distance from point to a line

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The distance from a point P with coordinates $(x_p, y_p) \in \mathbb{R}^2$ to the line with equation $ax + by + c = 0$ is given by $|ax_p + by_p + c|/\sqrt{a^2 + b^2}$.

Proof Every point x, y on the line is at some distance $\sqrt{(x - x_p)^2 + (y - y_p)^2}$ from P. What we need to find is the minimum such distance. Our problem is

$$\min(x - x_p)^2 + (y - y_p)^2$$

subject to

$$ax + by + c = 0$$

This problem is solvable using the Lagrange multiplier method. We minimize

$$(x - x_p)^2 + (y - y_p)^2 + \lambda(ax + by + c)$$

Calculating the derivatives with respect to x, y and λ and setting them to zero we get three equations:

$$2x - 2x_p + \lambda a = 0 \tag{1}$$

$$2y - 2y_p + \lambda b = 0 \tag{2}$$

$$2ax + 2by + 2c = 0 \tag{3}$$

Solving these leads to $x_p - x = a\frac{ax_p + by_p + c}{a^2 + b^2}$ and $y_p - y = b\frac{ax_p + by_p + c}{a^2 + b^2}$. We can now substitute these expressions into $\sqrt{(x - x_p)^2 + (y - y_p)^2}$ and we get (after some simplification) the desired result.