planetmath.org

Math for the people, by the people.

homotopy double groupoid of a Hausdorff space

Canonical name	HomotopyDoubleGroupoidOfAHausdorffSpace
Date of creation	2013-03-22 18:14:56
Last modified on	2013-03-22 18:14:56
Owner	bci1 (20947)
Last modified by	bci1 (20947)
Numerical id	76
Author	bci1 (20947)
Entry type	Definition
Classification	msc 55N33
Classification	msc 55N20
Classification	msc 22A22
Classification	msc 18D05
Classification	msc 55U40
Synonym	homotopy double groupoid of a compact space
Related topic	HomotopyAdditionLemma
Related topic	GroupoidCategory
Related topic	ThinDoubleTracks
Related topic	HigherDimensionalAlgebraHDA
Related topic	FundamentalGroupoidFunctor
Related topic	FundamentalGroupoid2
Related topic	FundamentalGroupoid
Related topic	HomotopyGroups
Related topic	HigherDimensionalAlgebraHDA
Related topic	GeneralizedVanKampenTheoremsHigherDimensional
Related topic	Homot
Defines	double groupoid
Defines	homotopy double groupoid
Defines	thin equivalence
Defines	thinly equivalent
Defines	higher dimensional homotopy
Defines	double category

Homotopy double groupoid of a Hausdorff space

Let X be a Hausdorff space. Also consider the HDA concept of a `http://planetmath.org/HigherDimensi` groupoid, and how it can be completely specified for a Hausdorff space, X. Thus, in ref. [**?**] Brown et al. associated to X a double groupoid, $\boldsymbol{\rho}_2^{\square}(X)$, called the *homotopy double groupoid of* X which is completely defined by the data specified in Definitions 0.1 to 0.3 in this entry and related objects.

Generally, the geometry of squares and their compositions leads to a common representation of a *double groupoid* in the following form:

$$\mathsf{D} = \begin{tikzcd} S \arrow[r, shift left=2, "s^1"] \arrow[r, shift right=2, "t^1"'] \arrow[d, shift right=2, "s_2"'] \arrow[d, shift left=2, "t_2"] & H \arrow[l] \arrow[d, shift right=2, "s"'] \arrow[d, shift left=2, "t"] \\ V \arrow[u] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & M \arrow[l] \arrow[u] \end{tikzcd} , \tag{0.1}$$

where M is a set of 'points', H, V are 'horizontal' and 'vertical' groupoids, and S is a set of 'squares' with two compositions.

The laws for a double groupoid are also defined, more generally, for any topological space $\mathbb{T}$, and make it also describable as a groupoid internal to the category of groupoids. Further details of this general definition are provided next.

Given two groupoids H, V over a set M, there is a double groupoid $\square(H, V)$ with H, V as horizontal and vertical edge groupoids, and squares given by quadruples

$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \tag{0.2}$$

for which we assume always that $h, h' \in H$, $v, v' \in V$ and that the initial and final points of these edges match in M as suggested by the notation, that is for example $sh = sv, th = sv', \ldots$, etc. The compositions are to be inherited from those of H, V, that is:

$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_1 \begin{pmatrix} & h' & \\ w & & w' \\ & h'' & \end{pmatrix} = \begin{pmatrix} & h & \\ vw & & v'w' \\ & h'' & \end{pmatrix}, \quad \begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_2 \begin{pmatrix} & k & \\ v' & & v'' \\ & k' & \end{pmatrix} = \begin{pmatrix} & hk & \\ v & & v'' \\ & h'k' & \end{pmatrix} . \tag{0.3}$$

Alternatively, the data for the above double groupoid D can be specified as a triple of groupoid structures:

$$(D_2, D_1, \partial_1^-, \partial_1^+, +_1, \varepsilon_1), (D_2, D_1, \partial_2^-, \partial_2^+, +_2, \varepsilon_2), (D_1, D_0, \partial_1^-, \partial_1^+, +, \varepsilon),$$

where:

$$D_0 = M \ , \ D_1 = V = H \ , \ D_2 = S,$$

$$s^1 = \partial_2^- \ , \ t^1 = \partial_2^+ \ , \ s_2 = s = \partial_1^-$$

and

$$t_2 = t = \partial_1^+ .$$

Then, as a first step, consider this data for the homotopy double groupoid specified in the following definition; in order to specify completely such data one also needs to define the related concepts of *thin equivalence* and the *relation of cubically thin homotopy*, as provided in the two definitions following the homotopy double groupoid data specified above and in the (main) Definition 0.1.

Definition 0.1. The data for the homotopy double groupoid, $\boldsymbol{\rho}^{\square}(X)$, will be denoted by :

$$(\boldsymbol{\rho}_2^{\square}(X), \boldsymbol{\rho}_1^{\square}(X), \partial_1^-, \partial_1^+, +_1, \varepsilon_1), \boldsymbol{\rho}_2^{\square}(X), \boldsymbol{\rho}_1^{\square}(X), \partial_2^-, \partial_2^+, +_2, \varepsilon_2)$$
$$(\boldsymbol{\rho}_1^{\square}(X), X, \partial^-, \partial^+, +, \varepsilon).$$

Here $\boldsymbol{\rho}_1(X)$ denotes the *path groupoid* of X from ref. [**?**] where it was defined as follows. The objects of $\boldsymbol{\rho}_1(X)$ are the points of X. The morphisms of $\boldsymbol{\rho}_1^{\square}(X)$ are the equivalence classes of paths in X with respect to the following (thin) equivalence relation $\sim_T$, defined as follows. The data for $\boldsymbol{\rho}_2^{\square}(X)$ is defined last; furthermore, the symbols specified after the thin square symbol specify both the sides (or the groupoid 'dimensions') of the square which are involved (i.e., 1 and 2, respectively), and also the order in which the shown operations (∂_1^-, ε_2... , etc) are to be performed relative to the thin square specified for each groupoid, ρ_1 *or* ρ_2; moreover, all such symbols are explicitly and precisely defined in the related entries of the concepts involved in this definition. These two groupoids can also be pictorially represented as the (H, V) pair depicted in the large Diagram (0.1), or D, shown at the top of this page.

Definition 0.2. *Thin equivalence*

Let $a, a' : x \simeq y$ be paths in X. Then a is *thinly equivalent* to a', denoted $a \sim_T a'$, if there is a thin relative homotopy between a and a'.

We note that $\sim_T$ is an equivalence relation, see [**?**]. We use $\langle a \rangle : x \simeq y$ to denote the $\sim_T$ class of a path $a : x \simeq y$ and call $\langle a \rangle$ the *semitrack* of a. The groupoid structure of $\boldsymbol{\rho}_1^{\square}(X)$ is induced by concatenation, $+$, of paths. Here one makes use of the fact that if $a : x \simeq x'$, $a' : x' \simeq x''$, $a'' : x'' \simeq x'''$ are paths then there are canonical thin relative homotopies

$$(a + a') + a'' \simeq a + (a' + a'') : x \simeq x''' \ (\textit{rescale})$$
$$a + e_{x'} \simeq a : x \simeq x'; \ e_x + a \simeq a : x \simeq x' \ (\textit{dilation})$$
$$a + (-a) \simeq e_x : x \simeq x \ (\textit{cancellation}).$$

The source and target maps of $\boldsymbol{\rho}_1^{\square}(X)$ are given by

$$\partial_1^- \langle a \rangle = x, \ \ \partial_1^+ \langle a \rangle = y,$$

if $\langle a \rangle : x \simeq y$ is a semitrack. Identities and inverses are given by

$$\varepsilon(x) = \langle e_x \rangle \quad \text{resp.} - \langle a \rangle = \langle -a \rangle .$$

At the next step, in order to construct the groupoid $\boldsymbol{\rho}_2^{\square}(X)$ data in Definition 0.1, R. Brown et al. defined as follows a *relation of cubically thin homotopy* on the set $R_2^{\square}(X)$ of squares.

Definition 0.3. *Cubically thin homotopy*
Let u, u' be squares in X with common vertices.

1. A *cubically thin homotopy* $U : u \equiv_T^{\square} u'$ between u and u' is a cube $U \in R_3^{\square}(X)$ such that

   (i) U is a homotopy between u and u',

   $$\text{i.e. } \partial_1^-(U) = u, \;\; \partial_1^+(U) = u',$$

   (ii) U is rel. vertices of I^2,

   $$\text{i.e. } \partial_2^-\partial_2^-(U), \;\; \partial_2^-\partial_2^+(U), \;\; \partial_2^+\partial_2^-(U), \;\; \partial_2^+\partial_2^+(U) \text{ are constant,}$$

   (iii) the faces $\partial_i^{\alpha}(U)$ are thin for $\alpha = \pm 1, \; i = 1, 2$.

2. The square u is *cubically T-equivalent* to u', denoted $u \equiv_T^{\square} u'$ if there is a cubically thin homotopy between u and u'.

Remark By removing from the above double groupoid construction the condition that all morphisms must be invertible one obtains the prototype of a *double category*.

References

[1] K.A. Hardie, K.H. Kamps and R.W. Kieboom., A homotopy 2-groupoid of a Hausdorff *Applied Categorical Structures*, **8** (2000): 209-234.

[2] R. Brown, K.A. Hardie, K.H. Kamps and T. Porter., `http://www.tac.mta.ca/tac/volumes/10/2/10-02.pdf`A homotopy double groupoid of a Hausdorff space , *Theory and Applications of Categories* **10**,(2002): 71-93.