betweenness in rays

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Defines	crossbar theorem

Let S be a linear ordered geometry. Fix a point p and consider the pencil $\Pi(p)$ of all rays emanating from it. Let $\alpha \neq \beta \in \Pi(p)$. A point q is said to be an *interior point* of α and β if there are points $s \in \alpha$ and $t \in \beta$ such that

1. q and s are on the same side of line $\overleftrightarrow{pt}$, and
2. q and t are on the same side of line $\overleftrightarrow{ps}$.

A point q is said to be *between* α and β if there are points $s \in \alpha$ and $t \in \beta$ such that q is between s and t. A point that is between two rays is an interior point of these rays, but not vice versa in general. A ray $\rho \in \Pi(p)$ is said to be *between* rays α and β if there is an interior point of α and β lying on ρ.

Properties

1. Suppose $\alpha, \beta, \rho \in \Pi(p)$ and ρ is between α and β. Then
   (a) any point on ρ is an interior point of α and β;
   (b) any point on the line containing ρ that is an interior point of α and β must be a point on ρ;
   (c) there is a point q on ρ that is between α and β. This is known as the **Crossbar Theorem**, the two "crossbars" being ρ and a line segment joining a point on α and a point on β;
   (d) if q is defined as above, then any point between p and q is between α and β.
2. There are no rays between two opposite rays.
3. If ρ is between α and β, then β is not between α and ρ.
4. If $\alpha, \beta \in \Pi(p)$ has a ray ρ between them, then α and β must lie on the same half plane of some line.
5. The converse of the above statement is true too: if $\alpha, \beta \in \Pi(p)$ are distinct rays that are not opposite of one another, then there exist a ray $\rho \in \Pi(p)$ such that ρ is between α and β.
6. Given $\alpha, \beta \in \Pi(p)$ with $\alpha \neq \beta$ and $\alpha \neq -\beta$. We can write $\Pi(p)$ as a disjoint union of two subsets:
   (a) $A = \{\rho \in \Pi(p) \mid \rho \text{ is between } \alpha \text{ and } \beta\}$,

(b) $B = \Pi(p) - A$.

Let $\rho, \sigma \in \Pi(p)$ be two rays distinct from both α and β. Suppose $x \in \rho$ and $y \in \sigma$. Then ρ, σ belong to the same subset if and only if $\overline{xy}$ does not intersect either α or β.

References

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[3] M. J. Greenberg, *Euclidean and Non-Euclidean Geometries, Development and History*, W. H. Freeman and Company, San Francisco (1974)