f-vector

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Defines	flag f-vector
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Let P be a polytope of dimension d. The *f-vector* of P is the finite integer sequence $(f_0, \ldots, f_{d-i})$, where the component in position i is the number of i-dimensional faces of P. For some purposes it is convenient to view the empty face and the polytope itself as improper faces, so $f_{-1} = f_d = 1$.

For example, a cube has 8 vertices, 12 edges, and 6 faces, so its f-vector is (8, 12, 6).

The entries in the f-vector of a convex polytope satisfy the Euler–Poincaré–Schläfli formula:

$$\sum_{-1 \leq i \leq d} (-1)^i f_i = 0.$$

Consequently, the face lattice of a polytope is Eulerian. For any graded poset with maximum and minimum elements there is an extension of the f-vector called the *flag f-vector*. For any subset S of $\{0, 1, \ldots, d-1\}$, the f_S entry of the flag f-vector of P is the number of chains of faces in $\mathcal{L}(P)$ with dimensions coming only from S.

The flag f-vector of a three-dimensional cube is given in the following table. For simplicity we drop braces and commas.

S	f_S
$\emptyset$	1
0	8
1	12
2	6
01	$8 \cdot 3 = 24$
02	$8 \cdot 3 = 24$
12	$12 \cdot 2 = 24$
012	$8 \cdot 3 \cdot 2 = 48$

For example, $f_{\{1,2\}} = 24$ because each of the 12 edges meets exactly two faces.

Although the flag f-vector of a d-polytope has 2^d entries, most of them are redundant, as they satisfy a collection of identities generalizing the Euler–Poincaré–Schläfli formula and called the generalized Dehn-Sommerville relations. Interestingly, the number of nonredundant entries in the flag f-vector of a d-polytope is one less than the Fibonacci number F_{d-1}.

References

[1] Bayer, M. and L. Billera, *Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets*, Invent. Math. 79 (1985), no. 1, 143–157.

[2] Bayer, M. and A. Klapper, *A new index for polytopes*, Discrete Comput. Geom. 6 (1991), no. 1, 33–47.

[3] Ziegler, G., *Lectures on polytopes*, Springer-Verlag, 1997.