proof of area of a quadrilateral

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One should think of the following diagram as representing a general quadrilateral.

$$\begin{tikzcd} B \arrow[r, dash, "b"] \arrow[d, dash, "a"'] & C \arrow[d, dash, "c"] \arrow[dl, dash, dotted] \\ A \arrow[r, dash, "d"'] & D \end{tikzcd}$$

$$AC^2 = a^2 + b^2 - 2ab\cos B = c^2 + d^2 - 2cd\cos D \tag{1}$$

Hence,

$$a^2 + b^2 - c^2 - d^2 = 2ab\cos B - 2cd\cos D \tag{2}$$

Add $2ab + 2cd$ to each side of (2):

$$a^2 + 2ab + b^2 - (c^2 - 2cd + d^2) = 2ab(1 + \cos B) - 2cd(-1 + \cos D) \tag{3}$$

Now use

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

and

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

to get

$$(a+b)^2 - (c-d)^2 = 4ab\cos^2\frac{B}{2} + 4cd\sin^2\frac{D}{2} \tag{4}$$

Similarly

$$(c+d)^2 - (a-b)^2 = 4ab\sin^2\frac{B}{2} + 4cd\cos^2\frac{D}{2} \tag{5}$$

Let s be the semiperimeter, so $s = \frac{a+b+c+d}{2}$. Then

$$LHS(4) = (a+b-c+d)(a+b+c-d) = 4(s-c)(s-d) \tag{6}$$

and

$$LHS(5) = (c+d-a+b)(c+d+a-b) = 4(s-a)(s-b) \tag{7}$$

Let R be the product of the left hand side of (4) times the left hand side of (5). Then

$$R = 16(ab)^2\cos^2\frac{B}{2}\sin^2\frac{B}{2} + 16(cd)^2\sin^2\frac{D}{2}\cos^2\frac{D}{2} + 16abcd(\cos^2\frac{B}{2}\cos^2\frac{D}{2} + \sin^2\frac{B}{2}\sin^2\frac{D}{2}) \tag{8}$$

$$= 4(ab)^2(2\sin\frac{B}{2}\cos\frac{B}{2})^2 + 4(cd)^2(2\sin\frac{D}{2}\cos\frac{D}{2})^2 + 16abcd(\cos^2\frac{B}{2}\cos^2\frac{D}{2} + \sin^2\frac{B}{2}\sin^2\frac{D}{2}) \tag{9}$$

$$= 4(ab)^2\sin^2 B + 4(cd)^2\sin^2 D + 16abcd(\cos^2\frac{B}{2}\cos^2\frac{D}{2} + \sin^2\frac{B}{2}\sin^2\frac{D}{2}). \tag{10}$$

We have:

$$8abcd\sin B\sin D = 32abcd\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{D}{2}\cos\frac{D}{2}.$$

which we can add and subtract to R:

$$R = (2ab\sin B + 2cd\sin D)^2 + 16abcd(\cos^2\frac{B}{2}\cos^2\frac{D}{2} - 2\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{D}{2}\cos\frac{D}{2} + \sin^2\frac{B}{2}\sin^2\frac{D}{2}) \tag{11}$$

$$= (2ab\sin B + 2cd\sin D)^2 + 16abcd(\cos\frac{B}{2}\cos\frac{D}{2} - \sin\frac{B}{2}\sin\frac{D}{2})^2. \tag{12}$$

But

$$(2ab\sin B + 2cd\sin D = 4K$$

so we have:

$$R = 16K^2 + 16abcd\cos^2\left(\frac{B+D}{2}\right). \tag{13}$$

Hence by (6), (7) and (13)

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{B+D}{2}\right). \tag{14}$$