converting between the Poincaré disc model and the upper half plane model

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If both the Poincaré disc model and the upper half plane model are considered as subsets of $\mathbb{C}$ rather than as subsets of $\mathbb{R}^2$ (that is, the Poincaré disc model is $\{z \in \mathbb{C} : |z| < 1\}$ and the upper half plane model is $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$), then one can use Möbius transformations to convert between the two models. The entry unit disk upper half plane conformal equivalence theorem yields that $f\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ defined by $f(z) = \dfrac{z-i}{z+i}$ maps the upper half plane model to the Poincaré disc model, and thus its inverse, $f^{-1}\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ defined by $f^{-1}(z) = \dfrac{-iz-i}{z-1}$, maps the Poincaré disc model to the upper half plane model.

Note that the Möbius transformation f^{-1} gives another justification of including ∞ in the boundary of the upper half plane model (see the entry on parallel lines in hyperbolic geometry for more details): 1 (or the ordered pair $(1,0)$) is on the boundary of the Poincaré disc model and $f^{-1}(1) = \infty$.

Note also that lines in the Poincaré disc model passing through 1 (or the ordered pair $(1,0)$) are in one-to-one correspondence with the lines that are vertical rays in the upper half plane model.