integral representation of length of smooth curve

Canonical name	IntegralRepresentationOfLengthOfSmoothCurve
Date of creation	2013-03-22 15:39:39
Last modified on	2013-03-22 15:39:39
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Numerical id	11
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Entry type	Derivation
Classification	msc 51N05
Related topic	ArcLength
Related topic	Rectifiable
Related topic	TotalVariation

Suppose $\gamma\colon [0,1] \to \mathbb{R}^m$ is a continuously differentiable curve. Then the definition of its length as a rectifiable curve

$$L = \sup\Bigl\{\sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|\colon 0 = t_0 < t_1 < \cdots < t_n = 1\,,\ n \in \mathbb{N}\Bigr\}$$

is equal to its length as computed in differential geometry:

$$\int_0^1 \|\gamma'(t)\|\,dt\,.$$

Proof. Let the partition $\{t_i\}$ of $[0,1]$ be arbitrary. Then

$$\begin{aligned}
\sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\| &= \sum_{i=1}^{n} \Bigl\| \int_{t_{i-1}}^{t_i} \gamma'(t)\,dt \Bigr\| && \text{(fundamental theorem of calculus)} \\
&\le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|\gamma'(t)\|\,dt && \text{(triangle inequality for integrals)} \\
&= \int_0^1 \|\gamma'(t)\|\,dt\,.
\end{aligned}$$

Hence $L \le \int_0^1 \|\gamma'(t)\|\,dt$. (By the way, this also shows that γ is rectifiable in the first place.)

The inequality in the other direction is more tricky. Given $\epsilon > 0$, we know that $\int_0^1 \|\gamma'(t)\|\,dt$ can be approximated up to ϵ by a Riemann sum of the form

$$\sum_{i=1}^{n} \|\gamma'(t_{i-1})\|(t_i - t_{i-1})$$

provided the partition $\{t_i\}$ is fine enough, i.e. has mesh width $\le \Delta$ for some small $\Delta > 0$. We want to approximate $\gamma'(t_{i-1})$ with $[\gamma(t_i) - \gamma(t_{i-1})]/(t_i - t_{i-1})$, but this only works if $t_i - t_{i-1}$ is small.

To get the precise estimates, use uniform continuity of γ' on $[0,1]$ to obtain a $\delta > 0$ such that $\|\gamma'(\tau) - \gamma'(t)\| \le \epsilon$ whenever $|\tau - t| \le \delta$. Then for all $0 < h \le \delta$ and $t \in [0,1]$,

$$\left\| \frac{\gamma(t+h) - \gamma(t)}{h} - \gamma'(t) \right\| \le \frac{1}{h} \int_t^{t+h} \|\gamma'(\tau) - \gamma'(t)\|\,d\tau \le \frac{h}{h}\,\epsilon = \epsilon\,.$$

Let the partition $\{t_i\}$ have a mesh width less than both δ and Δ. Then setting $h = t_i - t_{i-1}$ successively in each summand, we have

$$\begin{aligned}
\int_0^1 \|\gamma'(t)\| \, dt &\leq \sum_{i=1}^{n} \|\gamma'(t_{i-1})\|(t_i - t_{i-1}) + \epsilon \\
&\leq \sum_{i=1}^{n} \frac{\|\gamma(t_i) - \gamma(t_{i-1})\|}{t_i - t_{i-1}} (t_i - t_{i-1}) + \sum_{i=1}^{n} \epsilon(t_i - t_{i-1}) + \epsilon \\
&= \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\| + 2\epsilon \\
&\leq L + 2\epsilon \, .
\end{aligned}$$

Taking $\epsilon \to 0$ yields $\int_0^1 \|\gamma'(t)\| \, dt \leq L$. □

We remark that $L = \int_0^1 \|\gamma'(t)\| \, dt$ is true for piecewise smooth curves γ also, simply by adding together the results for each smooth segment of γ.