# pullback of a $k$-form

| | |
|---|---|
| Canonical name | PullbackOfAKform |
| Date of creation | 2013-03-22 14:00:34 |
| Last modified on | 2013-03-22 14:00:34 |
| Owner | bwebste (988) |
| Last modified by | bwebste (988) |
| Numerical id | 7 |
| Author | bwebste (988) |
| Entry type | Definition |
| Classification | msc 53-00 |
| Related topic | Pullback2 |
| Related topic | TangentMap |

If $X$ is a manifold, let $\Omega^k(X)$ be the vector space of $k$-forms on $X$.

**Definition** Suppose $X$ and $Y$ are smooth manifolds, and suppose $f$ is a smooth mapping $f : X \to Y$. Then the **pullback** induced by $f$ is the mapping $f^* : \Omega^k(Y) \to \Omega^k(X)$ defined as follows: If $\omega \in \Omega^k(Y)$, then $f^*(\omega)$ is the $k$-form on $X$ defined by the formula

$$(f^*\omega)_x(X_1, \dots, X_k) = \omega_{f(x)}\big((Df)_x(X_1), \dots, (Df)_x(X_k)\big)$$

where $x \in X$, $X_1, \dots, X_k \in T_x(X)$, and $Df$ is the tangent map $Df : TX \to TY$.

### 0.0.1 Properties

Suppose $X$ and $Y$ are manifolds.

- If $\mathrm{id}_X$ is the identity map on $X$, then $(\mathrm{id}_X)^*$ is the identity map on $\Omega^k(X)$.
- If $X, Y, Z$ are manifolds, and $f, g$ are mappings $f : X \to Y$ and $g : Y \to Z$, then
$$(g \circ f)^* = f^* \circ g^*.$$
- If $f$ is a diffeomorphism $f : X \to Y$, then $f^*$ is a diffeomorphism with inverse
$$(f^{-1})^* = (f^*)^*.$$
- If $f$ is a mapping $f : X \to Y$, and $\omega \in \Omega^k(Y)$, then
$$df^*\omega = f^*d\omega,$$
where $d$ is the exterior derivative.
- Suppose $f$ is a mapping $f : X \to Y$, $\omega \in \Omega^k(Y)$, and $\eta \in \Omega^l(Y)$. Then
$$f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta).$$
- If $g$ is a 0-form on $Y$, that is, $g$ is a real valued function $g : Y \to \mathbb{R}$, and $f$ is a mapping $f : X \to Y$, then $f^*(g) = f \circ g$.
- Suppose $U$ is a submanifold (or an open set) in an manifold $X$, and $\iota : U \hookrightarrow X$ is the inclusion mapping. Then $\iota^*$ restricts $k$-forms on $X$ to $k$-forms on $U$.