proof of Dehn's theorem

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We define the Dehn's invariant, which is a number given to any polyhedron which does not change under scissor-equivalence.

Choose an additive function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(\pi) = f(0) = 0$ and define for any polyhedron P the number (Dehn's invariant)

$$D(P) = \sum_{e \in \{\text{edges of } P\}} f(\theta_e)\ell(e)$$

where θ_e is the angle between the two faces of P joining in e, and $\ell(e)$ is the length of the edge e.

We want to prove that if we decompose P into smaller polyhedra $P_1, \dots, P_N$ as in the definition of scissor-equivalence, we have

$$D(P) = \sum_{k=1}^{N} D(P_k) \tag{1}$$

which means that if P is scissor equivalent to Q then $D(P) = D(Q)$.

Let $P_1, \dots, P_N$ be such a decomposition of P. Given any edge e of a piece P_k the following cases arise:

1. e is contained in the interior of P. Since an entire neighbourhood of e is contained in P the angles of the pieces which have e as an edge (or part of an edge) must have sum 2π. So in the right hand side of (**??**) the edge e gives a contribution of $f(2\pi)\ell(e)$ (recall that f is additive).

2. e is contained in a facet of P. The same argument as before is valid, only we find that the total contribution is $f(\pi)\ell(e)$.

3. e is contained in an edge e' of P. In this case the total contribution given by e to the right hand side of (**??**) is given by $f(\theta_{e'})\ell(e)$.

Since we have choosen f so that $f(\pi) = 0$ and hence also $f(2\pi) = 0$ (since f is additive) we conclude that the equivalence (**??**) is valid.

Now we are able to prove Dehn's Theorem. Choose T to be a regular tetrahedron with edges of length 1. Then $D(T) = 6f(\theta)$ where θ is the angle between two faces of T. We know that θ/π is irrational, hence there exists an additive function f such that $f(\theta) = 1$ while $f(\pi/2) = 0$ (as there exist additive functions which are not linear).

So if P is any parallelepiped we find that $D(P) = 0$ (since each angle between facets of P is $\pi/2$ and $f(\pi/2) = 0$) while $D(T) = 6$. This means that P and T cannot be scissor-equivalent.