Reynolds transport theorem

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Introduction *Reynolds transport theorem* [**?**] is a fundamental theorem used in formulating the basic laws of fluid mechanics. We will enunciate and demonstrate in this entry the referred theorem. For our purpose, let us consider a fluid flow, characterized by its streamlines, in the Euclidean vector space $(\mathbb{R}^3, \|\cdot\|)$ and embedded on it we consider, a continuum body $\mathscr{B}$ occupying a volume $\mathscr{V}$ whose particles are fixed by their material (Lagrangian) coordinates $\mathbf{X}$, and a region $\Re$ where a control volume $\mathfrak{v}$ is defined whose points are fixed by it spatial (Eulerian) coordinates $\mathbf{x}$ and bounded by the control surface $\partial\mathfrak{v}$. An arbitrary tensor field of any rank is defined over the fluid flow according to the following definition.

Definition 1. *We call an* extensive tensor property *to the expression*

$$\Psi(\mathbf{x}, t) := \int_{\mathfrak{v}} \psi(\mathbf{x}, t)\rho(\mathbf{x}, t)dv, \tag{1}$$

where $\psi(\mathbf{x}, t)$ *is the respective* intensive tensor property*.*

Theorem's hypothesis The kinematics of the continuum can be described by a diffeomorphism χ which, at any given instant $t \in [0, \infty) \subset \mathbb{R}$, gives the spatial coordinates $\mathbf{x}$ of the material particle $\mathbf{X}$,

$$\mathscr{V} \times [0, \infty) \to \mathfrak{v} \times [0, \infty), \qquad t \mapsto t, \qquad \mathbf{X} \mapsto \mathbf{x} = \chi(\mathbf{X}, t).$$

Indeed the above sentence corresponds to a change of coordinates which must verify

$$J = \left|\frac{\partial x_i}{\partial X_j}\right| \equiv \left|F_{ij}\right| \neq 0, \qquad F_{ij} := \frac{\partial x_i}{\partial X_j},$$

J being the Jacobian of transformation and F_{ij} the Cartesian components of the so-called *strain gradient tensor* $\mathbf{F}$.

Reynolds transport theorem 1. *The material rate of an extensive tensor property associate to a continuum body $\mathscr{B}$ is equal to the local rate of such property in a control volume $\mathfrak{v}$ plus the efflux of the respective intensive property across its control surface $\partial\mathfrak{v}$.*

Proof. By taking on Eq.(1) the material time derivative,

$$\frac{D\Psi}{Dt} = \dot{\Psi} = \overline{\int_{\mathfrak{v}} \psi\rho \, dv}^{\,\cdot} = \overline{\int_{\mathscr{V}} \psi\rho J dV}^{\,\cdot} = \int_{\mathscr{V}} \overline{\psi\rho J}^{\,\cdot} dV = \int_{\mathscr{V}} (\overline{\psi\rho}^{\,\cdot} J + \psi\rho\dot{J})dV =$$

$$\int_{\mathscr{V}} \Big\{ J \Big[\frac{\partial}{\partial t}(\psi\rho) + \mathbf{v}\cdot\nabla_x(\psi\rho) \Big] + \psi\rho \, (J \nabla_x \cdot \mathbf{v}) \Big\} dV = \int_{\mathscr{V}} \Big\{ \Big[\frac{\partial}{\partial t}(\psi\rho) \Big] + \Big[\mathbf{v}\cdot\nabla_x(\psi\rho) + (\psi\rho)\nabla_x\cdot\mathbf{v} \Big] \Big\} (JdV$$

$$= \int_{\mathfrak{v}} \frac{\partial}{\partial t}(\psi\rho) dv + \int_{\mathfrak{v}} \nabla_x \cdot (\psi\rho \, \mathbf{v}) dv = \frac{\partial}{\partial t} \int_{\mathfrak{v}} \psi\rho \, dv + \int_{\partial\mathfrak{v}} \psi\rho \, \mathbf{v}\cdot\mathbf{n} \, da,$$

since $\partial_t(dv) = 0$ ($\mathbf{x}$ fixed) on the first integral and by applying the Gauss-Green divergence theorem on the second integral at the left-hand side. Finally, by substituting Eq.(1) on the first integral at the right-hand side, we obtain

$$\dot{\Psi} = \frac{\partial \Psi}{\partial t} + \int_{\partial\mathfrak{v}} \psi\rho \, \mathbf{v}\cdot\mathbf{n} \, da, \tag{2}$$

endorsing the theorem statement. □

References

[1] O. Reynolds, *Papers on mechanical and physical subjects-the sub-mechanics of the Universe,* Collected Work, Volume III, Cambridge University Press, 1903.