equivalent statements to statement that sphere is not contractible

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Let V be a normed space. Recall the definition of the sphere and the ball in V:

$$\mathbb{S} = \{v \in V; \quad \|v\| = 1\}; \quad \mathbb{B} = \{v \in V; \quad \|v\| \leq 1\}.$$

Proposition. The following are equivalent:

(1) $\mathbb{S}$ is not contractible;

(2) for each continous map $F : \mathbb{B} \to \mathbb{B}$ there exists $x \in \mathbb{B}$ such that $F(x) = x$;

(3) there is no retraction from $\mathbb{B}$ onto $\mathbb{S}$.

Proof. The proof of this proposition probably can be found in some books about topology. I present here the proof from my lecture due to Prof. Górniewicz.

(1) $\Rightarrow$ (2) Assume there exists a continous map $F : \mathbb{B} \to \mathbb{B}$ such that for each $x \in \mathbb{B}$ we have $F(x) \neq x$. Define a map $H : \mathbb{S} \times [0, 1] \to \mathbb{S}$ as follows:

$$H(x,t) = \begin{cases} \frac{x - 2tF(x)}{\|x - 2tF(x)\|}, & \text{if } 0 \leq t \leq \frac{1}{2} \\ \\ \frac{(2-2t)x - F((2-2t)x)}{\|(2-2t)x - F((2-2t)x)\|} & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Thanks to the condition $F(x) \neq x$ this map is well defined and it is easy to check that this is a homotopy from the identity map to constant map. But $\mathbb{S}$ is not contractible. Contradiction.

(2) $\Rightarrow$ (3) Assume there exists a retraction $r : \mathbb{B} \to \mathbb{S}$. Define a map $F : \mathbb{B} \to \mathbb{B}$ by the formula $F(x) = -r(x)$. This map has no fixed point. Contradiction.

(3) $\Rightarrow$ (1) Assume that $\mathbb{S}$ is contractible and take any homotopy $H : \mathbb{S} \times [0, 1] \to \mathbb{S}$ from constant map to identity map, i.e. for all $x \in \mathbb{S}$ we have $H(x, 0) = x_0$ (for some $x_0 \in \mathbb{S}$) and $H(x, 1) = x$. Define a map $r : \mathbb{B} \to \mathbb{S}$ as follows:

$$r(x) = \begin{cases} x_0, & \text{if } \|x\| \leq \frac{1}{2} \\ \\ H(\frac{x}{\|x\|}, 2\|x\| - 1) & \text{if } \|x\| \geq \frac{1}{2} \end{cases}$$

It is easy to see that this formula defines a retraction from $\mathbb{B}$ onto $\mathbb{S}$. Contradiction. $\square$

Note that this proposition does not state that any of the conditions (1), (2), (3) hold. It only states that they are equivalent. It is well known that all of them are true if V is finite dimensional and all are false if V is infinite dimensional.