criterion for constructibility of regular polygon

Canonical name	CriterionForConstructibilityOfRegularPolygon
Date of creation	2013-03-22 17:18:40
Last modified on	2013-03-22 17:18:40
Owner	Wkbj79 (1863)
Last modified by	Wkbj79 (1863)
Numerical id	6
Author	Wkbj79 (1863)
Entry type	Theorem
Classification	msc 51M15
Classification	msc 12D15
Related topic	RegularPolygon
Related topic	RootOfUnity
Related topic	TheoremOnConstructibleAngles

Theorem 1. *Let n be an integer with $n \geq 3$. Then a* `http://planetmath.org/RegularPolygon`*re n-gon is* `http://planetmath.org/Constructible2`*constructible if and only if a* `http://planetmath.org/PrimitiveRootOfUnity`*primitive nth root of unity is a constructible number.*

Proof. First of all, note that a is a constructible number if and only if $\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ is a constructible number. See the entry on roots of unity for more details. Therefore, without loss of generality, only the constructibility of the number $\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ will be considered.

Sufficiency: If a regular n-gon is constructible, then so is the angle whose `http://planetmath.org/Vertex5`vertex is the `http://planetmath.org/Center9`center of the polygon and whose rays pass through adjacent vertices of the polygon. The `http://planetmath.org/AngleMeasure`measure of this angle is $\frac{2\pi}{n}$.

By the theorem on constructible angles, $\sin\left(\frac{2\pi}{n}\right)$ and $\cos\left(\frac{2\pi}{n}\right)$ are constructible numbers. Note that i is also a constructible number. Thus, $\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ is a constructible number.

Necessity: If $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ is a constructible number, then so is ω^m for any integer m.

On the complex plane, for every integer m with $0 \leq m < n$, construct the point corresponding to ω^m. Use line segments to connect the points corresponding to ω^m and ω^{m+1} for every integer m with $0 \leq m < n$. (Note that $\omega^0 = 1 = \omega^n$.) This forms a regular n-gon. □