formulas in Riemannian geometry

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The aim of this page is to collect frequently used formulas in Riemannian geometry.

Symbol conventions.

- g_{ij} : the components of the metric tensor;
- $\Gamma_{ijk} = \Gamma_{jik}$: the Christoffel symbols;
- $X_i = X^j g_{ij}$, and Y^i : rank 1 tensors;
- $T_{ij} = T_i{}^k g_{jk}$: a rank 2 tensor;
- indices i, j, k, l and subscripted versions thereof: components taken with respect to a local coordinate frame;
- x^i, y^j : systems of local coordinates;
- $\partial_i = \frac{\partial}{\partial x^i}$: local coordinate frame;
- boldfaced symbols: the actual geometric quantity, rather than components; e.g. $\boldsymbol{X} = X^i \partial_i$.

Formulas for the covariant derivative.

$$\begin{aligned}
\partial_k g_{ij} &= \Gamma_{kij} + \Gamma_{kji}, \\
\partial_k g^{ij} &= -(g^{jb}\Gamma_{bk}{}^i + g^{ia}\Gamma_{ak}{}^j), \\
\nabla_k g_{ij} &= 0, \\
\Gamma_{ijk} &= \tfrac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}), \\
\nabla_i X^j &= \partial_i X^j + \Gamma_{ik}{}^j X^k, \\
\nabla_{\boldsymbol{X}} \boldsymbol{Y} &= X^i \, \nabla_i Y^j \, \partial_j, \\
\nabla_i X_j &= \partial_i X_j - \Gamma_{ij}{}^k X_k, \\
\nabla_i T_{jk} &= \partial_i T_{jk} - \Gamma_{ij}{}^l T_{lk} - \Gamma_{ik}{}^l T_{jl}, \\
\nabla_i T^j{}_k &= \partial_i T^j{}_k + \Gamma_{il}{}^j T^l{}_k - \Gamma_{ik}{}^l T^j{}_l.
\end{aligned}$$

Formulas for geodesics

A geodesic is a curve $c\colon I \to M$ satisfying

$$\ddot{c}^i + \Gamma_{jk}{}^i\,\dot{c}^j\dot{c}^k = 0$$