if A is convex and f linear then $f(A)$ and $f^{-1}(A)$ are convex

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Proposition 1. *Suppose X, Y are vector spaces over $\mathbb{R}$ (or $\mathbb{C}$), and suppose $f\colon X \to Y$ is a linear map.*

1. *If $A \subseteq X$ is convex, then $f(A)$ is convex.*

2. *If $B \subseteq Y$ is convex, then $f^{-1}(B)$ is convex, where f^{-1} is the inverse image.*

Proof. For the first claim, suppose $y, y' \in f(A)$, say, $y = f(x)$ and $y' = f(x')$ for $x, x' \in A$, and suppose $\lambda \in (0, 1)$. Then

$$\begin{aligned} \lambda y + (1-\lambda)y' &= \lambda f(x) + (1-\lambda) f(x') \\ &= f(\lambda x + (1-\lambda)x'), \end{aligned}$$

so $\lambda y + (1-\lambda)y' \in f(A)$ as A is convex.

For the second claim, let us first recall that $x \in f^{-1}(B)$ if and only if $f(x) \in B$. Then, if $x, x' \in f^{-1}(B)$, and $\lambda \in (0, 1)$, we have

$$f(\lambda x + (1-\lambda)x') = \lambda f(x) + (1-\lambda) f(x').$$

As B is convex, the right hand side belongs to B, and $\lambda x + (1-\lambda)x' \in f^{-1}(B)$. $\square$