properties of an affine transformation

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In this entry, we prove some of the basic properties of affine transformations. Let $\alpha : A_1 \to A_2$ be an affine transformation and $[\alpha] : V_1 \to V_2$ its associated linear transformation.

Proposition 1. *α is one-to-one iff $[\alpha]$ is.*

Proof. Next, suppose α is one-to-one, and $T(v) = 0$ for some $v \in V_1$. Let $P, Q \in A_1$ with $f_1(P,Q) = v$. Then $0 = [\alpha](v) = [\alpha](f_1(P,Q)) = f_1(\alpha(P), \alpha(Q))$, which implies that $\alpha(P) = \alpha(Q)$, and therefore $P = Q$ by assumption. Conversely, suppose $[\alpha]$ is one-to-one, and $\alpha(P) = \alpha(Q)$. Then $[\alpha](f_1(P,Q)) = f_2(\alpha(P), \alpha(Q)) = 0$, so that $f_1(P,Q) = 0$, and consequently $P = Q$, showing that α is one-to-one. $\square$

Proposition 2. *α is onto iff $[\alpha]$ is.*

Proof. Suppose α is onto. Let $w \in V_2$, so there are $X, Y \in A_2$ such that $f_2(X,Y) = w$. Since α is onto, there are $P, Q \in A_1$ with $\alpha(P) = X$ and $\alpha(Q) = Y$. So $w = f_2(X,Y) = f_2(\alpha(P), \alpha(Q)) = [\alpha](f_1(P,Q))$. Hence $[\alpha]$ is onto. Conversely, assume $[\alpha]$ be onto, and pick $Y \in A_2$. Take an arbitrary point $P \in A_1$ and set $X = \alpha(P)$. There is $v \in V_1$ such that $[\alpha](v) = f_2(X,Y)$, since $[\alpha]$ is onto. Let $Q \in A_1$ such that $f_1(P,Q) = v$. Then $f_2(X, \alpha(Q)) = f_2(\alpha(P), \alpha(Q)) = [\alpha](f_1(P,Q)) = [\alpha](v) = f_2(X,Y)$. But $f_2(X, -)$ is a bijection, we must have $Y = \alpha(Q)$, showing that α is onto. $\square$

Corollary 1. *α is a bijection iff $[\alpha]$ is.*

Proposition 3. *A bijective affine transformation $\alpha : A_1 \to A_2$ is an affine isomorphism.*

Proof. Suppose an affine transformation $\alpha : A_1 \to A_2$ is a bijection. We want to show that $\alpha^{-1} : A_2 \to A_1$ is an affine transformation. Pick any $X, Y \in A_2$, then

$$[\alpha](f_1(\alpha^{-1}(X), \alpha^{-1}(Y))) = f_2(X,Y).$$

By the corollary above, $[\alpha]$ is bijective, and hence a linear isomorphism. So

$$f_1(\alpha^{-1}(X), \alpha^{-1}(Y)) = [\alpha]^{-1}(f_2(X,Y)).$$

This shows that α^{-1} is an affine transformation whose assoicated linear transformation is $[\alpha]^{-1}$. $\square$

Proposition 4. *Two affine spaces associated with the same vector space V are affinely isomorphic.*

Proof. In fact, all we need to do is to show that (A, f) is isomorphic to (V, g), where g is given by $g(v, w) = w - v$. Pick any $P \in A$, then $\alpha := f(P, -) : A \to V$ is a bijection. For any $v \in V$, there is a unique $Q \in A$ such that $v = f(P, Q)$. Then $1_V(f(X, Y)) = f(X, Y) = f(P, Y) - f(P, X) = \alpha(Y) - \alpha(X) = g(\alpha(X), \alpha(Y))$, showing that 1_V is the associated linear transformation of α. $\square$

Proposition 5. *Any affine transformation is a linear transformation between the corresponding induced vector spaces. In other words, if $\alpha : A \to B$ is affine, then $\alpha : A_P \to B_{\alpha(P)}$ is linear.*

Proof. Suppose $Q, R, S \in A$ are such that $Q+R = S$, or $f_1(P, Q)+f_1(P, R) = f_1(P, S)$. Then

$$\begin{aligned} f_2(\alpha(P), \alpha(S)) &= [\alpha](f_1(P, S)) \\ &= [\alpha](f_1(P, Q) + f_1(P, R)) \\ &= [\alpha](f_1(P, Q)) + [\alpha](f_1(P, R)) \\ &= f_2(\alpha(P), \alpha(Q)) + f_2(\alpha(P), \alpha(R)), \end{aligned}$$

which is equivalent to $\alpha(Q) + \alpha(R) = \alpha(S) = \alpha(Q + R)$.

Next, suppose $dQ = R$, or $df_1(P, Q) = f_1(P, R)$, where $d \in D$. Then

$$\begin{aligned} f_2(\alpha(P), \alpha(R)) &= [\alpha](f_1(P, R)) \\ &= [\alpha](df_1(P, Q)) \\ &= d[\alpha](f_1(P, Q)) \\ &= df_2(\alpha(P), \alpha(Q)), \end{aligned}$$

which is equivalent to $d\alpha(Q) = \alpha(R) = \alpha(dQ)$. $\square$

Proposition 6. *If (V, f) is an affine space associated with the vector space V, then the direction f is given by $f(v, w) = T(w - v)$ for some linear isomorphism (invertible linear transformation) T.*

Proof. By proposition 4, (V, f) is affinely isomorphic to (V, g) with $g(v, w) = w - v$. Suppose $\alpha : (V, f) \to (V, g)$ is the affine isomorphism. Then $[\alpha](f(v, w)) = g(\alpha(v), \alpha(w)) = \alpha(w) - \alpha(v)$. Since $[\alpha]$ is a linear isomorphism, $f(v, w) = [\alpha]^{-1}(\alpha(w)) - [\alpha]^{-1}(\alpha(v))$. By proposition 5, α itself is linear, so $f(v, w) = ([\alpha]^{-1} \circ \alpha)(w - v)$. Set $T = [\alpha]^{-1} \circ \alpha$. Then T is linear and invertible since α is, our assertion is proved. $\square$