proof of closed differential forms on a simple connected domain

Canonical name	ProofOfClosedDifferentialFormsOnASimpleConnectedDomain
Date of creation	2013-03-22 13:32:49
Last modified on	2013-03-22 13:32:49
Owner	paolini (1187)
Last modified by	paolini (1187)
Numerical id	9
Author	paolini (1187)
Entry type	Proof
Classification	msc 53-00
Related topic	SubstitutionNotation

lemma 1. Let γ_0 and γ_1 be two regular homotopic curves in D with the same end-points. Let $\sigma\colon [0,1]\times[0,1] \to D$ be the homotopy between γ_0 and γ_1 i.e.

$$\sigma(0,t) = \gamma_0(t), \qquad \sigma(1,t) = \gamma_1(t).$$

Notice that we may (and shall) suppose that σ is regular too. In fact $\sigma([0,1]\times[0,1])$ is a compact subset of D. Being D open this compact set has positive distance from the boundary ∂D. So we could regularize σ by mollification leaving its image in D.

Let $\omega(x,y) = a(x,y)\,dx + b(x,y)\,dy$ be our closed differential form and let $\sigma(s,t) = (x(s,t), y(s,t))$. Define

$$F(s) = \int_0^1 a(x(s,t),y(s,t))x_t(s,t) + b(x(s,t),y(s,t))y_t(s,t)\,dt;$$

we only have to prove that $F(1) = F(0)$.

We have

$$F'(s) = \frac{d}{ds}\int_0^1 ax_t + by_t\,dt$$

$$= \int_0^1 a_x x_s x_t + a_y y_s x_t + a x_{ts} + b_x x_s y_t + b_y y_s y_t + b y_{ts}\,dt.$$

Notice now that being $a_y = b_x$ we have

$$\frac{d}{dt}\left[ax_s + by_s\right] = a_x x_t x_s + a_y y_t x_s + a x_{st} + b_x x_t y_s + b_y y_t y_s + b y_{st}$$

$$= a_x x_s x_t + b_x x_s y_t + a x_{ts} + a_y y_s x_t + b_y y_s y_t + b y_{ts}$$

hence

$$F'(s) = \int_0^1 \frac{d}{dt}\left[ax_s + by_s\right]\,dt = \left[ax_s + by_s\right]_0^1.$$

Notice, however, that $\sigma(s,0)$ and $\sigma(s,1)$ are constant hence $x_s = 0$ and $y_s = 0$ for $t = 0,1$. So $F'(s) = 0$ for all s and $F(1) = F(0)$. $\square$

Lemma 2. Let us fix a point $(x_0,y_0)\in D$ and define a function $F\colon D\to\mathbb{R}$ by letting $F(x,y)$ be the integral of ω on any curve joining (x_0,y_0) with (x,y). The hypothesis assures that F is well defined. Let $\omega = a(x,y)\,dx + b(x,y)\,dy$. We only have to prove that $\partial F/\partial x = a$ and $\partial F/\partial y = b$.

Let $(x,y)\in D$ and suppose that $h\in\mathbb{R}$ is so small that for all $t\in[0,h]$ also $(x+t,y)\in D$. Consider the increment $F(x+h,y) - F(x,y)$. From the definition of F we know that $F(x+h,y)$ is equal to the integral of ω on a curve which starts from (x_0, y_0) goes to (x,y) and then goes to $(x+h,y)$ along the straight segment $(x+t,y)$ with $t \in [0,h]$. So we understand that

$$F(x+h,y) - F(x,y) = \int_0^h a(x+t,y)dt.$$

For the integral mean value theorem we know that the last integral is equal to $ha(x+\xi,y)$ for some $\xi \in [0,h]$ and hence letting $h \to 0$ we have

$$\frac{F(x+h,y) - F(x,y)}{h} = a(x+\xi,y) \to a(x,y) \quad h \to 0$$

that is $\partial F(x,y)/\partial x = a(x,y)$. With a similar argument (exchange x with y) we prove that also $\partial F/\partial y = b(x,y)$. $\square$

Theorem. Just notice that if D is simply connected, then any two curves in D with the same end points are homotopic. Hence we can apply Lemma 1 and then Lemma 2 to obtain the desired result. $\square$