a Kähler manifold is symplectic

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Let $\omega(X,Y) = g(JX,Y)$ on a Kähler manifold. We will prove that ω is a symplectic form.

- ω is anti-symmetric

  $\omega(X,Y) = g(JX,Y) = g(Y,JX) = g(JY,J^2X) = g(JY,-X) = -g(JY,X) = -\omega(Y,X)$. Here we used the fact that g is an Hermitian tensor on a Kähler manifold ($g(X,Y) = g(JX,JY)$)

- ω is linear

  Due to anti-symmetry, we just need to check linearity on the second slot. Since $g(JX,\cdot)$ is by definition linear, ω will also be linear.

- ω is non degenerate

  On a given point on the manifold, pick a non null vector X, $\alpha_X(\cdot) = \omega(X,\cdot) = g(JX,\cdot)$. Since g is non-degenerate[1], α is also non-degenerate (for all X). ω is thus non degenerate.

- ω is closed

  First note that

$$\begin{aligned} X(\omega(Y,Z)) &= \nabla_X(\omega(Y,Z)) \\ &= \nabla_X(g(JY,Z)) \\ &= g(\nabla_X(JY),Z) + g(JY,\nabla_X Z) \\ &= g(J\nabla_X Y,Z) + g(JY,\nabla_X Z) \\ &= \omega(\nabla_X Y,Z) + \omega(Y,\nabla_X Z) \end{aligned}$$

  Here we used the fact that both g and J are covariantly constant ($\nabla g = 0$ and $\nabla J = 0$)

  We aim to prove that $d\omega = 0$ which is equivalent to proving $(d\omega)(X,Y,Z) = 0$ for all vector fields X, Y, Z.

  Since this is a tensorial identity, WLOG we can assume that at a specific point p in the Kähler manifold $[X,Y]_p = [Y,Z]_p = [Z,X]_p = 0$ and prove the indentity for these vector fields[2].

[1] no vector but the null vector is orthogonal to every other vector

[2] in particular this works for the canonical base of T_pM associated with a local coordinate system

Consider X, Y, Z with the previous commutation relations at p, using the formulas for differential forms of small valence:

$$\begin{aligned}
(d\omega)(X,Y,Z) &= X(\omega(Y,Z)) + Y(\omega(Z,X) + Z(\omega(X,Y))) \\
&= \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z) + \\
&\quad \omega(\nabla_Y Z, X) + \omega(Z, \nabla_Y X) + \\
&\quad \omega(\nabla_Z X, Y) + \omega(X, \nabla_Z Y) \\
&= \omega(\nabla_X Y - \nabla_Y X, Z) + \omega(\nabla_Y Z - \nabla_Z Y, X) + \omega(\nabla_Z X - \nabla_X Z, Y)
\end{aligned}$$

The Levi-Civita connection is torsion-free, $\nabla_X Y - \nabla_Y X = [X, Y]$ thus:

$$(d\omega)(X,Y,Z) = \omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y)$$

And since all the commutators are null at p (by assumption) we get that:

$$(d\omega)(X,Y,Z) = 0$$

ω is therefore closed.