sectional curvature determines Riemann curvature tensor

Canonical name	SectionalCurvatureDeterminesRiemannCurvatureTensor
Date of creation	2013-03-22 15:55:09
Last modified on	2013-03-22 15:55:09
Owner	kerwinhui (11200)
Last modified by	kerwinhui (11200)
Numerical id	10
Author	kerwinhui (11200)
Entry type	Theorem
Classification	msc 53B21
Classification	msc 53B20

Theorem 1. *The sectional curvature operator* $\Pi \mapsto K(\Pi)$ *completely determines the Riemann curvature tensor.*

In fact, a more general result is true. Recall the Riemann $(1,3)$-curvature tensor $R\colon TM \otimes TM \otimes TM \to TM$ satisfies

$$\begin{aligned} (x,y,z,t)+(y,z,x,t)+(z,x,y,t) &= 0 \quad \text{First Bianchi identity} && (1)\\ (x,y,z,t)+(y,x,z,t) &= 0 && (2)\\ (x,y,z,t)-(z,t,x,y) &= 0, && (3) \end{aligned}$$

where $(x,y,z,t) := g(R(x,y,z),t)$, and the sectional curvature is defined by

$$K(\Pi = \operatorname{span}\{x,y\}) = \frac{g(R(x,y,x),y)}{g(x,x)g(y,y)-g(x,y)^2}. \qquad (4)$$

Thus Theorem **??** is implied by

Theorem 2. *Let* V *be a real inner product space, with inner product* $\langle -,- \rangle$. *Let* R *and* R' *be linear maps* $V^{\otimes 3} \to V$. *Suppose* R *and* R' *satisfies*

- *Equations* (**??**),(**??**),(**??**), *and*
- $K(\sigma) = K'(\sigma)$ *for all* 2*-planes* σ, *where* K, K' *are defined by* (**??**) *using* $\langle -,- \rangle$ *in of* $g(-,-)$.

Then $R = R'$.

Write

$$\begin{aligned} (x,y,z,t) &:= \langle R(x,y,z),t \rangle \\ (x,y,z,t)' &:= \langle R'(x,y,z),t \rangle. \end{aligned}$$

Proof of Theorem **??**. We need to prove, for all $x,y,z,t \in V$,

$$(x,y,z,t) = (x,y,z,t)'.$$

From $K = K'$, we get $(x,y,x,y) = (x,y,x,y)'$ for all $x,y \in V$. The first step is to use polarization identity to change this quadratic form (in x) into its associated symmetric bilinear form. Expand $(x+z,y,x+z,y) = (x+z,y,x+z,y)'$ and use (**??**), we get

$$(x,y,x,y)+2(x,y,z,y)+(z,y,z,y) = (x,y,x,y)'+2(x,y,z,y)'+(z,y,z,y)'.$$

So $(x,y,z,y) = (x,y,z,y)'$ for all $x,y,z \in V$.

Unfortunately, the form (x,y,z,t) is not symmetric in y and t, so we need to work harder. Expand $(x, y+t, z, y+t) = (x, y+t, z, y+t)'$, we get

$$(x,y,z,t) + (x,t,z,y) = (x,y,z,t)' + (x,t,z,y)'.$$

Now use (**??**) and (**??**), we get

$$\begin{aligned}(x,y,z,t) - (x,y,z,t)' &= (x,t,z,y)' - (x,t,z,y) \\ &= (z,y,x,t)' - (z,y,x,t) \\ &= (y,z,x,t) - (y,z,x,t)'.\end{aligned}$$

So $(x,y,z,t) - (x,y,z,t)'$ is invariant under cyclic permutation of x,y,z. But the cyclic sum is zero by (**??**). So

$$(x,y,z,t) = (x,y,z,t)' \quad \forall x,y,z,t \in V$$

as desired. $\square$