quadratic curves

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We want to determine the graphical representant of the general bivariate quadratic equation

$$Ax^2 + By^2 + 2Cxy + 2Dx + 2Ey + F = 0, \tag{1}$$

where A, B, C, D, E, F are known real numbers and $A^2 + B^2 + C^2 > 0$.

If $C \neq 0$, we will rotate the coordinate system, getting new coordinate axes x' and y', such that the equation (1) transforms into a new one having no more the mixed term $x'y'$. Let the rotation angle be α to the anticlockwise (positive) direction so that the x'- and y'-axes form the angles α and $\alpha + 90°$ with the original x-axis, respectively. Then there is the

$$\begin{aligned} x &= x' \cos\alpha - y' \sin\alpha \\ y &= x' \sin\alpha + y' \cos\alpha \end{aligned}$$

between the new and old coordinates (see rotation matrix). Substituting these expressions into (1) it becomes

$$Mx'^2 + Ny'^2 + 2Px'y' + 2Gx' + 2Hy' + F = 0, \tag{2}$$

where

$$\begin{cases} M = A\cos^2\alpha + B\sin^2\alpha + C\sin 2\alpha, \\ N = A\sin^2\alpha + B\cos^2\alpha - C\sin 2\alpha, \\ 2P = (B-A)\sin 2\alpha + 2C\cos 2\alpha. \end{cases} \tag{3}$$

It's always possible to determine α such that $(B-A)\sin 2\alpha = -2C\cos 2\alpha$, i.e. that

$$\tan 2\alpha = \frac{2C}{A-B}$$

for $A \neq B$ and $\alpha = 45°$ for the case $A = B$. Then the term $2Px'y'$ vanishes in (2), which becomes, dropping out the apostrophes,

$$Mx^2 + Ny^2 + 2Gx + 2Hy + F = 0. \tag{4}$$

- If none of the coefficients M and M equal zero, one can remove the first degree terms of (4) by first writing it as

$$M\left(x + \frac{G}{M}\right)^2 + N\left(y + \frac{H}{N}\right)^2 = \frac{G^2}{M} + \frac{H^2}{N} - F$$

and then translating the origin to the point $\left(-\frac{G}{M},\,-\frac{H}{N}\right)$, when we obtain the equation of the form

$$Mx^2+Ny^2 = K. \tag{5}$$

If M and M have the same http://planetmath.org/SignumFunctionsign, then in that (5) could have a counterpart in the plane, the sign must be the same as the sign of K; then the counterpart is the http://planetmath.org/Ellipse2ellip

$$\frac{x^2}{\left(\sqrt{|K/M|}\right)^2}+\frac{y^2}{\left(\sqrt{|K/N|}\right)^2} = 1.$$

If M and N have opposite signs and $K \neq 0$, then the curve (5) correspondingly is one of the http://planetmath.org/Hyperbola2hyperbolas

$$\frac{x^2}{\left(\sqrt{|K/M|}\right)^2}-\frac{y^2}{\left(\sqrt{|K/N|}\right)^2} = \pm 1,$$

which for $K = 0$ is reduced to a pair of intersecting lines.

- If one of M and N, e.g. the latter, is zero, the equation (4) may be written

  $$M\left(x+\frac{G}{M}\right)^2+2Hy+F-\frac{G^2}{M} = 0$$

  i.e.

  $$M\left(x+\frac{G}{M}\right)^2+2H\left(y+\frac{MF-G^2}{2HM}\right) = 0.$$

  Translating now the origin to the point $\left(-\frac{G}{M},\,\frac{G^2-MF}{2HM}\right)$ the equation changes to

  $$Mx^2+2Hy = 0. \tag{6}$$

  For $H \neq 0$, this is the equation $y = -\frac{M}{2H}x^2$ of a parabola, but for $H = 0$, of a *double line* $x^2 = 0$.

The kind of the quadratic curve (1) can also be found out directly from this original form of the equation. Namely, from the formulae (3) between the old and the new coefficients one may derive the connection

$$MN - P^2 = AB - C^2 \tag{7}$$

when one first adds and subtracts them obtaining

$$M + N = A + B,$$

$$M - N = (A - B)\cos 2\alpha + 2C \sin 2\alpha,$$
$$2P = (A - B)\sin 2\alpha + 2C \cos 2\alpha.$$

Two latter of these give

$$(M - N)^2 + 4P^2 = (A - B)^2 + 4C^2,$$

and when one subtracts this from the equation $(M + N)^2 = (A + B)^2$, the result is (7), which due to the choice of α is simply

$$MN = AB - C^2. \tag{8}$$

Thus the curve $Ax^2 + By^2 + 2Cxy + 2Dx + 2Ey + F = 0$ is, when it is real,

1. for $AB - C^2 > 0$ an `http://planetmath.org/Ellipse2`ellipse,
2. for $AB - C^2 < 0$ a `http://planetmath.org/Hyperbola2`hyperbola or two intersecting lines,
3. for $AB - C^2 = 0$ a `http://planetmath.org/Parabola2`parabola or a double line.

References

[1] L. Lindelöf: *Analyytisen geometrian oppikirja.* Kolmas painos. Suomalaisen Kirjallisuuden Seura, Helsinki (1924).