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transition to skew-angled coordinates

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Let the Euclidean plane $\mathbb{R}$ be equipped with the rectangular coordinate system with the x and y coordinate axes. We choose new coordinate axes through the old origin and http://planetmath.org/Projectionproject the new coordinates ξ, η of a point orthogonally on the x and y axes getting the old coordinates expressed as

$$\begin{cases} x = \xi\cos\alpha + \eta\cos\beta, \\ y = \xi\sin\alpha + \eta\sin\beta, \end{cases} \tag{1}$$

where α and β are the angles which the ξ-axis and η-axis, respectively, form with the x-axis (positive if x-axis may be rotated anticlocwise to ξ-axis, else negative; similarly for rotating the x-axis to the η-axis).

The of (1) are got by solving from it for ξ and η, getting

$$\xi = \frac{x\sin\beta - y\cos\beta}{\sin(\beta-\alpha)}, \quad \eta = \frac{-x\sin\alpha + y\cos\alpha}{\sin(\beta-\alpha)}.$$

Example. Let us consider the http://planetmath.org/Hyperbola2hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{2}$$

and take its asymptote $y = -\frac{b}{a}x$ for the ξ-axis and the asymptote $y = +\frac{b}{a}c$ for the η-axis. If ω is the angle formed by the latter asymptote with the x-axis, then $\alpha = -\omega$, $\beta = \omega$. By (1) we get first

$$\begin{cases} x = \xi\cos\omega + \eta\cos\omega = (\eta+\xi)\cos\omega, \\ y = -\xi\sin\omega + \eta\sin\omega = (\eta-\xi)\sin\omega. \end{cases}$$

Since $\tan\omega = \dfrac{b}{a}$, we see that $\cos\omega = \dfrac{a}{c}$, $\sin\omega = \dfrac{b}{c}$, where $c^2 = a^2 + c^2$, and accordingly

$$\frac{x}{a} = (\eta+\xi)\frac{a}{c} : a = \frac{\eta+\xi}{c}, \quad \frac{y}{b} = (\eta-\xi)\frac{b}{c} : b = \frac{\eta-\xi}{c}.$$

Substituting these quotients in the equation of the hyperbola we obtain

$$(\eta+\xi)^2 - (\eta-\xi)^2 = c^2,$$

and after simplifying,

$$\xi\eta = \frac{c^2}{4}. \tag{3}$$

This is the equation of the hyperbola (2) in the coordinate system of its asymptotes. Here, c is the distance of the `http://planetmath.org/Hyperbola2`focus from the nearer `http://planetmath.org/Hyperbola2`apex of the hyperbola.

If we, conversely, have in the rectangular coordinate system $(x,\, y)$ an equation of the form (3), e.g.

$$xy = \text{ constant}, \tag{4}$$

we can infer that it a hyperbola with asymptotes the coordinate axes. Since these are perpendicular to each other, it's clear that the hyperbola (4) is a `http://planetmath.org/Hyperbola2`rectangular one.

References

[1] L. Lindelöf: *Analyyttisen geometrian oppikirja*. Kolmas painos. Suomalaisen Kirjallisuuden Seura, Helsinki (1924).