Lie derivative (for vector fields)

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Defines	Lie derivative

Let M be a smooth manifold, and $X, Y \in \mathcal{T}(M)$ smooth vector fields on M. Let $\Theta : \mathcal{U} \to M$ be the flow of X, where $\mathcal{U} \subseteq \mathbb{R} \times M$ is an open neighborhood of $\{0\} \times M$. We make use of the following notation:

$$\mathcal{U}^p = \{t \in \mathbb{R} \,|\, (t, p) \in \mathcal{U}\}, \quad \forall p \in M,$$

$$\mathcal{U}_t = \{p \in M \,|\, (t, p) \in \mathcal{U}\}, \quad \forall t \in \mathbb{R},$$

and we introduce the auxiliary maps $\theta_t : \mathcal{U}_t \to M$ and $\theta^p : \mathcal{U}^p \to M$ defined as

$$\Theta(t, p) = \theta_t(p) = \theta^p(t), \quad \forall (t, p) \in \mathcal{U}.$$

The *Lie derivative* of Y along X is the vector field $\mathcal{L}_X Y \in \mathcal{T}(M)$ defined by

$$(\mathcal{L}_X Y)_p = \frac{d}{dt} \left(d(\theta_{-t})_{\theta_t(p)}(Y_{\theta_t(p)}) \right) \bigg|_{t=0} = \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)}(Y_{\theta_t(p)}) - Y_p}{t}, \quad \forall p \in M,$$

where $d(\theta_{-t})_{\theta_t(p)} \in \mathrm{Hom}(T_{\theta_t(p)}M, T_p M)$ if the push-forward of θ_{-t}, i.e.

$$d(\theta_{-t})_{\theta_t(p)}(v)(f) = v(f \circ \theta_{-t}), \quad \forall v \in T_{\theta_{-t}(p)}M,\ f \in C^\infty(p).$$

The following result is not immediate at all.

Theorem 1 *$\mathcal{L}_X Y = [X, Y]$, where $[X, Y] = XY - YX$ is the Lie bracket of X and Y.*