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curvature (space curve)

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Let $I \subset \mathbb{R}$ be an interval, and let $\gamma : I \to \mathbb{R}^3$ be an arclength parameterization of an oriented space curve, assumed to be regular, and free of points of inflection. We interpret $\gamma(t)$ as the trajectory of a particle moving through 3-dimensional space. Let $T(t), N(t), B(t)$ denote the corresponding moving trihedron. The speed of this particle is given by

$$v(t) = \|\gamma'(t)\|.$$

The quantity

$$\kappa(t) = \frac{\|T'(t)\|}{v(t)} = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$$

is called the *curvature* of the space curve. It is invariant with respect to reparameterization, and is therefore a measure of an intrinsic property of the curve, a real number geometrically assigned to the point $\gamma(t)$. If one parameterizes the curve with respect to the arclength s, one gets the more concise relation that

$$\kappa(s) = \frac{1 \cdot \|\gamma''(s)\| \cdot \sin \frac{\pi}{2}}{1^3} = \|\gamma''(s)\|.$$

Physically, curvature may be conceived as the ratio of the normal acceleration of a particle to the particle's speed. This ratio measures the degree to which the curve deviates from the straight line at a particular point. Indeed, one can show that of all the circles passing through $\gamma(t)$ and lying on the osculating plane, the one of radius $1/\kappa(t)$ serves as the best approximation to the space curve at the point $\gamma(t)$.

To treat curvature analytically, we take the derivative of the relation

$$\gamma'(t) = v(t)T(t).$$

This yields the following decomposition of the acceleration vector:

$$\gamma''(t) = v'(t)T(t) + v(t)T'(t) = v(t)\left\{(\log v)'(t)\, T(t) + \kappa(t)\, N(t)\right\}.$$

Thus, to change speed, one needs to apply acceleration along the tangent vector; to change heading the acceleration must be applied along the normal.