equation of plane

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The position of a plane τ can be fixed by giving the position vector $\overrightarrow{OQ}$ of the projection point Q of the origin on the plane.

Let the length of the position vector be r and the angles formed by the vector with the positive coordinate axes $\alpha,\, \beta,\, \gamma$. Let $P = (x,\, y,\, z)$ be an arbitrary point. Then P is in the plane τ iff its projection on the line OQ coincides with Q, i.e. http://planetmath.org/Iffiff the projection of the coordinate way of P is r. This may be expressed as the equation $x\cos\alpha + y\cos\beta + z\cos\gamma = r$ or

$$x\cos\alpha + y\cos\beta + z\cos\gamma - r = 0, \tag{1}$$

which thus is the equation of the plane.

Conversely, we may show that a first-degree equation

$$Ax + By + Cz + D = 0 \tag{2}$$

between the variables $x,\, y,\, z$ represents always a plane. In fact, we may without hurting generality suppose that $D \leqq 0$. Now $R := \sqrt{A^2+B^2+C^2} > 0$. Thus the length of the http://planetmath.org/PositionVectorradius vector of the point $(A,\, B,\, C)$ is R. Let the angles formed by the radius vector with the positive coordinate axes be $\alpha,\, \beta,\, \gamma$. Then we can write

$$A = R\cos\alpha, \quad B = R\cos\beta, \quad C = R\cos\gamma$$

(cf. direction cosines). Dividing (2) termwise by R gives us

$$x\cos\alpha + y\cos\beta + z\cos\gamma + \frac{D}{R} = 0,$$

where $\frac{D}{R} \leqq 0$. The last equation represents a plane whose distance from the origin is $-\frac{D}{R}$ and whose normal line forms the angles $\alpha,\, \beta,\, \gamma$ with the coordinate axes.

Since the coefficients $A,\, B,\, C$ are proportional to the direction cosines of the normal vector of this plane, they are direction numbers of the normal line of the plane.

Examples. The equations of the coordinate planes are
$x = 0$ (yz-plane), $y = 0$ (zx-plane), $z = 0$ (xy-plane);
the equation of the plane through the points $(1,\, 0,\, 0)$, $(0,\, 1,\, 0)$ and $(0,\, 0,\, 1)$

is
$x + y + z = 1.$

The plane can be represented also in a vectoral form, by using the position vector $\vec{r}_0$ of a point of the plane and two linearly independent vectors $\vec{u}$ and $\vec{v}$ parallel to the plane:

$$\vec{r} = \vec{r}_0 + s\vec{u} + t\vec{v}. \tag{3}$$

Here, $\vec{r}$ means the position vector of arbitrary point of the plane, s and t are real parameters. In the coordinate form, (3) may be e.g.

$$\begin{cases} x = x_0 + sa + td, \\ y = y_0 + sb + te, \\ z = z_0 + sc + tf. \end{cases}$$