example of cohomology and Mayer-Vietoris sequence

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Consider n-dimensional sphere $S^n = \{v \in R^{n+1} | \quad |v| = 1\}$.

Let $A = \{(x_0, ..., x_n) \in S^n | \quad x_0 > -1/2\}$ and $B = \{(x_0, ..., x_n) \in S^n | \quad x_0 < 1/2\}$.

Of course both A and B are open (in S^n) and their union is S^n. Furthermore, it can be easily seen, that their intersection can be contracted into "big circle", i.e. $A \cap B$ has homotopy type of S^{n-1}. Also both A and B are contractible (they are homeomorphic to R^n via stereographic projection). So, write part of a Meyer-Vietoris sequence (for the cohomology $H^m(X) = H^m(X, G)$, where G is a fixed Abelian group):

$\cdots \to H^m(A) \oplus H^m(B) \to H^m(A \cap B) \to H^{m+1}(S^n) \to H^{m+1}(A) \oplus H^{m+1}(B) \to \cdots$

Since both A and B are contractible and $A \cap B$ is homotopic to S^{n-1}, we have the following short exact sequence:

$0 \to H^m(S^{n-1}) \to H^{m+1}(S^n) \to 0$

which shows that $H^m(S^{n-1})$ is isomorphic to $H^m(S^n)$ for every $n > 0$ and $m > 0$. So, in order to calculate cohomology groups of spheres, we only need to know the cohomology groups of S^1. And those can be also calculated, if we once again apply previous schema. Note, that in the case of S^1 we have that $A \cap B$ has the homotopy type of a discrete space with two points. Therefore all their cohomology groups are trivial, except for H^0 (which can be easily calculated to be equal to $H^0(*) \oplus H^0(*)$, where * is a one-pointed space).

This schema can be used for other spaces like the torus (which can be also calculated from Kunneth's formula).