recession cone

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Defines	direction of a convex set

Let C be a convex set in $\mathbb{R}^n$. If C is bounded, then for any $x \in C$, any ray emanating from x will eventually "exit" C (that is, there is a point z on the ray such that $z \notin C$). If C is unbounded, however, then there exists a point $x \in C$, and a ray ρ emanating from x such that $\rho \subseteq C$. A *direction* d in C is a point in $\mathbb{R}^n$ such that for any $x \in C$, the ray $\{x + rd \mid r \geq 0\}$ is also in C (a subset of C).

The *recession cone* of C is the set of all directions in C, and is denoted by denoted by 0^+C. In other words,

$$0^+C = \{d \mid x + rd \in C,\ \forall x \in C,\ \forall r \geq 0\}.$$

If a convex set C is bounded, then the recession cone of C is pretty useless; it is $\{0\}$. The converse is not true, as illustrated by the convex set

$$C = \{(x, y) \mid 0 \leq x < 1,\ y \geq 1\} \cup \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$

Clearly, C is not bounded but $0^+C = \{0\}$. However, if the additional condition that C is closed is imposed, then we recover the converse.

Here are some other examples of recession cones of unbounded convex sets:

- If $C = \{(x, y) \mid |x| \leq y\}$, then $0^+C = C$.
- If $C = \{(x, y) \mid |x| < y\}$, then $0^+C = \overline{C}$, the closure of C.
- If $C = \{(x, y) \mid |x|^n \leq y, n > 1\}$, then $0^+C = \{(0, y) \mid y \geq 0\}$.

Remark. The recession cone of a convex set is convex, and, if the convex set is closed, its recession cone is closed as well.