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if the algebra of functions on a manifold is a Poisson ring then the manifold is symplectic

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Let M be a smooth manifold and let A be the algebra of smooth functions from M to $\mathbb{R}$. Suppose that there exists a bilinear operation $[,] \colon A \times A \to A$ which makes A a Poisson ring.

For this proof, we shall use the fact that $T^*(M)$ is the sheafification of the A-module generated by the set $\{df | f \in A\}$ modulo the relations

- $d(f + g) = df + dg$
- $df g = g\, df + f\, dg$

Let us define a map $\omega \colon T^*(M) \to T(M)$ by the following conditions:

- $\omega(df)(g) = [f, g]$ for all $f g \in A$
- $\omega(fX + gY) = f\omega(X) + g\omega(Y)$ for all $f, g \in A$ and all $X, Y \in T^*(M)$

For this map to be well-defined, it must respect the relations:

$$\omega(f + g)(h) = [f + g, h] = [f, h] + [g, h] = \omega(f)(h) + \omega(g)(h)$$

$$\omega(fg)(h) = [fg, h] = f[g, h] + g[f, h] = f\omega(g)(h) + g\omega(g)(h)$$

These two equations show that ω is a well-defined map from the presheaf hence, by general nonsense, a well defined map from the sheaf. The fact that $\omega(f\, dg)$ is a derivation readily follows from the fact that $[,]$ is a derivation in each slot.

Since $[,]$ is non-degenerate, ω is invertible. Denote its inverse by Ω. Since our manifold is finite-dimensional, we may naturally regard Ω as an element of $T^*(M) \otimes T^*(M)$. The fact that Ω is an antisymmetric tensor field (in other words, a 2-form) follows from the fact that $\Omega(df)(g) = [f, g] = -[g, f] = -\Omega(dg)(f)$.

Finally, we will use the Jacobi identity to show that Ω is closed. If $u, v, w \in T(M)$ then, by a general identity of differential geometry,

$$\langle d\Omega, u \wedge v \wedge w \rangle = \langle u, d\langle \Omega, v \wedge w \rangle \rangle + \langle v, d\langle \Omega, w \wedge u \rangle \rangle + \langle w, d\langle \Omega, u \wedge v \rangle \rangle$$

Since this identity is trilinear in u, v, w, we can restrict attention to a generating set. Because of the non-degeneracy assumption, vector fields of the form ad_f where f is a function form such a set.

By the definition of Ω, we have $\langle \Omega, ad_f \wedge ad_g \rangle = [f, g]$. Then $\langle ad_f, d\, \langle \Omega, ad_g \wedge ad_h \rangle = [f, [g, h]]$ so the Jacobi identity is satisfied.