proof of Mollweide's equations

Canonical name	ProofOfMollweidesEquations
Date of creation	2013-03-22 12:50:10
Last modified on	2013-03-22 12:50:10
Owner	mathwizard (128)
Last modified by	mathwizard (128)
Numerical id	5
Author	mathwizard (128)
Entry type	Proof
Classification	msc 51-00

We transform the equation

$$(a+b)\sin\frac{\gamma}{2} = c\cos\left(\frac{\alpha-\beta}{2}\right)$$

to

$$a\cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) + b\cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) = c\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + c\sin\frac{\alpha}{2}\sin\frac{\beta}{2},$$

using the fact that $\gamma = \pi-\alpha-\beta$. The left hand side can be further expanded, so that we get:

$$a\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) + b\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) = c\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}.$$

Collecting terms we get:

$$(a+b-c)\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - (a+b+c)\sin\frac{\alpha}{2}\sin\frac{\beta}{2} = 0.$$

Using $s := \frac{a+b+c}{2}$ and using the equations

$$\begin{aligned}\sin\frac{\alpha}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}}\\ \cos\frac{\beta}{2} &= \sqrt{\frac{s(s-a)}{bc}}\end{aligned}$$

we get:

$$2\frac{s(s-c)}{c}\sqrt{\frac{(s-a)(s-b)}{ab}} - 2\frac{s(s-c)}{c}\sqrt{\frac{(s-a)(s-b))}{ab}} = 0,$$

which is obviously true. So we can prove the first equation by going backwards. The second equation can be proved in quite the same way.