meromorphic function on projective space must be rational

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To define a rational function on complex projective space $\mathbb{P}^n$, we just take two homogeneous polynomials of the same degree p and q on $\mathbb{C}^{n+1}$, and we note that p/q induces a meromorphic function on $\mathbb{P}^n$. In fact, every meromorphic function on $\mathbb{P}^n$ is rational.

Theorem. *Let f be a meromorphic function on $\mathbb{P}^n$. Then f is rational.*

Proof. Note that the zero set of f and the pole set are analytic subvarieties of $\mathbb{P}^n$ and hence algebraic by Chow's theorem. f induces a meromorphic function $\tilde{f}$ on $\mathbb{C}^{n+1} \setminus \{0\}$. Let p and q be two homogeneous polynomials such that $q = 0$ are the poles and $p = 0$ are the zeros of $\tilde{f}$. We can assume we can take p and q such that if we multiply $\tilde{f}$ by q/p we have a holomorphic function outside the origin. Hence $(q/p)\tilde{f}$ extends through the origin by Hartogs' theorem. Further since $\tilde{f}$ was constant on complex lines through the origin, it is not hard to see that $(q/p)\tilde{f}$ is homogeneous and hence a homogeneous polynomial, by the same argument as in the proof of Chow's theorem. Since it is not zero outside the origin, it can't be zero at the origin, and hence $(q/p)\tilde{f}$ must be a constant, and the proof is finished. $\square$