coadjoint orbit

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Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. Then G has a natural action on $\mathfrak{g}^*$ called the coadjoint action, since it is dual to the adjoint action of G on $\mathfrak{g}$. The orbits of this action are submanifolds of $\mathfrak{g}^*$ which carry a natural symplectic structure, and are in a certain sense, the minimal symplectic manifolds on which G acts. The orbit through a point $\lambda \in \mathfrak{g}^*$ is typically denoted $\mathcal{O}_\lambda$.

The tangent space $T_\lambda \mathcal{O}_\lambda$ is naturally idenified by the action with $\mathfrak{g}/\mathfrak{r}_\lambda$, where $\mathfrak{r}_\lambda$ is the Lie algebra of the stabilizer of λ. The symplectic form on $\mathcal{O}_\lambda$ is given by $\omega_\lambda(X,Y) = \lambda([X,Y])$. This is obviously anti-symmetric and non-degenerate since $\lambda([X,Y]) = 0$ for all $Y \in \mathfrak{g}$ if and only if $X \in \mathfrak{r}_\lambda$. This also shows that the form is well-defined.

There is a close association between coadoint orbits and the representation theory of G, with irreducible representations being realized as the space of sections of line bundles on coadjoint orbits. For example, if $\mathfrak{g}$ is compact, coadjoint orbits are partial flag manifolds, and this follows from the Borel-Bott-Weil theorem.