circle

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Defines	unit circle
Defines	radius
Defines	radii
Defines	perimeter
Defines	circumference
Defines	three point formula for the circle
Defines	center

A *circle* is the locus of points which are equidistant from some fixed point. It is in the plane is determined by a *center* and a *radius*. The center is a point in the plane, and the length of the radius is a positive real number, the radius being a line segment from the center to the circumference. The circle consists of all points whose distance from the center equals the radius.

Another way of defining the circle is thus: Given A and P as two points and O as another point, the circle with center O is the locus of points X with OX congruent to AP. (Hilbert, 1927)

Notice there is no definition of distance needed to make that definition and so it works in many geometries, even ones with no distance function. Hilbert uses it in his Foundations of Geometry book Also used by Forder in his Foundations of Euclidean geometry book, c 1927

In this entry, we only work with the standard Euclidean norm in the plane. A circle has one center only.

A circle determines a closed curve in the plane, and this curve is called the *perimeter* or *circumference* of the circle. If the radius of a circle is r, then the length of the perimeter is $2\pi r$. Also, the area of the circle is πr^2. More precisely, the interior of the perimeter has area πr^2. The *diameter* of a circle is defined as $d = 2r$.

The circle is a special case of an ellipse. Also, in three dimensions, the analogous geometric object to a circle is a sphere.

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Let us next derive an analytic equation for a circle in Cartesian coordinates (x, y). If the circle has center (a, b) and radius $r > 0$, we obtain the following condition for the points of the sphere,

$$(x-a)^2 + (y-b)^2 = r^2. \tag{1}$$

In other words, the circle is the set of all points (x, y) that satisfy the above equation. In the special case that $a = b = 0$, the equation is simply $x^2 + y^2 = r^2$. The *unit circle* is the circle $x^2 + y^2 = 1$.

It is clear that equation **??** can always be reduced to the form

$$x^2 + y^2 + Dx + Ey + F = 0, \tag{2}$$

where D, E, F are real numbers. Conversely, suppose that we are given an equation of the above form where D, E, F are arbitrary real numbers. Next we derive conditions for these constants, so that equation (**??**) determines a circle [**?**]. Completing the squares yields

$$x^2 + Dx + \frac{D^2}{4} + y^2 + Ey + \frac{E^2}{4} = -F + \frac{D^2}{4} + \frac{E^2}{4},$$

whence

$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 - 4F + E^2}{4}.$$

There are three cases:

1. If $D^2 - 4F + E^2 > 0$, then equation (**??**) determines a circle with center $(-\frac{D}{2}, -\frac{E}{2})$ and radius $\frac{1}{2}\sqrt{D^2 - 4F + E^2}$.
2. If $D^2 - 4F + E^2 = 0$, then equation (**??**) determines the point $(-\frac{D}{2}, -\frac{E}{2})$.
3. If $D^2 - 4F + E^2 < 0$, then equation (**??**) has no (real) solution in the (x, y) - plane.

2 The circle in polar coordinates

Using polar coordinates for the plane, we can parameterize the circle. Consider the circle with center (a, b) and radius $r > 0$ in the plane $\mathbb{R}^2$. It is then natural to introduce polar coordinates (ρ, ϕ) for $\mathbb{R}^2 \setminus \{(a, b)\}$ by

$$x(\rho, \phi) = a + \rho \cos \phi,$$
$$y(\rho, \phi) = b + \rho \sin \phi,$$

with $\rho > 0$ and $\phi \in [0, 2\pi)$. Since we wish to parameterize the circle, the point (a, b) does not pose a problem; it is not part of the circle. Plugging these expressions for x, y into equation (**??**) yields the condition $\rho = r$. The given circle is thus parameterization by $\phi \mapsto (a + \rho \cos \phi, b + \rho \sin \phi)$, $\phi \in [0, 2\pi)$. It follows that a circle is a closed curve in the plane.

3 Three point formula for the circle

Suppose we are given three points on a circle, say (x_1, y_1), (x_2, y_2), (x_3, y_3). We next derive expressions for the parameters D, E, F in terms of these points. We also derive equation (**??**), which gives an equation for a circle in terms of a determinant.

First, from equation (**??**), we have

$$\begin{aligned}
x_1^2 + y_1^2 + Dx_1 + Ey_1 + F &= 0, \\
x_2^2 + y_2^2 + Dx_2 + Ey_2 + F &= 0, \\
x_3^2 + y_3^2 + Dx_3 + Ey_3 + F &= 0.
\end{aligned}$$

These equations form a linear set of equations for D, E, F, i.e.,

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \cdot \begin{pmatrix} D \\ E \\ F \end{pmatrix} = - \begin{pmatrix} x_1^2 + y_1^2 \\ x_2^2 + y_2^2 \\ x_3^2 + y_3^2 \end{pmatrix}.$$

Let us denote the matrix on the left hand side by Λ. Also, let us assume that $\det \Lambda \neq 0$. Then, using Cramer's rule, we obtain

$$\begin{aligned}
D &= -\frac{1}{\det \Lambda} \det \begin{pmatrix} x_1^2 + y_1^2 & y_1 & 1 \\ x_2^2 + y_2^2 & y_2 & 1 \\ x_3^2 + y_3^2 & y_3 & 1 \end{pmatrix}, \\
E &= -\frac{1}{\det \Lambda} \det \begin{pmatrix} x_1 & x_1^2 + y_1^2 & 1 \\ x_2 & x_2^2 + y_2^2 & 1 \\ x_3 & x_3^2 + y_3^2 & 1 \end{pmatrix}, \\
F &= -\frac{1}{\det \Lambda} \det \begin{pmatrix} x_1 & y_1 & x_1^2 + y_1^2 \\ x_2 & y_2 & x_2^2 + y_2^2 \\ x_3 & y_3 & x_3^2 + y_3^2 \end{pmatrix}.
\end{aligned}$$

These equations give the parameters D, E, F as functions of the three given points. Substituting these equations into equation (**??**) yields

$$\begin{aligned}
(x^2 + y^2) \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} &- x \det \begin{pmatrix} x_1^2 + y_1^2 & y_1 & 1 \\ x_2^2 + y_2^2 & y_2 & 1 \\ x_3^2 + y_3^2 & y_3 & 1 \end{pmatrix} \\
&- y \det \begin{pmatrix} x_1 & x_1^2 + y_1^2 & 1 \\ x_2 & x_2^2 + y_2^2 & 1 \\ x_3 & x_3^2 + y_3^2 & 1 \end{pmatrix} \\
&- \det \begin{pmatrix} x_1 & y_1 & x_1^2 + y_1^2 \\ x_2 & y_2 & x_2^2 + y_2^2 \\ x_3 & y_3 & x_3^2 + y_3^2 \end{pmatrix} = 0.
\end{aligned}$$

Using the cofactor expansion, we can now write the equation for the circle passing through $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ as [**?**, **?**]

$$\det \begin{pmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{pmatrix} = 0. \tag{3}$$

References

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