arc length of parabola

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The parabola is one of the quite few plane curves, the arc length of which is expressible in closed form; other ones are line, `http://planetmath.org/Circle`circle, semicubical parabola, `http://planetmath.org/NaturalLogarithm2`logarithmic curve, catenary, tractrix, cycloid, clothoid, astroid, Nielsen's spiral, logarithmic spiral. Determining the `http://planetmath.org/PerimeterOfEllipse`arc length of ellipse and hyperbola leads to elliptic integrals.

We evaluate the of the parabola

$$y\;=\;ax^2 \qquad (a > 0) \tag{1}$$

from the apex (the origin) to the point $(x,\,ax^2)$.

The usual arc length

$$s\;=\;\int_0^x\sqrt{1+y'^2}\,dx\;=\;\int_0^x\sqrt{1+4a^2x^2}\,dx\;=\;\frac{1}{2a}\int_0^{2ax}\sqrt{t^2+1}\,dt.$$

where one has made the `http://planetmath.org/ChangeOfVariableInDefiniteIntegral`substit $2ax =: t$. Then one can utilise the result in the entry `http://planetmath.org/IntegrationOfSqr` of $\sqrt{x^2+1}$, whence

$$s\;=\;\frac{1}{4a}\left(2ax\sqrt{4a^2x^2+1}+\operatorname{arsinh}2ax\right). \tag{2}$$

This expression for the parabola arc length becomes especially when the arc is extended from the apex to the end point $(\frac{1}{2a},\,\frac{1}{4a})$ of the parametre, i.e. the latus rectum; this arc length is

$$\frac{1}{4a}(\sqrt{2}+\operatorname{arsinh}1)\;=\;\frac{1}{4a}\left(\sqrt{2}+\ln(1+\sqrt{2})\right).$$

Here, $\sqrt{2}+\ln(1+\sqrt{2}) =: P$ is called the *universal parabolic constant*, since it is common to all parabolas; it is the ratio of the arc to the semiparametre. This constant appears also for example in the areas of some surfaces of revolution (see `http://mathworld.wolfram.com/UniversalParabolicConstant.html`Reese and Sondow).