exact trigonometry tables

Canonical name	ExactTrigonometryTables
Date of creation	2013-03-22 15:31:26
Last modified on	2013-03-22 15:31:26
Owner	stevecheng (10074)
Last modified by	stevecheng (10074)
Numerical id	11
Author	stevecheng (10074)
Entry type	Example
Classification	msc 51M04
Classification	msc 33B10
Related topic	ConstructibleAnglesWithIntegerValuesInDegrees
Related topic	Cotangent2
Related topic	Trigonometry
Related topic	TheoremOnConstructibleAngles
Related topic	GraphOfEquationXyConstant

Table 1: Basic angles encountered in trigonometry

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	1	0	∞

0.1 Basic angles

Since the trigonometric ratios for most angles cannot be calculated exactly in closed algebraic form, a few well-known angles that can be calculated often comprise the bulk of textbook exercises involving trigonometry.

The basic angles are given in Table **??**.

0.2 Other angles by addition and halving

These basic angles can be easily extended to obtain more angles of interest. Adding multiples of 90° merely rotates these angles into other quadrants; the appropriate values of sin and cos can be obtained through symmetry.

The values for 15° can be obtained by using the `http://planetmath.org/AngleSumIdentity`for for the difference of angles:

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\,.\end{aligned}$$

Likewise, we can find that

$$\begin{aligned}\cos(15^\circ) &= \frac{\sqrt{6}+\sqrt{2}}{4}\\ \sin(75^\circ) &= \sin(45^\circ + 30^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}\\ \cos(75^\circ) &= \frac{\sqrt{6}-\sqrt{2}}{4}\,.\end{aligned}$$

More exact angles can be obtained by solving the double angle identity:

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}\,, \quad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}\,.$$

So for example, $\sin 7.5^\circ = \sqrt{(4-\sqrt{6}-\sqrt{2})/8}$. These angles can be further added and subdivided to obtain a dense subset of exactly known angles. However, such effort is not generally useful. Computers and calculators use a combination of lookup-tables and numeric iteration to obtain their values.

0.3 The angles 18°, 36°, 54°, 72°

The 18°-36°-54°-72° series of angles cannot be obtained by halving, doubling, adding or subtracting the previous angles. Nevertheless, they are constructible, and their exact values can be derived by the following elementary procedure:

Consider an isosceles triangle with the angles 72°, 54° and 54°. From the triangle we derive the relation:

$$\sin\frac{72^\circ}{2} = \cos 54^\circ$$

Notice that $72 = 4\times 18$ and $54 = 3\times 18$, so if $x = 18^\circ$, then

$$\begin{aligned}
\sin 2x &= \cos 3x \\
2\sin x\cos x &= 4\cos^3 x - 3\cos x \\
2\sin x &= 4\cos^2 x - 3 \\
2\sin x &= 4(1-\sin^2 x) - 3
\end{aligned}$$

The last equation is a quadratic equation that can be solved for $\sin 18^\circ$. Carrying out the calculations, we obtain the values in Table **??**.

Table 2: Other constructible angles in trigonometry

θ	$\sin\theta$	$\cos\theta$
18°	$\dfrac{\sqrt{5}-1}{4}$	$\dfrac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$
36°	$\dfrac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$	$\dfrac{\sqrt{5}+1}{4}$
54°	$\dfrac{\sqrt{5}+1}{4}$	$\dfrac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$
72°	$\dfrac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$	$\dfrac{\sqrt{5}-1}{4}$