sheaf of sections

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0.1 Presheaf Definition

Consider a rank r vector bundle $E \to M$, whose typical fibre is defined with respect to a field k. Let $\{U_\alpha\}$ constitute a cover for M. Then, sections of the bundle over some $U \subset M$ are defined as continuous functions $U \to E$, which commute with the natural projection map $\pi : E \to M$; $\pi \circ s = id_M$. Denote the space of sections of the bundle over U to be $\Gamma(U, E)$. The space of sections is a vector space over the field k by defining addition and scalar multiplication pointwise: for $s, t \in \Gamma(U, E)$, $p \in U$ and $a \in k$

$$(s + t)(p) \equiv s(p) + t(p) \qquad\qquad (a \cdot s)(p) \equiv a \cdot s(p).$$

Then, this forms a presheaf $\mathcal{E}$, a functor from $((\mathrm{top}_M))$ to the category of vector spaces, with restriction maps the natural restriction of functions.

0.2 Sheaf Axioms

It is easy to see that it satisfies the sheaf axioms: for U open and $\{V_i\}$ a cover of U,

1. if $s \in \mathcal{E}(U)$ and $s|_{V_i} = 0$ for all i, then $s = 0$.
2. if $s_i \in \mathcal{E}(V_i)$ for all i, such that for each i, j with $V_i \cap V_j \neq \emptyset$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an $s \in \mathcal{E}(U)$ with $s|_{V_i} = s_i$ for all i.

The first follows from the fact that for any U, there is always at least one element of $\mathcal{E}(U)$, the zero section, and that the transition functions of the bundle are linear maps. The second follows by the construction of the bundle.

1 Sheafification

We may also see the vector bundle by applying associated sheaf construction to the presheaf $U \mapsto \Gamma(U, E)$. First though, we show that the stalk of the sheaf $\mathcal{E}$ at a point is isomorphic to the fibre of the bundle E at the point. Let $[s, U]$ be a germ at $p \in M$ ($p \in U \subset M$), and define a map $\psi : \mathcal{E}_p \to E_p$ by

$$\psi : [s, U] \mapsto s_p.$$

First, we show that the map is a vector space homomorphism. Consider two germs $[s, U]$ and $[t, V]$ in $\mathcal{E}_p$. These map to s_p and t_p respectively. We add the germs by finding an open set $W \in U \cap V$ and adding the restrictions of the sections;

$$[s, U] + [t, V] \equiv [s|_W + t|_W, W].$$

Of course, $p \in W$, so we have $\psi(s|_W + t|_W) = s_p + t_p$, since the restriction maps are simply restriction of functions. Now, it is easy to show that ψ is injective. Assume $\psi([t, V]) = \psi([s, U]) = s_p$. Then

$$\begin{aligned}
\psi([t, V]) - \psi([s, U]) &= s_p - s_p \\
\psi([t, V] - [s, U]) &= 0 \\
[t, V] &= [s, U]
\end{aligned}$$

Now, we show that ψ is surjective. For $s_p \in E_p$, let $U \subset M$ open be isomorphic to some subset $U_{\mathbb{R}}$ of $\mathbb{R}^m$. Then, $\Gamma(U, E)$ is the set of continuous maps $U \to V_E$, where V_E is the typical fibre of E;

$$\Gamma(U, E) = \bigoplus_{i=1}^{r} \mathcal{C}^{\infty}_{U_{\mathbb{R}}}.$$

Then let $[s, U]$ be the constant function $s : U_{\mathbb{R}} \mapsto s_x$, and we have constructed an isomorphism ψ between $\mathcal{E}_p$ and E_p.

To construct the Étalé space, take the disjoint union of stalks, $\text{Spé}(\mathcal{E}) = \coprod_{p \in M} \mathcal{E}_p$, and endow it with the following topology: the open sets shall be of the form

$$U_s = \left\{ s_p | s \in \Gamma(U, \mathcal{E}), p \in U \subset M \right\},$$

collection of germs of sections at points in $U \subset M$.

Then, the associated sheaf to $\mathcal{E}$ is the presheaf which assigns continuous maps $\Gamma(U, \text{Spé}(\mathcal{E}))$ to each open U. These are maps where the preimage of U_s is open. Clearly, this implies that $\Gamma(U, E) \subset \Gamma(U, \text{Spé}(\mathcal{E}))$. To go the other way, note that open sets of $\text{Spé}(\mathcal{E})$ are the images of continuous maps $U \to E$. An open subset of $\text{Spé}(\mathcal{E})$ may be written as a union of U_t; $U_{ts} \equiv \{t_p, s_p | p \in U\}$. Then, by single-valuedness of maps, a continuous map $U \to \text{Spé}(\mathcal{E})$ must map to U_t for some $t \in \Gamma(U, E)$, so we have $\Gamma(U, E) \supset \Gamma(U, \text{Spé}(\mathcal{E}))$.