Poisson bracket

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Let M be a symplectic manifold with symplectic form Ω. The *Poisson bracket* is a bilinear operation on the set of differentiable functions on M. In terms of local Darboux coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$, the Poisson bracket of two functions is defined as follows:

$$[f, g] = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}$$

It can be shown that the value of $[f, g]$ does not depend on the choice of Darboux coordinates. Therefore, the Poisson bracket is a well-defined operation on the symplectic manifold. Also, some authors use a different sign convention — what they call $[f, g]$ is what would be referred to as $-[f, g]$ here.

The Poisson bracket can be defined without reference to a special coordinate system as follows:

$$[f, g] = \Omega^{-1}(df, dg) = \sum_{i=1}^{2n} \Omega^{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

Here Ω^{-1} is the inverse of the symplectic form, and its components in an arbitrary coordinate system are denoted Ω^{ij}.

The Poisson bracket sastisfies several important algebraic identities. It is antisymmetric:

$$[f, g] = -[g, f]$$

It is a derivation:

$$[fg, h] = f[g, h] + g[f, h]$$

It satisfies Jacobi's identitity:

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

The Hamilton equations can be expressed elegantly in terms of the Poisson bracket. If X is a smooth function on M, we can describe the time-evolution of X by the equation

$$\frac{dX}{dt} = [X, H]$$

If X is a smooth function on $\mathbb{R} \times M$, we can describe the time-evolution of X by the more general equation

$$\frac{dX}{dt} = \frac{\partial X}{\partial t} - [X, H]$$