alternate characterization of curl

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Let $\mathbf{F}$ be a smooth vector field on (an open subset of) $\mathbb{R}^3$.

We show that $\operatorname{curl}\mathbf{F}$ defined using the `http://planetmath.org/curl`coordinate-free definition given on the parent entry is the same as the curl defined by $\nabla \times \mathbf{F}$ in Cartesian coordinates.

The case for spherical surfaces

This will be done by directly computing the limit $\mathbf{L}$ of surface integrals defining $\operatorname{curl}\mathbf{F}(\mathbf{p})$, using spheres $S^2(r,\mathbf{p})$ centered at $\mathbf{p}$ of radius r. The formula is:

$$\begin{aligned}\operatorname{curl}\mathbf{F}(\mathbf{p}) = \mathbf{L} &= \lim_{r\to 0} \frac{3}{4\pi r^3} \iint_{S^2(r,\mathbf{p})} \mathbf{n}\times\mathbf{F}\, dA \\ &= \lim_{r\to 0} \frac{3r^2}{4\pi r^3} \iint_{S^2} \mathbf{n}\times\mathbf{F}(r\mathbf{n}+\mathbf{p})\, dA\,,\end{aligned}$$

where $\mathbf{n}$ is the outward unit normal to the surface (at each point of the surface), and S^2 is the unit sphere at the origin.

We simplify the last integral. Expanding $\mathbf{F}(r\mathbf{n}+\mathbf{p})$ in a first-degree Taylor polynomial about $\mathbf{p}$, we have

$$\begin{aligned}\iint_{S^2} \mathbf{n}\times\mathbf{F}(r\mathbf{n}+\mathbf{p})\, dA &= \iint_{S^2} \mathbf{n}\times\mathbf{F}(\mathbf{p})\, dA \\ &\quad + \iint_{S^2} \mathbf{n}\times \mathrm{D}\,\mathbf{F}(\mathbf{p}) r\mathbf{n}\, dA + \iint_{S^2} \mathbf{n}\times o(\|r\mathbf{n}\|)\, dA\,.\end{aligned}$$

The integral $\iint_{S^2} \mathbf{n}\times\mathbf{F}(\mathbf{p})\, dA$ vanishes by symmetry of the sphere, while

$$\left\| \iint_{S^2} \mathbf{n}\times o(\|r\mathbf{n}\|)\, dA \right\| \leq \iint_{S^2} \|\mathbf{n}\|\, o(r)\, dA = o(r)\,.$$

Combining these facts, we obtain

$$\begin{aligned}\mathbf{L} &= \lim_{r\to 0}\left[0 + \frac{3}{4\pi r}\iint_{S^2} \mathbf{n}\times \mathrm{D}\,\mathbf{F}(\mathbf{p}) r\mathbf{n}\, dA + o(1)\right] \\ &= \frac{3}{4\pi}\iint_{S^2} \mathbf{n}\times \mathrm{D}\,\mathbf{F}(\mathbf{p})\mathbf{n}\, dA\,.\end{aligned}$$

Notice that $\mathbf{L}$ depends only on the derivative of $\mathbf{F}$ at $\mathbf{p}$.

We want to evaluate the last integral in Cartesian coordinates. Let $\mathbf{e}_k$ be an orthonormal basis of $\mathbb{R}^3$ oriented positively, and let B be the matrix of the derivative $\mathrm{D}\,\mathbf{F}(p)$ in this basis. Then the kth coordinate of $\mathbf{L}$ with respect to the same basis is

$$\left(\iint_{S^2} \mathbf{n} \times B\mathbf{n}\, dA\right) \cdot \mathbf{e}_k = \iint_{S^2} (\mathbf{n} \times B\mathbf{n}) \cdot \mathbf{e}_k \, dA$$

The kth coordinate of the integrand is

$$(\mathbf{n} \times B\mathbf{n}) \cdot \mathbf{e}_k = n^i \, (B\mathbf{n})^j \, \epsilon_{ijk} = n^i \, B^j_l n^l \, \epsilon_{ijk} \,,$$

where to lessen the writing, we employ the Einstein summation convention, along with the Levi-Civita permutation symbol ϵ_{ijk}, and B^j_l denotes the entry at the jth row, lth column of B.

In the summation above, if a summmand has $i \neq l$, then the integral of that summand over the sphere is zero, by symmetry. This means that in the summation the index l may be set to i, and thus

$$\left(\iint_{S^2} \mathbf{n} \times B\mathbf{n}\, dA\right) \cdot \mathbf{e}_k = \iint_{S^2} n^i B^j_i n^i \epsilon_{ijk} \, dA = B^j_i \epsilon_{ijk} \iint_{S^2} (n^i)^2 \, dA \,.$$

Now there is a formula for the evaluation of integrals of polynomials over $S^{m-1} \subset \mathbb{R}^m$, in terms of the gamma function; in our case ($m = 3$) the formula reads:

$$\iint_{S^2} (n^i)^2 \, dA = \frac{2\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2} + \frac{1}{2} + \frac{1}{2})} = \frac{2\Gamma(\frac{3}{2})\,\sqrt{\pi}\,\sqrt{\pi}}{\frac{3}{2}\Gamma(\frac{3}{2})} = \frac{4\pi}{3} \,.$$

(If you do not know this formula, the integral in our case can be computed directly using spherical coordinates.) Therefore the kth component of $\mathbf{L}$ is

$$\mathbf{L} \cdot \mathbf{e}_k = \frac{3}{4\pi} \, B^j_i \epsilon_{ijk} \iint_{S^2} (n^i)^2 \, dA = B^j_i \epsilon_{ijk} = \left.\frac{\partial F^j}{\partial x^i}\right|_{\mathbf{p}} \epsilon_{ijk} \,.$$

But this is just $(\nabla \times \mathbf{F}(\mathbf{p})) \cdot \mathbf{e}_k$.

The case for arbitrary surfaces

Although we have only computed

$$\mathbf{L} = \operatorname{curl} \mathbf{F}(\mathbf{p}) = \lim_{V \to 0} \frac{1}{V} \iint_S \mathbf{n} \times \mathbf{F} \, dA$$

only for spheres $S = S^2(r, \mathbf{p})$, this formula holds for arbitrary closed surfaces S that shrink nicely to $\mathbf{p}$. It is hardly obvious, especially since our computation before depended on the symmetry of the sphere extensively.

To show the general result, consider the triple scalar product $(\mathbf{v} \times \mathbf{F}) \cdot \mathbf{e}_k$. This is a linear functional in the vector $\mathbf{v}$, so there exists a unique vector function $\mathbf{g}_k$ such that $(\mathbf{v} \times \mathbf{F}) \cdot \mathbf{e}_k = \mathbf{g}_k \cdot \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$. We can find the components of this $\mathbf{g}_k$ by evaluating the functional at $\mathbf{v} = \mathbf{e}_i$:

$$g_k^i = \mathbf{g}_k \cdot \mathbf{e}_i = (\mathbf{e}_i \times \mathbf{F}) \cdot \mathbf{e}_k = \det(\mathbf{e}_i, \mathbf{F}, \mathbf{e}_k) = F^j \epsilon_{ijk}\,.$$

The reason for considering such expressions is that, putting $\mathbf{v} = \mathbf{n}$, we have

$$\iint_S (\mathbf{n} \times \mathbf{F}) \cdot \mathbf{e}_k \, dA = \iint_S \mathbf{g}_k \cdot \mathbf{n} \, dA = \iint_S \mathbf{g}_k \cdot d\mathbf{A}\,.$$

So we have converted the original integral into an ordinary surface integral. And this surface integral can be changed into a volume integral, by using the divergence theorem:

$$\iint_S \mathbf{g}_k \cdot d\mathbf{A} = \iiint_M \operatorname{div} \mathbf{g}_k \, dV = \iiint_M \frac{\partial F^j}{\partial x^i} \, \epsilon_{ijk} \, dV\,,$$

where M is the volume whose boundary is S. Hence

$$\begin{aligned}
\mathbf{L} \cdot \mathbf{e}_k &= \lim_{V \to 0} \frac{1}{V} \iint_S (\mathbf{n} \times \mathbf{F}) \cdot \mathbf{e}_k \, dA \\
&= \lim_{V \to 0} \frac{1}{V} \iiint_M \frac{\partial F^j}{\partial x^i} \, \epsilon_{ijk} \, dV \\
&= \left. \frac{\partial F^j}{\partial x^i} \right|_{\mathbf{p}} \epsilon_{ijk} = (\nabla \times \mathbf{F}(\mathbf{p})) \cdot \mathbf{e}_k\,.
\end{aligned}$$

Definition in terms of differential forms

We mention, in passing, a computational, yet coordinate-free, alternative to the definition of the curl, using differential forms. If ω is a 1-form on $\mathbb{R}^3$ such that $\omega(\mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle$, then the curl of $\mathbf{F}$ is defined as the vector function $\mathbf{g} = g^k \, e_k$ such that

$$d\omega(\mathbf{u}, \mathbf{v}) = \langle \mathbf{g}, \mathbf{u} \times \mathbf{v} \rangle\,.$$

In Cartesian coordinates, we have

$$\begin{aligned} \omega &= F^1\,dx^1 + F^2\,dx^2 + F^3\,dx^3 \\ d\omega &= g^1\,dx^2 \wedge dx^3 + g^2\,dx^3 \wedge\ dx^1 + g^3\,dx^1 \wedge dx^2\,, \end{aligned}$$

If we take the exterior derivative of the first equation for ω, and then equate components with the second equation for $d\omega$, we find that $g^k = (\nabla \times \mathbf{F}) \cdot \mathbf{e}_k$, so our new definition is equivalent to the others.