tangent of hyperbola

Canonical name	TangentOfHyperbola
Date of creation	2013-03-22 19:10:31
Last modified on	2013-03-22 19:10:31
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	12
Author	pahio (2872)
Entry type	Derivation
Classification	msc 53A04
Classification	msc 51N20
Classification	msc 51-00
Related topic	Slope
Related topic	TangentOfConicSection

Let us derive the equation of the tangent line of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1 \tag{1}$$

having $(x_0,\, y_0)$ as the tangency point $(y_0 \neq 0)$.

If $(x_1,\, y_1)$ is another point of the hyperbola $(x_1 \neq x_0)$, the secant line through both points is

$$y-y_0 = \frac{y_1-y_0}{x_1-x_0}(x-x_0). \tag{2}$$

Since both points satisfy the equation (1) of the hyperbola, we have

$$0 = 1–1 = \left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right)-\left(\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}\right) = \frac{(x_1-x_0)(x_1+x_0)}{a^2}-\frac{(y_1-y_0)(y_1+y_0)}{b^2},$$

which implies the proportion equation

$$\frac{y_1-y_0}{x_1-x_0} = \frac{b^2(x_1+x_0)}{a^2(y_1+y_0)}.$$

Thus the equation (2) may be written

$$y-y_0 = \frac{b^2(x_1+x_0)}{a^2(y_1+y_0)}(x-x_0). \tag{3}$$

When we let here $x_1 \to x_0,\;\; y_1 \to y_0,$ this changes to the equation of the tangent:

$$y-y_0 = \frac{b^2x_0}{a^2y_0}(x-x_0). \tag{4}$$

A little simplification allows to write it as

$$\frac{x_0x}{a^2}-\frac{y_0y}{b^2} = \frac{x_0^2}{a^2}-\frac{y_0^2}{b^2},$$

i.e.

$$\frac{x_0x}{a^2}-\frac{y_0y}{b^2} = 1. \tag{5}$$

Limiting position of tangent

Putting first $y := 0$ and then $x := 0$ into (5) one obtains the values

$$x = \frac{a^2}{x_0} \quad \text{and} \quad y = -\frac{b^2}{y_0}$$

on which the tangent line intersects the coordinate axes. From these one sees that when the point of tangency unlimitedly moves away from the origin ($x_0 \to \infty,\ y_0 \to \infty$), both intersection points tend to the origin. At the same time, the slope $\frac{b^2x_0}{a^2y_0}$ tends to a certain limit $\frac{b}{a}$, because

$$\frac{y_0}{x_0} = \frac{b}{a}\sqrt{x_0^2-a^2} : x_0 = \frac{b}{a}\sqrt{1-\frac{a^2}{x_0^2}} \longrightarrow \frac{b}{a}.$$

Thus one infers that the limiting position of the tangent line is the `http://planetmath.org/Hyperb` $y = \frac{b}{a}x$ of the hyperbola.

Consequently, one can say the asymptotes of a hyperbola to be whose tangency points are infinitely far.

The tangent (5) halves the angle between the focal radii of the hyperbola drawn from $(x_0,\, y_0)$.