curvature determines the curve

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The http://planetmath.org/CurvaturePlaneCurvecurvature of plane curve determines uniquely the form and of the curve, i.e. one has the

Theorem. If $s \mapsto k(s)$ is a continuous real function, then there exists always plane curves satisfying the equation

$$\kappa \;=\; k(s) \tag{1}$$

between their curvature κ and the arc length s. All these curves are http://planetmath.org/Congr

Proof. Suppose that a curve C satisfies the condition (1). Let the value $s=0$ correspond to the point P_0 of this curve. We choose O as the origin of the plane. The tangent and the normal of C in O are chosen as the x-axis and the y-axis, with positive directions the directions of the tangent and normal vectors of C, respectively. According to (1) and the definition of curvature, the equation

$$\frac{d\theta}{ds} \;=\; k(s)$$

for the direction angle θ of the tangent of C is valid in this coordinate system; the initial condition is

$$\theta \;=\; 0 \quad \text{when} \quad s \;=\; 0.$$

Thus we get

$$\theta \;=\; \int_0^s k(t)dt \;:=\; \vartheta(s), \tag{2}$$

which implies

$$\frac{dx}{ds} \;=\; \cos\vartheta(s), \qquad \frac{dy}{ds} \;=\; \sin\vartheta(s). \tag{3}$$

Since $x=y=0$ when $s=0$, we obtain

$$x \;=\; \int_0^s \cos\vartheta(t)\,dt, \qquad y \;=\; \int_0^s \sin\vartheta(t)\,dt. \tag{4}$$

Thus the function $s \mapsto k(s)$ determines uniquely these functions x and y of the parameter s, and (4) represents a curve with definite form and .

The above reasoning shows that every curve which satisfies (1) is http://planetmath.org/Cong with the curve (4).

We have still to show that the curve (4) satisfies the condition (1). By http://planetmath.org/HigherOrderDerivativesdifferentiating the equations (4) we get the equations (3), which imply $(\frac{dx}{ds})^2+(\frac{dy}{ds})^2 = 1$, or $ds^2 = dx^2+dy^2$ which means that the parameter s represents the arc length of the curve (4), counted from the origin. Differentiating (3) we get, because $\vartheta'(s) = k(s)$ by (2),

$$\frac{d^2x}{ds^2} = -k(s)\sin\vartheta(s), \qquad \frac{d^2y}{ds^2} = k(s)\cos\vartheta(s). \tag{5}$$

The equations (3) and (5) then yield

$$\frac{dx}{ds}\frac{d^2y}{ds^2}-\frac{dy}{ds}\frac{d^2x}{ds^2} = k(s),$$

i.e. the curvature of (4), according the http://planetmath.org/CurvaturePlaneCurveparent entry, satisfies

$$\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix} = k(s).$$

Thus the proof is settled.

References

[1] Ernst Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset I.* WSOY. Helsinki (1950).