Riemann curvature tensor

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Let $\mathcal{X}$ denote the vector space of smooth vector fields on a smooth Riemannian manifold (M, g). Note that $\mathcal{X}$ is actually a $\mathcal{C}^\infty(M)$ module because we can multiply a vector field by a function to obtain another vector field. The *Riemann curvature tensor* is the tri-linear $\mathcal{C}^\infty$ mapping

$$R : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X},$$

which is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where $X, Y, Z \in \mathcal{X}$ are vector fields, where ∇ is the Levi-Civita connection attached to the metric tensor g, and where the square brackets denote the Lie bracket of two vector fields. The tri-linearity means that for every smooth $f\colon M \to \mathbb{R}$ we have

$$fR(X, Y)Z = R(fX, Y)Z = R(X, fY)Z = R(X, Y)fZ.$$

In components this tensor is classically denoted by a set of four-indexed components $R^i{}_{jkl}$. This means that given a basis of linearly independent vector fields X_i we have

$$R(X_j, X_k)X_l = \sum_s R^s{}_{jkl} X_s.$$

In a two dimensional manifold it is known that the Gaussian curvature it is given by

$$K_g = \frac{R_{1212}}{g_{11} g_{22} - {g_{12}}^2}$$