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equal arc length and area

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We want to determine the nonnegative differentiable real functions $x \mapsto y$ whose graph has the property that the arc length between any two points of it is the same as the http://planetmath.org/AreaOfPlaneRegionarea by the curve, the x-axis and the ordinate lines of those points.

The requirement leads to the equation

$$\int_a^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \;=\; \int_a^x y\,dx. \tag{1}$$

By the fundamental theorem of calculus, we infer from (1) the differential equation

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} \;=\; y, \tag{2}$$

whence $\frac{dy}{dx} = \sqrt{y^2-1}$. In the case $y \not\equiv 1$, the separation of variables yields

$$\int dx \;=\; \int \frac{dy}{\sqrt{y^2-1}},$$

i.e.

$$x+C \;=\; \operatorname{arcosh} y.$$

Consequently, the equation (2) has the general solution

$$y \;=\; \cosh(x+C) \tag{3}$$

and the singular solution

$$y \;\equiv\; 1. \tag{4}$$

The functions defined by (3) and (4) are the only satisfying the given requirement. The graphs are a chain curve (which may be translated in the horizontal direction) and a line parallel to the x-axis. Evidently, the line is the envelope of the integral curves given be the general solution.