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Euclidean space as a manifold

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Let $\mathbb{E}^n$ be n-dimensional Euclidean space, and let $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ be the corresponding n-dimensional inner product space of translation isometries. Alternatively, we can consider Euclidean space as an inner product space that has forgotten which point is its origin. Forgetting even more information, we have the structure of $\mathbb{E}^n$ as a differential manifold. We can obtain an atlas with just one coordinate chart, a Cartesian coordinate system $(x^1, \ldots, x^n)$ which gives us a bijection between $\mathbb{E}^n$ and $\mathbb{R}^n$. The tangent bundle is trivial, with $\mathrm{T}\,\mathbb{E}^n \cong \mathbb{E}^n \times \mathbb{V}$. Equivalently, every tangent space $\mathrm{T}_p\,\mathbb{E}^n$, $p \in \mathbb{E}^n$. is isomorphic to $\mathbb{V}$.

We can retain a bit more structure, and consider $\mathbb{E}^n$ as a Riemannian manifold by equipping it with the metric tensor

$$\begin{aligned} g &= dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n \\ &= \delta_{ij} dx^i \otimes dx^j. \end{aligned}$$

We can also describe g in a coordinate-free fashion as

$$g(u,v) = \langle u, v\rangle, \quad u, v \in \mathbb{V}.$$

Properties

1. Geodesics are straight lines in $\mathbb{R}^n$.
2. The Christoffel symbols vanish identically.
3. The Riemann curvature tensor vanish identically.

Conversely, we can characterize Eucldiean space as a connected, complete Riemannian manifold with vanishing curvature and trivial fundamental group.