deck transformation

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Defines	deck transformation
Defines	covering transformation
Defines	equivalence
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Defines	Galois group of a cover

Let $p\colon E \to X$ be a covering map. A *deck transformation* or *covering transformation* is a map $D\colon E \to E$ such that $p \circ D = p$, that is, such that the following diagram commutes.

$$\begin{tikzcd} E \arrow[rr, "D"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

It is straightforward to check that the set of deck transformations is closed under compositions and the operation of taking inverses. Therefore the set of deck transformations is a subgroup of the group of homeomorphisms of E. This group will be denoted by $\mathrm{Aut}(p)$ and referred to as the *group of deck transformations* or as the *automorphism group* of p. It is worth noting that an alternative name for the group of deck transformations is the *Galois group* of the covering. This terminology arises from an analogy with the fundamental theorem of Galois theory which gives the inclusion-reversing identification addressed in the classification of covering spaces.

In the more general context of fiber bundles deck transformations correspond to *isomorphisms over the identity* since the above diagram could be expanded to:

$$\begin{tikzcd} E \arrow[r, "D"] \arrow[d, "p"'] & E \arrow[d, "p"] \\ X \arrow[r, "\mathrm{id}"'] & X \end{tikzcd}$$

An isomorphism not necessarily over the identity is called an *equivalence*. In other words an equivalence between two covering maps $p\colon E \to X$ and $p'\colon E' \to X'$ is a pair of maps $(\tilde{f}, f)$ that make the following diagram commute

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

i.e. such that $p \circ \tilde{f} = f \circ p'$.

Deck transformations should be perceived as the symmetries of p (hence the notation $\mathrm{Aut}(p)$), and therefore they should be expected to preserve any concept that is defined in terms of P. Most of what follows is an instance of this meta–principle.

Properties of deck transformations

For this section we assume that the total space E is connected and locally path connected. Notice that a deck transformation is a lifting of $p\colon E \to X$ and therefore (according to the lifting theorem) it is uniquely determined by the image of a point. In other words:

Proposition 1. *Let $D_1, D_2 \in \mathrm{Aut}(p)$. If there is $e \in E$ such that $D_1(e) = D_2(e)$ then $D_1 = D_2$. In particular if $D_1(e) = e$ for some $e \in E$ then $D_1 = id$.*

Another simple (or should I say double?) application of the lifting theorem gives

Proposition 2. *Given $e, e' \in E$ with $p(e) = p(e')$, there is a $D \in \mathrm{Aut}(p)$ such that $D(e) = e'$ if and only if $p_*(\pi_1(E, e)) = p_*(\pi_1(E, e'))$, where p_* denotes $\pi_1(p)$.*

Proposition 3. *Deck transformations commute with the monodromy action. That is if $* \in X$, $e \in p^{-1}(*)$, $\gamma \in \pi_1(X, *)$ and $D \in \mathrm{Aut}(p)$ then*

$$D(x \cdot \gamma) = D(x) \cdot \gamma$$

where $\cdot$ denotes the monodromy action.

Proof. If $\tilde{\gamma}$ is a lifting of γ starting at e, then $D \circ \tilde{\gamma}$ is a lifting of γ staring at $D(e)$. $\square$

We simplify notation by using π_e to denote the fundamental group $\pi_1(E, e)$ for $e \in E$.

Theorem 4. *For all $e \in E$*

$$\mathrm{Aut}(p) \cong N\left(p_*(\pi_e)\right)/p_*(\pi_e)$$

where, $N(p_*\pi_e)$ denotes the normalizer of $p_*\pi_e$ inside $\pi_1(X, p(e))$.

Proof. Denote $N(p_*\pi_e)$ by N. Note that if $\gamma \in N$ then $p_*(\pi_{e\cdot\gamma}) = p_*(\pi_e)$. Indeed, recall that $p_*(\pi_e)$ is the stabilizer of e under the momodromy action and therefore we have

$$p * (\pi_{e\cdot\gamma}) = \mathrm{Stab}(e \cdot \gamma) = \gamma\,\mathrm{Stab}(e)\gamma^{-1} = \gamma p_*(\pi_e)\gamma^{-1} = p_*(\pi_e)$$

where, the last equality follows from the definition of normalizer. One can then define a map

$$\varphi\colon\ N \to \mathrm{Aut}(p)$$

as follows: For $\gamma \in N$ let $\varphi(\gamma)$ be the deck transformation that maps e to $e \cdot \gamma$. Notice that Proposition **??** ensures the existence of such a deck transformation while Proposition **??** guarantees its uniqueness. Now

- φ is a homomorphism.
  Indeed $\varphi(\gamma_1\gamma_2)$ and $\varphi(\gamma_1) \circ \varphi(\gamma_2)$ are deck transformations that map e to $e \cdot (\gamma_1\gamma_2)$.

- φ is onto.
  Indeed given $D \in \mathrm{Aut}(p)$ since E is path connected one can find a path α in E connecting e and $D(e)$. Then $p \circ \alpha$ is a loop in X and $D = \varphi(p \circ \alpha)$.

- $\ker(\varphi) = p_*(\pi_e)$.
  Obvious.

  Therefore the theorem follows by the first isomorphism theorem.

□

Corollary 5. *If p is regular covering then*

$$\mathrm{Aut}(p) \cong \pi_1(X, *)/p_*\left(\pi_1(E, e)\right).$$

Corollary 6. *If p is the universal cover then*

$$\mathrm{Aut}(p) \cong \pi_1(X, *)\,.$$