Cauchy invariance rule

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If $f(u,\,v,\,w),\ u(x,\,y),\ v(x,\,y),\ w(x,\,y)$ are differentiable functions and

$$\bar{f}(x,\,y)\ :=\ f(u(x,\,y),\,v(x,\,y),\,w(x,\,y)) \tag{1}$$

their composite function, then according to the http://planetmath.org/node/2798chain rule, we have the partial derivatives

$$\begin{cases} \bar{f}'_x(x,\,y)\ =\ f'_u(u,\,v,\,w)\,u'_x(x,\,y)+f'_v(u,\,v,\,w)\,v'_x(x,\,y)+f'_w(u,\,v,\,w)\,w'_x(x,\,y), \\ \bar{f}'_y(x,\,y)\ =\ f'_u(u,\,v,\,w)\,u'_y(x,\,y)+f'_v(u,\,v,\,w)\,v'_y(x,\,y)+f'_w(u,\,v,\,w)\,w'_y(x,\,y). \end{cases} \tag{2}$$

Multiplying these two equations by dx and dy, respectively, and then adding them, we obtain for the total differential of the composite function the expression

$$\begin{aligned} d\bar{f}(x,\,y)\ &=\ \bar{f}'_x(x,\,y)\,dx+\bar{f}'_y(x,\,y)\,dy \\ &=\ (f'_u u'_x+f'_v v'_x+f'_w w'_x)\,dx+(f'_u u'_y+f'_v v'_y+f'_w w'_y)\,dy \\ &=\ f'_u[u'_x\,dx+u'_y\,dy]+f'_v[v'_x\,dx+v'_y\,dy]+f'_w[w'_x\,dx+w'_y\,dy]. \end{aligned}$$

But the sums in the brackets the total differentials of the inner functions, whence we may write

$$d\bar{f}(x,\,y)\ =\ f'_u(u,\,v,\,w)\,du(x,\,y)+f'_v(u,\,v,\,w)\,dv(x,\,y)+f'_w(u,\,v,\,w)\,dw(x,\,y) \tag{3}$$

where one must still substitute $u:=u(x,\,y),\ \ v:=v(x,\,y),\ \ w:=w(x,\,y)$. Comparing (3) with the expression of the total differential

$$df(u,\,v,\,w)\ =\ f'_u(u,\,v,\,w)\,du+f'_v(u,\,v,\,w)\,dv+f'_w(u,\,v,\,w)\,dw \tag{4}$$

of the outer function, we infer the following

Rule. The total differential of the composite function (1) is directly obtained from the expression of the total differential of the outer function, when one replaces in it the variables $u,\,v,\,w$ with the corresponding inner functions and the differentials $du,\,dv,\,dw$ with the total differentials of those inner functions.

This rule of Cauchy is analogical for any number of inner functions and their variables. The rule also offers the simplest way to form the partial derivatives of the composite function.

References

[1] Ernst Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset II.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1932).