angle

Canonical name	Angle
Date of creation	2013-03-22 15:32:36
Last modified on	2013-03-22 15:32:36
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	22
Author	CWoo (3771)
Entry type	Definition
Classification	msc 51F20
Classification	msc 51G05
Synonym	supplement
Related topic	PaschsTheorem
Related topic	BetweennessInRays
Related topic	SupplementaryAngles
Defines	supplementary
Defines	right angle
Defines	between rays
Defines	crossbar theorem
Defines	free angle
Defines	acute angle
Defines	obtuse angle
Defines	angle measure
Defines	side
Defines	vertex

1 Definition

In an ordered geometry S, given a point p let $\Pi(p)$ be the family of all rays emanating from it. Let $\alpha, \beta \in \Pi(p)$ such that $\alpha \neq \beta$ and $\alpha \neq -\beta$. The *angle* `http://planetmath.org/BetweennessInRays`between rays α and β at p is

$$\{\rho \in \Pi(p) \mid \rho \text{ is between } \alpha \text{ and } \beta\}.$$

This angle is denoted by $\angle \alpha p \beta$. The two rays α and β are the *sides* of the angle, and p the *vertex* of the angle. Since any point (other than the source p) on a ray uniquely determines the ray, we may also write the angle by $\angle apb$, whenever we have points $a \in \alpha$ and $b \in \beta$.

The notational device given for the angle suggests the possibility of defining an angle between two line segments satisfying certain conditions: let $\overline{pq}$ and $\overline{qr}$ be two open line segments with a common endpoint q. The angle between the two open line segments is the angle between the rays $\overrightarrow{qp}$ and $\overrightarrow{qr}$. In this case, we may denote the angle by $\angle pqr$.

Suppose ℓ is a line and p a point lying on ℓ. We have two opposite rays emanating from p that lie on ℓ. Call them σ and $-\sigma$. Any ray ρ emanating from a point p that does not lie on ℓ produces two angles at p, one between ρ and σ and the other between ρ and $-\sigma$. These two angles are said to be *supplement* of one another, or that $\angle \sigma p \rho$ is *supplementary* of $\angle(-\sigma) p \rho$. Every angle has exactly two supplements.

2 Ordering of Angles

Let S be an ordered geometry and ρ a ray in S with source point p. Consider the set E of all angles whose one side is ρ. Define an ordering on E by the following rule: for $\angle \alpha p \rho, \angle \beta p \rho \in E$,

1. $\angle \alpha p \rho = \angle \beta p \rho$ if $\alpha = \beta$,
2. $\angle \alpha p \rho < \angle \beta p \rho$ if $\alpha \in \angle \beta p \rho$, and
3. $\angle \alpha p \rho > \angle \beta p \rho$ if $\beta \in \angle \alpha p \rho$.

The ordering relation above is well-defined. However, it is quite limited, because there is no way to compare angles if the pair (of angles) do not share a common side. This can be remedied with an additional set of axioms on the geometry: the axioms of congruence.

In an ordered geometry satisfying the congruence axioms, we have the concept of angle congruence. This binary relation turns out to be an equivalence relation, so we can form the set of equivalence classes on angles. Each equivalence class of angles is called a *free angle*. Any member of a free angle $\mathfrak{a}$ is called a representative of $\mathfrak{a}$, which is simply an angle of form $\angle abc$, where b is the source of two rays $\overrightarrow{ba}$ and $\overrightarrow{bc}$. We write $\mathfrak{a} = [\angle abc]$. It is easy to see that given a point p and a ray ρ emanating from p, we can find, in each free angle, a representative whose one side is ρ. In other words, for any free angle $\mathfrak{a}$, it is possible to write $\mathfrak{a} = [\angle \alpha p \rho]$ for some ray α.

Now we are ready to define orderings on angles in general. In fact, this this done via free angles. Let $\mathfrak{A}$ be the set of all free angles in an ordered geometry satisfying the congruence axioms, and $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$. Write $\mathfrak{a} = [\angle \alpha p \rho]$ and $\mathfrak{b} = [\angle \beta p \rho]$. We say that $\mathfrak{a} < \mathfrak{b}$ if ray α is between β and ρ. The other inequality is dually defined. This is a well-defined binary relation. Given the ordering on free angles, we define $\angle \alpha p \beta < \angle \gamma q \delta$ if $[\angle \alpha p \beta] < [\angle \gamma q \delta]$.

Let ℓ be a line and p a point lying on ℓ. The point p determines two opposite rays ρ and $-\rho$ lying on ℓ. Any ray σ emanating from p that is distinct from either ρ and $-\rho$ determines exactly two angles: $\angle \rho p \sigma$ and $\angle (-\rho) p \sigma$. These two angles are said to be supplements of one another, or that one is supplementary of the other.

In an ordered geometry satisfying the congruence axioms, supplementary free angles are defined if each contains a representative that is supplementary to one another. Given two supplementary free angles $\mathfrak{a}, \mathfrak{b}$, we may make comparisons of the two:

- if $\mathfrak{a} = \mathfrak{b}$, then we say that $\mathfrak{a}$ is a *right free angle*, or simple a *right angle*. Clearly $\mathfrak{b}$ is a right angle if $\mathfrak{a}$ is;
- if $\mathfrak{a} > \mathfrak{b}$, then $\mathfrak{a}$ is called an *obtuse free angle*, or an *obtuse angle*. The supplement of an obtuse angle is called an *acute free angle*, or an *acute angle*. Thus, $\mathfrak{b}$ is acute if $\mathfrak{a}$ is obtuse.

Given any two free angles, we can always compare them. In other words, the law of trichotomy is satisfied by the ordering of free angles: for any $\mathfrak{a}$ and $\mathfrak{b}$, exactly one of

$$\mathfrak{a} > \mathfrak{b} \qquad\qquad \mathfrak{a} = \mathfrak{b} \qquad\qquad \mathfrak{a} < \mathfrak{b}$$

is true.

3 Operations on Angles

Let S be an ordered geometry satisfying the congruence axioms and $\mathfrak{a}$ and $\mathfrak{b}$ are two free angles. Write $\mathfrak{a} = [\angle\alpha p\beta]$ and $\mathfrak{b} = [\angle\beta p\gamma]$. If β is between α and γ, we define an "addition" of $\mathfrak{a}$ and $\mathfrak{b}$, written $\mathfrak{a} + \mathfrak{b}$ as the free angle $\mathfrak{c}$ with representative $\angle\alpha p\gamma$. In symbol, this says that if β is between α and γ, then

$$[\angle\alpha p\beta] + [\angle\beta p\gamma] = [\angle\alpha p\gamma].$$

This is a well-defined binary operation, provided that *one free angle is between the other two.* Therefore, the sum of a pair of supplementary angles is not defined! In addition, if $\mathfrak{a}$ and $\mathfrak{c}$ are two free angles, such that there exists a free angle $\mathfrak{b}$ with $\mathfrak{a} + \mathfrak{b} = \mathfrak{c}$, then $\mathfrak{b}$ is unique and we denote it by $\mathfrak{c} - \mathfrak{a}$. It is also possible to define the multiplication of a free angle by a positive integer, provided that the resulting angle is a well-defined free angle. Finally, division of a free angle by positive integral powers of 2 can also be defined.

4 Angle Measurement

An angle measure $\mathcal{A}$ is a function defined on free angles of an ordered geometry S with the congruence axioms, such that

1. $\mathcal{A}$ is real-valued and positive,
2. $\mathcal{A}$ is additive; in other words, $\mathcal{A}(\mathfrak{a}+\mathfrak{b}) = \mathcal{A}(\mathfrak{a})+\mathcal{A}(\mathfrak{b})$, if $\mathfrak{a}+\mathfrak{b}$ is defined;

Here are some properties:

- if $\mathcal{A}(\mathfrak{a}) = \mathcal{A}(\mathfrak{b})$, then $\mathfrak{a} = \mathfrak{b}$.
- $\mathfrak{a} > \mathfrak{b}$ iff $\mathcal{A}(\mathfrak{a}) > \mathcal{A}(\mathfrak{b})$.
- for any free angle $\mathfrak{a}$, denote its supplement by $\mathfrak{a}^s$. Then $\mathcal{A}(\mathfrak{a}) + \mathcal{A}(\mathfrak{a}^s)$ is a positive constant $r_{\mathcal{A}}$ that does not depend on $\mathfrak{a}$.

- $\mathcal{A}$ is bounded above by $r_{\mathcal{A}}$.
- if $\mathcal{A}$ and $\mathcal{B}$ are angle measures, then $\mathcal{A}+\mathcal{B}$ defined by $(\mathcal{A}+\mathcal{B})(\mathfrak{a}) = \mathcal{A}(\mathfrak{a}) + \mathcal{B}(\mathfrak{a})$ is an angle measure too.
- if $\mathcal{A}$ is an angle measure, then for any positive real number r, $r\mathcal{A}$ defined by $(r\mathcal{A})(\mathfrak{a}) = r(\mathcal{A}(\mathfrak{a}))$ is also an angle measure. In the event that r is an integer such that $r\mathfrak{a}$ makes sense, we also have $r(\mathcal{A}(\mathfrak{a})) = \mathcal{A}(r\mathfrak{a})$.

If S is a neutral geometry, then we impose a third requirement for a function to be an angle measure:

3. for any real number r with $0 < r < r_{\mathcal{A}}$, there is a free angle $\mathfrak{a}$ such that $\mathcal{A}(\mathfrak{a}) = r$.

Once the measure of a free angle is defined, one can next define the measure of an angle: let $\mathcal{A}$ be a measure of the free angles, define $\mathcal{A}'$ on angles by $\mathcal{A}'(\angle\alpha p\beta) = \mathcal{A}([\angle\alpha p\beta])$. This is a well-defined function. It is easy to see that $\mathcal{A}'(\angle\alpha p\beta) = \mathcal{A}'(\angle\gamma q\delta)$ iff $\angle\alpha p\beta \cong \angle\gamma q\delta$, and $\mathcal{A}'(\angle\alpha p\beta) > \mathcal{A}'(\angle\gamma q\delta)$ iff $\angle\alpha p\beta > \angle\gamma q\delta$.

Two popular angle measures are the degree measure and the radian measure. In the degree measure, $r_{\mathcal{A}} = 180°$. In the radian measure, $r_{\mathcal{A}} = \pi$.

References

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