Hamilton equations

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The Hamilton equations are a formulation of the equations of motion in classical mechanics.

Local formulation

Suppose $U \subseteq \mathbb{R}^n$ is an open set, suppose I is an interval (representing time), and $H\colon U \times \mathbb{R}^n \times I \to \mathbb{R}$ is a smooth function. Then the equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j}(q(t), p(t), t), \tag{1}$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}(q(t), p(t), t), \tag{2}$$

are the *Hamilton equations* for the curve

$$(q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)\colon I \to U \times \mathbb{R}^n.$$

Such a solution is called a *bicharacteristic*, and H is called a *Hamiltonian function*. Here we use classical notation; the q_i's represent the location of the particles, the p_i's represent the momenta of the particles.

Global formulation

Suppose P is a symplectic manifold with symplectic form ω and that $H\colon P \to \mathbb{R}$ is a smooth function. Then X_H, the Hamiltonian vector field corresponding to H is determined by

$$dH = \omega(X_H, \cdot).$$

The most common case is when P is the cotangent bundle of a manifold Q equipped with the canonical symplectic form $\omega = -d\alpha$, where α is the `http://planetmath.org/Poincare1Form`Poincaré 1-form. (Note that other authors may have different sign convention.) Then Hamilton's equations are the equations for the flow of the vector field X_H. Given a system of coordinates $x^1, \dots x^{2n}$ on the manifold P, they can be written as follows:

$$\dot{x}^i = (X_H)^i(x_1, \dots x_{2n}, t)$$

The relation with the former definition is that in canonical local coordinates (q_i, p_j) for T^*Q, the flow of X_H is determined by equations (**??**)-(**??**).

Also, the following terminology is frequently encountered — the manifold P is known as the phase space, the manifold Q is known as the configuration space, and the product $P \times \mathbb{R}$ is known as state space.