proof of Poincaré lemma

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Let X be a smooth manifold, and let ω be a closed differential form of degree $k > 0$ on X. For any $x \in X$, there exists a contractible neighbourhood $U \subset X$ of x (i.e. U is homotopy equivalent to a single point), with inclusion map

$$\iota \colon U \hookrightarrow X.$$

To construct such a neighbourhood, take for example an open ball in a coordinate chart around x. Because of the homotopy invariance of de Rham cohomology, the kth de Rham cohomology group $\mathrm{H}^k(U)$ is isomorphic to that of a point; in particular,

$$\mathrm{H}^k(U) = 0 \quad \text{for all } k > 0.$$

Since $d(\iota^*\omega) = \iota^*(d\omega) = 0$, this implies that there exists a $(k-1)$-form η on U such that $d\eta = \iota^*\omega$. In the case where X is a contractible manifold, such an η exists globally since we can choose $U = X$ above.