curve

Canonical name	Curve
Date of creation	2013-03-22 12:54:17
Last modified on	2013-03-22 12:54:17
Owner	rmilson (146)
Last modified by	rmilson (146)
Numerical id	28
Author	rmilson (146)
Entry type	Definition
Classification	msc 53B25
Classification	msc 14H50
Classification	msc 14F35
Classification	msc 51N05
Synonym	parametrized curve
Synonym	parameterized curve
Synonym	path
Synonym	trajectory
Related topic	FundamentalGroup
Related topic	TangentSpace
Related topic	RealTree
Defines	closed curve
Defines	Jordan curve
Defines	regular curve
Defines	simple closed curve
Defines	simple curve
Defines	plane curve
Defines	planar curve
Defines	convex curve
Defines	locally convex curve
Defines	local multiplicity
Defines	globally convex
Defines	global multiplicity

Summary.

The term *curve* is associated with two closely related notions. The first notion is kinematic: a parameterized curve is a function of one real variable taking values in some ambient geometric setting. This variable can be interpreted as time, in which case the function describes the evolution of a moving particle. The second notion is geometric; in this sense a curve is an arc, a 1-dimensional subset of an ambient space. The two notions are related: the image of a parameterized curve describes the trajectory of a moving particle. Conversely, a given arc admits multiple parameterizations. A trajectory can be traversed by moving particles at different speeds.

In algebraic geometry, the term curve is used to describe a 1-dimensional variety relative to the complex numbers or some other ground field. This can be potentially confusing, because a curve over the complex numbers refers to an object which, in conventional geometry, one would refer to as a surface. In particular, a Riemann surface can be regarded as as complex curve.

Kinematic definition

Let $I \subset \mathbb{R}$ be an `http://planetmath.org/Interval`interval of the real line. A parameterized curve is a continuous mapping $\gamma : I \to X$ taking values in a topological space X. We say that γ is a *simple curve* if it has no self-intersections, that is if the mapping γ is injective.

We say that γ is a *closed curve*, or a *`http://planetmath.org/loop`loop* whenever $I = [a, b]$ is a closed interval, and the endpoints are mapped to the same value; $\gamma(a) = \gamma(b)$. Equivalently, a loop may be defined to be a continuous mapping $\gamma \colon \mathbb{S}^1 \to X$ whose domain $\mathbb{S}^1$ is the unit circle. A simple closed curve is often called a *Jordan curve*.

If $X = \mathbb{R}^2$ then γ is called a *plane curve* or *planar curve*.

A smooth closed curve γ in $\mathbb{R}^n$ is *locally* if the local multiplicity of intersection of γ with each hyperplane at of each of the intersection points does not exceed n. The *global multiplicity* is the sum of the local multiplicities. A simple smooth curve in $\mathbb{R}^n$ is called (or *globally*) if the global multiplicity of its intersection with any affine hyperplane is less than or equal to n. An example of a closed convex curve in $\mathbb{R}^{2n}$ is the normalized generalized ellipse:

$$(\sin t, \cos t, \frac{\sin 2t}{2}, \frac{\cos 2t}{2}, \dots, \frac{\sin nt}{n}, \frac{\cos nt}{n}).$$

In odd dimension there are no closed convex curves.

In many instances the ambient space X is a differential manifold, in which case we can speak of differentiable curves. Let I be an open interval, and let $\gamma : I \to X$ be a differentiable curve. For every $t \in I$ can regard the http://planetmath.org/RelatedRatesderivative, $\dot{\gamma}(t)$, as the http://planetmath.org/RelatedRatesvelocity of a moving particle, at time t. The velocity $\dot{\gamma}(t)$ is a http://planetmath.org/TangentSpacetangent vector, which belongs to $T_{\gamma(t)}X$, the tangent space of the manifold X at the point $\gamma(t)$. We say that a differentiable curve $\gamma(t)$ is *regular*, if its velocity, $\dot{\gamma}(t)$, is non-vanishing for all $t \in I$.

It is also quite common to consider curves that take values in $\mathbb{R}^n$. In this case, a parameterized curve can be regarded as a vector-valued function $\vec{\gamma} : I \to \mathbb{R}^n$, that is an n-tuple of functions

$$\vec{\gamma}(t) = \begin{pmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{pmatrix},$$

where $\gamma_i : I \to \mathbb{R}$, $i = 1, \dots, n$ are scalar-valued functions.

Geometric definition.

A (non-singular) curve C, equivalently, an arc, is a connected, 1-dimensional submanifold of a differential manifold X. This means that for every point $p \in C$ there exists an open neighbourhood $U \subset X$ of p and a chart $\alpha : U \to \mathbb{R}^n$ such that

$$\alpha(C \cap U) = \{(t, 0, \dots, 0) \in \mathbb{R}^n : -\epsilon < t < \epsilon\}$$

for some real $\epsilon > 0$.

An alternative, but equivalent definition, describes an arc as the image of a regular parameterized curve. To accomplish this, we need to define the notion of reparameterization. Let $I_1, I_2 \subset \mathbb{R}$ be intervals. A reparameterization is a continuously differentiable function

$$s : I_1 \to I_2$$

whose derivative is never vanishing. Thus, s is either monotone increasing, or monotone decreasing. Two regular, parameterized curves

$$\gamma_i : I_i \to X, \quad i = 1, 2$$

are said to be related by a reparameterization if there exists a reparameterization $s : I_1 \to I_2$ such that

$$\gamma_1 = \gamma_2 \circ s.$$

The inverse of a reparameterization function is also a reparameterization. Likewise, the composition of two parameterizations is again a reparameterization. Thus the reparameterization relation between curves, is in fact an equivalence relation. An arc can now be defined as an equivalence class of regular, simple curves related by reparameterizations. In order to exclude pathological embeddings with wild endpoints we also impose the condition that the arc, as a subset of X, be homeomorphic to an open interval.