relative interior

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Let S be a subset of the n-dimensional Euclidean space $\mathbb{R}^n$. The *relative interior* of S is the interior of S considered as a subset of its affine hull $\mathrm{Aff}(S)$, and is denoted by $\mathrm{ri}(S)$.

The difference between the interior and the relative interior of S can be illustrated in the following two examples. Consider the closed unit square

$$I^2 := \{(x, y, 0) \mid 0 \leq x, y \leq 1\}$$

in $\mathbb{R}^3$. Its interior is $\varnothing$, the empty set. However, its relative interior is

$$\mathrm{ri}(I^2) = \{(x, y, 0) \mid 0 < x, y < 1\},$$

since $\mathrm{Aff}(I^2)$ is the x-y plane $\{(x, y, 0) \mid x, y \in \mathbb{R}\}$. Next, consider the closed unit cube

$$I^3 := \{(x, y, z) \mid 0 \leq x, y, z \leq 1\}$$

in $\mathbb{R}^3$. The interior and the relative interior of I^3 are the same:

$$\mathrm{int}(I^3) = \mathrm{ri}(I^3) = \{(x, y, z) \mid 0 < x, y, z < 1\}.$$

Remarks.

- As another example, the relative interior of a point is the point, whereas the interior of a point is $\varnothing$.
- It is true that if $T \subseteq S$, then $\mathrm{int}(T) \subseteq \mathrm{int}(S)$. However, this is not the case for the relative interior operator ri, as shown by the above two examples: $\varnothing \neq I^2 \subseteq I^3$, but $\mathrm{ri}(I^2) \cap \mathrm{ri}(I^3) = \varnothing$.
- The companion concept of the relative interior of a set S is the *relative boundary* of S: it is the boundary of S in $\mathrm{Aff}(S)$, denoted by $\mathrm{rbd}(S)$. Equivalently, $\mathrm{rbd}(S) = \overline{S} - \mathrm{ri}(S)$, where $\overline{S}$ is the closure of S.
- S is said to be *relatively open* if $S = \mathrm{ri}(S)$.
- All of the definitions above can be generalized to convex sets in a topological vector space.