neutral geometry

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Dedekind Cuts. Let ℓ be a line in a linear ordered geometry S and let A, B be two subsets on ℓ. A point p is said to be between A and B if for any pair of points $q \in A$ and $r \in B$, p is between q and r. Note that p necessarily lies on ℓ.

For example, given a ray ρ on a line ℓ. If p is the source of ρ, then p is a point between ρ and its opposite ray $-\rho$, regardless whether the ray is defined to be open or closed. It is easy to see that p is the unique point between ρ and $-\rho$.

Given a line ℓ, a Dedekind cut on ℓ is a pair of subsets $A, B \subseteq \ell$ such that $A \cup B = \ell$ and there is a unique point p between A and B. A ray ρ on a line ℓ and its compliment $\overline{\rho}$ constitute a Dedekind cut on ℓ.

If A, B form a Dedekind cut on ℓ, then A and B have two additional properties:

1. no point on A is strictly between two points on B, and
2. no point on B is strictly between two points on A.

Conversely, if A, B satisfy the above two conditions, can we say that A and B constitute a Dedekind cut? In a neutral geometry, the answer is yes.

Neutral Geometry. A neutral geometry is a linear ordered geometry satisfying

1. the congruence axioms, and
2. the continuity axiom: given any line ℓ with $\ell = A \cup B$ such that
   (a) no point on A is (strictly) between two points on B, and
   (b) no point on B is (strictly) between two points on A.

   then A and B constitute a Dedekind cut on ℓ. In other words, there is a unique point o between A and B.

Clearly, $A \cap B$ contains at most one point. The continuity axiom is also known as Dedekind's Axiom.

Properties.

1. Let $\ell = A \cup B$ be a line, satisfying (a) and (b) above and let $p \in A$. Suppose ρ lying on ℓ is a ray emanating from p. Then either $\rho \subseteq A$ or $B \subseteq \rho$.

2. Let $\ell = A \cup B$ be a line, satisfying (a) and (b) above and let o be the unique point as mentioned above. Then a closed ray emanating from o is either A or B. This implies that every Dedekind cut on a line ℓ consists of at least one ray.

3. We can similarly propose a continuity axiom on a ray as follows: given any ray ρ with $\rho = A \cup B$ such that

   - no point on A is strictly between two points on B, and
   - no point on B is strictly between two points on A.

   then there is a unique point o on ρ between A and B. It turns out that the two continuity axioms are equivalent.

4. **Archimedean Property** Given two line segments $\overline{pq}$ and $\overline{rs}$, then there is a unique natural number n and a unique point t, such that

   (a) t lies on the line segment $n \cdot \overline{rs} \subseteq \overrightarrow{rs}$,

   (b) t does not lie on the line segment $(n-1) \cdot \overline{rs}$, and

   (c) $\overline{pq} \cong \overline{rt}$.

   This property usually appears in the study of ordered fields.

5. For any given line ℓ and any point p, there exists a line m passing through p that is perpendicular to ℓ.

6. Consequently, for any given line ℓ and any point p not lying on ℓ, there exists at leaast one line passing through p that is parallel to ℓ. If there is more than one line passing through p parallel to ℓ, then there are infinitely many of these lines.

Examples.

- A Euclidean geometry is a neutral geometry satisfying the Euclid's parallel axiom: for any given line and any given point not lying on the line, there is a unique line passing through the point and parallel to the given line.

- A hyperbolic geometry (or Bolyai-Lobachevsky geometry) is a neutral geometry satisfying the hyperbolic axiom: for any given line and any given point not lying on the line, there are at least two distinct (hence infinitely many) lines passing through the point and parallel to the given line.

- In fact, one can replace the *indefinite article* "a" in the first letter of each of the above examples by the *definite article* "the". It can be shown that any two Euclidean geometries are geometrically isomorphic (preserving incidence, order, congruence, and continuity). Similarly, any two hyperbolic geometries are isomorphic. Such geometries are said to be *categorical.*

- An elliptic geometry is not a neutral geometry, because pairwise distinct parallel lines do not exist.