Hodge star operator

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Defines	hodge star operator

Let V be a n-dimensional (n finite) vector space with inner product g. The *Hodge star operator* (denoted by $*$) is a linear operator mapping http://planetmath.org/node/3050p-forms on V to $(n-p)$-forms, i.e.,

$$*:\Omega^p(V)\to\Omega^{n-p}(V).$$

In terms of a basis $\{e^1,\dots,e^n\}$ for V and the corresponding dual basis $\{e_1,\dots,e_n\}$ for V^* (the star used to denote the dual space is not to be confused with the Hodge star!), with the inner product being expressed in terms of components as $g=\sum_{i,j=1}^n g_{ij}e^i\otimes e^j$, the $*$-operator is defined as the linear operator that maps the basis elements of $\Omega^p(V)$ as

$$*(e^{i_1}\wedge\cdots\wedge e^{i_p})=\frac{\sqrt{|g|}}{(n-p)!}g^{i_1l_1}\cdots g^{i_pl_p}\varepsilon_{l_1\cdots l_p\,l_{p+1}\cdots l_n}e^{l_{p+1}}\wedge\cdots\wedge e^{l_n}.$$

Here, $|g|=\det g_{ij}$, and ε is the Levi-Civita permutation symbol

This operator may be defined in a coordinate-free manner by the condition

$$u\wedge *v=g(u,v)\ \mathbf{Vol}(g)$$

where the notation $g(u,v)$ denotes the inner product on p-forms (in coordinates, $g(u,v)=g_{i_1j_1}\cdots g_{i_pj_p}u^{i_1\dots i_p}v^{j_1\dots j_p}$) and $\mathbf{Vol}(g)$ is the unit volume form associated to the metric. (in coordinates, $\mathbf{Vol}(g)=\sqrt{\det(g)}e^1\wedge\cdots\wedge e^n$)

Generally $**=(-1)^{p(n-p)}\,\mathrm{id}$, where id is the identity operator in $\Omega^p(V)$. In three dimensions, $**=\mathrm{id}$ for all $p=0,\dots,3$. On $\mathbb{R}^3$ with Cartesian coordinates, the metric tensor is $g=dx\otimes dx+dy\otimes dy+dz\otimes dz$, and the Hodge star operator is

$$*dx=dy\wedge dz,\qquad *dy=dz\wedge dx,\qquad *dz=dx\wedge dy.$$

The Hodge star operation occurs most frequently in differential geometry in the case where M^n is a n-dimensional orientable manifold with a Riemannian (or pseudo-Riemannian) tensor g and V is a cotangent vector space of M^n. Also, one can extend this notion to antisymmetric tensor fields by computing Hodge star pointwise.