differential operator

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Roughly speaking, a *differential operator* is a mapping, typically understood to be linear, that transforms a function into another function by means of partial derivatives and multiplication by other functions.

On $\mathbb{R}^n$, a differential operator is commonly understood to be a linear transformation of $\mathcal{C}^\infty(\mathbb{R}^n)$ having the form

$$f \mapsto \sum_I a^I f_I, \quad f \in \mathcal{C}^\infty(\mathbb{R}^n),$$

where the sum is taken over a finite number of multi-indices $I = (i^1, \ldots, i^n) \in \mathbb{N}^n$, where $a^I \in \mathcal{C}^\infty(\mathbb{R}^n)$, and where f_I denotes a partial derivative of f taken i_1 times with respect to the first variable, i_2 times with respect to the second variable, etc. The *order* of the operator is the maximum number of derivatives taken in the above formula, i.e. the maximum of $i_1 + \ldots + i_n$ taken over all the I involved in the above summation.

On a $\mathcal{C}^\infty$ manifold M, a differential operator is commonly understood to be a linear transformation of $\mathcal{C}^\infty(M)$ having the above form relative to some system of coordinates. Alternatively, one can equip $\mathcal{C}^\infty(M)$ with the limit-order topology, and define a differential operator as a continuous transformation of $\mathcal{C}^\infty(M)$.

The order of a differential operator is a more subtle notion on a manifold than on $\mathbb{R}^n$. There are two complications. First, one would like a definition that is independent of any particular system of coordinates. Furthermore, the order of an operator is at best a local concept: it can change from point to point, and indeed be unbounded if the manifold is non-compact. To address these issues, for a differential operator T and $x \in M$, we define $\operatorname{ord}_x(T)$ the order of T at x, to be the smallest $k \in \mathbb{N}$ such that

$$T[f^{k+1}](x) = 0$$

for all $f \in \mathcal{C}^\infty(M)$ such that $f(x) = 0$. For a fixed differential operator T, the function $\operatorname{ord}(T) : M \to \mathbb{N}$ defined by

$$x \mapsto \operatorname{ord}_x(T)$$

is lower semi-continuous, meaning that

$$\operatorname{ord}_y(T) \geq \operatorname{ord}_x(T)$$

for all $y \in M$ sufficiently close to x.

The global order of T is defined to be the maximum of $\text{ord}_x(T)$ taken over all $x \in M$. This maximum may not exist if M is non-compact, in which case one says that the order of T is infinite.

Let us conclude by making two remarks. The notion of a differential operator can be generalized even further by allowing the operator to act on sections of a bundle.

A differential operator T is a local operator, meaning that

$$T[f](x) = T[g](x), \quad f, g \in \mathcal{C}^\infty(M),\ x \in M,$$

if $f \equiv g$ in some neighborhood of x. A theorem, proved by Peetre states that the converse is also true, namely that every local operator is necessarily a differential operator.

References

1. Dieudonné, J.A., *Foundations of modern analysis*

2. Peetre, J. , "Une caractérisation abstraite des opérateurs différentiels", *Math. Scand.*, v. 7, 1959, p. 211