Serret-Frenet equations

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Let $I \subset \mathbb{R}$ be an interval, and let $\gamma : I \to \mathbb{R}^3$ be an arclength parameterization of an oriented space curve, assumed to be http://planetmath.org/Curveregular, and free of points of inflection. Let $T(s)$, $N(s)$, $B(s)$ denote the corresponding moving trihedron, and $\kappa(s), \tau(s)$ the corresponding http://planetmath.org/CurvatureOfACur and http://planetmath.org/Torsiontorsion functions. The following differential relations, called the Serret-Frenet equations, hold between these three vectors.

$$\begin{aligned} T'(s) &= \kappa(s)N(s); && (1) \\ N'(s) &= -\kappa(s)T(s) + \tau(s)B(s); && (2) \\ B'(s) &= -\tau(s)N(s). && (3) \end{aligned}$$

Equation (**??**) follows directly from the http://planetmath.org/MovingFramedefinition of the normal $N(s)$ and from the http://planetmath.org/CurvatureAndTorsiondefinition of the curvature, $\kappa(s)$. Taking the derivative of the relation

$$N(s) \cdot T(s) = 0,$$

gives

$$N'(s) \cdot T(s) = -T'(s) \cdot N(s) = -\kappa(s).$$

Taking the derivative of the relation

$$N(s) \cdot N(s) = 1,$$

gives

$$N'(s) \cdot N(s) = 0.$$

By the http://planetmath.org/CurvatureAndTorsiondefinition of torsion, we have

$$N'(s) \cdot B(s) = \tau(s).$$

This proves equation (**??**). Finally, taking derivatives of the relations

$$\begin{aligned} T(s) \cdot B(s) &= 0, \\ N(s) \cdot B(s) &= 0, \\ B(s) \cdot B(s) &= 1, \end{aligned}$$

and making use of (**??**) and (**??**) gives

$$\begin{aligned} B'(s) \cdot T(s) &= -T'(s) \cdot B(s) = 0, \\ B'(s) \cdot N(s) &= -N'(s) \cdot B(s) = -\tau(s), \\ B'(s) \cdot B(s) &= 0. \end{aligned}$$

This proves equation (**??**).

It is also convenient to describe the Serret-Frenet equations by using matrix notation. Let $F : I \to \mathrm{SO}(3)$ (see - special orthogonal group), the mapping defined by

$$F(s) = (T(s), N(s), B(s)), \quad s \in I$$

represent the Frenet frame as a 3×3 orthonormal matrix. Equations (**??**) (**??**) (**??**) can be succinctly given as

$$F(s)^{-1}F'(s) = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}$$

In this formulation, the above relation is also known as the structure equations of an oriented space curve.