geodesic

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Defines	focal point
Defines	minimizing geodesic
Defines	geodesic curve

Let M be a differentiable manifold (at least two times differentiable) with affine connection ∇. The solution to the equation

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

defined in the interval $[0, a]$, is called a *geodesic* or a *geodesic curve*. It can be shown that if ∇ is a Levi-Civita connection and a is 'small enough', then the curve γ is the shortest possible curve between the points $\gamma(0)$ and $\gamma(a)$, and is often referred to as a *minimizing geodesic* between these points.

Conversely, any curve which minimizes the between two arbitrary points in a manifold, is a geodesic.

examples of geodesics includes straight lines in Euclidean space ($\mathbb{R}^n$) and great circles on spheres (such as the equator of earth). The latter of which is not minimizing if the geodesic from the point p is extended beyond its antipodal point. This example also points out to us that between any two points there may be more than one geodesic. In fact, between a point and its antipodal point on the sphere, there are an infinite number of geodesics. Given a p, it is also a property for a point q (known as a *focal point* of p) where different geodesics issuing from p intersects, to be the point where any given geodesic from p ceases to be minimizing.

Coordinates In coordinates the equation is given by the system

$$\frac{d^2x_k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt} = 0 \qquad 1 \le k \le n$$

where Γ^k_{ij} is the Christoffel symbols (see entry about connection), t is the parameter of the curve and $\{x_1, \ldots, x_n\}$ are coordinates on M.

The formula follows since if $\dot{\gamma} = \sum_i \frac{dx_i}{dt}\partial_{x_i}$, where $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$ are the corresponding coordinate vectors, we have

$$\begin{aligned}
\nabla_{\dot{\gamma}}\dot{\gamma} &= \nabla_{\sum_i \frac{dx_i}{dt}\partial_{x_i}} \sum_j \frac{dx_j}{dt}\partial_{x_j} \\
&= \sum_k \dot{\gamma}\left(\frac{dx_k}{dt}\right)\partial_{x_k} + \sum_{i,j} \frac{dx_j}{dt}\frac{dx_i}{dt}\nabla_{\partial_{x_i}}\partial_{x_j} \\
&= \sum_k \left(\frac{d^2x_k}{dt^2} + \sum_{i,j} \frac{dx_i}{dt}\frac{dx_j}{dt}\Gamma^k_{ij}\right)\partial_{x_k}.
\end{aligned}$$

Metric spaces A geodesic in a metric space (X, d) is simply a continuous $f : [0, a] \to X$ such that the `http://planetmath.org/LengthOfCurveInAMetricSpace`length of f is a. Of course, the may be infinite. A geodesic metric space is a metric space where the distance between two points may be realized by a geodesic.