example needing two Lagrange multipliers

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Find the semi-axes of the ellipse of intersection, formed when the plane $z = x+y$ intersects the ellipsoid

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25} = 1.$$

Let $(x,\,y,\,z)$ be any point of the ellipsoid. The http://planetmath.org/SquareOfNumbersqua $x^2+y^2+z^2$ of the distance of this point from the http://planetmath.org/Midpoint3midpoint $(0,\,0,\,0)$ has under the constraints

$$\begin{cases} g \;:=\; \frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1 \;=\; 0,\\ h \;:=\; x+y-z \;=\; 0 \end{cases} \tag{1}$$

the minimum and maximum values at the end points of the semi-axes of the ellipse. Since we have two constraints, we must take equally many Lagrange multipliers, λ and μ. A necessary condition of the extremums of

$$f \;:=\; x^2+y^2+z^2$$

is that in to (1), also the equations

$$\begin{cases} \frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}+\mu\frac{\partial h}{\partial x} \;=\; 2x+\frac{1}{2}x\lambda+\mu \;=\; 0,\\ \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}+\mu\frac{\partial h}{\partial y} \;=\; 2y+\frac{2}{5}y\lambda+\mu \;=\; 0,\\ \frac{\partial f}{\partial z}+\lambda\frac{\partial g}{\partial z}+\mu\frac{\partial h}{\partial z} \;=\; 2z+\frac{2}{25}z\lambda-\mu \;=\; 0, \end{cases} \tag{2}$$

are satisfied. I.e., we have five equations (1), (2) and five unknowns λ, μ, x, y, z.

The equations (2) give

$$x \;=\; -\frac{2\mu}{\lambda+4}, \quad y \;=\; -\frac{5\mu}{2\lambda+10}, \quad z \;=\; \frac{25\mu}{2\lambda+50},$$

which expressions may be put into the equation $h = 0$, and so on. One obtains the values

$$\lambda_1 = -10, \quad \lambda_2 = -\frac{75}{17}, \quad \mu_1 = \pm 6\sqrt{\frac{5}{19}}, \quad \mu_2 = \pm\frac{140}{17\sqrt{646}}$$

with which the extremum points $(x,\,y,\,z)$ can be evaluated. The corresponding values of $f$are 10 and $\frac{75}{17}$, whence the major semi-axis is $\sqrt{10}$ and the minor semi-axis $\frac{5\sqrt{255}}{17}$.