n-dimensional isoperimetric inequality

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Isoperimetric inequalities for 2 and 3 dimensions are generalized here to n dimensions. First, the n-dimensional ball is shown to have the greatest volume for a given (n-1)-surface area. Then, the volume and area of the n-ball are used to establish the n-dimensional isoperimetric inequality.

1 THE GREATEST N-VOLUME

We shall use cartesian coordinates based on the ortho-normal vector system $\overline{B}_1, \overline{B}_2$, etc... An (n-1) surface S is defined by a function of n coordinates equated to zero. On this surface, any coordinate can be considered as a function of all the others. Let us take the last one x_n, as a function of $x_1, x_2...x_{n-1}$, and call it z for brevity. This surface is the envelope of an n-dimensional solid of volume V:

$$V = \int_V dx_1 dx_2 ... dz = \int_S z dx_1 dx_2 ... dx_{n-1}$$

We are going to maximize V, subject to the condition that the surface S has a given area A:

$$S = \int_S ds = A$$

The infinitesimal surface element ds is (see the annex):

$$S = \int_S \sqrt{1 + Z_1^2 + ... + Z_{n-1}^2} dx_1 ... dx_{n-1}$$

Z_i are the partial derivatives of z with respect to x_i. This surface constraint is handled with the help of a Lagrange multiplier R which allows us to maximize a single function F:

$$I = \int_S F dx_1 ... dx_{n-1} = \int_S \left(z + R\sqrt{1 + Z_1^2 + ... + Z_{n-1}^2} \right) dx_1 ... dx_{n-1}$$

The solution to this variational problem is given by the n-1 Euler-Lagrange equations:

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial Z_i} \right) = 0$$

In our case, they turn to be:

$$\frac{\partial}{\partial x_i} \left(\frac{Z_i}{\sqrt{1 + Z_1^2 + ... + Z_{n-1}^2}} \right) = \frac{1}{R}$$

After a first integration, and squaring, we have:

$$\frac{Z_i^2}{1+Z_1^2+...+Z_{n-1}^2} = \frac{(x_i-a_i)^2}{R^2}$$

Summing all these equations together, after some algebra, we get:

$$Z_i = \frac{\partial z}{\partial x_i} = \frac{x_i - a_i}{\sqrt{R^2-(x_1-a_1)^2-...-(x_{n-1}-a_{n-1})^2}}$$

This system is easily integrated and gives exactly the equation on an n-ball, which is therefore a stationary point of the functional I. Since the minimum volume is obviously zero for a flat solid, the n-ball has necessarily the maximum volume.

2 THE ISOPERIMETRIC INEQUALITY

The volume BV_n of an n-ball of radius R is (ref 1):

$$BV_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} R^n$$

Γ is Euler's gamma function. Since this volume is obviously the integral of the surface from 0 to R, the surface is the derivative of the volume with respect to R:

$$BA_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} R^{n-1}$$

Eliminating the radius R between these two equations, we get:

$$\frac{(BV_n)^{n-1}}{(BA_n)^n} = \frac{\pi^{-\frac{n}{2}}\left[\Gamma\left(\frac{n}{2}\right)\right]^n}{2^n\left[\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right]^{n-1}} = \frac{\Gamma\left(\frac{n}{2}+1\right)}{\left(n\sqrt{\pi}\right)^n}$$

This equality holds for an n-ball. The volume V_n of an arbitrary solid of area A_n cannot be greater than the volume BV_n of an n-ball with the same area; therefore the following inequality holds:

$$\frac{(V_n)^{n-1}}{(A_n)^n} \le \frac{\Gamma\left(\frac{n}{2}+1\right)}{\left(n\sqrt{\pi}\right)^n}$$

This is the so-called isoperimetric inequality for n dimensions.

3 ANNEX: N-DIMENSIONAL PARALLELEPIPED

The infinitesimal surface element of an n-dimensional solid is in fact the volume of an infinitesimal (n-1)-parallelepiped. This volume (ref 2) is the square root of the Gram determinant of the edge vectors $\overline{U}_1, \overline{U}_2 ... \overline{U}_{n-1}$. The elements of this determinant are the dot products of the edge vectors:

$$G_{ij} = \overline{U}_i \cdot \overline{U}_j$$

Let P be the position vector of a point in the (n-1) dimensional enveloppe of the solid. $\overline{\delta}_i$ is the infinitesimal displacement we get by varying the coordinate x_i by dx_i and keeping all the other (n-2) independent variables fixed. Only the last coordinate z varies to maintain the new position into the envelope:

$$\overline{\delta}_i = dx_i \overline{B}_i + dz \overline{B}_n = dx_i (\overline{B}_i + \frac{\partial z}{\partial x_i} \overline{B}_n) = dx_i (\overline{B}_i + Z_i \overline{B}_n)$$

Z_i is a shortcut for the partial derivative of z with respect to x_i. The (n-1) infinitesimal vectors $\overline{\delta}_i$ define an (n-1)-parallelepiped and its Gram determinant is:

$$G_{ij} = \overline{\delta}_i \cdot \overline{\delta}_j = (\delta_{ij} + Z_i Z_j) dx_i dx_j$$

δ_{ij} is the Kronecker symbol. In the determinant, dx_i appears in one row and one column, so that it can be factored out twice. Therefore, the volume of the (n-1)-parallelepiped $\overline{\delta}_i$, or the surface element ds is:

$$ds = \sqrt{\|\delta_{ij} + Z_i Z_j\|} dx_1 dx_2 ... dx_{n-1}$$

The determinant of the matrix H defined by $H_{ij} = \delta_{ij} + Z_i Z_j$ is the product of its eigenvalues. For any (n-1)-vector $\overline{W}$ we have:

$$H\overline{W} = \overline{W} + (\overline{Z} \cdot \overline{W}) \overline{Z}$$

$\overline{Z}$ being the (n-1)-vector $(Z_1, Z_2 ... Z_{n-1})$. If $\overline{W}$ is orthogonal to $\overline{Z}$, $H\overline{W} = \overline{W}$ and its eigenvalue is 1. But there are (n-2) such vectors, so the determiant is the last eignevalue for $\overline{W} = \overline{Z}$: $H\overline{Z} = (1 + |Z|^2)\overline{Z}$. Finally:

$$ds = \sqrt{Z_1^2 + Z_2^2 + ... + Z_{n-1}^2} dx_1 dx_2 ... dx_{n-1}$$

References

[1] Eric Weinstein - *Hypersphere*
http://mathworld.wolfram.com/Hypersphere.html
An elegant proof of the hypersphere volume formula.

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http://www.jyi.org/volumes/volume2/issue1/articles/barth.html