normal curvatures

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Let us determine the http://planetmath.org/NormalSectionnormal curvatures $\varkappa$ of the surface

$$z \;=\; z(x,\,y) \tag{1}$$

in the origin, when (1) has the continuous 1st and 2nd order partial derivatives in a neighbourhood of $(0,\,0)$ and satisfies

$$z(0,\,0) \;=\; z'_x(0,\,0) \;=\; z'_y(0,\,0) = 0. \tag{2}$$

It's a question of the http://planetmath.org/CurvaturePlaneCurvecurvature of the intersection curves of the surface (1) and planes containing the z-axis, which is the normal of the surface in the origin.

If the angle between the zx-plane and a plane τ containing the z-axis is denoted by φ, when the line of intersection of the plane τ and the xy-plane is represented by the equations

$$x \;=\; \varrho\cos\varphi, \quad y \;=\; \varrho\sin\varphi \qquad (-\infty \;<\; \varrho \;<\; \infty),$$

then equation of the the normal section curve C_φ is

$$z \;=\; z(\varrho\cos\varphi,\; \varrho\sin\varphi),$$

where ϱ is the abscissa and z the ordinate. It follows that

$$\frac{dz}{d\varrho} \;=\; \frac{\partial z}{\partial x}\cos\varphi + \frac{\partial z}{\partial y}\sin\varphi,$$

$$\frac{d^2z}{d\varrho^2} \;=\; \frac{\partial^2 z}{\partial x^2}\cos^2\!\varphi + 2\frac{\partial^2 z}{\partial x\partial y}\sin\varphi\cos\varphi + \frac{\partial^2 z}{\partial y^2}\sin^2\!\varphi;$$

thus by (2), in the origin we have

$$\frac{dz}{d\varrho} \;=\; 0, \quad \frac{d^2z}{d\varrho^2} \;=\; a\cos^2\!\varphi + 2b\sin\varphi\cos\varphi + c\sin^2\!\varphi,$$

where $a,\,b,\,c$ the values of the derivatives $\frac{\partial^2 z}{\partial x^2},\,\frac{\partial^2 z}{\partial x\partial y},\,\frac{\partial^2 z}{\partial y^2}$ in the origin.

Using those values, we obtain for the normal curvature of C_φ in the origin the value

$$\varkappa(\varphi) \;=\; \left[\frac{\frac{d^2z}{d\varrho^2}}{\left(1+\left(\frac{dz}{d\varrho}\right)^2\right)^{3/2}}\right]_{\varrho=0} \;=\; a\cos^2\!\varphi+2b\sin\varphi\cos\varphi+c\sin^2\!\varphi. \quad (3)$$

This result gets a more illustrative form when we try to express it by using the least and the greatest value of $\varkappa(\varphi)$. Instead to utilize the zeros of the derivative of the sum in (3), it's simpler first to transfer to the http://planetmath.org/DoubleAngleIdentitydouble angle,

$$\varkappa(\varphi) \;=\; \frac{a\!+\!c}{2}+\frac{a\!-\!c}{2}\cos2\varphi+b\sin2\varphi, \quad (4)$$

and here to introduce an auxiliary angle α $(0\,\leq\,\alpha\,<\,\pi)$ such that

$$\frac{a\!-\!c}{2} \;:=\; k\cos2\alpha, \quad b \;:=\; k\sin2\alpha.$$

This allows us to write (4) as

$$\varkappa(\varphi) \;=\; \frac{a\!+\!c}{2}+k\,\cos2(\varphi\!-\!\alpha). \quad (5)$$

From this we see immediately that the curvature attains its greatest and least value $\dfrac{a\!+\!c}{2}\pm k$ when $\varphi=\alpha$ and $\varphi=\alpha\!+\!\frac{\pi}{2}$.

Accordingly, the corresponding τ, the *principal normal planes*, are perpendicular to each other; their normal section curves on the surface (1) in the origin are briefly called the *principal sections*.

The expression (5) of the normal curvature may still be edited. Let us take a new parameter angle $\varphi-\alpha:=\theta$. One can write

$$\varkappa(\varphi) \;=\; \frac{a\!+\!c}{2}(\cos^2\theta\!+\!\sin^2\theta)\!+\!k(\cos^2\theta\!-\!\sin^2\theta) \;=\; \left(\frac{a\!+\!c}{2}+k\right)\cos^2\theta\!+\!\left(\frac{a\!+\!c}{2}-k\right)\sin^2\theta \;:=\; \varkappa_\theta.$$

So the final result, the so-called *http://planetmath.org/SecondFundamentalFormEuler's theorem*, can be expressed in the form

$$\varkappa_\theta \;=\; \varkappa_1\cos^2\theta+\varkappa_2\sin^2\theta. \quad (6)$$

Here, the *principal curvatures* $\varkappa_1$ and $\varkappa_2$ are the greatest and the least value of the normal curvature, respectively, and θ is the `http://planetmath.org/AngleBetweenTwoPlanes` between the normal section plane corresponding $\varkappa_1$ and the normal section plane corresponding $\varkappa_\theta$. As it becomes clear in the `http://planetmath.org/NormalSection`parent entry, the result (6) is true not only in the origin but at any point on a surface when the given function has the continuous 1st and 2nd derivatives in some neighbourhood of the point.

References

[1] Ernst Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset II.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1932).