# proof of criterion for convexity II

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| Canonical name | ProofOfCriterionForConvexityII |
| Date of creation | 2013-03-22 18:25:25 |
| Last modified on | 2013-03-22 18:25:25 |
| Owner | yesitis (13730) |
| Last modified by | yesitis (13730) |
| Numerical id | 5 |
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| Entry type | Proof |
| Classification | msc 52A41 |
| Classification | msc 26A51 |
| Classification | msc 26B25 |

If $f$ was not convex, then there was a point $\xi \in (a, b)$ such that $f(\xi) > h(x) = \frac{f(v)-f(u)}{v-u}(x-u) + f(u)$ for some $u < v$ in $(a, b)$. Since $f$ is continuous, there would be a neighborhood $(\xi-\delta, \xi+\delta), \delta > 0$, of $\xi$ such that $f(x) > h(x)$ for all $x$ in this neighborhood. (I.e., $f(x)$ was "above" the line segment joining $f(u)$ and $f(v)$.) Let $s = \xi - \delta, t = \xi + \delta$.

Using the two points $A = (s, f(s)), B = (t, f(t))$, we construct another line segment $\overline{AB}$ whose equation is given by $g(x) = \frac{f(s)-f(t)}{2\delta}(x-s) + f(s)$; we have $f(x) > g(x)$ for $x \in (s, t)$. In particular,

$$f(\xi) = f\left(\frac{s+t}{2}\right) > g(\xi) = \frac{f(s)+f(t)}{2}. \tag{1}$$

(One easily verifies $g(\xi) = (f(s) + f(t))/2$.) This contradicts hypothesis.

Note that we have tacitly used the fact that $h(x) = \lambda f(v) + (1-\lambda) f(u)$ for some $\lambda$ and $g(x) = \lambda f(s) + (1-\lambda) f(t)$ for some $\lambda$.