every symplectic manifold has even dimension

Canonical name	EverySymplecticManifoldHasEvenDimension
Date of creation	2013-03-22 15:44:05
Last modified on	2013-03-22 15:44:05
Owner	cvalente (11260)
Last modified by	cvalente (11260)
Numerical id	18
Author	cvalente (11260)
Entry type	Theorem
Classification	msc 53D05
Related topic	AlternatingForm

All we need to prove is that every finite dimensional vector space V with an anti-symmetric non-degenerate linear form ω has an even dimension $2k$. This is only a linear algebra result. In the case of a symplectic manifold V is just the tangent space at a point, and thus its dimension equals the manifold's dimension.

Pick any not null vector $v_0 \in V$. Since ω is non-degenerate $\omega(v_0, \cdot)$ is a not null linear form. Therefore there exists a not null vector u_0 such that $\omega(v_0, u_0) = 1$

Now v_0 and u_0 are linearly independent because if $v_0 = \lambda u_0$ then $\omega(v_0, u_0) = \omega(\lambda u_0, u_0) = \lambda\omega(u_0, u_0) = 0$ (by anti-symmetry).

Let $V_0 = \text{span}\{v_0, u_0\}$. Consider a space V_1 of "orthogonal" elements to V_0 under ω. That is:

$V_1 = \{v_1 \in V : \text{ for all } v \in V_0,\ \omega(v, v_1) = 0\}$

We now prove $V = V_0 \bigoplus V_1$:

- $V_0 \bigcap V_1 = \{0\}$

  Suppose $w \in V_0 \bigcap V_1$ is not null, then it can be written $w = \alpha v_0 + \beta u_0$ because it belongs to V_0. Since it also belongs to V_1 is is "orthogonal" to both v_0 and u_0. That is:

  $\omega(v_0, w) = 0 \implies \beta\omega(v_0, u_0) = 0 \implies \beta = 0$ similarly

  $\omega(u_0, w) = 0 \implies \alpha\omega(u_0, v_0) = 0 \implies \alpha = 0$

  So w must be null.

- $V = V_0 \bigoplus V_1$

  Suppose $w \in V$. Let $\alpha = \omega(v_0, w)$, $\beta = \omega(u_0, w)$, $w_0 = \alpha u_0 - \beta v_0$.

  Then $\omega(v_0, w_0) = \alpha = \omega(v_0, w)$ and $\omega(u_0, w_0) = \beta = \omega(u_0, w)$.

  Considering $w_1 = w - w_0$ we have $w = w_0 + w_1$ (by construction) and $\omega(v_0, w_1) = \omega(v_0, w - w_0) = \omega(v_0, w) - \omega(v_0, w_0) = \omega(v_0, w) - \omega(v_0, w) = 0$ and similarly for $\omega(u_0, w_1)$

  So $w_1 \in V_1$, $w_0 \in V_0$ and $w = w_0 + w_1$ and thus $V = V_0 \bigoplus V_1$

So the matrix representation of ω is block-diagonal in $V_0 \bigoplus V_1$ and a restriction anti-symmetric bilinear for of ω to V_1 exists.

If V_1 is not null we can repeat the procedure with the restriction. Since $\dim(V) = \dim(V_0) + \dim(V_1)$ and V is finite dimensional the procedure must stop at a finite step.

At the end we get a decomposition $V = \bigoplus_{i=0}^{k-1} V_i$, where $\dim(V_i) = 2$ and $\dim(V) = 2k$ is even.