evolute

Canonical name	Evolute
Date of creation	2013-03-22 17:35:06
Last modified on	2013-03-22 17:35:06
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	8
Author	pahio (2872)
Entry type	Topic
Classification	msc 53A04
Related topic	ConditionOfOrthogonality
Related topic	ArcLength
Related topic	BolzanosTheorem
Related topic	SubstitutionNotation
Defines	evolute

The locus of the center of curvature of a plane curve is called the *evolute* of this curve.

The coordinates of the center of curvature belonging to the point $P = (x,\, y)$ of the curve γ are

$$\xi \;=\; x-\varrho\,\sin\alpha, \qquad \eta \;=\; y+\varrho\,\cos\alpha, \tag{1}$$

where ϱ is the radius of curvature in P and α is the slope angle of the tangent line of the curve in P. So (1) may be regarded as the equations of the evolute of γ.

If the plane curve is given in the parametric form $x = x(t),\;\; y = y(t)$, the corresponding parametric equations of the evolute are

$$\xi \;=\; x-\frac{(x'^2+y'^2)y'}{x'y''-x''y'}, \qquad \eta \;=\; y+\frac{(x'^2+y'^2)x'}{x'y''-x''y'}.$$

In the spexial case that the curve is given in the form $y = y(x)$ these equations can be written

$$\xi \;=\; x-\frac{(1+y'^2)y'}{y''}, \qquad \eta \;=\; y+\frac{1+y'^2}{y''}.$$

For examining the properties of the evolute we choose for parameter the arc length s, measured from a certain point of the curve; then in (1) the quantities x, y, ϱ, α and thus ξ and η are functions of s. We assume that all needed derivatives exist and are continuous.

Differentiating (1) with respect to s, we obtain

$$\frac{d\xi}{ds} \;=\; \frac{dx}{ds}-\varrho\frac{d\alpha}{ds}\cos\alpha-\frac{d\varrho}{ds}\sin\alpha, \qquad \frac{d\eta}{ds} \;=\; \frac{dy}{ds}-\varrho\frac{d\alpha}{ds}\sin\alpha+\frac{d\varrho}{ds}\cos\alpha,$$

and recalling that $\frac{dx}{ds} = \cos\alpha$, $\frac{dy}{ds} = \sin\alpha$ and $\varrho\frac{d\alpha}{ds} = 1$ it yields

$$\frac{d\xi}{ds} \;=\; -\frac{d\varrho}{ds}\sin\alpha, \qquad \frac{d\eta}{ds} \;=\; \frac{d\varrho}{ds}\cos\alpha. \tag{2}$$

If $\frac{d\varrho}{ds} \neq 0$ in the point $(x,\, y)$ of γ, the derivatives $\frac{d\xi}{ds}$ and $\frac{d\eta}{ds}$ do not vanish simultaneously, and so the evolute has in the corresponding point $(\xi,\, \eta)$ a tangent line with the slope

$$\frac{d\eta}{ds}:\frac{d\xi}{ds} \;=\; -\frac{1}{\tan\alpha}.$$

Since the of this is the slope of the normal line of the given curve γ, we have the

Theorem 1. The normal line of the curve in a point (x, y), where $\dfrac{d\varrho}{ds} \neq 0$, is the tangent line of the evolute, having as tangency point the corresponding center of curvature (ξ, η). Thus the evolute is the envelope of the normal lines of the curve.

We shall calculate the arc length σ of the evolute corresponding the arc of the curve γ which is passed through when the parameter s grows from s_1 to s_2; we assume that ϱ and $\frac{d\varrho}{ds}$ are then continuous and distinct from zero. According the arc length formula,

$$\sigma = \int_{s_1}^{s_2} \sqrt{\left(\frac{d\xi}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2}\, ds.$$

Using the equations (2) and the fact that the sign of $\frac{d\varrho}{ds}$ does not change, we can write

$$\sigma = \int_{s_1}^{s_2} \sqrt{\left(\frac{d\varrho}{ds}\right)^2}\, ds = \int_{s_1}^{s_2} \left|\frac{d\varrho}{ds}\right| ds = \left|\int_{s_1}^{s_2} \frac{d\varrho}{ds}\, ds\right| = \left| \Big/_{s_1}^{s_2} \varrho \right| = |\varrho_2 - \varrho_1|,$$

where ϱ_1 and ϱ_2 are the corresponding of γ. We have proved the

Theorem 2. The of an arc of the evolute is equal to the difference of the of the given curve touching the arc of the evolute in its end points, provided that ϱ and $\frac{d\varrho}{ds}$ are continuous and do not change their sign on the arc of the curve.

References

[1] Ernst Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset I.* Toinen painos. WSOY, Helsinki (1950).