Weizenbock's inequality

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In a triangle with sides a, b, c, and with area A, the following inequality holds:

$$a^2 + b^2 + c^2 \geq 4A\sqrt{3}$$

The proof goes like this: if $s = \frac{a+b+c}{2}$ is the semiperimeter of the triangle, then from Heron's formula we have:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

But by squaring the latter and expanding the parentheses we obtain:

$$16A^2 = 2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)$$

Thus, we only have to prove that:

$$(a^2 + b^2 + c^2)^2 \geq 3[2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)]$$

or equivalently:

$$4(a^4 + b^4 + c^4) \geq 4(a^2b^2 + a^2c^2 + b^2c^2)$$

which is trivially equivalent to:

$$(a^2 - b^2)^2 + (a^2 - c^2)^2 + (b^2 - c^2)^2 \geq 0$$

Equality is achieved if and only if $a = b = c$ (i.e. when the triangle is equilateral) .

See also the Hadwiger-Finsler inequality, from which this result follows as a corollary.