congruence axioms

Canonical name	CongruenceAxioms
Date of creation	2013-03-22 15:31:59
Last modified on	2013-03-22 15:31:59
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	16
Author	CWoo (3771)
Entry type	Axiom
Classification	msc 51F20
Synonym	axioms of congruence
Defines	congruence relation

General Congruence Relations. Let A be a set and $X = A \times A$. A relation on X is said to be a *congruence relation* on X, denoted $\cong$, if the following three conditions are satisfied:

1. $(a, b) \cong (b, a)$, for all $a, b \in A$,
2. if $(a, a) \cong (b, c)$, then $b = c$, where $a, b, c \in A$,
3. if $(a, b) \cong (c, d)$ and $(a, b) \cong (e, f)$, then $(c, d) \cong (e, f)$, for any $a, b, c, d, e, f \in A$.

By applying $(b, a) \cong (a, b)$ twice, we see that $\cong$ is reflexive according to the third condition. From this, it is easy to that $\cong$ is symmetric, since $(a, b) \cong (c, d)$ and $(a, b) \cong (a, b)$ imply $(c, d) \cong (a, b)$. Finally, $\cong$ is transitive, for if $(a, b) \cong (c, d)$ and $(c, d) \cong (e, f)$, then $(c, d) \cong (a, b)$ because $\cong$ is symmetric and so $(a, b) \cong (e, f)$ by the third condition. Therefore, the congruence relation is an equivalence relation on pairs of elements of A.

Congruence Axioms in Ordered Geometry. Let (A, B) be an ordered geometry with strict betweenness relation B. We say that the ordered geometry (A, B) satisfies the *congruence axioms* if

1. there is a congruence relation $\cong$ on $A \times A$;
2. if $(a, b, c) \in B$ and $(d, e, f) \in B$ with
   - $(a, b) \cong (d, e)$, and
   - $(b, c) \cong (e, f)$,

   then $(a, c) \cong (d, f)$;
3. given (a, b) and a ray ρ emanating from p, there exists a unique point q lying on ρ such that $(p, q) \cong (a, b)$;
4. given the following:
   - three rays emanating from p_1 such that they intersect with a line ℓ_1 at a_1, b_1, c_1 with $(a_1, b_1, c_1) \in B$, and
   - three rays emanating from p_2 such that they intersect with a line ℓ_2 at a_2, b_2, c_2 with $(a_2, b_2, c_2) \in B$,
   - $(a_1, b_1) \cong (a_2, b_2)$ and $(b_1, c_1) \cong (b_2, c_2)$,

- $(p_1, a_1) \cong (p_2, a_2)$ and $(p_1, b_1) \cong (p_2, b_2)$,

then $(p_1, c_1) \cong (p_2, c_2)$;

5. given three distinct points a, b, c and two distinct points p, q such that $(a, b) \cong (p, q)$. Let H be a closed half plane with boundary $\overleftrightarrow{pq}$. Then there exists a unique point r lying on H such that $(a, c) \cong (p, r)$ and $(b, c) \cong (q, r)$.

Congruence Relations on line segments, triangles, and angles. With the above five congruence axioms, one may define a congruence relation (also denoted by $\cong$ by abuse of notation) on the set S of closed line segments of A by

$$\overline{ab} \cong \overline{cd} \qquad \text{iff} \qquad (a, b) \cong (c, d),$$

where $\overline{ab}$ (in this entry) denotes the closed line segment with endpoints a and b.

It is obvious that the congruence relation defined on line segments of A is an equivalence relation. Next, one defines a congruence relation on triangles in A: $\triangle abc \cong \triangle pqr$ if their sides are congruent:

1. $\overline{ab} \cong \overline{pq}$,
2. $\overline{bc} \cong \overline{qr}$, and
3. $\overline{ca} \cong \overline{rp}$.

With this definition, Axiom 5 above can be restated as: given a triangle $\triangle abc$, such that $\overline{ab}$ is congruent to a given line segment $\overline{pq}$. Then there is exactly one point r on a chosen side of the line $\overleftrightarrow{pq}$ such that $\triangle abc \cong \triangle pqr$. Not surprisingly, the congruence relation on triangles is also an equivalence relation.

The last major congruence relation in an ordered geometry to be defined is on angles: $\angle abc$ is *congruent to* $\angle pqr$ if there are

1. a point a_1 on $\overrightarrow{ba}$,
2. a point c_1 on $\overrightarrow{bc}$,
3. a point p_1 on $\overrightarrow{qp}$, and

4. a point r_1 on $\overrightarrow{qr}$

such that $\triangle a_1bc_1 \cong \triangle p_1qr_1$.

It is customary to also write $\angle abc \cong \angle pqr$ to mean that $\angle abc$ is congruent to $\angle pqr$. Clearly for any points $x \in \overrightarrow{ba}$ and $y \in \overrightarrow{bc}$, we have $\angle xby \cong \angle abc$, so that $\cong$ is reflexive. $\cong$ is also symmetric and transitive (as the properties are inherited from the congruence relation on triangles). Therefore, the congruence relation on angles also defines an equivalence relation.

References

[1] D. Hilbert, *Foundations of Geometry*, Open Court Publishing Co. (1971)

[2] K. Borsuk and W. Szmielew, *Foundations of Geometry*, North-Holland Publishing Co. Amsterdam (1960)

[3] M. J. Greenberg, *Euclidean and Non-Euclidean Geometries, Development and History*, W. H. Freeman and Company, San Francisco (1974)