area of the n-sphere

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The area of S^n the unit n-sphere (or hypersphere) is the same as the total solid angle it subtends at the origin. To calculate it, consider the following integral

$$I(n) = \int_{\mathbb{R}^{n+1}} e^{-\sum_{i=1}^{n+1} x_i^2}\, d^{n+1}x.$$

Switching to polar coordinates we let $r^2 = \sum_{i=1}^{n+1} x_i^2$ and the integral becomes

$$I(n) = \int_{S^n} d\Omega \int_0^\infty r^n e^{-r^2}\, dr.$$

The first integral is the integral over all solid angles and is exactly what we want to evaluate. Let us denote it by $A(n)$. With the change of variable $t = r^2$, the second integral can be evaluated in terms of the gamma function $\Gamma(x)$:

$$I(n)/A(n) = \frac{1}{2}\int_0^\infty t^{\frac{n-1}{2}} e^{-t}\, dt = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right).$$

We can also evaluate $I(n)$ directly in Cartesian coordinates:

$$I(n) = \left[\int_{-\infty}^{\infty} e^{-x^2}\, dx\right]^{n+1} = \pi^{\frac{n+1}{2}},$$

where we have used the standard Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

Finally, we can solve for the area

$$A(n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

If the radius of the sphere is R and not 1, the correct area is $A(n)R^n$.

Note that this formula works only for $n \geq 0$. The first few special cases are

- $n = 0$ $\Gamma(1/2) = \sqrt{\pi}$, hence $A(0) = 2$ (in this case, the area just counts the number of points in $S^0 = \{+1, -1\}$);
- $n = 1$ $\Gamma(1) = 1$, hence $A(1) = 2\pi$ (this is the familiar result for the circumference of the unit circle);
- $n = 2$ $\Gamma(3/2) = \sqrt{\pi}/2$, hence $A(2) = 4\pi$ (this is the familiar result for the area of the unit sphere);
- $n = 3$ $\Gamma(2) = 1$, hence $A(3) = 2\pi^2$;
- $n = 4$ $\Gamma(5/2) = 3\sqrt{\pi}/4$, hence $A(4) = 8\pi^2/3$.