total differential

Canonical name	TotalDifferential
Date of creation	2013-03-22 19:11:24
Last modified on	2013-03-22 19:11:24
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	10
Author	pahio (2872)
Entry type	Definition
Classification	msc 53A04
Classification	msc 26B05
Classification	msc 01A45
Synonym	exact differential
Synonym	differential
Related topic	ExactDifferentialForm
Related topic	ExactDifferentialEquation
Related topic	Differential
Related topic	DifferntiableFunction
Defines	differentiable

There is the generalisation of the theorem in http://planetmath.org/Differentialthe parent entry concerning the real functions of several variables; here we formulate it for three variables:

Theorem. Suppose that S is a ball in $\mathbb{R}^3$, the function $f\colon S \to \mathbb{R}$ is continuous and has partial derivatives $f'_x,\, f'_y,\, f'_z$ in S and the partial derivatives are continuous in a point $(x,\, y,\, z)$ of S. Then the increment

$$\Delta f \;:=\; f(x{+}\Delta x,\, y{+}\Delta y,\, z{+}\Delta z) - f(x,\, y,\, z),$$

which f gets when one moves from $(x,\, y,\, z)$ to another point $(x{+}\Delta x,\, y{+}\Delta y,\, z{+}\Delta z)$ of S, can be split into two parts as follows:

$$\Delta f \;=\; [f'_x(x,\, y,\, z)\Delta x + f'_y(x,\, y,\, z)\Delta y + f'_z(x,\, y,\, z)\Delta z] + \langle\varrho\rangle\varrho. \qquad (1)$$

Here, $\varrho := \sqrt{\Delta x^2{+}\Delta y^2{+}\Delta z^2}$ and $\langle\varrho\rangle$ is a quantity tending to 0 along with ϱ.

The former part of Δx is called the (*total*) *differential* or the *exact differential* of the function f in the point $(x,\, y,\, z)$ and it is denoted by $df(x,\, y,\, z)$ of briefly df. In the special case $f(x,\, y,\, z) \equiv x$, we see that $df = \Delta x$ and thus $\Delta x = dx$; similarly $\Delta y = dy$ and $\Delta z = dz$. Accordingly, we obtain for the general case the more consistent notation

$$df \;=\; f'_x(x,\, y,\, z)dx + f'_y(x,\, y,\, z)dy + f'_z(x,\, y,\, z)dz, \qquad (2)$$

where $dx,\, dy,\, dz$ may be thought as independent variables.

We now assume conversely that the increment of a function f in $\mathbb{R}^3$ can be split into two parts as follows:

$$f(x{+}\Delta x,\, y{+}\Delta y,\, z{+}\Delta z) - f(x,\, y,\, z) \;=\; [A\Delta x + B\Delta y + C\Delta z] + \langle\varrho\rangle\varrho \qquad (3)$$

where the coefficients $A,\, B,\, C$ are independent on the quantities $\Delta x,\, \Delta y,\, \Delta z$ and $\varrho,\, \langle\varrho\rangle$ are as in the above theorem. Then one can infer that the partial derivatives $f'_x,\, f'_y,\, f'_z$ exist in the point $(x,\, y,\, z)$ and have the values $A,\, B,\, C$, respectively. In fact, if we choose $\Delta y = \Delta z = 0$, then $\varrho = |\Delta x|$ whence (3) attains the form

$$f(x{+}\Delta x,\, y{+}\Delta y,\, z{+}\Delta z) - f(x,\, y,\, z) \;=\; A\Delta x + \langle\Delta x\rangle\Delta x$$

and therefore

$$A = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,\, y+\Delta y,\, z+\Delta z) - f(x,\, y,\, z)}{\Delta x} = f'_x(x,\, y,\, z).$$

Similarly we see the values of f'_y and f'_z.

The last consideration showed the uniqueness of the total differential.

Definition. A function f in $\mathbb{R}^3$, satisfying the conditions of the above theorem is said to be *differentiable* in the point $(x,\, y,\, z)$.

Remark. The differentiability of a function f of two variables in the point $(x,\, y)$ means that the surface $z = f(x,\, y)$ has a tangent plane in this point.

References

[1] Ernst Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset II.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1932).