intersection of quadratic surface and plane

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The http://planetmath.org/IntersectionOfSphereAndPlaneintersection of a sphere with a plane is a circle, similarly the intersection of any surface of revolution formed by the revolution of an ellipse or a hyperbola about its axis with a plane perpendicular to the axis of revolution is a circle of latitude.

We can get as intersection curves of other quadratic surfaces and a plane also other quadratic curves (conics). If for example the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \;=\; 1 \tag{1}$$

is cut with the plane $z = 0$ (i.e. the xy-plane), we substitute $z = 0$ to the equation of the ellipsoid, and thus the intersection curve satisfies the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} \;=\; 1,$$

which an ellipse. Actually, all plane intersections of the ellipsoid are ellipses, which may be in special cases circles.

As another exaple of quadratic surface we take the hyperbolic paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} \;=\; 2z. \tag{2}$$

Cutting it e.g. with the plane $y = b$, which is parallel to the zx-plane, the substitution yields the equation

$$2z \;=\; \frac{x^2}{a^2}-1$$

meaning that the intersection curve in the plane $y = b$ has the http://planetmath.org/Projectic parabola in the zx-plane with such an equation, and accordingly is such a parabola.

If we cut the surface (2) with the plane $z = \frac{1}{2}$, the result is the hyperbola having the projection

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} \;=\; 1$$

in the xy-plane. But cutting with $z = 0$ gives $\frac{x^2}{a^2}-\frac{y^2}{b^2} \;=\; 0$, i.e. the pair of lines $y = \pm\frac{b}{a}x$ which is a degenerate conic.

Let us then consider the general equation

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy+2A''x+2B''y+2C''z+D \;=\; 0 \tag{3}$$

of quadratic surface and an arbitrary plane

$$ax+by+cz+d \;=\; 0 \tag{4}$$

where at least one of the coefficients a, b, c is distinct from zero. Their intersection equation is obtained, supposing that e.g. $c \neq 0$, by substituting the solved form

$$z \;=\; -\frac{ax+by+d}{c}$$

of (4) to the equation (3). We then apparently have the equation of the form

$$\alpha x^2+\beta y^2+2\gamma xy+2\delta x+2\varepsilon y+\zeta \;=\; 0,$$

which a http://planetmath.org/QuadraticCurvesquadratic curve or some of the degenerated cases of them.