geometry as the study of invariants under certain transformations

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An approach to geometry first formulated by Felix Klein in his Erlangen lectures is to describe it as the study of invariants under certain allowed transformations. This involves taking our space as a set S, and considering a subgroup G of the group $Bij(S)$, the set of bijections of S. Objects are subsets of S, and we consider two objects $A, B \subset S$ to be equivalent if there is an $f \in G$ such that $f(A) = B$.

A property P of subsets of S is said to be a *geometric property* if it is invariant under the action of the group G, which is to say that $P(S)$ is true (or false) if and only if $P(g(S))$ is true (or false) for every transformation $g \in G$. For example, the property of being a straight line is a geometric property in Euclidean geometry. Note that the question whether or not a certin property is geometric depends on the choice of group. For instance, in the case of Euclidean geometry, the property of orthogonality is geometric because, given two lines L_1 and L_2 and any transformation g which belongs to the Euclidean group, the lines $g(L_1)$ and $g(L_2)$ are orthogonal if and only if L_1 and L_2 are orthogonal. However, if we consider affine geometry, orthogonality is no longer a geometric property because, given two orthogonal lines L_1 and L_2, one can find a transformation f which belongs to the affine group such that $f(L_1)$ is not orthogonal to $f(L_2)$.

Invariants can also be numbers. A real-valued function f whose domain consists of subsets of S is an invariant, or a *geometrical quantity* if the domain of X is invariant under the action of G and $f(X) = f(g(X))$ for all subsets X in the domain of f and all transformations $g \in G$. Familiar examples from Euclidean geometry are the length of line segments, areas of triangles, and angles. An important feature of the group-theoretic approach to geometry is that one one can use the techniques of invariant theory to systematically find and classify the invariants of a geometrical system. Using this approach, one can start with the description of a geometrical system in terms of a set and a group and rediscover geometric quantities which were originally found by trial and error.

One is not always interested in considering all possible subsets of S. For instance, in algebraic geometry, one only cares about subsets which can be defined by sytems of algebraic equations. To accommodate this desire, one may revise Klein's definition by replacing the set S with a suitable category (such as the category of algebraic subsets) to obtain the definition "geometry is the study of the invariants of a category C under the action of a group G which acts upon this category." Not only is such an approach popular in contemporary algebraic geometry, it is also useful when discussing such phenomena as duality transforms which map a point in one space to a line in another space and vice-versa. Such a phenomenon is not easily accomodated in a set-theoretic framework, but in terms of category theory, the duality transform can be described as a contravariant functor.

Klein's definition provides an organizing principle for classifying geometries. Ever since the discovery of non-Euclidean geometry, geometers have been defined and studied many different geometries. Without an organizing principle, the discussion and comparison of these geometries could become confusing. In the next section, we shall describe several familiar geometric systems from the standpoint of Klein's definition.

0.1 Basic examples

0.1.1 Euclidean geometry

Euclidean geometry deals with $\mathbb{R}^n$ as a vector space along with a metric d. The allowed transformations are bijections $f\colon \mathbb{R}^n \to \mathbb{R}^n$ that preserve the metric, that is, $d(\boldsymbol{x}, \boldsymbol{y}) = d(f(\boldsymbol{x}), f(\boldsymbol{y}))$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Such maps are called *isometries*, and the group is often denoted by $\mathrm{Iso}(\mathbb{R}^n)$. Defining a norm by $|x| = d(\boldsymbol{x}, \boldsymbol{0})$, for $\boldsymbol{x} \in \mathbb{R}^n$, we obtain a notion of length or distance. We can also define an inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x} \cdot \boldsymbol{y}$ on $\mathbb{R}^n$ using the standard dot product (this induces the same norm which can now be defined as $|x| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$). An inner product leads to a definition of the angle between two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ to be $\angle \boldsymbol{x}\boldsymbol{y} = \cos^{-1}\left(\dfrac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{|\boldsymbol{x}| \cdot |\boldsymbol{y}|} \right)$. It is clear that since isometries preserve the metric, they preserve distance and angle. As an example, it can be shown that the group $\mathrm{Iso}(\mathbb{R}^2)$ consists of translations, reflections, glides, and rotations. In general, a member f of $\mathrm{Iso}(\mathbb{R}^n)$ has the form $f(\boldsymbol{x}) = \boldsymbol{U}\boldsymbol{x} + \boldsymbol{c}$, where $\boldsymbol{U}$ is an orthogonal $n \times n$ matrix and $\boldsymbol{c} \in \mathbb{R}^n$.

0.1.2 Affine geometry

Unlike Euclidean geometry, we are no longer bound to "rigid motion" transformations in affine geometry. Here, we are interested in what happens to geometric objects when they undergo a finite series of "parallel projections". For example, imagine two Euclidean planes ($\mathbb{R}^2$) in $\mathbb{R}^3$. Loosely speaking, Euclidean geometry deals with transformations that take objects from one plane to the other, when the planes are *parallel* to each other. In affine geometry, the transformation is between two copies of $\mathbb{R}^2$, but they are no longer required to be parallel to each other anymore. Objects from one plane will appear to be "stretched" in the other. A circle will turn into an ellipse, etc...

For $\mathbb{R}^2$, in terms of the Kleinian view of geometry, affine geometry consists of the ordinary Euclidean plane, together with a group of transformations that

1. map straight lines to straight lines,
2. map parallel lines to parallel lines, and
3. preserve ratios of lengths of line segments along a given straight line.

Of course, the properties can be generalized to $\mathbb{R}^n$ and $n-1$ dimensional hyperplanes. A typical tranformation in an affine geometry is called an `http://planetmath.org/AffineTransformation`affine transformation: $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, where $x \in \mathbb{R}^n$ and $\boldsymbol{A}$ is an invertible $n \times n$ real matrix.

0.1.3 Projective geometry

Projective geometry was motivated by how we see objects in everyday life. For example, parallel train tracks appear to meet at a point far away, even though they are always the same distance apart. In projective geometry, the primary invariant is that of incidence. The notion of parallelism and distance is not present as with Euclidean geometry. There are different ways of approaching projective geometry. One way is to add points of infinity to Euclidean space. For example, we may form the projective line by adding a point of infinity ∞, called the ideal point, to $\mathbb{R}$. We can then create the projective plane where for each line $l \in \mathbb{R}^2$, we attach an ideal point, and two ordinary lines have the same ideal point if and only if they are parallel. The projective plane then consists of the regular plane $\mathbb{R}^2$ along with the ideal line, which consists of all ideal points of all ordinary lines. The idea here is to make central projections from a point sending a line to another a bijective map.

Another approach is more algebraic, where we form $P(V)$ where V is a vector space. When $V = \mathbb{R}^n$, we take the quotient of $\mathbb{R}^{n+1} - \{0\}$ where $v \sim \lambda \cdot v$ for $v \in \mathbb{R}^n, \lambda \in \mathbb{R}$. The allowed transformations is the group $PGL(\mathbb{R}^{n+1})$, which is the general linear group modulo the subgroup of scalar matrices.

0.1.4 Spherical geometry

Spherical geometry deals with restricting our attention in Euclidean space to the unit sphere S^n. The role of straight lines is taken by great circles. Notions of distance and angles can be easily developed, as well as spherical laws of cosines, the law of sines, and spherical triangles.