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example of rewriting a differential equation as a Pfaffian system

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To show how one may reformulate a differential equation as Pfaff's problem for a set of differential forms, consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

The first step is to rewrite the equation as a system of first-order equations

$$\frac{\partial a}{\partial t} - \frac{\partial b}{\partial x} - \frac{\partial c}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} - a = 0$$

$$\frac{\partial u}{\partial x} - b = 0$$

$$\frac{\partial u}{\partial y} - c = 0$$

To translate these equations into the language of differential forms, we shall use the fact that

$$du = \frac{\partial u}{\partial t}\, dt + \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy$$

from which it follows that

$$du \wedge dx \wedge dy = \frac{\partial u}{\partial t}\, dt \wedge dx \wedge dy$$

$$du \wedge dy \wedge dt = \frac{\partial u}{\partial x}\, dt \wedge dx \wedge dy$$

$$du \wedge dt \wedge dx = \frac{\partial u}{\partial y}\, dt \wedge dx \wedge dy$$

We can do likewise with a or b or c in the place of u; there is no point in repeating the formulas for each of these variables.

Multiplying the differential equations through by the form $dt \wedge dx \wedge dy$ and using the above identities to eliminate partial derivatives, we obtain the following system of differential forms:

$$da \wedge dx \wedge dy - db \wedge dy \wedge dt - dc \wedge dt \wedge dx$$

$$du \wedge dx \wedge dy - a\, dt \wedge dx \wedge dy$$

$$du \wedge dy \wedge dt - b\, dt \wedge dx \wedge dy$$

$$du \wedge dt \wedge dx - c\, dt \wedge dx \wedge dy$$

From the way these forms were constructed, it is clear that a three dimensional surface in the seven dimensional space with coordinates x, y, t, a, b, c, u which solves Pfaff's problem and can be parameterized by x, y, t corresponds to the graph of a solution to the system of differential equations, and hence to a solution of the wave equation.

Note: These considerations are purely local. The global topology of the seven-dimensional space will depend on the domain on which the original wave equation was formulated and on the boundary conditions.