# Steiner system

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**Definition**. An $S(\tau,\kappa,\nu)$ **Steiner system** is a $\tau$-$(\nu,\kappa,1)$ design (i.e. $\lambda=1$). The values $\tau,\kappa,\nu$ are the parameters of the Steiner system.

Since $\lambda=1$, a Steiner system is a simple design, and therefore we may interpret a block to *be* a set of points ($B=\mathcal{P}_B$), which we will do from now on.

Given parameters $\tau,\kappa,\nu$, there may be several non-isomorphic systems, or no systems at all.

Let $\mathcal{S}$ be an $S(\tau,\kappa,\nu)$ system with point set $\mathcal{P}$ and block set $\mathcal{B}$, and choose a point $P\in\mathcal{P}$ (often, the system is so symmetric that it makes no difference which point you choose). The choice uniquely induces an $S(\tau-1,\kappa-1,\nu-1)$ system $\mathcal{S}_1$ with point set $\mathcal{P}_1=\mathcal{P}\setminus\{P\}$ and block set $\mathcal{B}_1$ consisting of $B\setminus\{P\}$ for only those $B\in\mathcal{B}$ that contained $P$. This works because for any $T_1\subseteq\mathcal{P}_1$ with $|T_1|=\tau-1$ there was a unique $B\in\mathcal{B}$ that contained $T=T_1\cup\{P\}$.

This recurses down all the way to $\tau=1$ (a partition of $\nu-\tau+1$ into blocks of $\kappa-\tau+1$) and finally to $\tau=0$ (one arbitrary block of $\kappa-\tau$). If any of the divisibility conditions (see the entry `http://planetmath.org/Design`design for more detail) on the way there do not hold, there cannot exist a Steiner system with the original parameters either.

For instance, **Steiner triple systems** $S(2,3,\nu)$ (the first Steiner systems studied, by Kirkman, before Steiner) exist for $\nu=0$ and all $\nu\equiv 1$ or $3 \pmod 6$, and no other $\nu$.

The reverse construction, turning an $S(\tau,\kappa,\nu)$ into an $S(\tau+1,\kappa+1,\nu+1)$, need not be unique and may be impossible. Famously an $S(4,5,11)$ and a $S(5,6,12)$ have the Mathieu groups $M_{11}$ and $M_{12}$ as their automorphism groups, while $M_{22}$, $M_{23}$ and $M_{24}$ are those of an $S(3,6,22)$, $S(4,7,23)$ and $S(5,8,24)$, with connexions to the binary Golay code and the Leech lattice.

**Remark**. A *Steiner system* $S(t,k,n)$ can be equivalently characterized as a $k$-uniform hypergraph on $n$ vertices such that every set of $t$ vertices is contained in exactly one edge. Notice that any $S(2,k,n)$ is just a $k$-uniform linear space.