derivation of Pappus's centroid theorem

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I. Let s denote the arc rotating about the x-axis (and its length) and R be the y-coordinate of the centroid of the arc. If the arc may be given by the equation

$$y = y(x)$$

where $a \le x \le b$, the area of the formed surface of revolution is

$$A = 2\pi \int_a^b y(x)\sqrt{1+[y'(x)]^2}\,dx.$$

This can be concisely written

$$A = 2\pi \int_s y\,ds \tag{1}$$

since differential-geometrically, the product $\sqrt{1+[y'(x)]^2}\,dx$ is the arc-element. We rewrite (1) as

$$A = s \cdot 2\pi \cdot \frac{1}{s}\int_s y\,ds.$$

Here, the last factor is the ordinate of the centroid of the rotating arc, whence we have the result

$$A = s \cdot 2\pi R$$

which states the first Pappus's centroid theorem.

II. For deriving the second Pappus's centroid theorem, we suppose that the region defined by

$$a \le x \le b, \quad 0 \le y_1(x) \le y \le y_2(x),$$

having the area A and the centroid with the ordinate R, rotates about the x-axis and forms the solid of revolution with the volume V. The centroid of the area-element between the arcs $y = y_1(x)$ and $y = y_2(x)$ is $[y_2(x)+y_1(x)]/2$ when the abscissa is x; the area of this element with the width dx is $[y_2(x)-y_1(x)]\,dx$. Thus we get the equation

$$R = \frac{1}{A}\int_a^b \frac{y_2(x)+y_1(x)}{2}[y_2(x)-y_1(x)]\,dx$$

which may be written shortly

$$R = \frac{1}{2A}\int_a^b (y_2^2-y_1^2)\,dx. \tag{2}$$

The volume of the solid of revolution is

$$V \;=\; \pi\int_a^b (y_2^2 - y_1^2)\,dx \;=\; A\cdot 2\pi\cdot\frac{1}{2A}\int_a^b (y_2^2 - y_1^2)\,dx.$$

By (2), this attains the form

$$V \;=\; A\cdot 2\pi R.$$