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# affine combination

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## Definition

Let $V$ be a vector space over a division ring $D$. An *affine combination* of a finite set of vectors $v_1, \ldots, v_n \in V$ is a linear combination of the vectors

$$k_1 v_1 + \cdots + k_n v_n$$

such that $k_i \in D$ subject to the condition $k_1 + \cdots + k_n = 1$. In effect, an affine combination is a weighted average of the vectors in question.

For example, $v = \frac{1}{2}v_1 + \frac{1}{2}v_2$ is an affine combination of $v_1$ and $v_2$ provided that the characteristic of $D$ is not 2. $v$ is known as the midpoint of $v_1$ and $v_2$. More generally, if $\operatorname{char}(D)$ does not divide $m$, then

$$v = \frac{1}{m}(v_1 + \cdots + v_m)$$

is an affine combination of the $v_i$'s. $v$ is the barycenter of $v_1, \ldots, v_n$.

## Relations with Affine Subspaces

Assume now $\operatorname{char}(D) = 0$. Given $v_1, \ldots, v_n \in V$, we can form the set $A$ of all affine combinations of the $v_i$'s. We have the following

> $A$ is a finite dimensional affine subspace. Conversely, a finite dimensional affine subspace $A$ is the set of all affine combinations of a finite set of vectors in $A$.

*Proof.* Suppose $A$ is the set of affine combinations of $v_1, \ldots, v_n$. If $n = 1$, then $A$ is a singleton $\{v\}$, so $A = 0 + v$, where $0$ is the null subspace of $V$. If $n > 1$, we may pick a non-zero vector $v \in A$. Define $S = \{a - v \mid a \in A\}$. Then for any $s \in S$ and $d \in D$, $ds = d(a - v) = da + (1 - d)v - v$. Since $da + (1 - d)v \in A$, $ds \in S$. If $s_1, s_2 \in S$, then $\frac{1}{2}(s_1 + s_2) = \frac{1}{2}((a_1 - v) + (a_2 - v)) = \frac{1}{2}(a_1 + a_2) - v \in S$, since $\frac{1}{2}(a_1 + a_2) \in A$. So $\frac{1}{2}(s_1 + s_2) \in S$. Therefore, $s_1 + s_2 = 2(\frac{1}{2}(s_1 + s_2)) \in S$. This shows that $S$ is a vector subspace of $V$ and that $A = S + v$ is an affine subspace.

Conversely, let $A$ be a finite dimensional affine subspace. Write $A = S + v$, where $S$ is a subspace of $V$. Since $\dim(S) = \dim(A) = n$, $S$ has a basis $\{s_1, \ldots, s_n\}$. For each $i = 1, \ldots, n$, define $v_i = ns_i + v$. Given $a \in A$, we have

$$\begin{aligned} a &= s + v = k_1 s_1 + \cdots + k_n s_n + v \\ &= \frac{k_1}{n}(v_1 - v) + \cdots + \frac{k_n}{n}(v_n - v) + v \\ &= \frac{k_1}{n}v_1 + \cdots + \frac{k_n}{n}v_n + (1 - \frac{k_1}{n} - \cdots - \frac{k_n}{n})v. \end{aligned}$$

From this calculation, it is evident that $a$ is an affine combination of $v_1, \ldots, v_n$, and $v$. $\square$

When $A$ is the set of affine combinations of two distinct vectors $v, w$, we see that $A$ is a line, in the sense that $A = S + v$, a translate of a one-dimensional subspace $S$ (a line through 0). Every element in $A$ has the form $dv + (1-d)w$, $d \in D$. Inspecting the first part of the proof in the previous proposition, we see that the argument involves no more than two vectors at a time, so the following useful corollary is apparant:

> $A$ is an affine subspace iff for every pair of vectors in $A$, the line formed by the pair is also in $A$.

Note, however, that the $A$ in the above corollary is not assumed to be finite dimensional.

**Remarks**.

- If one of $v_1, \ldots, v_n$ is the zero vector, then $A$ coincides with $S$. In other words, an affine subspace is a vector subspace if it contains the zero vector.
- Given $A = \{k_1 v_1 + \cdots + k_n v_n \mid v_i \in V, k_i \in D, \sum k_i = 1\}$, the subset
$$\{k_1 v_1 + \cdots + k_n v_n \in A \mid k_i = 0\}$$
is also an affine subspace.

## Affine Independence

Since every element in a finite dimensional affine subspace $A$ is an affine combination of a finite set of vectors in $A$, we have the similar concept of a spanning set of an affine subspace. A minimal spanning set $M$ of an affine subspace is said to be *affinely independent*. We have the following three equivalent characterization of an affinely independent subset $M$ of a finite dimensional affine subspace:

1. $M = \{v_1, \ldots, v_n\}$ is affinely independent.

2. every element in $A$ can be written as an affine combination of elements in $M$ in a *unique* fashion.

3. for every $v \in M$, $N = \{v_i - v \mid v \neq v_i\}$ is linearly independent.

*Proof.* We will proceed as follows: (1) implies (2) implies (3) implies (1).

(1) implies (2). If $a \in A$ has two distinct representations $k_1 v_1 + \cdots + k_n v_n = a = r_1 v_1 + \cdots + r_n v_n$, we may assume, say $k_1 \neq r_1$. So $k_1 - r_1$ is invertible with inverse $t \in D$. Then

$$v_1 = t(r_2 - k_2)v_2 + \cdots + t(r_n - k_n)v_n.$$

Furthermore,

$$\sum_{i=2}^{n} t(r_i - k_i) = t(\sum_{i=2}^{n} r_i - \sum_{i=2}^{n} k_i) = t(1 - r_1 - 1 + k_1) = 1.$$

So for any $b \in A$, we have

$$b = s_1 v_1 + \cdots + s_n v_n = s_1(t(r_2 - k_2)v_2 + \cdots + t(r_n - k_n)v_n) + \cdots + s_n v_n.$$

The sum of the coefficients is easily seen to be 1, which implies that $\{v_2, \ldots, v_n\}$ is a spanning set of $A$ that is smaller than $M$, a contradiction.

(2) implies (3). Pick $v = v_1$. Suppose $0 = s_2(v_2 - v_1) + \cdots + s_n(v_n - v_1)$. Expand and we have $0 = (-s_2 - \cdots - s_n)v_1 + s_2 v_2 + \cdots + s_n v_n$. So $(1 - s_2 - \cdots - s_n)v_1 + s_2 v_2 + \cdots + s_n v_n = v_1 \in A$. By assumption, there is exactly one way to express $v_1$, so we conclude that $s_2 = \cdots = s_n = 0$.

(3) implies (1). If $M$ were not minimal, then some $v \in M$ could be expressed as an affine combination of the remaining vectors in $M$. So suppose $v_1 = k_2 v_2 + \cdots + k_n v_n$. Since $\sum k_i = 1$, we can rewrite this as $0 = k_2(v_2 - v_1) + \cdots + k_n(v_n - v_1)$. Since not all $k_i = 0$, $N = \{v_2 - v_1, \ldots, v_n - v_1\}$ is not linearly independent. $\square$

**Remarks**.

- If $\{v_1, \ldots, v_n\}$ is affinely independent set spanning $A$, then $\dim(A) = n - 1$.

- More generally, a set $M$ (not necessarily finite) of vectors is said to be affinely independent if there is a vector $v \in M$, such that $N = \{w - v \mid v \neq w \in M\}$ is linearly independent (every finite subset of $N$ is linearly independent). The above three characterizations are still valid in this general setting. However, one must be careful that an affine combination is a finitary operation so that when we take the sum of an infinite number of vectors, we have to realize that only a finite number of them are non-zero.

- Given any set $S$ of vectors, the *affine hull* of $S$ is the smallest affine subspace $A$ that contains every vector of $S$, denoted by $\mathrm{Aff}(S)$. Every vector in $\mathrm{Aff}(S)$ can be written as an affine combination of vectors in $S$.