A_1 := B \cup E \cup \{0\}$$

where 0 is the origin. B and $C \cup D$ are trivially equi-decomposable. Since C and B as well as D and C are congruent, $C \cup D$ and $B \cup C$ are equi-decomposable. Finally, B and $C \cup D$ as well as C and B are congruent, so $B \cup C$ and $B \cup C \cup D$ are equi-decomposable. In total, B and $B \cup C \cup D$ are equi-decomposable by theorem **??**, so A_1 and U are equi-decomposable by theorem **??**. Similarly, we conclude that C, D and $B \cup C \cup D$ are equi-decomposable.

Since E' is only countable but there are uncountably many rotations of S, there exists a rotation θ such that $\theta(E') \subsetneq B' \cup C' \cup D'$, so $F := \theta(E)$ is a proper subset of $B \cup C \cup D$. Since C and $B \cup C \cup D$ are equi-decomposable, there exists by theorem **??** a proper subset $G \subsetneq C$ such that G and F (and thus E) are equi-decomposable. Finally, let $p \in C \setminus G$ an arbitrary point and set

$$A_2 := D \cup G \cup \{p\},$$

a disjoint union by construction. Since D and $B \cup C \cup D$, G and E as well as $\{0\}$ and $\{p\}$ are equi-decomposable, A_2 and U are equi-decomposable by theorem **??**. A_1 and A_2 are disjoint (but $A_1 \cup A_2 \neq U$!).

Now U and U' are congruent, so U' and A_2 are equi-decomposable by theorem **??**. If U and U' are disjoint, set $H := A_2$. Otherwise we may use theorem **??** and choose $H \subsetneq A_2$ such that H and $U' \setminus U$ are equi-decomposable. By theorem **??**, $A_1 \cup H$ and $U \cup (U' \setminus U) = U \cup U'$ are equi-decomposable. Now we have

$$A_1 \cup H \subseteq U \subseteq U \cup U',$$

so by theorem **??**, U and $U \cup U'$ are equi-decomposable.

References

[BT] St. Banach, A. Tarski, Sur la décomposition des ensembles de points en parties respectivement congruentes, *Fund. math.* 6, 244–277, (1924).