

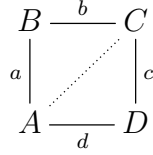


Math for the people, by the people.

proof of area of a quadrilateral

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One should think of the following diagram as representing a general quadrilateral.



$$AC^2 = a^2 + b^2 - 2ab \cos B = c^2 + d^2 - 2cd \cos D \quad (1)$$

Hence,

$$a^2 + b^2 - c^2 - d^2 = 2ab \cos B - 2cd \cos D \quad (2)$$

Add $2ab + 2cd$ to each side of (2):

$$a^2 + 2ab + b^2 - (c^2 - 2cd + d^2) = 2ab(1 + \cos B) - 2cd(-1 + \cos D) \quad (3)$$

Now use

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

and

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

to get

$$(a + b)^2 - (c - d)^2 = 4ab \cos^2 \frac{B}{2} + 4cd \sin^2 \frac{D}{2} \quad (4)$$

Similarly

$$(c + d)^2 - (a - b)^2 = 4ab \sin^2 \frac{B}{2} + 4cd \cos^2 \frac{D}{2} \quad (5)$$

Let s be the semiperimeter, so $s = \frac{a+b+c+d}{2}$. Then

$$LHS(4) = (a + b - c + d)(a + b + c - d) = 4(s - c)(s - d) \quad (6)$$

and

$$LHS(5) = (c + d - a + b)(c + d + a - b) = 4(s - a)(s - b) \quad (7)$$

Let R be the product of the left hand side of (4) times the left hand side of (5). Then

$$\begin{aligned}
R &= 16(ab)^2 \cos^2 \frac{B}{2} \sin^2 \frac{B}{2} + 16(cd)^2 \sin^2 \frac{D}{2} \cos^2 \frac{D}{2} + 16abcd(\cos^2 \frac{B}{2} \cos^2 \frac{D}{2} + \sin^2 \frac{B}{2} \sin^2 \frac{D}{2}) \\
&\quad (8) \\
&= 4(ab)^2 (2 \sin \frac{B}{2} \cos \frac{B}{2})^2 + 4(cd)^2 (2 \sin \frac{D}{2} \cos \frac{D}{2})^2 + 16abcd(\cos^2 \frac{B}{2} \cos^2 \frac{D}{2} + \sin^2 \frac{B}{2} \sin^2 \frac{D}{2}) \\
&\quad (9) \\
&= 4(ab)^2 \sin^2 B + 4(cd)^2 \sin^2 D + 16abcd(\cos^2 \frac{B}{2} \cos^2 \frac{D}{2} + \sin^2 \frac{B}{2} \sin^2 \frac{D}{2}). \\
&\quad (10)
\end{aligned}$$

We have:

$$8abcd \sin B \sin D = 32abcd \sin \frac{B}{2} \cos \frac{B}{2} \sin \frac{D}{2} \cos \frac{D}{2}.$$

which we can add and subtract to R :

$$\begin{aligned}
R &= (2ab \sin B + 2cd \sin D)^2 + 16abcd(\cos^2 \frac{B}{2} \cos^2 \frac{D}{2} - 2 \sin \frac{B}{2} \cos \frac{B}{2} \sin \frac{D}{2} \cos \frac{D}{2} + \sin^2 \frac{B}{2} \sin^2 \frac{D}{2}) \\
&\quad (11) \\
&= (2ab \sin B + 2cd \sin D)^2 + 16abcd(\cos \frac{B}{2} \cos \frac{D}{2} - \sin \frac{B}{2} \sin \frac{D}{2})^2. \quad (12)
\end{aligned}$$

But

$$(2ab \sin B + 2cd \sin D = 4K$$

so we have:

$$R = 16K^2 + 16abcd \cos^2 \left(\frac{B+D}{2} \right). \quad (13)$$

Hence by (6), (7) and (13)

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \left(\frac{B+D}{2} \right). \quad (14)$$