Hessian and inflexion points

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Theorem 1. *Suppose that C is a curve in the real projective plane $\mathbb{RP}^2$ given by a homogeneous equation $F(x, y, z) = 0$ of* `http://planetmath.org/HomogeneousFunction`*degr of homogeneity n. If F has continuous first derivatives in a neighborhood of a point P and the gradient of F is non-zero at P and P is an inflection point of C, then $H(P) = 0$, where H is the Hessian determinant:*

$$H = \begin{vmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial z \partial x} & \frac{\partial^2 F}{\partial z \partial y} & \frac{\partial^2 F}{\partial z^2} \end{vmatrix}$$

Proof. We may choose a system x, y, z of homogenous coordinates such that the point P lies at $(0, 0, 1)$ and the equation of the tangent to C at P is $y = 0$. Using the implicit function theorem, we may conclude that there exists an interval $(-\epsilon, \epsilon)$ and a function $f\colon (-\epsilon, \epsilon) \to \mathbb{R}$ such that $F(t, f(t), 1) = 0$ when $-\epsilon < t < \epsilon$. In other words, the portion of curve near P may be described in non-homogenous coordinates by $y = f(x)$. By the way the coordinates were chosen, $f(0) = 0$ and $f'(0) = 0$. Because P is an inflection point, we also have $f''(0) = 0$.

Differentiating the equation $F(t, f(t), 1) = 0$ twice, we obtain the following:

$$\begin{aligned} 0 = \frac{d}{dt} F(t, f(t), 1) &= \frac{\partial F}{\partial x}(t, f(t), 1) + f'(t) \frac{\partial F}{\partial y}(t, f(t), 1) \\ 0 = \frac{d^2}{dt^2} F(t, f(t), 1) &= \frac{\partial^2 F}{\partial x^2}(t, f(t), 1) + f'(t) \frac{\partial^2 F}{\partial x \partial y}(t, f(t), 1) \\ &\quad + (f'(t))^2 \frac{\partial^2 F}{\partial y^2}(t, f(t), 1) + f''(t) \frac{\partial F}{\partial y}(t, f(t), 1) \end{aligned}$$

We will now put $t = 0$ but, for reasons which will be explained later, we do not yet want to make use of the fact that $f''(0) = 0$:

$$\begin{aligned} &\frac{\partial F}{\partial x}(0, 0, 1) = 0 \\ &\frac{\partial^2 F}{\partial x^2}(0, 0, 1) = -f''(0) \frac{\partial F}{\partial y}(0, 0, 1) \end{aligned}$$

Since F is homogenous, Euler's formula holds:

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z} = nF$$

Taking partial derivatives, we obtain the following:

$$
\begin{aligned}
x\frac{\partial^2 F}{\partial x^2} + y\frac{\partial^2 F}{\partial x \partial y} + z\frac{\partial^2 F}{\partial x \partial z} &= (n-1)\frac{\partial F}{\partial x} \\
x\frac{\partial^2 F}{\partial x \partial y} + y\frac{\partial^2 F}{\partial y^2} + z\frac{\partial^2 F}{\partial y \partial z} &= (n-1)\frac{\partial F}{\partial y} \\
x\frac{\partial^2 F}{\partial x \partial z} + y\frac{\partial^2 F}{\partial y \partial z} + z\frac{\partial^2 F}{\partial z^2} &= (n-1)\frac{\partial F}{\partial z}
\end{aligned}
$$

Evaluating at $(0,0,1)$ and making use of the equations deduced above, we obtain the following:

$$
\begin{aligned}
\frac{\partial F}{\partial z}(0,0,1) &= 0 \\
\frac{\partial^2 F}{\partial x \partial z}(0,0,1) &= 0 \\
\frac{\partial^2 F}{\partial y \partial z}(0,0,1) &= (n-1)\frac{\partial F}{\partial y}(0,0,1) \\
\frac{\partial^2 F}{\partial z^2}(0,0,1) &= 0
\end{aligned}
$$

Making use of these facts, we may now evaluate the determinant:

$$
\begin{aligned}
H(0,0,1) &= \begin{vmatrix} -f''(0)\frac{\partial F}{\partial y}(0,0,1) & \frac{\partial^2 F}{\partial x \partial y}(0,0,1) & 0 \\ \frac{\partial^2 F}{\partial x \partial y}(0,0,1) & \frac{\partial^2 F}{\partial^2 y}(0,0,1) & (n-1)\frac{\partial F}{\partial y}(0,0,1) \\ 0 & (n-1)\frac{\partial F}{\partial y}(0,0,1) & 0 \end{vmatrix} \\
&= (n-1)^2 \left(\frac{\partial F}{\partial y}(0,0,1)\right)^2 f''(0)
\end{aligned}
$$

Since P is an inflection point, $f''(0) = 0$, so we have $H(0,0,1) = 0$. $\square$

Actually, we proved slightly more than what was stated. Because the gradient is assumed not to vanish at P, but $\partial F/\partial x = 0$ and $\partial F/\partial z = 0$ by the way we set up our coordinate system, we must have $\partial F/\partial y \neq 0$. Thus, we see that, if $n \neq 1$, then $H(0,0,1) = 0$ if and only if $f''(0)$. However, note that this does not mean that the Hessian vanishes if and only if P is an inflection point since the definition of inflection point not only requires that $f''(0) = 0$ but that the sign of $f''(t)$ change as t passes through 0.

This result is used quite often in algebraic geometry, where F is a homogenous polynomial. In such a context, it is desirable to keep demonstrations purely algebraic and avoid introducing analysis where possible, so a variant of this result is preferred. The theorem may be restated as follows:

Theorem 2. *Suppose that C is a curve in the real projective plane $\mathbb{RP}^2$ given by an equation $F(x, y, z) = 0$ where F is a homogenous polynomial of degree n. If C is regular at a point P and P is an inflection point of C, then $H(P) = 0$, where H is the Hessian determinant.*

To make our proof purely algebraic, we replace the use of the implicit function theorem to obtain f with an expansion in a formal power series. As above, we choose our x, y, z coordinates so as to place P at $(0, 0, 1)$ and make C tangent to the line $y = 0$ at P. Then, since P is a regular point of C, we may parameterize C by a formal power series $f(t) = \sum_{k=0}^{\infty} c_k t^k$ such that $F(t, f(t), 1) = 0$. Then, if we `http://planetmath.org/DerivativeOfPolynomial`define derivatives algebraically, we may proceed with the rest of the proof exactly as above.