Hesse configuration

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A Hesse configuration is a set P of nine non-collinear points in the projective plane over a field K such that any line through two points of P contains exactly three points of P. Then there are 12 such lines through P. A Hesse configuration exists if and only if the field K contains a primitive third root of unity. For such K the projective automorphism group $\mathrm{PGL}(3, K)$ acts transitively on all possible Hesse configurations.

The configuration P with its intersection structure of 12 lines is isomorphic to the affine space $A = \mathbb{F}^2$ where $\mathbb{F}$ is a field with three elements.

The group $\Gamma \subset \mathrm{PGL}(3, K)$ of all symmetries that map P onto itself has order 216 and it is isomorphic to the group of affine transformations of A that have determinant 1. The stabilizer in Γ of any of the 12 lines through P is a cyclic subgroup of order three and Γ is generated by these subgroups.

The symmetry group Γ is isomorphic to $G(K)/Z(K)$ where $G(K) \subset \mathrm{GL}(3, K)$ is a group of order 648 generated by reflections of order three and $Z(K)$ is its cyclic center of order three. The reflection group $G(\mathbb{C})$ is called the Hesse group which appears as G_{25} in the classification of finite complex reflection groups by Shephard and Todd.

If K is algebraically closed and the characteristic of K is not 2 or 3 then the nine inflection points of an elliptic curve E over K form a Hesse configuration.