Riemannian manifold

Canonical name	RiemannianManifold
Date of creation	2013-03-22 13:02:54
Last modified on	2013-03-22 13:02:54
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Last modified by	djao (24)
Numerical id	31
Author	djao (24)
Entry type	Definition
Classification	msc 53B20
Classification	msc 53B21
Synonym	Riemann space and metric
Related topic	QuantumGeometry2
Related topic	Gradient
Related topic	CategoryOfRiemannianManifolds
Related topic	HomotopyCategory
Related topic	CWComplexDefinitionRelatedToSpinNetworksAndSpinFoams
Related topic	QuantumAlgebraicTopologyOfCWComplexRepresentationsNewQATResultsF
Defines	Riemannian metric
Defines	Riemannian structure
Defines	metric tensor

A *Riemannian metric tensor* is a covariant, type $(0,2)$ tensor field $g \in \Gamma(\mathrm{T}^* M \otimes \mathrm{T}^* M)$ such that at each point $p \in M$, the bilinear form $g_p : \mathrm{T}_p M \times \mathrm{T}_p M \to \mathbb{R}$ is symmetric and positive definite. Here T^*M is the cotangent bundle of M (defined as a sheaf), Γ is the set of global sections of $T^*M \otimes T^*M$, and g_p is the value of the function g at the point $p \in M$.

Let $(x^1, \ldots, x^n)$ be a system of local coordinates on an open subset $U \subset M$, let $dx^i,\ i = 1, \ldots, n$ be the corresponding coframe of 1-forms, and let $\dfrac{\partial}{\partial x^i},\ i = 1, \ldots, n$ be the corresponding dual frame of vector fields. Using the local coordinates, the metric tensor has the unique expression

$$g = \sum_{i,j=1}^{n} g_{ij}\, dx^i \otimes dx^j,$$

where the metric tensor components

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

are smooth functions on U.

Once we fix the local coordinates, the functions g_{ij} completely determine the Riemannian metric. Thus, at each point $p \in U$, the matrix $(g_{ij}(p))$ is symmetric, and positive definite. Indeed, it is possible to define a Riemannian structure on a manifold M by specifying an atlas over M together with a matrix of functions g_{ij} on each coordinate chart which are symmetric and positive definite, with the proviso that the g_{ij}'s must be compatible with each other on overlaps.

A manifold M together with a Riemannian metric tensor g is called a *Riemannian manifold*.

Note: A Riemannian metric tensor on M is not a distance metric on M. However, on a connected manifold every Riemannian metric tensor on M induces a distance metric on M, given by

$$d(x,y) := \inf \left\{ \int_0^1 \left[g\!\left(\frac{dc}{dt}, \frac{dc}{dt}\right)_{c(t)} \right]^{1/2} dt \right\}, \quad x, y \in M,$$

where the infimum is taken over all rectifiable curves $c : [0,1] \to M$ with $c(0) = x$ and $c(1) = y$.

Often, it is the g_{ij} that are referred to as the "Riemannian metric". This, however, is a misnomer. Properly speaking, the g_{ij} should be called local coordinate components of a metric tensor, where as "Riemannian metric" should refer to the distance function defined above. However, the practice of calling the collection of g_{ij}'s by the misnomer "Riemannian metric" appears to have stuck.

Remarks:

- Both the Riemannian manifold and Riemannian metric tensor are fundamental concepts in Einstein's General Relativity (GR) theory where the "Riemannian metric" and curvature of the physical Riemannian space-time are changed by the presence of massive bodies and energy according to `http://planetmath.org/EinsteinFieldEquations`Einstein's fundamental GR field equations.
- The category of Riemannian manifolds (or 'spaces') provides an alternative framework for GR theories as well as algebraic quantum field theories (AQFTs);
- The category of 'pseudo-Riemannian' manifolds, deals in fact with extensions of Minkowski spaces, does not possess the Riemannian metric defined in this entry on Riemannian manifolds, and is claimed as a useful approach to defining 4D-spacetimes in relativity theories.