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linear function

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Let $\mathscr{S}_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $\mathscr{S}_2 = (\mathcal{P}_2, \mathcal{L}_2)$ be two near-linear spaces.

Definition. A *linear function* from $\mathscr{S}_1$ to $\mathscr{S}_2$ is a mapping on the points that sends lines of $\mathscr{S}_1$ to lines of $\mathscr{S}_2$. In other words, a linear function is a function $\sigma : \mathcal{P}_1 \to \mathcal{P}_2$ such that

$$\sigma(\ell) \in \mathcal{L}_2 \text{ for every } \ell \in \mathcal{L}_1.$$

Here, $\sigma(\ell)$ is the set $\{\sigma(P) \mid P \in \ell\}$. A linear function is also called a homomorphism.

When both $\mathscr{S}_1$ and $\mathscr{S}_2$ are linear spaces, then σ being a linear function is equivalent to saying that P, Q are collinear iff $\sigma(P), \sigma(Q)$ are collinear.

If $\mathscr{S}_1$ is a linear space, then so is $(\sigma(\mathcal{P}_1), \sigma(\mathcal{L}_1))$. This shows that if $\sigma : \mathscr{S}_1 \to \mathscr{S}_1$ is onto, $\mathscr{S}_2$ is a linear space if $\mathscr{S}_1$ is.

Let $\sigma : \mathscr{S}_1 \to \mathscr{S}_2$ be a one-to-one linear function. If points $P_1 \neq P_2$ lie on line ℓ, then $\sigma(P_1) \neq \sigma(P_2)$ lie on $\sigma(\ell)$. This also shows that three collinear points in $\mathscr{S}_1$ are mapped to three collinear points in $\mathscr{S}_2$. In addition, we have

$$|\ell| = |\sigma(\ell)| \text{ for any line } \ell \text{ in } \mathscr{S}_1.$$

Definition. When $\sigma : \mathscr{S}_1 \to \mathscr{S}_2$ is a bijection whose inverse σ^{-1} is also linear, we say that σ is an isomorphism. When $\mathscr{S}_1 = \mathscr{S}_2 = \mathscr{S}$, we call σ an automorphism, or more commonly among geometers, a *collineation*, of the space $\mathscr{S}$.

Suppose $\sigma : \mathscr{S}_1 \to \mathscr{S}_2$ is an isomorphism. For every point P, let P^* be the set of all lines passing through P. Then

$$|P^*| = |\sigma(P)^*| \text{ for any point } P \text{ in } \mathscr{S}_1.$$

It is possible to have a bijective linear function whose inverse is not linear. For example, let $\mathscr{S}_1$ be the space with two points P, Q with no lines, and $\mathscr{S}_2$ the space with the same two points with line $\{P, Q\}$. Then the identity function on $\{P, Q\}$ is a bijective linear function whose inverse is not linear. On the other hand, if the both spaces are linear, then the inverse is always linear.

Remark. The usage of the term "linear function" differs from its more usual meaning as a linear transformation between vector spaces in the study of linear algebra.

References

[1] L. M. Batten, *Combinatorics of Finite Geometries*, 2nd edition, Cambridge University Press (1997)