Poincaré lemma

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The Poincaré lemma states that every closed differential form is locally http://planetmath.org/ExactDifferentialFormexact.

Theorem. *(Poincaré Lemma) [**?**] Suppose X is a smooth manifold, $\Omega^k(X)$ is the set of smooth differential k-forms on X, and suppose ω is a closed form in $\Omega^k(X)$ for some $k > 0$.*

- *Then for every $x \in X$ there is a neighbourhood $U \subset X$, and a $(k-1)$-form $\eta \in \Omega^{k-1}(U)$, such that*

$$d\eta = \iota^*\omega,$$

  where ι is the inclusion $\iota : U \hookrightarrow X$.

- *If X is contractible, this η exists globally; there exists a $(k-1)$-form $\eta \in \Omega^{k-1}(X)$ such that*

$$d\eta = \omega.$$

Notes

Despite the name, the Poincaré lemma is an extremely important result. For instance, in algebraic topology, the definition of the kth de Rham cohomology group

$$H^k(X) = \frac{\operatorname{Ker}\{d\colon \Omega^k(X) \to \Omega^{k+1}(X)\}}{\operatorname{Im}\{d\colon \Omega^{k-1}(X) \to \Omega^k(X)\}}$$

can be seen as a measure of the degree in which the Poincaré lemma fails. If $H^k(X) = 0$, then every k form is exact, but if $H^k(X)$ is non-zero, then X has a non-trivial topology (or "holes") such that k-forms are not globally exact. For instance, in $X = \mathbb{R}^2 \setminus \{0\}$ with polar coordinates (r, ϕ), the 1-form $\omega = d\phi$ is not globally exact.

References

[1] L. Conlon, *Differentiable Manifolds: A first course*, Birkhäuser, 1993.