Poisson ring

Canonical name	PoissonRing
Date of creation	2013-03-22 14:46:12
Last modified on	2013-03-22 14:46:12
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	9
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Entry type	Definition
Classification	msc 53D05
Defines	Poisson algebra

A *Poisson ring* A is a commutative ring on which a binary operation $[,]$, known as the Poisson bracket is defined. This operation must satisfy the following identities:

1. $[f, g] = -[g, f]$

2. $[f + g, h] = [f, h] + [g, h]$

3. $[fg, h] = f[g, h] + g[f, h]$

4. $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$

If, in addition, A is an algebra over a field, then we call A a *Poisson algebra*. In this case, we may wish to add the extra requirement

$$[sf, g] = s[f, g]$$

for all scalars s.

Because of properties 2 and 3, for each $g \in A$, the operation ad_g defined as $ad_g(f) = [f, g]$ is a derivation. If the set $\{ad_g | g \in A\}$ generates the set of derivations of A, we say that A is *non-degenerate*.

It can be shown that, if A is non-degenerate and is isomorphic as a commutative ring to the algebra of smooth functions on a manifold M, then M must be a symplectic manifold and $[,]$ is the Poisson bracket defined by the symplectic form.

Many important operations and results of symplectic geometry and Hamiltonian mechanics may be formulated in terms of the Poisson bracket and, hence, apply to Poisson algebras as well. This observation is important in studying the classical limit of quantum mechanics — the non-commutative algebra of operators on a Hilbert space has the Poisson algebra of functions on a symplectic manifold as a singular limit and properties of the non-commutative algebra pass over to corresponding properties of the Poisson algebra.

In addition to their use in mechanics, Poisson algebras are also used in the study of Lie groups.