contact manifold

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Defines	contact structure
Defines	contact form
Defines	contactomorphism

Let M be a smooth manifold and α a one form on M. Then α is a *contact form* on M if

1. for each point $m \in M$, $\alpha_m \neq 0$ and
2. the restriction $d\alpha_m|_{\ker \alpha_m}$ of the differential of α is nondegenerate.

Condition 1 ensures that $\xi = \ker \alpha$ is a subbundle of the vector bundle TM. Condition 2 equivalently says $d\alpha$ is a symplectic structure on the vector bundle $\xi \to M$. A *contact structure* ξ on a manifold M is a subbundle of TM so that for each $m \in M$, there is a contact form α defined on some neighborhood of m so that $\xi = \ker \alpha$. A co-oriented contact structure is a subbundle of TM of the form $\xi = \ker \alpha$ for some globally defined contact form α.

A (co-oriented) *contact manifold* is a pair (M, ξ) where M is a manifold and ξ is a (co-oriented) contact structure. Note, symplectic linear algebra implies that $\dim M$ is odd. If $\dim M = 2n + 1$ for some positive integer n, then a one form α is a contact form if and only if $\alpha \wedge (d\alpha)^n$ is everywhere nonzero.

Suppose now that $(M_1, \xi_1 = \ker \alpha_1)$ and $(M_2, \xi_2 = \ker \alpha_2)$ are co-oriented contact manifolds. A diffeomorphism $\phi : M_1 \to M_2$ is called a *contactomorphism* if the pullback along ϕ of α_2 differs from α_1 by some positive smooth function $f : M_1 \to \mathbb{R}$, that is, $\phi^* \alpha_2 = f \alpha_1$.

Examples:

1. $\mathbb{R}^3$ is a contact manifold with the contact structure induced by the one form $\alpha = dz + xdy$.
2. Denote by $\mathbb{T}^2$ the two-torus $\mathbb{T}^2 = S^1 \times S^1$. Then, $\mathbb{R} \times \mathbb{T}^2$ (with coordinates t, θ_1, θ_2) is a contact manifold with the contact structure induced by $\alpha = \cos t\theta_1 + \sin t\theta_2$.