proof of $d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$ (local coordinates)

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Since this result is local (in other words, the identity holds on the whole manifold if and only if its restriction to every coordinate patch of the manifold holds), it suffices to demonstrate it in a local coordinate system. To do this, we shall compute coordinate expressions for the various terms and verify that the sum of terms on the right-hand side equals the left-hand side.

$$d\alpha(X,Y) = (\alpha_{j,i} - \alpha_{i,j})X^i Y^j = \alpha_{j,i} X^i Y^j - \alpha_{i,j} X^i Y^j$$

$$X(\alpha(Y)) = X^i \partial_i (\alpha_j Y^j) = X^i \alpha_{j,i} Y^j + X^i \alpha_j Y^j{}_{,i}$$

$$Y(\alpha(X)) = Y^j \partial_j (\alpha_i X^i) = Y^j \alpha_{i,j} X^i + Y^j \alpha_i X^i{}_{,j}$$

$$\alpha([X,Y]) = \alpha_i (X^j Y^i{}_{,j} - Y^j X^i{}_{,j}) = \alpha_i X^j Y^i{}_{,j} - \alpha_i Y^j X^i{}_{,j}$$

Upon combining the right-hand sides of the last three equations and cancelling common terms, we obtain

$$X^i \alpha_{j,i} Y^j + X^i \alpha_j Y^j{}_{,i} - Y^j \alpha_{i,j} X^i - \alpha_i X^j Y^i{}_{,j}$$

Upon renaming dummy indices (switching i with j), the second and fourth terms cancel. What remains is exactly the right-hand side of the first equation. Hence, we have

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$$