polarities and forms

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Through out this article we assume $\dim V \neq 2$. This is not a true constraint as there are only trivial dualities for $\dim V \leq 2$.

Proposition 1. *Every duality gives rise to a non-degenerate sesquilinear form, and visa-versa.*

Proof. To see this, let $d : PG(V) \to PG(V)$ be a duality. We may express this as an order preserving map $d : PG(V) \to PG(V^*)$. Then by the fundamental theorem of projective geometry it follows d is induced by a bijective semi-linear transformation $\hat{d} : V \to V^*$.

An semi-linear isomorphism of V to V^* is equivalent to specifying a non-degenerate sesquilinear form. In particular, define the form $b : V \times V \to k$ by $b(v, w) = (v)(w\hat{d})$ (notice $w\hat{d} \in V^*$ so $w\hat{d} : V \to k$).

Now, if $b : V \times V \to k$ is a non-degenerate sesquilinear form. Then define

$$\hat{b} : V \to V^* : v \mapsto b(-, v) : V \to k$$

which is semi-linear, as b is sesquilinear, and bijective, since b is non-degenerate. Therefore $\hat{b}$ induces an order preserving bijection $PG(V) \to PG(V^*)$, that is, a duality. $\square$

We write $W^\perp$ for the image of the induced duality of a non-degenerate sesquilinear form b. Notice that $W^\perp = \{w \in V : b(v, W) = 0\}$. (Although the form may not be reflexive, we still use the $\perp$ notation, but we now demonstrate that we can indeed specialize to the reflexive case.) Notice then that

$$\dim W^\perp = \dim V - \dim W.$$

Corollary 2. *Every polarity gives rise to a reflexive non-degenerate sesquilinear form, and visa-versa.*

Proof. Let b be the sesquilinear form induced by the polarity p. Then suppose we have $v, w \in V$ such that $0 = b(v, w) = (v)(w\hat{p})$. So $\langle v\rangle \leq \langle w\rangle^\perp = \langle w\rangle p$. But p has order 2 so $\langle v\rangle^\perp = \langle v\rangle p \geq \langle w\rangle$. But this implies $b(w, v) = 0$ so b is reflexive.

Likewise, given a reflexive non-degenerate sesquilinear form b it gives rise do a duality d induced by $\hat{b}$. By the reflexivity, $b(W, W^\perp) = 0$ implies $b(W^\perp, W) = 0$ also. As $(W^\perp)^\perp = \{v \in V : b(v, (W^\perp)^\perp) = 0\}$ it follows $W \leq (W^\perp)^\perp$. But by dimension arguments:

$$\dim(W^\perp)^\perp = \dim V - \dim W^\perp = \dim V - (\dim V - \dim W) = \dim W$$

we conclude $W = (W^\perp)^\perp$. Thus d is a polarity. $\square$

From the fundamental theorem of projective geometry it follows if $\dim V \neq 2$ then every order preserving map is induced by a semi-linear transformation of V. In similar fashion we have

Proposition 3. *$P\Gamma L^*(V) = P\Gamma L(V) \rtimes \mathbb{Z}_2$, meaning that every order reversing map $f : PG(V) \to PG(V)$ can be decomposed as a $f = sr$ where s is induced from a semi-linear transformation and r is a polarity.*

Proof. Let d be any duality of $PG(V)$. Then d^2 is order preserving. Thus d^2 is a projectivity so by the fundamental theorem of projective geometry d^2 is induced by a semi-linear transformation s. Therefore $P\Gamma L(V)$ has index 2 in $P\Gamma L^*(V)$. Finally it suffices to provide any polarity of $PG(V)$ to prove $P\Gamma L^*(V) = P\Gamma L(V) \rtimes \mathbb{Z}_2$. For this use any reflexive non-degenerate sesquilinear form. □

Remark 4. *The group $P\Gamma L^*(V)$ is the automorphism group of $PSL(V)$. In particular, the polarities account for the graph automorphisms of the Dynkin diagram of A_{d-1}, $d = \dim V$. When $\dim V = 2$ there is no graph automorphism, just as there are no dualities (points are hyperplanes when $\dim V = 2$.)*

References

[1] Gruenberg, K. W. and Weir, A.J. *Linear Geometry 2nd Ed.* (English) [B] Graduate Texts in Mathematics. 49. New York - Heidelberg - Berlin: Springer-Verlag. X, 198 p. DM 29.10; $ 12.80 (1977).

[2] Kantor, W. M. *Lectures notes on Classical Groups.*