Lie bracket

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The Lie bracket is an anticommutative, bilinear, first order differential operator on vector fields. It may be defined either in terms of local coordinates or in a global, coordinate-free fashion. Though both defintions are prevalent, it is perhaps easier to formulate the Lie Bracket without the use of coordinates at all, as a commutator:

Definition (Global, coordinate-free) Suppose X and Y are vector fields on a smooth manifold M. Regarding these vector fields as operators on functions, the Lie bracket is their commutator:

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Definition (Local coordinates) Suppose X and Y are vector fields on a smooth n-dimensional manifold M, suppose $(x^1, \ldots, x^n)$ are local coordinates around some point $x \in M$, and suppose that in these local coordinates

$$\begin{aligned} X(x) &= X^i(x) \frac{\partial}{\partial x^i}\Big|_x, \\ Y(x) &= Y^i(x) \frac{\partial}{\partial x^i}\Big|_x. \end{aligned}$$

Then the *Lie bracket* of the above vector fields is the locally defined vector field

$$[X, Y](x) = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j}\Big|_x - Y^i \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j}\Big|_x.$$

(The Einstein summation convention employed in the above equations — repeated indices are to be summed from the range 1 to n.)

Properties

Suppose X, Y, Z are smooth vector fields on a smooth manifold M.

- $[X, Y] = \mathcal{L}_X Y$ where $\mathcal{L}_X Y$ is the Lie derivative.
- $[\cdot, \cdot]$ is anti-symmetric and bi-linear.
- Vector fields on M with the Lie bracket is a Lie algebra. That is to say, the Lie bracket satisfies the Jacobi identity:

  $$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

- The Lie bracket is covariant with respect to changes of coordinates.