tangent space

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Summary The tangent space of differential manifold M at a point $x \in M$ is the vector space whose elements are velocities of trajectories that pass through x. The standard notation for the tangent space of M at the point x is $T_x M$.

Definition (Standard). Let M be a differential manifold and x a point of M. Let

$$\gamma_i : I_i \to M, \quad I_i \subset \mathbb{R}, \quad i = 1, 2$$

be two differentiable trajectories passing through x at times $t_1 \in I_1, t_2 \in I_2$, respectively. We say that these trajectories are in first order contact at x if for all differentiable functions $f : U \to \mathbb{R}$ defined in some neighbourhood $U \subset M$ of x, we have

$$(f \circ \gamma_1)'(t_1) = (f \circ \gamma_2)'(t_2).$$

First order contact is an equivalence relation, and we define $T_x M$, the *tangent space* of M at x, to be the set of corresponding equivalence classes.

Given a trajectory

$$\gamma : I \to M, \quad I \subset \mathbb{R}$$

passing through x at time $t \in I$, we define $\dot{\gamma}(t)$ the *tangent vector* , a.k.a. the *velocity*, of γ at time t, to be the equivalence class of γ modulo first order contact. We endow $T_x M$ with the structure of a real vector space by identifying it with $\mathbb{R}^n$ relative to a system of local coordinates. These identifications will differ from chart to chart, but they will all be linearly compatible.

To describe this identification, consider a coordinate chart

$$\alpha : U_\alpha \to \mathbb{R}^n, \quad U_\alpha \subset M, \quad x \in U.$$

We call the real vector

$$(\alpha \circ \gamma)'(t) \in \mathbb{R}^n$$

the representation of $\dot{\gamma}(t)$ relative to the chart α. It is a simple exercise to show that two trajectories are in first order contact at x if and only if their velocities have the same representation. Another simple exercise will show that for every $\mathbf{u} \in \mathbb{R}^n$ the trajectory

$$t \to \alpha^{-1}(\alpha(x) + t\mathbf{u})$$

has velocity $\mathbf{u}$ relative to the chart α. Hence, every element of $\mathbb{R}^n$ represents some actual velocity, and therefore the mapping $T_x M \to \mathbb{R}^n$ given by

$$[\gamma] \to (\alpha \circ \gamma)'(t), \quad \gamma(t) = x,$$

is a bijection.

Finally if $\beta : U_\beta \to \mathbb{R}^n, \quad U_\beta \subset M, \quad x \in U_\beta$ is another chart, then for all differentiable trajectories $\gamma(t) = x$ we have

$$(\beta \circ \gamma)'(t) = J(\alpha \circ \gamma)'(t),$$

where J is the Jacobian matrix at $\alpha(x)$ of the suitably restricted mapping $\beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^n$. The linearity of the above relation implies that the vector space structure of $T_x M$ is independent of the choice of coordinate chart.

Definition (Classical). Historically, tangent vectors were specified as elements of $\mathbb{R}^n$ relative to some system of coordinates, a.k.a. a coordinate chart. This point of view naturally leads to the definition of a tangent space as $\mathbb{R}^n$ modulo changes of coordinates.

Let M be a differential manifold represented as a collection of parameterization domains

$$\{V_\alpha \subset \mathbb{R}^n : \alpha \in \mathcal{A}\}$$

indexed by labels belonging to a set $\mathcal{A}$, and transition function diffeomorphisms

$$\sigma_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}, \quad \alpha, \beta \in \mathcal{A}, \quad V_{\alpha\beta} \subset V_\alpha$$

Set

$$\hat{M} = \{(\alpha, x) \in \mathcal{A} \times \mathbb{R}^n : x \in V_\alpha\},$$

and recall that a points of the manifold are represented by elements of $\hat{M}$ modulo an equivalence relation imposed by the transition functions [see Manifold — Definition (Classical)]. For a transition function $\sigma_{\alpha\beta}$, let

$$\mathrm{J}\sigma_{\alpha\beta} : V_{\alpha\beta} \to \mathrm{Mat}_{n,n}(\mathbb{R})$$

denote the corresponding Jacobian matrix of partial derivatives. We call a triple

$$(\alpha, x, \mathbf{u}), \quad \alpha \in \mathcal{A}, \; x \in V_\alpha, \; \mathbf{u} \in \mathbb{R}^n$$

the representation of a tangent vector at x relative to coordinate system α, and make the identification

$$(\alpha, x, \mathbf{u}) \simeq (\beta, \sigma_{\alpha\beta}(x), [\mathrm{J}\sigma_{\alpha\beta}](x)(\mathbf{u})), \quad \alpha, \beta \in \mathcal{A},\ x \in V_{\alpha\beta},\ \mathbf{u} \in \mathbb{R}^n.$$

to arrive at the definition of a tangent vector at x.

Notes. The notion of tangent space derives from the observation that there is no natural way to relate and compare velocities at different points of a manifold. This is already evident when we consider objects moving on a surface in 3-space, where the velocities take their value in the tangent planes of the surface. On a general surface, distinct points correspond to distinct tangent planes, and therefore the velocities at distinct points are not commensurate.

The situation is even more complicated for an abstract manifold, where absent an ambient Euclidean setting there is, apriori, no obvious “tangent plane” where the velocities can reside. This point of view leads to the definition of a velocity as some sort of equivalence class.

See also: tangent bundle, connection, parallel translation