# examples of symplectic manifolds


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Examples of symplectic manifolds: The most basic example of a symplectic manifold is $\mathbb{R}^{2n}$. If we choose coordinate functions $x_1, \dots, x_n, y_1, \dots y_n$, then

$$\omega = \sum_{m=1}^{n} dx_m \wedge dy_m$$

is a symplectic form, and one can easily check that it is closed.

Any orientable 2-manifold is symplectic. Any volume form is a symplectic form.

If $M$ is any manifold, then the cotangent bundle $T^*M$ is symplectic. If $x_1, \dots, x_n$ are coordinates on a coordinate patch $U$ on $M$, and $\xi_1, \dots, \xi_n$ are the functions $T^*(U) \to \mathbb{R}$

$$\xi_i(m, \eta) = \eta(\frac{\partial}{\partial x_i})(m)$$

at any point $(m, \eta) \in T^*(M)$, then

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i.$$

(Equivalently, using the notation $\alpha$ from the entry Poincare 1-form, we can define $\omega = -d\alpha$.)

One can check that this behaves well under coordinate transformations, and thus defines a form on the whole manifold. One can easily check that this is closed and non-degenerate.

All orbits in the coadjoint action of a Lie group on the dual of it Lie algebra are symplectic. In particular, this includes complex Grassmannians and complex projective spaces.

Examples of non-symplectic manifolds: Obviously, all odd-dimensional manifolds are non-symplectic.

More subtlely, if $M$ is compact, $2n$ dimensional and $M$ is a closed 2-form, consider the form $\omega^n$. If this form is exact, then $\omega^n$ must be 0 somewhere, and so $\omega$ is somewhere degenerate. Since the wedge of a closed and an exact form is exact, no power $\omega^m$ of $\omega$ can be exact. In particular, $H^{2m}(M) \neq 0$ for all $0 \leq m \neq n$, for any compact symplectic manifold.

Thus, for example, $S^n$ for $n > 2$ is not symplectic. Also, this means that any symplectic manifold must be orientable.

Finally, it is not generally the case that connected sums of compact symplectic manifolds are again symplectic: Every symplectic manifold admits an almost complex structure (a symplectic form and a Riemannian metric on a manifold are sufficient to define an almost complex structure which is compatible with the symplectic form in a nice way). In the case of a connected sum of two symplectic manifolds, there does not necessarily exist such an almost complex structure, and hence connected sums cannot be (generically) symplectic.