convex function

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Defines	strictly convex function
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Definition Suppose Ω is a convex set in a vector space over $\mathbb{R}$ (or $\mathbb{C}$), and suppose f is a function $f : \Omega \to \mathbb{R}$. If for any $x, y \in \Omega$, $x \neq y$ and any $\lambda \in (0, 1)$, we have

$$f\Big(\lambda x + (1-\lambda)y\Big) \leq \lambda f(x) + (1-\lambda)f(y),$$

we say that f is a *convex function*. If for any $x, y \in \Omega$ and any $\lambda \in (0, 1)$, we have

$$f\Big(\lambda x + (1-\lambda)y\Big) \geq \lambda f(x) + (1-\lambda)f(y),$$

we say that f is a *concave function*. If either of the inequalities are strict, then we say that f is a *strictly convex function*, or a *strictly concave function*, respectively.

Properties

- A function f is a (strictly) convex function if and only if $-f$ is a (strictly) concave function. For this reason, most of the below discussion only focuses on convex functions. Analogous result holds for concave functions.

- On $\mathbb{R}$, a continuous function is convex if and only if for all $x, y \in \mathbb{R}$, we have
$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

- On $\mathbb{R}$, a once differentiable function is convex if and only if f' is monotone increasing.

- Suppose f is twice continuously differentiable function on $\mathbb{R}$. Then f is convex if and only if $f'' \geq 0$. If $f'' > 0$, then f is strictly convex.

- A local minimum of a convex function is a global minimum. See `http://planetmath.org/ExtremalValueOfConvexconcaveFunctions`this page.

Examples

- e^x,e^{-x}, and x^2 are convex functions on $\mathbb{R}$. Also, x^4 is strictly convex, but $12x^2$ vanishes at $x = 0$.

- A `http://planetmath.org/NormedVectorSpace`norm is a convex function.

Remark.

We may generalize the above definition of a convex function to an that of an extended real-valued function whose domain is not necessarily a convex set. First, we define what an *epigraph* of a function is.

Let Ω be a subset of a vector space over the reals, and f an extended real-valued function defined on Ω. The *epigraph* of f, denoted by $\operatorname{epi}(f)$, is the set

$$\{(x,r) \mid x \in \Omega,\ r \geq f(x)\}.$$

An extended real-valued function f defined on a subset Ω of a vector space V over the reals is said to be *convex* if its *epigraph* is a convex subset of $V \times \mathbb{R}$. With this definition, the domain Ω of f need not be convex. However, its subset $\{x \in \Omega \mid f(x) < \infty\}$, called the *effective domain* and denoted by $\operatorname{eff.dom}(f)$, is convex. To see this, suppose $x, y \in \operatorname{eff.dom}(f)$ and $z = \lambda x + (1-\lambda)y$ with $0 \leq \lambda \leq 0$. Then $(z, \overline{z}) = \lambda(x, f(x)) + (1-\lambda)(y, f(y)) \in \operatorname{epi}(f)$, where $\overline{z} = \lambda f(x) + (1-\lambda) f(y)$, since $\operatorname{epi}(f)$ is convex by definition. Therefore, $z \in \operatorname{dom}(f)$. In fact, $f(z) \leq \overline{z} = \lambda f(x) + (1-\lambda) f(y) < \infty$, which implies that $z \in \operatorname{eff.dom}(f)$.