curvature of a circle

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Let C_r be a circle of radius r centered at the origin.

A canonical parameterization of the curve is (counterclockwise)

$$g(s) = r\left(\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right)\right)$$

for $s \in (0, 2\pi r)$ (actually this leaves out the point $(r, 0)$ but this could be treated via another parameterization taking $s \in (-\pi r, \pi r)$)

Differentiating the parameterization we get

$$\mathbf{T} = g'(s) = \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\right)$$

and this results in the normal

$$\mathbf{N} = J \cdot \mathbf{T} = -\left(\cos\left(\frac{s}{r}\right),\ \sin\left(\frac{s}{r}\right)\right) = -\frac{g(s)}{r}$$

Differentiating g a second time we can calculate the curvature

$$\mathbf{T}' = -\frac{1}{r}\left(\cos\left(\frac{s}{r}\right),\ \sin\left(\frac{s}{r}\right)\right) = \frac{1}{r}\mathbf{N}$$

and by definition

$$\mathbf{T}' = k\mathbf{N} \quad \therefore\ k = \frac{1}{r}$$

and thus the curvature of a circle of radius r is $\frac{1}{r}$ provided that the positive direction on the circle is anticlockwise; otherwise it is $-\frac{1}{r}$.