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hyperbolic rotation

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Let $\mathbb{E}$ be the Euclidean plane equipped with the Cartesian coordinate system. Recall that given a circle C centered at the origin O, one can define an "ordinary" rotation R to be a linear transformation that takes any point on C to another point on C. In other words, $R(C) \subseteq C$.

Similarly, given a rectangular hyperbola (the counterpart of a circle) H centered at the origin, we define a *hyperbolic rotation* (with respect to H) as a linear transformation T (on $\mathbb{E}$) such that $T(H) \subseteq H$.

Since a hyperbolic rotation is defined as a linear transformation, let us see what it looks like in matrix form. We start with the simple case when a rectangular hyperbola H has the form $xy = r$, where r is a non-negative real number.

Suppose T denotes a hyperbolic rotation such that $T(H) \subseteq H$. Set

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix representation of T, and $xy = x'y' = r$. Solving for a, b, c, d and we get $ad = 1$ and $b = c = 0$. In other words, with respect to rectangular hyperbolas of the form $xy = r$, the matrix representation of a hyperbolic rotation looks like

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

Since the matrix is non-singular, we see that in fact $T(H) = H$.

Now that we know the matrix form of a hyperbolic rotation when the rectangular hyperbolas have the form $xy = r$, it is not hard to solve the general case. Since the two asymptotes of any rectangular hyperbola H are perpendicular, by an appropriate change of bases (ordinary rotation), H can be transformed into a rectangular hyperbola H' whose asymptotes are the x and y axes, so that H' has the algebraic form $xy = r$. As a result, the matrix representation of a hyperbolic rotation T with respect to H has the form

$$P \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} P^{-1}$$

for some $0 \neq a \in \mathbb{R}$ and some orthogonal matrix P. In other words, T is diagonalizable with a and a^{-1} as eigenvalues (T is non-singular as a result).

Below are some simple properties:

- Unlike an ordinary rotation R, where R fixes any circle centered at O, a hyperbolic rotation T fixing one rectangular hyperbola centered at O may not fix another hyperbola of the same kind (as implied by the discussion above).

- Let P be the pencil of all rectangular hyperbolas centered at O. For each $H \in P$, let $[H]$ be the subset of P containing all hyperbolas whose asymptotes are same as the asymptotes for H. If a hyperbolic rotation T fixing H, then $T(H') = H'$ for any $H' \in [H]$.

- $[\cdot]$ defined above partitions P into disjoint subsets. Call each of these subset a sub-pencil. Let A be a sub-pencil of P. Call T fixes A if T *fixes* any element of A. Let $A \neq B$ be sub-pencils of P. Then T fixes A iff T does not fix B.

- Let A, B be sub-pencils of P. Let T, S be hyperbolic rotations such that T fixes A and S fixes B. Then $T \circ S$ is a hyperbolic rotation iff $A = B$.

- In other words, the set of all hyperbolic rotations fixing a sub-pencil is closed under composition. In fact, it is a group.

- Let T be a hyperbolic rotation fixing the hyperbola $xy = r$. Then T fixes its branches (connected components) iff T has positive eigenvalues.

- T preserves area.

- Suppose T fixes the unit hyperbola H. Let $P, Q \in H$. Then T fixes the (measure of) hyperbolic angle between P and Q. In other words, if α is the measure of the hyperbolic angle between P and Q and, by abuse of notation, let $T(\alpha)$ be the measure of the hyperbolic angle between $T(P)$ and $T(Q)$. Then $\alpha = T(\alpha)$.

The definition of a hyperbolic rotation can be generalized into an arbitrary two-dimensional vector space: it is any diagonalizable linear transformation with a pair of eigenvalues a, b such that $ab = 1$.