second fundamental form

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In classical differential geometry the *second fundamental form* is a symmetric bilinear form defined on a differentiable surface M embedded in $\mathbb{R}^3$, which in some sense measures the curvature of M in space.

To construct the second fundamental form requires a small digression. After the digression we will discuss how it relates to the curvature of M.

Construction of the second fundamental form

Consider the tangent planes $\mathrm{T}_p M$ of the surface M for each point $p \in M$. There are two unit normals to $\mathrm{T}_p M$. Assuming M is orientable, we can choose one of these unit `http://planetmath.org/MutualPositionsOfVectors`normals, $n(p)$, so that $n(p)$ varies smoothly with p.

Since $n(p)$ is a unit vector in $\mathbb{R}^3$, it may be considered as a point on the sphere $S^2 \subset \mathbb{R}^3$. Then we have a map $n\colon M \to S^2$. It is called the *normal map* or *Gauss map*.

The *second fundamental form* is the tensor field $\mathcal{II}$ on M defined by

$$\mathcal{II}_p(\xi, \eta) = -\langle \mathrm{D}\, n_p(\xi), \eta \rangle\,, \quad \xi, \eta \in \mathrm{T}_p M\,, \tag{1}$$

where $\langle , \rangle$ is the dot product of $\mathbb{R}^3$, and we consider the tangent planes of surfaces in $\mathbb{R}^3$ to be subspaces of $\mathbb{R}^3$.

The linear transformation $\mathrm{D}\, n_p$ is in reality the tangent mapping $\mathrm{D}\, n_p \colon \mathrm{T}_p M \to \mathrm{T}_{n(p)} S^2$, but since $\mathrm{T}_{n(p)} S^2 = \mathrm{T}_p M$ by the definition of n, we prefer to think of $\mathrm{D}\, n_p$ as $\mathrm{D}\, n_p \colon \mathrm{T}_p M \to \mathrm{T}_p M$.

The tangent map $\mathrm{D}\, n$, is often called the *Weingarten map*.

Proposition 1. *The second fundamental form is a symmetric form.*

Proof. This is a computation using a coordinate chart σ for M. Let u, v be the corresponding names for the coordinates. From the equation

$$\left\langle n, \frac{\partial \sigma}{\partial v} \right\rangle = 0\,,$$

differentiating with respect to u using the product rule gives

$$\begin{aligned}\left\langle n, \frac{\partial^2\sigma}{\partial u \partial v} \right\rangle &= -\left\langle \frac{\partial n}{\partial u}, \frac{\partial \sigma}{\partial v} \right\rangle \\ &= -\left\langle \mathrm{D}\, n \left(\frac{\partial \sigma}{\partial u}\right), \frac{\partial \sigma}{\partial v} \right\rangle \\ &= \mathcal{II}\left(\frac{\partial \sigma}{\partial u}, \frac{\partial \sigma}{\partial v}\right) .\end{aligned} \tag{2}$$

(The second equality follows from the definition of the tangent map $\mathrm{D}\, n$.) Reversing the roles of u, v and repeating the last derivation, we obtain also:

$$\left\langle n, \frac{\partial^2\sigma}{\partial u \partial v} \right\rangle = \left\langle n, \frac{\partial^2\sigma}{\partial v \partial u} \right\rangle = \mathcal{II}\left(\frac{\partial \sigma}{\partial v}, \frac{\partial \sigma}{\partial u}\right) . \tag{3}$$

Since $\partial\sigma/\partial u$ and $\partial\sigma/\partial v$ form a basis for $\mathrm{T}_p M$, combining (**??**) and (**??**) proves that $\mathcal{II}$ is symmetric. □

In view of Proposition **??**, it is customary to regard the second fundamental form as a quadratic form, as it done with the first fundamental form. Thus, the second fundamental form is referred to with the following expression[1]:

$$L\, du^2 + 2M\, dudv + N\, dv^2 .$$

Compare with the tensor notation

$$\mathcal{II} = L\, du \otimes du + M\, du \otimes dv + M\, dv \otimes du + N dv \otimes dv .$$

Or in matrix form (with respect to the coordinates u, v),

$$\mathcal{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} .$$

[1]Unfortunately the coefficient M here clashes with our use of the letter M for the surface (manifold), but whenever we write M, the context should make clear which meaning is intended. The use of the symbols L, M, N for the coefficients of the second fundamental form is standard, but probably was established long before anyone thought about manifolds.

Curvature of curves on a surface

Let γ be a curve lying on the surface M, parameterized by arc-length. Recall that the curvature $\kappa(s)$ of γ at s is $\gamma''(s)$. If we want to measure the curvature of the surface, it is natural to consider the component of $\gamma''(s)$ in the normal $n(\gamma(s))$. Precisely, this quantity is

$$\langle \gamma''(s), n(\gamma(s)) \rangle \,,$$

and is called the *normal curvature* of γ on M.

So to study the curvature of M, we ignore the component of the curvature of γ in the tangent plane of M. Also, physically speaking, the normal curvature is proportional to the acceleration required to keep a moving particle *on* the surface M.

We now come to the motivation for defining the second fundamental form:

Proposition 2. *Let γ be a curve on M, parameterized by arc-length, and $\gamma(s) = p$. Then*

$$\langle \gamma''(s), n(p) \rangle = \mathcal{II}\big(\gamma'(s), \gamma'(s)\big) \,.$$

Proof. From the equation

$$\langle n(\gamma(s)), \gamma'(s) \rangle = 0 \,,$$

differentiate with respect to s:

$$\begin{aligned} \langle n(\gamma(s)), \gamma''(s) \rangle &= -\left\langle \frac{d}{ds} n(\gamma(s)), \gamma'(s) \right\rangle \\ &= -\big\langle \mathrm{D}\, n(\gamma'(s)), \gamma'(s) \big\rangle \\ &= \mathcal{II}\big(\gamma'(s), \gamma'(s)\big) \,. \end{aligned}$$

□

It is now time to mention an important consequence of Proposition **??**: the fact that $\mathcal{II}$ is symmetric means that $-\mathrm{D}\, n$ is *self-adjoint* with respect to the inner product $\mathcal{I}$ (the first fundamental form). So, if $-\mathrm{D}\, n$ is expressed as a matrix with orthonormal coordinates (with respect to $\mathcal{I}$), then the matrix is symmetric. (The minus sign in front of $\mathrm{D}\, n$ is to make the formulas work out nicely.)

Certain theorems in linear algebra tell us that, $-\mathrm{D}\, n_p$ being self-adjoint, it has an orthonormal basis of eigenvectors e_1, e_2 with corresponding eigenvalues $\kappa_1 \leq \kappa_2$. These eigenvalues are called the *principal curvatures* of M at p. The eigenvectors e_1, e_2 are the *principal directions*. The terminology is justified by the following theorem:

Theorem 1 (Euler's Theorem)**.** *The normal curvature of a curve γ has the form*

$$\langle \gamma''(s), n(p) \rangle = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta\,, \quad p = \gamma(s)\,.$$

It follows that the minimum possible normal curvature is κ_1, and the maximum possible is κ_2.

Proof. Since e_1, e_2 form an orthonormal basis for $\mathrm{T}_p M$, we may write

$$\gamma'(s) = \cos\theta\, e_1 + \sin\theta\, e_2$$

for some angle θ. Then

$$\begin{aligned}\langle \gamma''(s), n(p) \rangle &= \mathcal{II}\big(\gamma'(s), \gamma'(s)\big) \\ &= \big\langle -\mathrm{D}\, n_p(\gamma'(s)), \gamma'(s) \big\rangle \\ &= \langle \kappa_1 \cos\theta\, e_1 + \kappa_2 \sin\theta\, e_2, \cos\theta\, e_1 + \sin\theta\, e_2 \rangle \\ &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta\,.\end{aligned}$$

□

Matrix representations of second fundamental form and Weingarten map

At this point, we should find the explicit prescriptions for calculating the second fundamental form and the Weingarten map.

Let σ be a coordinate chart for M, and u, v be the names of the coordinates. For a test vector $\xi \in \mathrm{T}_p M$, we write ξ_u and ξ_v for the u, v coordinates of ξ.

We compute the matrix W for $-\mathrm{D}\, n$ in u, v-coordinates. We have

$$\begin{aligned}\begin{pmatrix} \xi_u & \xi_v \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \xi_u \\ \xi_v \end{pmatrix} &= \mathcal{II}(\xi, \xi) = \langle -\mathrm{D}\, n(\xi), \xi \rangle \\ &= \left(Q \begin{pmatrix} \xi_u \\ \xi_v \end{pmatrix} \right)^{\mathrm{T}} QW \begin{pmatrix} \xi_u \\ \xi_v \end{pmatrix} \\ &= \begin{pmatrix} \xi_u & \xi_v \end{pmatrix} (Q^{\mathrm{T}} Q)\, W \begin{pmatrix} \xi_u \\ \xi_v \end{pmatrix}\,,\end{aligned}$$

where Q is the matrix that changes from u, v-coordinates to orthonormal coordinates for $\mathrm{T}_p M$ — this is necessary to compute the inner product. But

$$Q^{\mathrm{T}} Q = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \mathcal{I} \quad \text{(the first fundamental form)},$$

because Q is the matrix with columns $\partial\sigma/\partial u$ and $\partial\sigma/\partial v$ expressed in orthonormal coordinates.

(More to be written...)

References

[1] Michael Spivak. *A Comprehensive Introduction to Differential Geometry*, volumes I and II. Publish or Perish, 1979.

[2] Andrew Pressley. *Elementary Differential Geometry*. Springer-Verlag, 2003.