manifold

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Entry type	Definition
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Synonym	differentiable manifold
Synonym	differential manifold
Synonym	smooth manifold
Related topic	NotesOnTheClassicalDefinitionOfAManifold
Related topic	LocallyEuclidean
Related topic	3Manifolds
Related topic	Surface
Related topic	TopologicalManifold
Related topic	ProofOfLagrangeMultiplierMethodOnManifolds
Related topic	Submanifold
Defines	coordinate chart
Defines	chart
Defines	local coordinates
Defines	atlas
Defines	change of coordinates
Defines	differential structure
Defines	transition function
Defines	smooth structure
Defines	diffeomorphism
Defines	diffeomorphic
Defines	topological manifold
Defines	real-analytic manifold

Summary. A *manifold* is a space that is locally like $\mathbb{R}^n$, however lacking a preferred system of coordinates. Furthermore, a manifold can have global topological properties, such as non-contractible http://planetmath.org/Curveloops, that distinguish it from the topologically trivial $\mathbb{R}^n$.

Standard Definition. An n-dimensional topological manifold M is a second countable, Hausdorff topological space[1] that is locally homeomorphic to open subsets of $\mathbb{R}^n$.

A differential manifold is a topological manifold with some additional structure information. A *chart*, also known as a *system of local coordinates*, is a mapping $\alpha : U \to \mathbb{R}^n$, such that the domain $U \subset M$ is an open set, and such that U is homeomorphic to the image $\alpha(U)$. Let $\alpha : U_\alpha \to \mathbb{R}^n$, and $\beta : U_\beta \to \mathbb{R}^n$ be two charts with overlapping http://planetmath.org/Functiondomains. The continuous injection

$$\beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^n$$

is called a *transition function*, and also called a *a change of coordinates*. An *atlas* $\mathcal{A}$ is a collection of charts $\alpha : U_\alpha \to \mathbb{R}^n$ whose domains cover M, i.e.

$$M = \bigcup_\alpha U_\alpha.$$

Note that each transition function is really just n real-valued functions of n real variables, and so we can ask whether these are continuously differentiable. The atlas $\mathcal{A}$ defines a differential structure on M, if every transition function is continuously differentiable.

More generally, for $k = 1, 2, \ldots, \infty, \omega$, the atlas $\mathcal{A}$ is said to define a $\mathcal{C}^k$ differential structure, and M is said to be of class $\mathcal{C}^k$, if all the transition functions are k-times continuously differentiable, or real analytic in the case of $\mathcal{C}^\omega$. Two differential structures of class $\mathcal{C}^k$ on M are said to be isomorphic if the union of the corresponding atlases is also a $\mathcal{C}^k$ atlas, i.e. if all the new

[1] For connected manifolds, the assumption that M is second-countable is logically equivalent to M being paracompact, or equivalently to M being metrizable. The topological hypotheses in the definition of a manifold are needed to exclude certain counter-intuitive pathologies. Standard illustrations of these pathologies are given by the *long line* (lack of paracompactness) and the *forked line* (points cannot be separated). These pathologies are fully described in Spivak. See http://planetmath.org/BibliographyForDifferentialGeometrythis page.

transition functions arising from the merger of the two atlases remain of class $\mathcal{C}^k$. More generally, two $\mathcal{C}^k$ manifolds M and N are said to be diffeomorphic, i.e. have equivalent differential structure, if there exists a homeomorphism $\phi : M \to N$ such that the atlas of M is equivalent to the atlas obtained as ϕ-pullbacks of charts on N.

The atlas allows us to define differentiable mappings to and from a manifold. Let

$$f : U \to \mathbb{R}, \quad U \subset M$$

be a continuous function. For each $\alpha \in \mathcal{A}$ we define

$$f_\alpha : V \to \mathbb{R}, \quad V \subset \mathbb{R}^n,$$

called the representation of f relative to chart α, as the suitably restricted composition

$$f_\alpha = f \circ \alpha^{-1}.$$

We judge f to be differentiable if all the representations f_α are differentiable. A path

$$\gamma : I \to M, \quad I \subset \mathbb{R}$$

is judged to be differentiable, if for all differentiable functions f, the suitably restricted composition $f \circ \gamma$ is a differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Finally, given manifolds M, N, we judge a continuous mapping $\phi : M \to N$ between them to be differentiable if for all differentiable functions f on N, the suitably restricted composition $f \circ \phi$ is a differentiable function on M.