proof of Darboux's theorem (symplectic geometry)

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We first observe that it suffices to prove the theorem for symplectic forms defined on an open neighbourhood of $0 \in \mathbb{R}^{2n}$.

Indeed, if we have a symplectic manifold (M, η), and a point x_0, we can take a (smooth) coordinate chart about x_0. We can then use the coordinate function to push η forward to a symplectic form ω on a neighbourhood of 0 in $\mathbb{R}^{2n}$. If the result holds on $\mathbb{R}^{2n}$, we can compose the coordinate chart with the resulting symplectomorphism to get the theorem in general.

Let $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$. Our goal is then to find a (local) diffeomorphism Ψ so that $\Psi(0) = 0$ and $\Psi^*\omega_0 = \omega$.

Now, we recall that ω is a non–degenerate two–form. Thus, on $T_0\mathbb{R}^{2n}$, it is a non–degenerate anti–symmetric bilinear form. By a linear change of basis, it can be put in the standard form. So, we may assume that $\omega(0) = \omega_0(0)$.

We will now proceed by the "Moser trick". Our goal is to find a diffeomorphism Ψ so that $\Psi(0) = 0$ and $\Psi^*\omega = \omega_0$. We will obtain this diffeomorphism as the time–1 map of the flow of an ordinary differential equation. We will see this as the result of a deformation of ω_0.

Let $\omega_t = t\omega_0 + (1-t)\omega$. Let Ψ_t be the time t map of the differential equation

$$\frac{d}{dt}\Psi_t(x) = X_t(\Psi_t(x))$$

in which X_t is a vector field determined by a condition to be stated later.

We will make the ansatz

$$\Psi_t^*\omega = \omega_t.$$

Now, we differentiate this :

$$0 = \frac{d}{dt}\Psi_t^*\omega_t = \Psi_t^*(L_{X_t}\omega_t + \frac{d}{dt}\omega_t).$$

($L_{X_t}\omega_t$ denotes the Lie derivative of ω_t with respect to the vector field X_t.)

By applying Cartan's identity and recalling that ω is closed, we obtain :

$$0 = \Psi_t^*(d\iota_{X_t}\omega_t + \omega - \omega_0)$$

Now, $\omega - \omega_0$ is closed, and hence, by Poincaré's Lemma, locally exact. So, we can write $\omega - \omega_0 = -d\lambda$.

Thus

$$0 = \Psi_t^*(d(i_{X_t}\omega_t - \lambda))$$

We want to require then

$$i_{X_t}\omega_t = \lambda.$$

Now, we observe that $\omega_0 = \omega$ at 0, so $\omega_t = \omega_0$ at 0. Then, as ω_0 is non–degenerate, ω_t will be non–degenerate on an open neighbourhood of 0. Thus, on this neighbourhood, we may use this to define X_t (uniquely!).

We also observe that $X_t(0) = 0$. Thus, by choosing a sufficiently small neighbourhood of 0, the flow of X_t will be defined for time greater than 1.

All that remains now is to check that this resulting flow has the desired properties. This follows merely by reading our of the ODE, backwards.