polytope

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Related topic	Polyhedron
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Related topic	EulersPolyhedronTheorem
Defines	V-polytope
Defines	H-polytope
Defines	d-polytope
Defines	dimension

A *polytope* is the convex hull of finitely many points in Euclidean space. A polytope constructed in this way is the convex hull of its vertices and is called a $\mathcal{V}$*-polytope*. An $\mathcal{H}$*-polytope* is a bounded intersection of upper halfspaces. By the Weyl–Minkowski theorem, these descriptions are equivalent, that is, every $\mathcal{V}$-polytope is an $\mathcal{H}$-polytope, and vice versa. This shows that our intuition, based on the study of low-dimensional polytopes, that one can describe a polytope either by its vertices or by its facets is essentially correct.

The *dimension* of P is the smallest d such that P can be embedded in $\mathbb{R}^d$. A d-dimensional polytope is also called a d*-polytope*.

A face of a polytope is the intersection of the polytope with a supporting hyperplane. Intuitively, a supporting hyperplane is a hyperplane that "just touches" the polytope, as though the polytope were just about to pass through the hyperplane. Note that this intuitive picture does not cover the case of the empty face, where the supporting hyperplane does not touch the polytope at all, or the fact that a polytope is a face of itself. The faces of a polytope, when partially ordered by set inclusion, form a geometric lattice, called the face lattice of the polytope.

The Euler polyhedron formula, which states that if a 3-polytope has V vertices, E edges, and F faces, then

$$V - E + F = 2,$$

has a generalization to all d-polytopes. Let $(f_{-1} = 1, f_0, \ldots, f_{d-1}, f_d = 1)$ be the f-vector of a d-polytope P, so f_i is the number of i-dimensional faces of P. Then these numbers satsify the Euler–Poincaré–Schläfli formula:

$$\sum_{i=-1}^{d} (-1)^i f_i = 0. \tag{1}$$

This is the first of many relations among entries of the f-vector satisfied by all polytopes. These relations are called the *Dehn–Sommerville relations*. Any poset which satisfies these relations is `http://planetmath.org/EulerianPoset`Eulerian, so the face lattice of any polytope is Eulerian.

References

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