planetmath.org

Math for the people, by the people.

volume of the n-sphere

Canonical name	VolumeOfTheNsphere
Date of creation	2013-03-22 13:47:09
Last modified on	2013-03-22 13:47:09
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Derivation
Classification	msc 51M05
Related topic	AreaOfTheNSphere

The volume contained inside S^n, the n-sphere (or hypersphere), is given by the integral

$$V(n) = \int_{\sum_{i=1}^{n+1} x_i^2 \leq 1} d^{n+1}x.$$

Going to polar coordinates ($r^2 = \sum_{i=1}^{n+1} x_i^2$) this becomes

$$V(n) = \int_{S^n} d\Omega \int_0^1 r^n \, dr.$$

The first integral is the integral over all solid angles subtended by the sphere and is equal to its area $A(n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$, where $\Gamma(x)$ is the gamma function. The second integral is elementary and evaluates to $\int_0^1 r^n \, dr = 1/(n+1)$.

Finally, the volume is

$$V(n) = \frac{\pi^{\frac{n+1}{2}}}{\frac{n+1}{2}\Gamma\left(\frac{n+1}{2}\right)} = \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+3}{2}\right)}.$$

If the sphere has radius R instead of 1, then the correct volume is $V(n)R^{n+1}$.

Note that this formula works for $n \geq 0$. The first few cases are

$n = 0$ $\Gamma(3/2) = \sqrt{\pi}/2$, hence $V(0) = 2$ (this is the length of the interval $[-1, 1]$ in $\mathbb{R}$);

$n = 1$ $\Gamma(2) = 1$, hence $V(1) = \pi$ (this is the familiar result for the area of the unit circle);

$n = 2$ $\Gamma(5/2) = 3\sqrt{\pi}/4$, hence $V(2) = 4\pi/3$ (this is the familiar result for the volume of the sphere);

$n = 3$ $\Gamma(3) = 2$, hence $V(3) = \pi^2/2$.