# Euclidean transformation

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| Defines | glide reflection |
| Defines | angle of rotation |

Let $V$ and $W$ be Euclidean vector spaces. A *Euclidean transformation* is an affine transformation $E : V \to W$, given by

$$E(v) = L(v) + w$$

such that $L$ is an http://planetmath.org/OrthogonalTransformationorthogonal linear transformation.

As an affine transformation, all affine properties, such as incidence and parallelism are preserved by $E$. In addition, since $E(u-v) = L(u-v)$ and $L$ is an , $E$ preserves lengths of line segments and http://planetmath.org/AngleBetweenTwoLinesang between two line segments. Because of this, a Euclidean transformation is also called a *rigid motion*, which is a popular term used in mechanics.

### Types of Euclidean transformations

There are three main types of Euclidean transformations:

1. **translation**. If $L = I$, then $E$ is just a translation. Any Euclidean transformation can be decomposed into a product of an orthogonal transformation $L(v)$, followed by a translation $T(v) = v + w$.

2. **rotation**. If $w = 0$, then $E$ is just an orthogonal transformation. If $\det(E) = 1$, then $E$ is called a *rotation*. The orientation of any basis (of $V$) is preserved under a rotation. In the case where $V$ is two-dimensional, a rotation is conjugate to a matrix of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{1}$$

   where $\theta \in \mathbb{R}$. Via this particular (unconjugated) map, any vector emanating from the origin is rotated in the counterclockwise direction by an angle of $\theta$ to another vector emanating from the origin. Thus, if $E$ is conjugate to the matrix given above, then $\theta$ is the *angle of rotation* for $E$.

3. **reflection**. If $w = 0$ but $\det(E) = -1$ instead, then $E$ is a called *reflection*. Again, in the two-dimensional case, a reflection is  to a matrix of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \tag{2}$$

where $\theta \in \mathbb{R}$. Any vector is reflected by this particular (unconjugated) map to another by a "mirror", a line of the form $y = x \tan(\frac{\theta}{2})$.

**Remarks**.

- In general, an orthogonal transformation can be represented by a matrix of the form
$$\begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_n \end{pmatrix},$$
where each $A_i$ is either $\pm 1$ or a rotation matrix (1) (or reflection matrix (2)) given above. When its determinant is -1 (a reflection), it has an invariant subspace of $V$ of codimension 1. One can think of this hyperplane as the mirror.

- Another common rigid motion is the *glide reflection*. It is a Euclidean transformation that is expressible as a product of a reflection, followed by a translation.