Radon's lemma

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Every set $A \subset \mathbb{R}^d$ of $d+2$ or more points can be partitioned into two disjoint sets A_1 and A_2 such that the convex hulls of A_1 and A_2 intersect.

Proof. Without loss of generality we assume that the set A consists of exactly $d+2$ points which we number $a_1, a_2, \ldots, a_{d+2}$. Denote by $a_{i,j}$ the j'th component of i'th vector, and write the components in a matrix as

$$M = \begin{bmatrix} a_{1,1} & a_{2,1} & \ldots & a_{d+2,1} \\ a_{1,2} & a_{2,2} & \ldots & a_{d+2,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,d} & a_{2,d} & \ldots & a_{d+2,d} \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Since M has fewer rows than columns, there is a non-zero column vector λ such that $M\lambda = 0$, which is equivalent to the existence of a solution to the system

$$\begin{aligned} 0 &= \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_{d+2} a_{d+2} \qquad (1) \\ 0 &= \lambda_1 + \lambda_2 + \cdots + \lambda_{d+2} \end{aligned}$$

Let A_1 be the set of those a_i for which λ_i is positive, and A_2 be the rest. Set s to be the sum of positive λ_i's. Then by the system (**??**) above

$$\frac{1}{s} \sum_{a_i \in A_1} \lambda_i a_i = \frac{1}{s} \sum_{a_i \in A_2} (-\lambda_i) a_i$$

is a point of intersection of convex hulls of A_1 and A_2. $\square$