face of a convex set

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Defines	extreme point
Defines	improper face

Let C be a convex set in $\mathbb{R}^n$ (or any topological vector space). A *face* of C is a subset F of C such that

1. F is convex, and
2. given any line segment $L \subseteq C$, if $\mathrm{ri}(L) \cap F \neq \varnothing$, then $L \subseteq F$.

Here, $\mathrm{ri}(L)$ denotes the relative interior of L (open segment of L).

A zero-dimensional face of a convex set C is called an *extreme point* of C.

This definition formalizes the notion of a face of a convex polygon or a convex polytope and generalizes it to an arbitrary convex set. For example, any point on the boundary of a closed unit disk in $\mathbb{R}^2$ is its face (and an extreme point).

Observe that the empty set and C itself are faces of C. These faces are sometimes called *improper faces*, while other faces are called *proper faces*.

Remarks. Let C be a convex set.

- The intersection of two faces of C is a face of C.
- A face of a face of C is a face of C.
- Any proper face of C lies on its relative boundary, $\mathrm{rbd}(C)$.
- The set $\mathrm{Part}(C)$ of all relative interiors of the faces of C partitions C.
- If C is compact, then C is the convex hull of its extreme points.
- The set $F(C)$ of faces of a convex set C forms a lattice, where the meet is the intersection: $F_1 \wedge F_2 := F_1 \cap F_2$; the join of F_1, F_2 is the smallest face $F \in F(C)$ containing both F_1 and F_2. This lattice is bounded lattice (by $\varnothing$ and C). And it is not hard to see that $F(C)$ is a complete lattice.
- However, in general, $F(C)$ is not a modular lattice. As a counterexample, consider the unit square $[0,1] \times [0,1]$ and faces $a = (0,0)$, $b = \{(0,y) \mid y \in [0,1]\}$, and $c = (1,1)$. We have $a \leq b$. However, $a \vee (b \wedge c) = (0,0) \vee \varnothing = (0,0)$, whereas $(a \vee b) \wedge c = b \wedge \varnothing = \varnothing$.
- Nevertheless, $F(C)$ is a complemented lattice. Pick any face $F \in F(C)$. If $F = C$, then $\varnothing$ is a complement of F. Otherwise, form $\mathrm{Part}(C)$ and

Part(F), the partitions of C and F into disjoint unions of the relative interiors of their corresponding faces. Clearly $\mathrm{Part}(F) \subset \mathrm{Part}(C)$ strictly. Now, it is possible to find an extreme point p such that $\{p\} \in \mathrm{Part}(C) - \mathrm{Part}(F)$. Otherwise, all extreme points lie in Part(F), which leads to

$$\mathrm{Part}(F) = \mathrm{Part}(\text{convex hull of extreme points of } C) = \mathrm{Part}(C),$$

a contradiction. Finally, let G be the convex hull of extreme points of C not contained in Part(F). We assert that G is a complement of F. If $x \in G \cap F$, then $G \cap F$ is a proper face of G and of F, hence its extreme points are also extreme points of G, and of F, which is impossible by the construction of G. Therefore $F \cap G = \varnothing$. Next, note that the union of extreme points of G and of F is the collection of all extreme points of C, this is again the result of the construction of G, so any $y \in C$ is in the join of all its extreme points, which is equal to the join of F and G (since join is universally associative).

- Additionally, in $F(C)$, zero-dimensional faces are compact elements, and compact elements are faces with finitely many extreme points. The unit disk D is not compact in $F(D)$. Since every face is the convex hull (join) of all extreme points it contains, $F(C)$ is an algebraic lattice.

References

[1] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, 1996.