simplest common equation of conics

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In the plane, the locus of the points having the ratio of their distances from a certain point (the focus) and from a certain line (the directrix) equal to a given constant ε, is a conic section, which is an ellipse, a http://planetmath.org/ConicSectionpa or a hyperbola depending on whether ε is less than, equal to or greater than 1.

For showing this, we choose the y-axis as the directrix and the point $(q,\,0)$ as the focus. The locus condition reads then

$$\sqrt{(x-q)^2+y^2}\;=\;\varepsilon x.$$

This is simplified to

$$(1-\varepsilon^2)x^2-2qx+y^2+q^2\;=\;0. \tag{1}$$

If $\varepsilon=1,$ we obtain the parabola

$$y^2\;=\;2qx-q^2.$$

In the following, we thus assume that $\varepsilon\neq 1.$

Setting $y:=0$ in (1) we see that the x-axis cuts the locus in two points with the midpoint of the segment connecting them having the abscissa

$$x_0\;=\;\frac{q}{1-\varepsilon^2}.$$

We take this point as the new origin (replacing x by $x+x_0$); then the equation (1) changes to

$$(1-\varepsilon^2)x^2+y^2\;=\;\frac{\varepsilon^2q^2}{1-\varepsilon^2}. \tag{2}$$

From this we infer that the locus is

1. in the case $\varepsilon<1$ an http://planetmath.org/Ellipse2ellipse with the semiaxes
$$a\;=\;\frac{\varepsilon q}{1-\varepsilon^2},\qquad b\;=\;\frac{\varepsilon q}{\sqrt{1-\varepsilon^2}}$$
and with eccentricity ε;

2. in the case $\varepsilon>1$ a http://planetmath.org/Hyperbola2hyperbola with semiaxes
$$a\;=\;\frac{\varepsilon q}{\varepsilon^2-1},\qquad b\;=\;\frac{\varepsilon q}{\sqrt{\varepsilon^2-1}}$$
and also now with the eccentricity ε.

equation

the origin into a focus of a conic section (and in the cases of ellipse and hyperbola, the abscissa axis through the other focus). As before, let q be the distance of the focus from the corresponding directrix. Let r and φ be the polar coordinates of an arbitrary point of the conic. Then the locus condition may be expressed as

$$\frac{r}{q \pm r\cos\varphi} \;=\; \varepsilon.$$

Solving this equation for the `http://planetmath.org/node/6968`polar radius r yields the form

$$r \;=\; \frac{\varepsilon q}{1 \mp \varepsilon\cos\varphi} \tag{3}$$

for the common polar equation of the conic. The sign alternative ($\mp$) depends on whether the polar axis ($\varphi = 0$) intersects the directrix or not.