local dimension of a locally Euclidean space

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Let X be a locally Euclidean space. Recall that the local dimension of X in $y \in X$ is a natural number $n \in \mathbb{N}$ such that there is an open neighbourhood $U \subseteq X$ of y homeomorphic to $\mathbb{R}^n$. This number is well defined (please, see parent object for more details) and we will denote it by $\dim_y X$.

Proposition. Function $f : X \to \mathbb{N}$ defined by $f(y) = \dim_y X$ is continuous (where on $\mathbb{N}$ we have discrete topology).

Proof. It is enough to show that preimage of a point is open. Assume that $n \in \mathbb{N}$ and $y \in X$ is such that $f(y) = n$. Then there is an open neighbourhood $U \subseteq X$ of y such that U is homeomorphic to $\mathbb{R}^n$. Obviously for any $x \in U$ we have that U is an open neighbourhood of x homeomorphic to $\mathbb{R}^n$. Therefore $f(x) = n$, so $U \subseteq f^{-1}(n)$. Thus (since y was arbitrary) we've shown that around every point in $f^{-1}(n)$ there is an open neighbourhood of that point contained in $f^{-1}(n)$. This shows that $f^{-1}(n)$ is open, which completes the proof. □

Corollary. Assume that X is a connected, locally Euclidean space. Then local dimension is constant, i.e. there exists natural number $n \in \mathbb{N}$ such that for any $y \in X$ we have

$$\dim_y X = n.$$

Proof. Consider the mapping $f : X \to \mathbb{N}$ such that $f(y) = \dim_y X$. Proposition shows that f is continuous. Therefore $f(X)$ is connected, because X is. But $\mathbb{N}$ has discrete topology, so there are no other connected subsets then points. Thus there is $n \in \mathbb{N}$ such that $f(X) = \{n\}$, which completes the proof. □

Remark. Generally, local dimension need not be constant. For example consider $X_1, X_2 \subseteq \mathbb{R}^3$ such that

$$X_1 = \{(x, 0, 0) \mid x \in \mathbb{R}\} \quad X_2 = \{(x, y, 1) \mid x, y \in \mathbb{R}\}.$$

One can easily show that $X = X_1 \cup X_2$ (with topology inherited from $\mathbb{R}^3$) is locally Euclidean, but $\dim_{(0,0,0)} X = 1$ and $\dim_{(1,1,1)} X = 2$.