example of using Lagrange multipliers

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One to determine the perpendicular distance of the parallel planes

$$Ax+By+Cz+D \;=\; 0 \quad \text{and} \quad Ax+By+Cz+E \;=\; 0$$

is to use the Lagrange multiplier method. In this case we may to minimise the Euclidean distance of a point $(x,\,y,\,z)$ of the former plane to a (fixed) point $(x_0,\,y_0,\,z_0)$ of the latter plane.

Thus we have the equation $Ax_0+By_0+Cz_0+E \;=\; 0$ which we can subtract from the first plane equation, getting

$$g \;:=\; A(x-x_0)+B(y-y_0)+C(z-x_0)+D-E \;=\; 0. \tag{1}$$

This is the (only) constraint equation for minimising the `http://planetmath.org/SquareOfANumbe`

$$f \;:=\; (x-x_0)^2+(y-y_0)^2+(z-x_0)^2 \tag{2}$$

of the distance of the points.

The polynomial functions f and g satisfy the differentiability requirements. Accordingly, we can find the minimising point $(x,\,y,\,z)$ by considering the system of equations formed by (1) and

$$\begin{cases} \frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x} \;\equiv\; 2(x-x_0)+\lambda A \;=\; 0, \\ \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y} \;\equiv\; 2(y-y_0)+\lambda B \;=\; 0, \\ \frac{\partial f}{\partial z}+\lambda\frac{\partial g}{\partial z} \;\equiv\; 2(z-z_0)+\lambda C \;=\; 0. \end{cases} \tag{3}$$

We solve from (3) the differences

$$x-x_0 \;=\; -\frac{A\lambda}{2}, \quad y-y_0 \;=\; -\frac{B\lambda}{2}, \quad z-z_0 \;=\; -\frac{C\lambda}{2}$$

and set them into (1). It then yields the value

$$\lambda \;=\; \frac{2(D-E)}{A^2+B^2+C^2}$$

of the Lagrange multiplier, which we substitute into the preceding three equations obtaining

$$x-x_0 \;=\; \frac{A(D-E)}{A^2+B^2+C^2}, \quad y-y_0 \;=\; \frac{B(D-E)}{A^2+B^2+C^2}, \quad z-z_0 \;=\; \frac{C(D-E)}{A^2+B^2+C^2}.$$

These values give the minimal distance when put into the expression of $\sqrt{f}$:

$$d \;=\; \sqrt{\frac{(D-E)^2(A^2+B^2+C^2)}{(A^2+B^2+C^2)^2}}.$$

Hence we have gotten the distance

$$d \;=\; \frac{|D-E|}{\sqrt{A^2+B^2+C^2}}.$$