Legendre's theorem on angles of triangle

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Adrien-Marie Legendre has proved some theorems concerning the sum of the angles of triangle. Here we give one of them, being the inverse of the theorem in the entry "sum of angles of triangle in Euclidean geometry".

Theorem. If the sum of the interior angles of every triangle equals straight angle, then the parallel postulate is true, i.e., in the plane determined by a line and a point outwards it there is exactly one line through the point which does not intersect the line.

Proof. We consider a line a and a point B not belonging to a. Let BA be the normal line of a (with $A \in a$) and b be the normal line of BA through the point B. By the supposition of the theorem, b does not intersect a.

We will show that in the plane determined by the line a and the point B, there are through B no other lines than b not intersecting the line a. For this purpose, we choose through B a line b' which differs from b; let the line b' form with BA an acute angle β.

We determine on the line a a point A_1 such that $AA_1 = AB$. By the supposition of the theorem, in the isosceles right triangle BAA_1 we have

$$\alpha_1 \;=:\; \angle AA_1B \;=\; \frac{\pi}{4} \;=\; \frac{\pi}{2^2}.$$

Next we determine on a a second point A_2 such that $A_1A_2 = A_1B$. By the supposition of the theorem, in the isosceles triangle BA_1A_2 we have

$$\alpha_2 \;=:\; \angle AA_2B \;=\; \frac{\alpha_1}{2} \;=\; \frac{\pi}{2^3}.$$

We continue similarly by forming isosceles triangles using the points A_3, A_3, ..., A_n of the line a such that

$$A_2A_3 \;=\; BA_2,\;\; A_3A_4 \;=\; BA_3,\;\; \ldots,\;\; A_{n-1}A_n \;=\; BA_{n-1}.$$

Then the acute angles being formed beside the points are

$$\alpha_3 \;=\; \frac{\pi}{2^4},\;\; \alpha_4 \;=\; \frac{\pi}{2^5},\;\; \ldots,\;\; \alpha_n \;=\; \frac{\pi}{2^{n+1}}.$$

They form a geometric sequence with the common ratio $r = \frac{1}{2}$. When n is sufficiently great, the member α_n is less than any given positive angle. As we have so much triangles $BA_{n-1}A_n$ that $\alpha_n < \frac{\pi}{2} - \beta$, then

$$\angle ABA_n \;=\; \frac{\pi}{2} - \alpha_n \;>\; \beta.$$

Then the line b' falls after penetrating B into the inner territory of the triangle ABA_n. Thereafter it must leave from there and thus intersect the side AA_n of this triangle. Accordingly, b' intersects the line a.

The above reasoning is possible for each line $b' \neq b$ through B. Consequently, the parallel axiom is in force.

References

[1] Karl Ariva: *Lobatsevski geomeetria*. Kirjastus "Valgus", Tallinn (1992).