Archimedes' cylinders in cube

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The following problem has been solved by **Archimedes**:

Two distinct circular cylinders are inscribed in a cube; the axes thus intersect each other perpendicularly. Determine the volume common to both cylinders, when the radius of the base of the cylinders is r.

If the solid common to both cylinders is cut with a plane parallel to the axes of both cylinders, the figure of intersection is a square. Denote the distance of the plane from the center of the cube be x. By the Pythagorean theorem, half of the side of the square is $\sqrt{r^2-x^2}$ and the area of the square is $4(\sqrt{r^2-x^2})^2$. Accordingly, we have the function

$$A(x) \;:=\; 4(r^2-x^2)$$

for the area of the intersection square. If we let x here to grow from 0 to r, then half of the given solid is got. By the volume of the `http://planetmath.org/VolumeAsIntegr` entry, the half volume of the solid is

$$\frac{1}{2}V \;=\; \int_0^r 4(r^2-x^2)\,dx \;=\; 4\Big/_{x=0}^{r}\left(r^2x-\frac{x^3}{3}\right) \;=\; \frac{8}{3}r^3.$$

So the volume in the question is $\frac{16}{3}r^3$. It is $\dfrac{2}{3}$ of the volume of the cube.