face of a convex set, alternative definition of

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The following definition of a face of a convex set in a real vector space is sometimes useful.

Let C be a convex subset of $\mathbb{R}^n$. Before we define faces, we introduce oriented hyperplanes and supporting hyperplanes.

Given any vectors n and p in $\mathbb{R}^n$, define the hyperplane $H(n,p)$ by

$$H(n,p) = \{x \in \mathbb{R}^n \colon n \cdot (x-p) = 0\};$$

note that this is the degenerate hyperplane $\mathbb{R}^n$ if $n = 0$. As long as $H(n,p)$ is nondegenerate, its removal disconnects $\mathbb{R}^n$. The *upper halfspace* of $\mathbb{R}^n$ determined by $H(n,p)$ is

$$H(n,p)^+ = \{x \in \mathbb{R}^n \colon n \cdot (x-p) \geq 0\}.$$

A hyperplane $H(n,p)$ is a *supporting hyperplane* for C if its upper halfspace contains C, that is, if $C \subset H(n.p)^+$.

Using this terminology, we can define a *face* of a convex set C to be the intersection of C with a supporting hyperplane of C. Notice that we still get the empty set and C as improper faces of C.

Remarks. Let C be a convex set.

- If $F_1 = C \cap H(n_1, p_1)$ and $F_2 = C \cap H(n_2, p_2)$ are faces of C intersecting in a point p, then $H(n_1 + n_2, p)$ is a supporting hyperplane of C, and $F_1 \cap F_2 = C \cap H(n_1 + n_2, p)$. This shows that the faces of C form a meet-semilattice.

- Since each proper face lies on the base of the upper halfspace of some supporting hyperplane, each such face must lie on the relative boundary of C.