hyperplane arrangement

Canonical name	HyperplaneArrangement
Date of creation	2013-03-22 15:47:55
Last modified on	2013-03-22 15:47:55
Owner	mps (409)
Last modified by	mps (409)
Numerical id	6
Author	mps (409)
Entry type	Definition
Classification	msc 52B99
Classification	msc 52C35
Synonym	subspace arrangement
Synonym	central arrangement
Related topic	Zonotope
Defines	Grassmannian

Let V be a vector space over a field $\mathbb{K}$. A *hyperplane arrangment* in V is a family $\mathcal{A} = \{\mathcal{H}_i\}_{i \in I}$ of affine hyperplanes in V. If all of the hyperplanes pass through 0, $\mathcal{A}$ is called *central*; otherwise, it is *affine*. More generally, a *subspace arrangement* is a family of affine subspaces of V. The same distinction between central and affine subspace arrangement holds.

Example 1. *Let $V = \mathbb{K}^n$. Then the family*

$$\mathbb{K}P^n = \{S \subset V \mid \dim_{\mathbb{K}}(S) = 1\}$$

of 1-dimensional subspaces of V is a central subspace arrangement, the projective space of dimension n over $\mathbb{K}$.

Instead of considering all lines through a vector space, we could consider all k-dimensional subspaces of the space.

Example 2. *Again let $V = \mathbb{K}^n$, and suppose $0 \leq k \leq n$. Then the family*

$$\mathrm{Gr}(V, k) = \{S \subset V \mid \dim_{\mathbb{K}}(S) = k\}$$

of k-dimensional subspaces of V is a central subspace arrangement, the Grassmannian. Observe that $\mathbb{K}P^n = \mathrm{Gr}(\mathbb{K}^n, 1)$.

If V is a topological vector space and $\mathcal{A}$ is a hyperplane arrangement, then it makes sense to ask for the fundamental group of the complement $V \setminus \bigcup_{\mathcal{H} \in \mathcal{A}} \mathcal{H}$.

Example 3. *If $\mathcal{A}$ is a finite hyperplane arrangement over $V = \mathcal{R}^n$, then the arrangement `http://planetmath.org/Partition`partitions V into a finite number of contractible cells. By selecting a point in each cell and taking the convex hull of the result, we obtain a polytope combinatorially equivalent to the zonotope dual to the arrangement. Since the question of the fundamental group here is not interesting, we could also use the embedding $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$ to complexify $\mathcal{A}$. In this case the complement $\mathbb{C}^n \setminus \bigcup_{\mathcal{H} \in \mathcal{A}} \mathcal{H}$ usually has nontrivial fundamental group.*

References

[1] Klain, D. A., and G.-C. Rota, , *Introduction to geometric probability*, Cambridge University Press, 1997.

[2] Orlik, P., and H. Terao, *Arrangements of hyperplanes*, Springer-Verlag, 1992.