proof of tangents law

Canonical name	ProofOfTangentsLaw
Date of creation	2013-03-22 13:11:04
Last modified on	2013-03-22 13:11:04
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Last modified by	CWoo (3771)
Numerical id	6
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Entry type	Proof
Classification	msc 51-00

To prove that

$$\frac{a-b}{a+b} = \frac{\tan(\frac{A-B}{2})}{\tan(\frac{A+B}{2})}$$

we start with the sines law, which says that

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}.$$

This implies that

$$a\sin(B) = b\sin(A)$$

We can write $\sin(A)$ as

$$\sin(A) = \sin(\frac{A+B}{2})\cos(\frac{A-B}{2}) + \cos(\frac{A+B}{2})\sin(\frac{A-B}{2}).$$

and $\sin(B)$ as

$$\sin(B) = \sin(\frac{A+B}{2})\cos(\frac{A-B}{2}) - \cos(\frac{A+B}{2})\sin(\frac{A-B}{2}).$$

Therefore, we have

$$a(\sin(\frac{A+B}{2})\cos(\frac{A-B}{2}) - \cos(\frac{A+B}{2})\sin(\frac{A-B}{2})) = b(\sin(\frac{A+B}{2})\cos(\frac{A-B}{2}) + \cos(\frac{A+B}{2})\sin($$

Dividing both sides by $\cos(\frac{A-B}{2})\cos(\frac{A+B}{2})$, we have,

$$a(\tan(\frac{A+B}{2}) - \tan(\frac{A-B}{2})) = b(\tan(\frac{A+B}{2}) + \tan(\frac{A-B}{2}))$$

This gives us

$$\frac{a}{b} = \frac{\tan(\frac{A+B}{2}) + \tan(\frac{A-B}{2})}{\tan(\frac{A+B}{2}) - \tan(\frac{A-B}{2})}$$

Hence we find that

$$\frac{a-b}{a+b} = \frac{\frac{a}{b} - 1}{\frac{a}{b} + 1} = \frac{\tan(\frac{A-B}{2})}{\tan(\frac{A+B}{2})}.$$