pencil of lines

Canonical name	PencilOfLines
Date of creation	2013-03-22 18:09:03
Last modified on	2013-03-22 18:09:03
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	8
Author	pahio (2872)
Entry type	Theorem
Classification	msc 51N20
Related topic	LineInThePlane
Related topic	Determinant2
Related topic	HomogeneousLinearProblem
Related topic	Pencil2

Let

$$A_i x + B_i y + C_i = 0 \tag{1}$$

be equations of some lines. Use the short notations $A_i x + B_i y + C_i := L_i$.

If the lines $L_1 = 0$ and $L_2 = 0$ have an intersection point P, then, by the `http://planetmath.org/LineThroughAnIntersectionPoint`parent entry, the equation

$$k_1 L_1 + k_2 L_2 = 0 \tag{2}$$

with various real values of k_1 and k_2 can any line passing through the point P; this set of lines is called a *pencil of lines*.

Theorem. A necessary and sufficient condition in to three lines

$$L_1 = 0, \quad L_2 = 0, \quad L_3 = 0$$

pass through a same point, is that the determinant formed by the coefficients of their equations (1) vanishes:

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = 0.$$

Proof. If the line $L_3 = 0$ belongs to the fan of lines determined by the lines $L_1 = 0$ and $L_2 = 0$, i.e. all the three lines have a common point, there must be the identity

$$L_3 \equiv L_1 + L_2,$$

i.e. there exist three real numbers k_1, k_2, k_3, which are not all zeroes, such that the equation

$$k_1 L_1 + k_2 L_2 + k_3 L_3 \equiv 0 \tag{3}$$

is satisfied identically by all real values of x and y. This means that the group of homogeneous linear equations

$$\begin{cases} k_1 A_1 + k_2 A_2 + k_3 A_3 = 0, \\ k_1 B_1 + k_2 B_2 + k_3 B_3 = 0, \\ k_1 C_1 + k_2 C_2 + k_3 C_3 = 0 \end{cases}$$

has nontrivial solutions k_1, k_2, k_3. By linear algebra, it follows that the determinant of this group of equations has to vanish.

Suppose conversely that the determinant vanishes. This implies that the above group of equations has a nontrivial solution k_1, k_2, k_3. Thus we can write the identic equation (3). Let e.g. $k_1 \neq 0$. Solving (3) for L_1 yields

$$L_1 \equiv -\frac{k_2 L_2 + k_3 L_3}{k_1},$$

which shows that the line $L_1 = 0$ belongs to the fan determined by the lines $L_2 = 0$ and $L_3 = 0$; so the lines pass through a common point.

References

[1] Lauri Pimiä: *Analyyttinen geometria.* Werner Söderström Osakeyhtiö, Porvoo and Helsinki (1958).