planetmath.org

Math for the people, by the people.

translation plane

Canonical name	TranslationPlane
Date of creation	2013-03-22 19:15:15
Last modified on	2013-03-22 19:15:15
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Definition
Classification	msc 51A40
Classification	msc 51A35
Related topic	MoufangPlane
Defines	translation line
Defines	dual translation plane
Defines	translation point
Defines	affine translation plane

Let π be a projective plane. Recall that a central collineation on π is a collineation ρ with a center C and an axis ℓ. It is well-known that C and ℓ are uniquely determined. We also call ρ a (C, ℓ)-collineation.

Definition. Let π be a projective plane. We say that π is (C, ℓ)*-transitive* if there is a point C and a line ℓ, such that for any points P, Q where

- P, Q and C are collinear and pairwise distinct,
- $P, Q \notin \ell$,

there is a (C, ℓ)-collineation ρ such that $\rho(P) = Q$.

It can be shown that π if (C, ℓ)-transitive iff it is (C, ℓ)-Desarguesian; that is, if two triangles are perspective from point C, then they are perspective from line ℓ. From this, it is easy to see that π is a Desarguesian plane iff it is (C, ℓ)-transitive for any point C and any line ℓ, of π.

Now, suppose that C lies on ℓ. Then one can show that π is (C, ℓ)-transitive iff it can be coordinatized by a linear ternary ring R such that R is a group with respect to the derived operation $+$ (addition). When π is so coordinatized, ℓ is the line at infinity, and C is the point whose coordinate is (∞).

This group is not necessarily abelian. So what condition(s) must be imposed on π so that $(R, +)$ is an abelian group? The answer lies in the next definition:

Definition. Let π be a projective plane. π is said to be (m, ℓ)-transitive if there are lines m, ℓ such that π is (C, ℓ)-transitive for all $C \in m$.

Definition. A projective plane π is a *translation plane* if there is a line ℓ such that π is (ℓ, ℓ)-transitive. We also say that π is a translation plane with respect to ℓ. The line ℓ is called a *translation line* of π.

It can be shown that π is a translation plane with respect to ℓ iff it can be coordinatized by a Veblen-Wedderburn system (thus implying that $(R, +)$ is abelian).

When π is a translation plane with respect to two distinct lines ℓ and m, then it is not hard to see that it is a translation plane with respect to every line passing through $\ell \cap m$.

When π is a translation plane with respect to three non-concurrent lines, then it is a translation plane with respect to every line. A projective plane which is a translation plane with respect to every line is called a Moufang plane. An example of a translation plane that is not Moufang is the Hall plane, coordinatized by the Hall quasifield. An example of a projective plane that is not a translation plane is the Hughes plane.

Remark. There are also duals to the notions above: a projective plane π is

1. (P, Q)-*transitive* if there are points P, Q such that π is (P, m)-transitive for any line m passing through Q.

2. a *dual translation plane* if there is a point P such that π is (P, P)-transitive. We also say that π is a dual translation plane with respect to P, and that P is a *translation point* of π.

If π is a projective plane, then the following are true:

- π is translation plane with respect to some line ℓ and a dual translation plane with respect to some $P \in \ell$ iff π can be coordinatized by a semifield. In this coordinatization, ℓ is the line at infinity and P is the point with coordinate (∞).

- π is translation plane with respect to some line PQ and (P, Q)- and (Q, P)-transitive iff π can be coordinatized by a nearfield. In this coordinatization, PQ is the line at infinity where P and Q have coordinates (0) and (∞) (or vice versa).

Remark. By removing the line at infinity from a translation plane, we obtain an *affine translation plane*. By the definition of a translation plane, an affine translation plane can be characterized as an affine plane where the minor affine Desarguesian property holds.

References

[1] R. Casse, *Projective Geometry, An Introduction*, Oxford University Press (2006)