lengths of angle bisectors

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In any triangle, the w_a, w_b, w_c of the angle bisectors opposing the sides a, b, c, respectively, are

$$w_a = \frac{\sqrt{bc\,[(b+c)^2-a^2]}}{b+c}, \tag{1}$$

$$w_b = \frac{\sqrt{ca\,[(c+a)^2-b^2]}}{c+a}, \tag{2}$$

$$w_c = \frac{\sqrt{ab\,[(a+b)^2-c^2]}}{a+b}. \tag{3}$$

Proof. By the symmetry, it suffices to prove only (1).

According the angle bisector theorem, the bisector w_a divides the side a into the portions

$$\frac{b}{b+c}\cdot a \;=\; \frac{ab}{b+c}, \qquad \frac{c}{b+c}\cdot a \;=\; \frac{ca}{b+c}.$$

If the angle opposite to a is α, we apply the law of cosines to the half-triangles by w_a:

$$\begin{cases} 2w_a b\cos\frac{\alpha}{2} \;=\; w_a^2+b^2-\left(\frac{ab}{b+c}\right)^2 \\ 2w_a c\cos\frac{\alpha}{2} \;=\; w_a^2+c^2-\left(\frac{ca}{b+c}\right)^2 \end{cases} \tag{4}$$

For eliminating the angle α, the equations (4) are divided sidewise, when one gets

$$\frac{b}{c} \;=\; \frac{w_a^2+b^2-\left(\frac{ab}{b+c}\right)^2}{w_a^2+c^2-\left(\frac{ca}{b+c}\right)^2},$$

from which one can after some routine manipulations solve w_a, and this can be simplified to the form (1).