arc length of logarithmic curve

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The arc length of the graph of http://planetmath.org/NaturalLogarithm2logarithm function is expressible in closed form (other cases are listed in the entry arc length of parabola). The usual arc length

$$s \;=\; \int_a^b\sqrt{1+(f'(x))^2}\,dx$$

gives, if $0 < a < b$, for $f(x) := \ln x$, $f'(x) = \frac{1}{x}$, the expression

$$s \;=\; \int_a^b\frac{\sqrt{1\!+\!x^2}}{x}\,dx. \tag{1}$$

Here, finding a suitable substitution for integration may be a bit difficult. E.g. $x := \tan t$ leads to

$$\int\frac{\sqrt{1\!+\!x^2}}{x}\,dx \;=\; \int\frac{dt}{\sin t\,\cos^2 t},$$

the substitution $x := \sinh t$ to

$$\int\frac{\sqrt{1\!+\!x^2}}{x}\,dx \;=\; \int\frac{\cosh^2 t}{\sinh t}\,dt,$$

which both seem to require a new substitution. As well the Euler's substitutions (1st and 2nd ones) lead to awkward rational functions as integrands.

But there is the straightforward substitution

$$\sqrt{1\!+\!x^2} \;:=\; t, \quad x \;=\; \sqrt{t^2\!-\!1}, \quad dx \;=\; \frac{t\,dt}{\sqrt{t^2\!-\!1}}$$

yielding

$$\int\frac{\sqrt{1\!+\!x^2}}{x}\,dx \;=\; \int\frac{t^2\,dt}{t^2\!-\!1} \;=\; t+\frac{1}{2}\ln\frac{t\!-\!1}{t\!+\!1}+C \;=\; t-\operatorname{arcoth}t+C$$

(see area functions) and then

$$\int\frac{\sqrt{1\!+\!x^2}}{x}\,dx \;=\; \sqrt{1\!+\!x^2}\!+\!\frac{1}{2}\ln\frac{\sqrt{1\!+\!x^2}-1}{\sqrt{1\!+\!x^2}+1}\!+\!C \;=\; \sqrt{1\!+\!x^2}\!+\!\ln\frac{x}{1+\sqrt{1\!+\!x^2}}\!+\!C.$$

Using this antiderivative, one can obtain the arc length (1). For example, if $a = \sqrt{3}$ and $b = \sqrt{15}$, the result is $s = 2+\ln\frac{3}{\sqrt{5}}$.

As for finding the arc length of the graph of the http://planetmath.org/node/2541exponential function $x \mapsto e^x$, which actually is the same curve as the graph of the inverse function $x \mapsto \ln x$, one may write the expression

$$s \;=\; \int_{\alpha}^{\beta}\sqrt{1+e^{2x}}\,dx. \tag{2}$$

Since here the substitution

$$e^x \;:=\; t, \quad x \;=\; \ln t, \quad dx \;=\; \frac{dt}{t}$$

shows that

$$\int\sqrt{1+e^{2x}}\,dx \;=\; \int\frac{\sqrt{1+t^2}}{t}\,dt,$$

we see that it's really a question of the same task as above. The antiderivative is

$$\int\sqrt{1+e^{2x}}\,dx \;=\; \sqrt{1+e^{2x}}-\operatorname{arsinh}e^{-x}+C \;=\; \sqrt{1+e^{2x}}+\ln\frac{e^x}{1+\sqrt{1+e^{2x}}}+C.$$