determining envelope

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Theorem. Let c be the parameter of the family $F(x,\, y,\, c) = 0$ of curves and suppose that the function F has the partial derivatives F'_x, F'_y and F'_c in a certain domain of $\mathbb{R}^3$. If the family has an envelope E in this domain, then the coordinates x, y of an arbitrary point of E and the value c of the parameter determining the family member touching E in $(x,\, y)$ satisfy the pair of equations

$$\begin{cases} F(x,\, y,\, c) = 0, \\ F'_c(x,\, y,\, c) = 0. \end{cases} \tag{1}$$

I.e., one may in principle eliminate c from such a pair of equations and obtain the equation of an envelope.

Example 1. Let us determine the envelope of the the family

$$y = Cx + \frac{Ca}{\sqrt{1+C^2}} \tag{2}$$

of lines, with C the parameter (a is a positive constant). Now the pair (1) for the envelope may be written

$$F(x,\, y,\, C) := Cx - y + \frac{Ca}{\sqrt{1+C^2}} = 0, \quad F'_C(x,\, y,\, C) \equiv x + \frac{a}{(1+C^2)\sqrt{1+C^2}} = 0. \tag{3}$$

It's easier to first eliminate x by taking its expression from the second equation and putting it to the first equation. It follows the expression $y = \frac{C^3 a}{(1+C^2)\sqrt{1+C^2}}$, and so we have the parametric presentation

$$x = -\frac{a}{(1+C^2)\sqrt{1+C^2}}, \quad y = \frac{C^3 a}{(1+C^2)\sqrt{1+C^2}}$$

of the envelope. The parameter C can be eliminated from these equations by squaring both equations, then taking cube roots and adding both equations. The result is symmetric equation

$$\sqrt[3]{x^2} + \sqrt[3]{y^2} = \sqrt[3]{a^2},$$

which represents an astroid. But the parametric form tells, that the envelope consists only of the left half of the astroid.

Example 2. What is the envelope of the family

$$y - \frac{1}{2}a^2 = -\frac{1}{4}(x-a)^2, \tag{4}$$

of parabolas, with a the parameter?

With a fixed a, the equation presents a parabola which is `http://planetmath.org/Congruence` to the parabola $y = -\frac{1}{4}x^2$ and the apex of which is $(a,\, \frac{1}{2}a^2)$. When a is changed, the parabola is submitted to a translation such that the apex draws the parabola $y = \frac{1}{2}x^2$.

The pair (1) for the envelope of the parabolas (4) is simply

$$y - \frac{1}{2}a^2 + \frac{1}{4}(x-a)^2 = 0, \quad x = -a,$$

which allows immediately eliminate a, giving

$$y = -\frac{1}{2}x^2. \tag{5}$$

Thus the envelope of the parabolas is a "narrower" parabola. One infers easily, that a parabola (4) touches the envelope (5) in the point $(-a,\, -\frac{1}{2}a^2)$ which is symmetric with the apex of (4) with respect to the origin.

The converse of the above theorem is not true. In fact, we have the

Proposition. The curve

$$x = x(c), \quad y = y(c), \tag{6}$$

given in this parametric form and satisfying the condition (1), is not necessarily the envelope of the family $F(x,\, y,\, c) = 0$ of curves, but may as well be the locus of the special points of these curves, namely in the case that the functions (6) satisfy except (1) also both of the equations

$$F'_x(x,\, y,\, c) = 0, \quad F'_y(x,\, y,\, c) = 0.$$

Examples. Let's look some simple cases illustrating the above proposition.

a) The family $(x-c)^2 - y = 0$ consists of congruent parabolas having their vertices on the x-axis. Differentiating the equation with respect to

c gives $x - c = 0$, and thus the corresponding pair (1) yields the result $x = c$, $y = 0$, i.e. the x-axis, which also is the envelope.

b) In the case of the family $(x-c)^2 - y^3 = 0$ (or $y = \sqrt[3]{(x-c)^2}$) the pair (1) defines again the x-axis, which now isn't the envelope but the locus of the special points (sharp vertices) of the curves.

c) The third family $(x-c)^3 - y^2 = 0$ of the semicubical parabolas also gives from (1) the x-axis, which this time is simultaneously the envelope of the curves and the locus of the special points.