proof of Carathéodory's theorem

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The convex hull of P consists precisely of the points that can be written as convex combination of finitely many number points in P. Suppose that p is a convex combination of n points in P, for some integer n,

$$p = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$$

where $\alpha_1 + \ldots + \alpha_n = 1$ and $x_1, \ldots, x_n \in P$. If $n \leq d+1$, then it is already in the required form.

If $n > d+1$, the $n-1$ points $x_2 - x_1$, $x_3 - x_1, \ldots, x_n - x_1$ are linearly dependent. Let β_i, $i = 2, 3, \ldots, n$, be real numbers, which are not all zero, such that

$$\sum_{i=2}^{n} \beta_i (x_i - x_1) = 0.$$

So, there are n constants $\gamma_1, \ldots \gamma_n$, not all equal to zero, such that

$$\sum_{i=1}^{n} \gamma_i x_i = 0,$$

and

$$\sum_{i=1}^{n} \gamma_i = 0.$$

Let $\mathcal{I}$ be a subset of indices defined as

$$\{i \in \{1, 2, \ldots, n\} : \gamma_i > 0\}.$$

Since $\sum_{i=1}^{n} \gamma_i = 0$, the subset $\mathcal{I}$ is not empty. Define

$$a = \max_{i \in I} \alpha_i / \gamma_i.$$

Then we have

$$p = \sum_{i=1}^{n} (\alpha_i - a\gamma_i) x_i,$$

which is a convex combination with at least one zero coefficient. Therefore, we can assume that p can be written as a convex combination of $n-1$ points in P, whenever $n > d+1$.

After repeating the above process several times, we can express p as a convex combination of at most $d+1$ points in P.