relation between germ space and generalized germ space

Canonical name	RelationBetweenGermSpaceAndGeneralizedGermSpace
Date of creation	2013-03-22 19:18:23
Last modified on	2013-03-22 19:18:23
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	5
Author	joking (16130)
Entry type	Theorem
Classification	msc 53B99

Let X, Y be a topological spaces and $x \in X$. Consider `http://planetmath.org/GermSpace`the germ space and `http://planetmath.org/GermSpace`the generalized germ space at x:

$$G_x(X,Y); \quad G^*_x(X,Y).$$

If $f : X \to Y$ is a continuous function, then we have an induced element $[f] \in G^*_x(X,Y)$. It can be easily seen, that if $[f] = [g] \in G_x(X,Y)$, then $[f] = [g] \in G^*_x(X,Y)$. In particular we have a well-defined mapping

$$\tau : G_x(X,Y) \to G^*_x(X,Y);$$

$$\tau([f]) = [f].$$

Proposition 1. τ is injective.

Proof. Indeed, assume that $\tau([f]) = \tau([g])$ for some $f, g : X \to Y$. Let $f' : U \to Y$ and $g' : U' \to Y$ be a representatives of $\tau([f])$ and $\tau([g])$ respectively. It follows, that there exists an open neighbourhood $V \subseteq X$ of x such that

$$f_{|V} = f'_{|V} = g'_{|V} = g_{|V}.$$

In particular $[f] = [g]$ in $G_x(X,Y)$, which completes the proof. □

Proposition 2. If X is a normal space and Y is a normal absolute retract (for example $Y = \mathbb{R}$), then τ is onto.

Proof. Assume that $[f] \in G^*_x(X,Y)$ for some $f : U \to Y$. Since X is regular (because it is normal) then there exists an open neighbourhood $V \subseteq X$ such that the closure $\overline{V} \subseteq U$. Now since X is normal and Y is a normal absolute retract, then $f_{|\overline{V}}$ can be extended to entire X (by the generalized Tietze extension theorem). It is easily seen that any such extension gives the same element in $G_x(X,Y)$ (and it is independent on the choice of the representative f) and if $F : X \to Y$ is an extension of $f_{|\overline{V}}$, then

$$\tau([F]) = [f]$$

because $F_{|V} = f_{|V}$. This completes the proof. □

Remark. If in addition Y is a topological ring (for example $Y = \mathbb{R}$), then it can be easily checked that τ preserves ring structures. In particular if X is normal and $Y = \mathbb{R}$ or $Y = \mathbb{C}$, then τ is an isomorphism of rings. Also it is a good question whether the assumptions in proposition 2 can be weakened.