projectivity

Canonical name	Projectivity
Date of creation	2013-03-22 15:58:00
Last modified on	2013-03-22 15:58:00
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	11
Author	CWoo (3771)
Entry type	Definition
Classification	msc 51A10
Classification	msc 51A05
Related topic	PolaritiesAndForms
Related topic	SesquilinearFormsOverGeneralFields
Related topic	Perspectivity
Related topic	ProjectiveSpace
Related topic	LinearFunction
Related topic	Collineation
Defines	projective transformation
Defines	projective property

Let $PG(V)$ and $PG(W)$ be projective geometries, with V, W vector spaces over a field K. A function p from $PG(V)$ to $PG(W)$ is called a *projective transformation*, or simply a *projectivity* if

1. p is a bijection, and

2. p is order preserving.

A *projective property* is any geometric property, such as incidence, linearity, etc... that is preserved under a projectivity.

From the definition, we see that a projectivity p carries 0 to 0, V to W. Furthermore, it carries points to points, lines to lines, planes to planes, etc.. In short, p preserves linearity. Because p is a bijection, p also preserves dimensions, that is $\dim(S) = \dim(p(S))$, for any subspace S of V. In particular, $\dim(V) = \dim(W)$. Other properties preserved by p are incidence: if $S \cap T \neq \varnothing$, then $p(S) \cap p(T) \neq \varnothing$; and `http://planetmath.org/CrossRatio`cross ratios.

Every bijective semilinear transformation defines a projectiviity. To see this, let $f : V \to W$ be a semilinear transformation. If S is a subspace of V, then $f(S)$ is a subspace of W, as $x, y \in f(S)$, then $x + y = f(a) + f(b) = f(a + b) \in f(S)$, and $\alpha x = \beta^\theta x = \beta^\theta f(a) = f(\beta a) \in f(S)$, where θ is an automorphism of the common underlying field K. Also, if S is a subspace of a subspace T of V, then $f(S)$ is a subspace of $f(T)$. Now if we define $f^* : PG(V) \to PG(W)$ by $f^*(S) = f(S)$, it is easy to see that f^* is a projectivity.

Conversely, if V and W are of finite dimension greater than 2, then a projectivity $p : PG(V) \to PG(W)$ induces a semilinear transformation $\hat{p} : V \to W$. This highly non-trivial fact is the (first) fundamental theorem of projective geometry.

If the semilinear transformation induced by the projectivity p is in fact a linear transformation, then p is a *collineation*: three distinct collinear points are mapped to three distinct collinear points.

Remark. The definition given in this entry is a generalization of the definition typically given for a projective transformation. In the more restictive definition, a projectivity p is defined merely as a bijection between two projective spaces that is induced by a linear isomorphism. More precisely, if $P(V)$ and $P(W)$ are projective spaces induced by the vector spaces V and W, if $L : V \to W$ is a bijective linear transformation, then $p = P(L) :$

$P(V) \to P(W)$ defined by

$$P(L)[v] = [Lv]$$

is the corresponding projective transformation. $[v]$ is the homogeneous coordinate representation of v. In this definition, a projectiity is always a collineation. In the case where the vector spaces are finite dimensional with specified bases, p is expressible in terms of an invertible matrix ($Lv = Av$ where A is an invertible matrix).