hyperkähler manifold

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Definition: Let M be a smooth manifold, and *I,J,K* $\in$ *End(TM)* endomorphisms of the tangent bundle satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -Id_{TM}.$$

The manifold *(M,I,J,K)* is called **hypercomplex** if the almost complex structures *I, J, K* are **integrable**. If, in addition, M is equipped with a Riemannian metric g which is Kähler with respect to *I,J,K*, the manifold *(M,I,J,K,g)* is called **hyperkähler**.

Since g is Kähler with respect to *(I,J,K)*, we have

$$\nabla I = \nabla J = \nabla K = 0$$

where ∇ denotes the Levi-Civita connection. This means that the holonomy of ∇ lies inside the group *Sp(n)* of quaternionic-Hermitian endomorphisms. The converse is also true: a Riemannian manifold is hyperkähler if and only if its holonomy is contained in *Sp(n)*. This definition is standard in differential geometry.

In physics literature, one sometimes assumes that the holonomy of a hyperkähler manifold is precisely *Sp(n)*, and not its proper subgroup. In mathematics, such hyperkähler manifolds are called **simple hyperkähler manifolds**.

The following splitting theorem (due to F. Bogomolov) is implied by Berger's classification of irreducible holonomies.

Theorem: Any hyperkähler manifold has a finite covering which is a product of a hyperkähler torus and several simple hyperkähler manifolds.

Consider the Kähler forms $\omega_I, \omega_J, \omega_K$ on M:

$$\omega_I(\cdot,\cdot) := g(\cdot, I\cdot), \quad \omega_J(\cdot,\cdot) := g(\cdot, J\cdot), \quad \omega_K(\cdot,\cdot) := g(\cdot, K\cdot).$$

An elementary linear-algebraic calculation implies that the 2-form $\omega_J + \sqrt{-1}\omega_K$ is of Hodge type (2,0) on *(M,I)*. This form is clearly closed and non-degenerate, hence it is a holomorphic symplectic form.

In algebraic geometry, the word "hyperkähler" is essentially synonymous with "holomorphically symplectic", due to the following theorem, which is implied by Yau's solution of Calabi conjecture (the famous Calabi-Yau theorem).

Theorem: Let *(M,I)* be a compact, Kähler, holomorphically symplectic manifold. Then there exists a unique hyperkähler metric on *(M,I)* with the same Kähler class.

Remark: The hyperkähler metric is unique, but there could be several hyperkähler structures compatible with a given hyperkähler metric on *(M,I)*.

References

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