antipodal map on S^n is homotopic to the identity if and only if n is odd

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Lemma. *If $X\colon S^n \to S^n$ is a unit vector field, then there is a homotopy between the antipodal map on S^n and the identity map.*

Proof. Regard S^n as a subspace of R^{n+1} and define $H\colon S^n \times [0,1] \to R^{n+1}$ by $H(v,t) = (\cos \pi t)v + (\sin \pi t)X(v)$. Since X is a unit vector field, $X(v) \perp v$ for any $v \in S^n$. Hence $\|H(v,t)\| = 1$, so H is into S^n. Finally observe that $H(v,0) = v$ and $H(v,1) = -v$. Thus H is a homotopy between the antipodal map and the identity map. □

Proposition. *The antipodal map $A\colon S^n \to S^n$ is homotopic to the identity if and only if n is odd.*

Proof. If n is even, then the antipodal map A is the composition of an odd of reflections. It therefore has degree -1. Since the degree of the identity map is $+1$, the two maps are not homotopic.

Now suppose n is odd, say $n = 2k-1$. Regard S^n has a subspace of $\mathbb{R}^{2k}$. So each point of S^n has coordinates $(x_1, \ldots, x_{2k})$ with $\sum_i x_i^2 = 1$. Define a map $X\colon \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ by $X(x_1, x_2, \ldots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1})$, pairwise swapping coordinates and negating the even coordinates. By construction, for any $v \in S^n$, we have that $\|X(v)\| = 1$ and $X(v) \perp v$. Hence X is a unit vector field. Applying the lemma, we conclude that the antipodal map is homotopic to the identity. □

References

[1] Hatcher, A. *Algebraic topology*, Cambridge University Press, 2002.

[2] Munkres, J. *Elements of algebraic topology*, Addison-Wesley, 1984.