proof of criterion for convexity

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Theorem 1. *Suppose $f\colon (a,b) \to \mathbb{R}$ is continuous and that, for all $x, y \in (a,b)$,*

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}.$$

Then f is convex.

Proof. We begin by showing that, for any natural numbers n and $m \le 2^n$,

$$f\left(\frac{mx+(2^n-m)y}{2^n}\right) \le \frac{mf(x)+(2^n-m)f(y)}{2^n}$$

by induction. When $n = 1$, there are three possibilities: $m = 1$, $m = 0$, and $m = 2$. The first possibility is a hypothesis of the theorem being proven and the other two possibilities are trivial.

Assume that

$$f\left(\frac{m'x+(2^n-m')y}{2^n}\right) \le \frac{m'f(x)+(2^n-m')f(y)}{2^n}$$

for some n and all $m' \le 2^n$. Let m be a number less than or equal to 2^{n+1}. Then either $m \le 2^n$ or $2^{n+1} - m \le 2^n$. In the former case we have

$$\begin{aligned} f\left(\frac{1}{2}\frac{mx+(2^n-m)y}{2^n}+\frac{y}{2}\right) &\le \frac{1}{2}\left(f\left(\frac{mx+(2^n-m)y}{2^n}\right)+f(y)\right) \\ &\le \frac{1}{2}\frac{mf(x)+(2^n-m)f(y)}{2^n}+\frac{f(y)}{2} \\ &= \frac{mf(x)+(2^{n+1}-m)f(y)}{2^{n+1}}. \end{aligned}$$

In the other case, we can reverse the roles of x and y.

Now, every real number s has a binary expansion; in other words, there exists a sequence $\{m_n\}$ of integers such that $\lim_{n\to\infty} m_n/2^n = s$. If $0 \le s \le 1$, then we also have $0 \le m_n \le 2^n$ so, by what we proved above,

$$f\left(\frac{m_nx+(2^n-m_n)y}{2^n}\right) \le \frac{m_nf(x)+(2^n-m_n)f(y)}{2^n}.$$

Since f is assumed to be continuous, we may take the limit of both sides and conclude

$$f(sx+(1-s)y) \le sf(x)+(1-s)f(y),$$

which implies that f is convex. $\square$