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Christoffel symbols

Canonical name	ChristoffelSymbols
Date of creation	2013-03-22 15:43:52
Last modified on	2013-03-22 15:43:52
Owner	juanman (12619)
Last modified by	juanman (12619)
Numerical id	24
Author	juanman (12619)
Entry type	Definition
Classification	msc 53B20
Classification	msc 53-01
Synonym	connection coefficients
Related topic	Connection

A vector field in $\mathbb{R}^n$ can be seen as a differentiable (C^∞) map $V\colon \mathbb{R}^n \to \mathbb{R}^n$.

Or as a section $\mathbb{R}^n \xrightarrow{V} T(\mathbb{R}^n)$ where $T\mathbb{R}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$ is the $\mathbb{R}^n$'s trivial tangent bundle obeying $p \mapsto (p, V(p) \in T_p(\mathbb{R}^n))$ with $T_p(\mathbb{R}^n) \equiv \mathbb{R}^n$ being the tangent space at p.

Another viewpoint about tangent vectors is that they are also linear operators called **derivations** and they act over scalars $f\colon \mathbb{R}^n \to \mathbb{R}$ via $p \mapsto Vf|_p = V(p) \cdot \nabla f|_p$.

Let X be one of them and $dX|_p$ its Jacobian matrix evaluated at the point $p \in \mathbb{R}^n$. Then, for any other vector field $Y\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$dX|_p(Y(p))$$

measures how X varies in the direction Y at p.

We have $dX|_p(Y(p)) = (Y(p) \cdot \nabla X^1|_p, ..., Y(p) \cdot \nabla X^n|_p)$, where $X = \sum_s X^s e_s$ in components. Also, it is obvious that $p \mapsto dX|_p(Y(p))$ defines a new vector field in $\mathbb{R}^n$ which is symbolized as

$$D_Y X$$

We can be consider it as a bilinear map

$$D : T(\mathbb{R}^n) \times T(\mathbb{R}^n) \to T(\mathbb{R}^n).$$

$$(X, Y) \mapsto D_X Y$$

Further, it is easy to see that for any scalar $f\colon \mathbb{R}^n \to \mathbb{R}$

1. $D_{fY} X = f D_Y X$
2. $D_Y(fX) = (Yf)X + f D_Y X$
3. $D_X Y - D_Y X = [X, Y]$
4. $X(Y \cdot Z) = D_X Y \cdot Z + X \cdot D_X Z$

Here we have abbreviated (as usual) $Yf = Y \cdot \nabla F$ and the operation $[X, Y]$ is the Lie bracket.

This D is called the **standard connection** of $\mathbb{R}^n$.

Now, let M be a n-dimensional differentiable manifold and let TM be its tangent bundle. The set of differentiable sections $\Gamma(M) = \{X\colon M \to TM\}$

is a differentiable Lie algebra which is endowed with a differentiable inner product $g\colon \Gamma(M)\times\Gamma(M)\to\mathbb{R}$ via

$$g(X,Y)|_p = X(p)\cdot Y(p)$$

in each $T_p(M)\equiv\mathbb{R}^n$.

It is possible construct a bilinear operator ∇

$$\nabla\colon \Gamma(M)\times\Gamma(M)\to\Gamma(M)$$

compatible with g and which satisfies the following properties

1. $\nabla_{fY}X = f\nabla_Y X$
2. $\nabla_Y(fX) = (Yf)X + f\nabla_Y X$
3. $\nabla_X Y - \nabla_Y X = [X,Y]$
4. $Xg(Y,Z) = g(\nabla_X Y, Z) + g(X,\nabla_X Z)$

The **Fundamental Theorem of Riemannian Geometry** establishes that this ∇ exists and it is unique, and it is called the **Levi-Civita connection** for the metric g on M.

Now, if one uses a coordinated patch in M one has a set of n-coordinated vector fields $\partial_1,..,\partial_n$ meaning $\partial_i = \frac{\partial}{\partial u^i}$ being u^i the coordinate functions. These are also dubbed holonomic derivations.

So it makes sense to speak about the derivatives $\nabla_{\partial_i}\partial_j$ and since the ∂_i are tangent which generate at a point $T_p(M)$, then $\nabla_{\partial_i}\partial_j$ is also tangent, so there are $n\times n$ numbers (functions if one varies position) Γ^s_{ij} which enters in the relation

$$\nabla_{\partial_i}\partial_j = \sum_s \Gamma^s_{ij}\partial_s.$$

These coefficients Γ^s_{ij} are called **Christoffel symbols** and an easy calculation shows that

$$\Gamma^k_{ij} = \frac{1}{2}\sum_s g^{ks}[g_{sj,i} + g_{is,j} - g_{ij,s}]$$

where $g_{ij} = g(\partial_i,\partial_j)$, g^{ij} are the entries of the matrix $[g_{ij}]^{-1}$ and $g_{ij,k} = \partial_k(g_{ij})$.

Routinely one can check that under a change of coordinates $u^i \to w^j$ these functions transform as

$$\bar{\Gamma}^i_{kl} = \frac{\partial w^i}{\partial u^m}\frac{\partial u^n}{\partial w^k}\frac{\partial u^p}{\partial w^l}\Gamma^m_{np} + \frac{\partial^2 u^p}{\partial w^k \partial w^l}\frac{\partial w^i}{\partial u^p}$$

here we have used Einstein's sum convention (m, n, p-sums) and the term

$$\frac{\partial^2 u^p}{\partial w^k \partial w_l}\frac{\partial w^i}{\partial u^p}$$

shows that the Γ^i_{kl} are not tensors.

For a proof please see the last part in: `http://planetmath.org/?op=getobj&from=collab&id=`

Connection with base vectors. Let us assume that coordinates u^i are referred to a right-handed orthogonal Cartesian system with attached constant base vectors $\mathbf{e}_i \equiv \mathbf{e}^i$ and coordinates w^j referred to a general curvilinear system attached to a local covariant base vectors $\mathbf{g}_j$ and local contravariant base vectors $\mathbf{g}^k$, both systems embedded in the Euclidean space $\mathbb{R}^n$. We shall also suppose diffeomorphic the transfomation $u^i \mapsto w^j$. Then, by definition

$$\mathbf{g}_j := \frac{\partial u^i}{\partial w^j}\mathbf{e}_i\ , \qquad \mathbf{g}^j := \frac{\partial w^j}{\partial u^i}\mathbf{e}^i\ , \tag{1}$$

and its inverses

$$\mathbf{e}_i = \mathbf{e}^i = \frac{\partial u^i}{\partial w^j}\mathbf{g}^j = \frac{\partial w^j}{\partial u^i}\mathbf{g}_j\ . \tag{2}$$

Let us consider differentiation of base vectors $\mathbf{g}_j$, which may be written from (1),(2)

$$\frac{\partial \mathbf{g}_j}{\partial w^k} = \frac{\partial^2 u^i}{\partial w^j \partial w^k}\mathbf{e}_i = \frac{\partial^2 u^i}{\partial w^j \partial w^k}\frac{\partial u^i}{\partial w^s}\mathbf{g}^s = \frac{\partial^2 u^i}{\partial w^j \partial w^k}\frac{\partial w^s}{\partial u^i}\mathbf{g}_s \equiv \frac{\partial \mathbf{g}_k}{\partial w^j}\ ,$$

and using the Christoffel symbols this becomes

$$\frac{\partial \mathbf{g}_j}{\partial w^k} = \Gamma_{jks}\mathbf{g}^s = \Gamma^r_{jk}\mathbf{g}_r\ , \tag{3}$$

where

$$\Gamma_{jks} = \frac{\partial^2 u^i}{\partial w^j \partial w^k}\frac{\partial u^i}{\partial w^s}\ , \qquad \Gamma^r_{jk} = g^{rs}\Gamma_{jks}\ . \tag{4}$$

Since the transformation of covariant and contravariant metric tensors are given by

$$g_{jk} = \frac{\partial u^i}{\partial w^j}\frac{\partial u^l}{\partial w^k}\delta_{il} \,, \qquad g^{jk} = \frac{\partial w^j}{\partial u^i}\frac{\partial w^k}{\partial u^l}\delta^{il} \,,$$

is easy to see from here that Christoffel symbol Γ_{jks} enjoy the property

$$\Gamma_{jks} = \frac{1}{2}\left(\frac{\partial g_{js}}{\partial w^k} + \frac{\partial g_{ks}}{\partial w^j} - \frac{\partial g_{jk}}{\partial w^s}\right) . \tag{5}$$

In a similar way we find for the derivative of the contravariant base vectors

$$\frac{\partial \mathbf{g}^j}{\partial w^k} = -\Gamma^j_{ks}\mathbf{g}^s \,. \tag{6}$$

Is easy to show the following results:

$$\Gamma_{jks} = \Gamma_{kjs} = \mathbf{g}_s \cdot \frac{\partial \mathbf{g}_k}{\partial w^j} = \mathbf{g}_s \cdot \frac{\partial \mathbf{g}_j}{\partial w^k} \,,$$

$$\Gamma^r_{jk} = \Gamma^r_{kj} = \mathbf{g}^r \cdot \frac{\partial \mathbf{g}_j}{\partial w^k} = \mathbf{g}^r \cdot \frac{\partial \mathbf{g}_k}{\partial w^j} = -\mathbf{g}_j \cdot \frac{\partial \mathbf{g}^r}{\partial w^k} \,,$$

$$\Gamma^i_{ir} = \frac{1}{2}g^{is}(g_{is,r} + g_{rs,i} - g_{ir,s}) = \frac{1}{2}g^{is}g_{is,r} = \frac{1}{2g}\frac{\partial g}{\partial g_{is}}\frac{\partial g_{is}}{\partial w^r} = \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial w^r} \,,$$

$$\Gamma_{jsk} + \Gamma_{ksj} = g_{jk,s} \,,$$

comma denoting differentiation with respect to the curvilinear coordinates w^j and $g = |g_{jk}|$. When the coordinate curves are orthogonal we have the following formulae for the Christoffel symbols: **(repeated indices are not to be summed)**

$$\Gamma_{jks} = 0 \,, \qquad \Gamma^s_{jk} = 0 \,, \qquad (j \neq k \neq s \neq j),$$

$$\Gamma_{iir} = -\frac{1}{2}\frac{\partial g_{ii}}{\partial w^r} \,, \qquad \Gamma^r_{ii} = -\frac{1}{2g_{rr}}\frac{\partial g_{ii}}{\partial w^r} \,, \qquad (r \neq i) \,,$$

$$\Gamma_{iri} = \Gamma_{rii} = \frac{1}{2}\frac{\partial g_{ii}}{\partial w^r} \,, \qquad \Gamma^r_{ri} = \Gamma^r_{ir} = \frac{1}{2g_{rr}}\frac{\partial g_{rr}}{\partial w^i} = \frac{1}{2}\frac{\partial \log g_{rr}}{\partial w^i} \,.$$