space curve

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Related topic	Torsion
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Defines	point of inflection
Defines	arclength parameterization
Defines	reparameterization

Kinematic definition. A *parameterized space curve* is a parameterized curve taking values in 3-dimensional Euclidean space. It may be interpreted as the trajectory of a particle moving through space. Analytically, a smooth space curve is represented by a sufficiently differentiable mapping $\gamma : I \to \mathbb{R}^3$, of an interval $I \subset \mathbb{R}$ into 3-dimensional Euclidean space $\mathbb{R}^3$. Equivalently, a parameterized space curve can be considered a 3-vector of functions:

$$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \quad t \in I.$$

Regularity hypotheses. To preclude the possibility of kinks and corners, it is necessary to add the hypothesis that the mapping be `http://planetmath.org/Curve`regular, that is to say that the derivative $\gamma'(t)$ never vanishes. Also, we say that $\gamma(t)$ is a point of inflection if the first and second derivatives $\gamma'(t), \gamma''(t)$ are linearly dependent. Space curves with points of inflection are beyond the scope of this entry. Henceforth we make the assumption that $\gamma(t)$ is both and lacks points of inflection.

Geometric definition. A *space curve*, per se, needs to be conceived of as a subset of $\mathbb{R}^3$ rather than a mapping. Formally, we could define a space curve to be the image of some parameterization $\gamma : I \to \mathbb{R}^3$. A more useful concept, however, is the notion of an *oriented space curve*, a space curve with a specified direction of motion. Formally, an oriented space curve is an equivalence class of parameterized space curves; with $\gamma_1 : I_1 \to \mathbb{R}^3$ and $\gamma_2 : I_2 \to \mathbb{R}^3$ being judged equivalent if there exists a smooth, monotonically increasing reparameterization function $\sigma : I_1 \to I_2$ such that

$$\gamma_1(t) = \gamma_2(\sigma(t)), \quad t \in I_1.$$

Arclength parameterization. We say that $\gamma : I \to \mathbb{R}^3$ is an arclength parameterization of an oriented space curve if

$$\|\gamma'(t)\| = 1, \quad t \in I.$$

With this hypothesis the length of the space curve between points $\gamma(t_2)$ and $\gamma(t_1)$ is just $|t_2 - t_1|$. In other words, the parameter in such a parameterization measures the relative distance along the curve.

Starting with an arbitrary parameterization $\gamma : I \to \mathbb{R}^3$, one can obtain an arclength parameterization by fixing a $t_0 \in I$, setting

$$\sigma(t) = \int_{t_0}^{t} \|\gamma'(x)\| \, dx,$$

and using the inverse function σ^{-1} to reparameterize the curve. In other words,

$$\hat{\gamma}(t) = \gamma(\sigma^{-1}(t))$$

is an arclength parameterization. Thus, every space curve possesses an arclength parameterization, unique up to a choice of additive constant in the arclength parameter.