Dehn's theorem

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We all know the elementary formula to compute the area of a triangle: basis times height divided by two. This formula can be justified with a scissor type argument: one divides the triangle into smaller polygons and rearranges these polygons to obtain a rectangle which should have the same area.

Can we use the same argument to compute the volume of a pyramid? This is the third Hilbert's problem. Quite surprisingly the answer is negative, as states the theorem below. This means that the formulae to compute the volume of polyhedra cannot be proved without a limiting process (for example using integrals).

Definition 1. *We say that two polyhedra P and Q are* scissor-equivalent *if there exists a finite number $P_1, \dots, P_N$ of polyhedra and $\theta_1, \dots, \theta_N$ isometries such that*

1. $P = \bigcup_{k=1}^{N} P_k$ *and* $Q = \bigcup_{k=1}^{N} \theta_k(P_k)$;
2. $P_j \cap P_k$ *and* $\theta_j(P_j) \cap \theta_k(P_k)$ *have empty interior for every* $k \neq j$

The properties given above assure that two scissor-equivalent polyhedra must have the same volume. It is also simple to prove that the scissor-equivalence is indeed an equivalence relation.

Theorem 1. *The regular tetrahedron is not* scissor-equivalent *to any parallelepiped.*