Cartan structural equations

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To deduce the Cartan structural equations in a coordinated frame we are going to use the definition of the Christoffel symbols (connection coefficients) and where we always are going to use the Einstein sum convention:

$$\nabla_{\partial_i}\partial_j = \Gamma^s{}_{ij}\partial_s$$

and the curvature tensor

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

where X, Y, Z are any three vector fields in a riemannian manifold $\mathcal{M}$ with the Levi-Civita connection ∇.

First, we define through the relation $\nabla_X\partial_i = \omega^s{}_i(X)\partial_s$ a set of scalar function $\omega^s{}_i$ which are easily to see that they actually are 1-forms. We observe that $\omega^s{}_i(\partial_j) = \Gamma^s{}_{ij}$.

They satisfy skew-symmetry rule: $\omega_{si} = -\omega_{is}$, which arises from the covariant constancy of the metric tensor g_{kl} i.e.

$$\begin{aligned}
0 &= \nabla_X g_{kl} \\
&= \nabla_X\langle\partial_k,\partial_l\rangle \\
&= \langle\nabla_X\partial_k,\partial_l\rangle + \langle\partial_k,\nabla_X\partial_l\rangle \\
&= \langle\omega^s{}_k(X)\partial_s,\partial_l\rangle + \langle\partial_k,\omega^s{}_l(X)\partial_s\rangle \\
&= \omega^s{}_k(X)g_{sl} + \omega^s{}_l(X)g_{ks} \\
0 &= \omega_{lk}(X) + \omega_{kl}(X)
\end{aligned}$$

that last equation is valid for each vector field X, then $\omega_{lk} = -\omega_{kl}$.

Next we define through the relation

$$R(X,Y)\partial_i = \Omega^s{}_i(X,Y)\partial_s$$

the scalars $\Omega^s{}_i(X,Y)$ which are the so called connection 2-forms. That they are really 2-forms is an easy caligraphic exercise.

Now by the use of the Riemann curvature tensor above we see

$$\begin{aligned}
R(X,Y)\partial_i &= \nabla_X\nabla_Y\partial_i - \nabla_Y\nabla_X\partial_i - \nabla_{[X,Y]}\partial_i \\
&= \nabla_X(\omega^s{}_i(Y)\partial_s) - \nabla_Y(\omega^s{}_i(X)\partial_s) - \omega^s{}_i[X,Y]\partial_s \\
&= X(\omega^s{}_i(Y))\partial_s + \omega^s{}_i(Y)\nabla_X\partial_s - Y(\omega^s{}_i(X)\partial_s - \omega^s{}_i(X)\nabla_Y\partial_s - \omega^s{}_i[X,Y]\partial_s \\
&= X(\omega^s{}_i(Y))\partial_s + \omega^s{}_i(Y)\omega^t{}_s(X)\partial_t - Y(\omega^s{}_i(X)\partial_s - \omega^s{}_i(X)\omega^t{}_s(Y)\partial_t - \omega^s{}_i[X,Y]\partial_s \\
&= [X(\omega^s{}_i(Y)) + \omega^t{}_i(Y)\omega^s{}_t(X) - Y(\omega^s{}_i(X)) - \omega^t{}_i(X)\omega^s{}_t(Y) - \omega^s{}_i[X,Y]]\partial_s \\
\Omega^s{}_i(X,Y)\partial_s &= [X(\omega^s{}_i(Y)) - Y(\omega^s{}_i(X)) - \omega^s{}_i[X,Y] + \omega^s{}_t(X)\omega^t{}_i(Y) - \omega^s{}_t(Y)\omega^t{}_i(X)]\partial_s
\end{aligned}$$

In this last relation we recognize -in the first three terms- the exterior derivative of ${\omega^s}_i$ evaluated at (X, Y) i.e.

$$d{\omega^s}_i(X, Y) = X({\omega^s}_i(Y)) - Y({\omega^s}_i(X)) - {\omega^s}_i[X, Y]$$

and in the last two terms its wedge product

$${\omega^s}_t \wedge {\omega^t}_i(X, Y) = {\omega^s}_t(X){\omega^t}_i(Y) - {\omega^s}_t(Y){\omega^t}_i(X)$$

all these for any two fields X, Y. Hence

$${\Omega^s}_i = d{\omega^s}_i + {\omega^s}_t \wedge {\omega^t}_i$$

which is called the second Cartan structural equation for the coordinated frame field ∂_i.

More interesting things happen in an an-holonomic basis.