curl

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Defines	curl of a vector field

The *curl* (also known as *rotor*) is a first order linear differential operator which acts on vector fields in $\mathbb{R}^3$.

Intuitively, the curl of a vector field measures the extent to which a vector field differs from being the gradient of a scalar field. The name "curl" comes from the fact that vector fields at a point with a non-zero curl can be seen as somehow "swirling around" said point. A mathematically precise formulation of this notion can be obtained in the form of the definition of curl as limit of an integral about a closed circuit.

Let F be a vector field in $\mathbb{R}^3$.

Pick an orthonormal basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ and write $\vec{F} = F^1\vec{e_1} + F^2\vec{e_2} + F^3\vec{e_3}$. Then the curl of F, notated $\operatorname{curl}\vec{F}$ or $\operatorname{rot}\vec{F}$ or $\vec{\nabla} \times \vec{F}$, is given as follows:

$$\begin{aligned}\operatorname{curl}\vec{F} &= \left[\frac{\partial F^3}{\partial q^2} - \frac{\partial F^2}{\partial q^3}\right]\vec{e_1} + \left[\frac{\partial F^1}{\partial q^3} - \frac{\partial F^3}{\partial q^1}\right]\vec{e_2} + \\ &\quad \left[\frac{\partial F^2}{\partial q^1} - \frac{\partial F^1}{\partial q^2}\right]\vec{e_3}\end{aligned}$$

By applying the chain rule, one can verify that one obtains the same answer irregardless of choice of basis, hence curl is well-defined as a function of vector fields. Another way of coming to the same conclusion is to exhibit an expression for the curl of a vector field which does not require the choice of a basis. One such expression is as follows: Let V be the volume of a closed surface S enclosing the point p. Then one has

$$\operatorname{curl}\vec{F}(p) = \lim_{V\to 0}\frac{1}{V}\int\!\!\int_S \vec{n} \times \vec{F} dS$$

Where n is the outward unit normal vector to S.

Curl is easily computed in an arbitrary orthogonal coordinate system by using the appropriate scale factors. That is

$$\begin{aligned}\operatorname{curl}\vec{F} &= \frac{1}{h_3h_2}\left[\frac{\partial}{\partial q^2}\left(h_3F^3\right) - \frac{\partial}{\partial q^3}\left(h_2F^2\right)\right]\vec{e_1} + \frac{1}{h_3h_1}\left[\frac{\partial}{\partial q^3}\left(h_1F^1\right) - \frac{\partial}{\partial q^1}\left(h_3F^3\right)\right]\vec{e_2} + \\ &\quad \frac{1}{h_1h_2}\left[\frac{\partial}{\partial q^1}\left(h_2F^2\right) - \frac{\partial}{\partial q^2}\left(h_1F^1\right)\right]\vec{e_3}\end{aligned}$$

for the arbitrary orthogonal curvilinear coordinate system (q^1, q^2, q^3) having scale factors (h_1, h_2, h_3). Note the scale factors are given by

$$h_i = \left(\frac{d}{dx_i}\right)\left(\frac{d}{dx_i}\right) \ni i \in \{1, 2, 3\}.$$

Non-orthogonal systems are more easily handled with tensor analysis or exterior calculus.

$$(\operatorname{curl} \vec{F})^i = \epsilon^{ijk} \nabla_j F_k$$

$$\operatorname{curl} \vec{F} = *d(F_1 dx^1 + F_2 dx^2 + F_3 dx^3)$$