projective plane

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Projective planes

A **projective plane** is a plane (in various senses) where not only

- for any two distinct POINTs, there is exactly one LINE through both of them

(as usual, in things we call a "plane"), but also

- for any two distinct LINEs, there is exactly one POINT on both of them

(in other words, no parallel LINEs). This gives **duality** between POINTs and LINEs: for any statement there is a corresponding statement that swaps the words POINT and LINE, and swaps the phrases *lies on* and *passes through* (for which we can use the neutral *is incident with*).

A third axiom is commonly added to avoid degenerate cases:

- there exist four points no three of which are collinear.

Both finite and infinite projective planes exist.

Here's an example, just to show that such things exist: let S be the unit sphere (the 2-dimensional surface $x^2 + y^2 + z^2 = 1$ embedded in $\mathbb{R}^3$). Call every great circle (circle with radius 1 whose centre coincides with that of the sphere) a LINE, and call every *pair of opposite points* on the sphere a POINT. The notion of a POINT lying on a LINE is well defined here, because each great circle (one of our LINEs) passing through some point on the sphere also passes through its opposite point (which together form one of our POINTs). Such a situation where different flavors of "point" are discussed will arise again, and it is the reason why the entities that act as POINT and LINE of a projective plane are typeset in their own distinctive way in this entry.

And here's a finite example, the Fano plane: the seven POINTs are labeled with the residue classes (mod 7) and the seven LINEs are numbered likewise. A POINT p and a LINE q are incident if and only if the equation $x^2 = q - p$ does not have a solution. Line q is incident with POINTs $q+1$, $q+2$, $q+4$ and POINT p is incident with LINEs $p-1$, $p-2$, $p-4$. Another way to get the same plane (but with a quite different numbering): let $\mathbb{F}_2$ be the finite field of order 2 and $\mathbb{F}_2^3$ the 3-dimensional vector space over $\mathbb{F}$. The seven non-null vectors label our POINTs, and the LINEs are numbered likewise. A POINT is incident with a LINE if and only if their dot product is zero (for example: POINT (0,1,1) lies on LINE (1,1,1) because $0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 2 = 0$).

Homogeneous coordinates

Historically, projective planes were formed by extending ordinary planes.

Let Π^* be a plane, a two-dimensional vector space over a field $\mathbb{F}$, where a point can be represented by its Cartesian coordinates (X, Y) with X and $Y \in \mathbb{F}$ (when $\mathbb{F} = \mathbb{R}$, this is the usual Euclidean plane). To specify a line ℓ, we can give a linear equation

$$pX + qY + r = 0 \tag{1}$$

satisfied by all the points on ℓ. The slope of the line is given by $-p/q$ (if $q \neq 0$) which is a single field element. Writing it as a **ratio** $-p : q$ allows us to define slope for vertical lines as well, where the ratio is $1 : 0$. This kind of extension of the field to include some kind of infinity will be a recurrent theme.

There are many asymmetries between points and lines here. First of all, the line is thought of as *being* a set of points (the points on the line) while a point is not the same thing as the set of lines through that point. This means we're really interpreting equation (**??**) as

$$\ell \stackrel{\text{def}}{=} \{(X, Y) \mid pX + qY + r = 0\}$$

where X and Y are variables (coordinates of any point $\in \ell$) and p, q and r parameters specifying which line. We can redress this conceptual imbalance by giving the line a kind of coordinates $[\,p, q, r\,]$. The linear equation (**??**) now takes on a different interpretation: it is the statement *that* the point (X, Y) and the line $[\,p, q, r\,]$ are incident, with both sets of coordinates on an equal footing.

Secondly, there are three coordinates for the line but only two for the point. This is caused by the fact that line coordinate triples are not unique; only the *ratio* of the coordinates matters. We can define equivalence classes

$$[\,p : q : r\,] \stackrel{\text{def}}{=} \{[fp, fq, fr] \mid f \in \mathbb{F},\ f \neq 0\}$$

of coordinate triples that represent the same line. The p, q and r used to label a $[\,p : q : r\,]$ are of course still just as non-unique (this is in the same spirit as labeling a residue class or other coset by one of its elements). One possible convention (for lines with $r \neq 0$ at least) is to choose the representative with $r = 1$.

To find more symmetry between points and lines we could first define coordinate triples for points with that same behaviour, so-called **homogeneous coordinates**

$$(x,\, y,\, z) \stackrel{\text{def}}{=} (x/z,\, y/z)$$

which means $(X, Y, 1)$ and more generally (fX, fY, f) for $f \neq 0$ are new names for the point (X, Y), and then define the equivalence classes

$$(x : y : z) \stackrel{\text{def}}{=} \{(fx, fy, fz) \mid f \in \mathbb{F},\ f \neq 0\}$$

to formally put all those different names for the same point back into a single box. This exercise gives the statement

$$px + qy + rz = 0 \tag{2}$$

that $(x : y : z)$ and $[\, p : q : r\,]$ are incident a pleasing symmetry.

The line and points at infinity

Thus far, we cannot have any point $(x : y : z)$ where $z = 0$ (it does not correspond to any point (X, Y) in the plane). By contrast, $[\,p : q : r\,]$ can have $r = 0$ (for a line through the arbitrary origin of the XY coordinate frame). For lines only the case $p = q = 0$ is missing, whereas $x = y = 0$ is fine for a point. Thus, by trying to make points and lines as similar as possible we have unearthed their essential difference algebraically.

The geometric difference is that lines can be parallel, and it is easy to see this is the same difference. For any two points (x, y, z) and (x', y', z') we can find a line $[\,yz' - zy',\, zx' - xz',\, xy' - yx'\,]$ that passes through both, and it is a valid line (first two coordinates not both zero) if the points are valid (z and z' nonzero) and distinct, but attempting the dual construction reveals pairs of valid lines that do not intersect in a valid point.

We now extend the plane $\mathit{\Pi}^*$ to a new kind of plane $\mathit{\Pi}$, inheriting all the points and lines of $\mathit{\Pi}^*$ as POINTs and LINEs of $\mathit{\Pi}$ and co-opting additionally

- the new LINE $[\,0 : 0 : 1\,]$ — note only one new LINE is needed as all triples $[\,0,\, 0,\, f]$ (with $f \neq 0$) fall in this equivalence class, and
- the new POINTs $(1 : f : 0)$ (comprising all $(x, y, 0)$ with nonzero x, using $f = y/x$), as well as $(0 : 1 : 0)$ (this class has those $(x, y, 0)$ where $x = 0$).

The only ratio excluded for a POINT is $(0 : 0 : 0)$ and for a LINE $[\,0 : 0 : 0\,]$, making the situation symmetric. Note all the new POINTs "lie on" the new LINE and none of the old ones do, which can be seen by applying (**??**). Any of the old LINEs acquires one of the new POINTs, LINEs that were parallel get the same new POINT and LINEs that weren't get different new POINTs. This prevents any pair of LINEs already intersecting in an old POINT intersecting in a new one as well, and provides any pair that didn't yet intersect a "place" where to do so.

The new POINTs are what is shared by parallel lines, so correspond to *directions* (pairs of opposite directions, in fact). They are called **points at infinity** and the new LINE comprising them the **line at infinity**.

The embedding in $\mathbb{F}^3$

The extension of $\mathit{\Pi}^*$ to $\mathit{\Pi}$ in the previous section has an immediate geometric interpretation. Interpret every (x, y, z) as a distinct point in $\mathbb{F}^3 \backslash (0, 0, 0)$. The equivalence classes $(x : y : z)$, or strictly speaking $(x : y : z) \cup \{(0, 0, 0)\}$, are now lines through the origin:

- the POINTs of $\mathit{\Pi}$ can be regarded as 1-dimensional subspaces of $\mathbb{F}^3$.

We could equate the $[\,p : q : r\,]$ with lines through the origin as well, but it will turn out to be more convenient to identify them with the planes through the origin $\perp$ those lines. Those planes consist of precisely those vectors $(x, y, z) \in \mathbb{F}^3$ that are $\perp$ the vector $(p, q, r) \in \mathbb{F}^3$. In this way

- the LINEs of $\mathit{\Pi}$ can be regarded as 2-dimensional subspaces of $\mathbb{F}^3$.

The reason this is more convenient is that now, by construction, all the points (vectors) in the 1-d subspace corresponding to a POINT (x, y, z) are $\perp$ the vector (p, q, r) if and only if equation (**??**) holds. In other words, those points lie inside the 2-d subspace corresponding to the LINE $[\,p : q : r\,]$ if and only if the POINT is incident with this LINE:

- POINT P is incident with LINE ℓ if and only if the line corresponding to P $\subset$ the plane corresponding to ℓ.

We can collapse back from 3 dimensions to 2 dimensions in various ways.

First, intersect the 1- and 2-dimensional subspaces representing POINTs and LINEs with the plane in $\mathbb{F}^3$ with $z = 1$; call this plane $\mathit{\Pi}^*_1$.

- ○ The POINTs (lines) $(x : y : z)$ with $z \neq 0$ each contain one point $(x/z, y/z, 1)$ in $\mathit{\Pi}^*_1$. The POINTs with $z = 0$ (lines in $\mathbb{F}^3$ parallel to $\mathit{\Pi}^*_1$) don't have any point in $\mathit{\Pi}^*_1$.
- ○ The LINEs (planes) $[\,p : q : r\,]$ with p and q not both zero intersect $\mathit{\Pi}^*_1$ in a line with equation (**??**). The only missing LINE (plane) is the one through the origin parallel to $\mathit{\Pi}^*_1$, which contains all the missing POINTs.

So the plane $\mathit{\Pi}^*_1$ is exactly the plane $\mathit{\Pi}^*$ we started off with, with an extra third coordinate 1 tacked on at the end.

Alternatively, intersect the 1- and 2-dimensional subspaces representing POINTs and LINEs with the sphere $x^2 + y^2 + z^2 = 1$.

- The intersection of a POINT (line through the origin) with this sphere is two opposite points.
- The intersection of a LINE (plane through the origin) with the sphere is a great circle.

When $\mathbb{F} = \mathbb{R}$, this is exactly the first example in this article.

The second example there (the finite one, the Fano plane) can be seen as the embedding in $\mathbb{F}^3$ when $\mathbb{F} = \mathbb{F}_2$. Its other representation showed it also has a cyclic symmetry modulo 7 (many more than one, in fact — its automorphism group has order 168).

A health warning is in order: the coordinatisation here gives the spurious idea there is some special relation between the POINT $(a : b : c)$ and the LINE $[\,a : b : c\,]$. In the sphere in $\mathbb{R}^3$ for example, such a POINT plays the rôle of poles relative to the LINE as equator. The whole point of projective geometry however, which we will not pursue further in this entry, is that only incidence between POINTs and LINEs is considered meaningful, and metric considerations (distances and angles) are ignored. If we redo the $\mathbb{R}^3$ example with any arbitrary set of three independent vectors as basis, we get projective planes all isomorphic with each other as far as incidence is concerned, but the pairs $(a : b : c)$ and $[\,a : b : c\,]$ that end up with the "same" coordinates are different in each version.

In the finite example too, if we choose a basis different from (1,0,0), (0,1,0), (0,0,1) we can find the same plane in many different guises: there are 7 ways to choose the first basis vector, 6 ways to find a second distinct from the first, and 4 ways for a third not in the plane of the first two, and $7 \cdot 6 \cdot 4 = 168$.

Classical finite planes

The constructions above can be carried out for $\mathbb{F}^2 = \mathbb{R}^2$ where they extend the usual Euclidean plane to "the" projective plane, but equally well for any other field, such as **finite fields** (aka Galois fields). Such fields have $q = p^d$ elements for p any prime and d any positive integer.

The Π^* that takes the place of the Euclidean plane is now called a (finite) **affine plane**. For $d = 1$ i.e. $q = p$ it is still fairly easy to visualise. Now the field is arithmetic (mod p) and the affine plane is a grid of $q \times q$ points, with lines connecting them at all possible integer ratio slopes (you need to wrap top to bottom and left to right, in modulo fashion, if you attempt to draw it). There are q slopes $f : 1$ (in a field, every nonzero element has a multiplicative inverse so every $a : b$ is some $ab^{-1} : 1$), and one slope $1 : 0$ for vertical lines. For each of the $q + 1$ slopes there are q parallel lines, making $q^2 + q$ lines in all.

The (finite) **projective plane** Π adds $q + 1$ POINTs at infinity (one for each slope, each direction shared by a bunch of parallel lines) to the q^2 points of the grid, $q^2 + q + 1$ POINTs in all. And it adds one LINE at infinity to the $q^2 + q$ lines we had, making $q^2 + q + 1$ LINEs as well. So

- the plane has $q^2 + q + 1$ POINTs and $q^2 + q + 1$ LINEs.

This is also evident from the embedding in $\mathbb{F}^3 \setminus \{(0, 0, 0)\}$ where each time $q - 1$ non-(0,0,0) points lie on the same POINT (line through the origin), and $(q^3 - 1)/(q - 1) = q^2 + q + 1$. Each LINE (plane through the origin) is $\perp$ such a line through the origin, so the numbers are the same. We also have that

- every LINE is incident with $q + 1$ POINTs; every POINT with $q + 1$ LINEs

because each LINE (plane through the origin) contains $q^2 - 1$ non-(0,0,0) points, and $(q^2 - 1)/(q - 1) = q + 1$. The simplest way to see the number the other way round is also $q + 1$ is calling POINTs LINEs and vice versa and embedding the thing in another $\mathbb{F}^3$; the incidence relation (**??**) is unchanged under this swap.

The field-based planes constructed here are the **classical** planes, also called **Desarguesian** because Desargues's theorem holds in them, and **Pappian** because Pappus's theorem holds. Πάππος (Pappus) was a 4th century Alexandrine mathematician and Desargues a 17th century French one.

It can be shown that Pappus's theorem holds in precisely those planes constructed in this way with $\mathbb{F}$ a field (commutative division ring), whereas

Desargues's theorem holds whenever $\mathbb{F}$ is a skew field (division ring). For finite planes both conditions are the same by Wedderburn's theorem, so Desarguesian and Pappian are synonyms. For infinite planes you can have Desarguesian non-Pappian ones, for instance if $\mathbb{F}$ is taken to be the quaternions.

The question arises what other algebraic structures other than (skew) fields can produce projective planes. See also the `http://planetmath.org/node/6943`**finite projective planes entry**.

Projective spaces

The same construction with homogeneous coordinates can be carried out in different numbers of dimensions. This extends $\mathbb{F}^1$ to the **projective line** with for a finite field $q+1$ elements, labeled by $\mathbb{F} \cup \{\infty\}$ where ∞ is any item not in $\mathbb{F}$. It likewise extends $\mathbb{F}^n$ for $n > 2$ to higher **projective spaces** with $(q^{n+1}-1)/(q-1)$ elements.

For $n \neq 2$ however these classical constructions give the only possible projective spaces; there is nothing corresponding to the wild variety of non-classical projective planes we find for $n = 2$.