simplicial complex

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An abstract *simplicial complex* K is a collection of nonempty finite sets with the property that for any element $\sigma \in K$, if $\tau \subset \sigma$ is a nonempty subset, then $\tau \in K$. An element of K of cardinality $n+1$ is called an *n-simplex*. An element of an element of K is called a *vertex*. In what follows, we may occasionally identify a vertex v with its corresponding singleton set $\{v\} \in K$; the reader will be alerted when this is the case.

The *standard n-complex*, denoted by Δ_n, is the simplicial complex consisting of all nonempty subsets of $\{0, 1, \dots, n\}$.

1 Geometry of a simplicial complex

Let K be a simplicial complex, and let V be the set of vertices of K. Although there is an established notion of infinite simplicial complexes, the geometrical treatment of simplicial complexes is much simpler in the finite case and so for now we will assume that V is a finite set of cardinality k.

We introduce the vector space $\mathbb{R}^V$ of formal $\mathbb{R}$–linear combinations of elements of V; i.e.,

$$\mathbb{R}^V := \{a_1 V_1 + a_2 V_2 + \cdots + a_k V_k \mid a_i \in \mathbb{R},\ V_i \in V\},$$

and the vector space operations are defined by formal addition and scalar multiplication. Note that we may regard each vertex in V as a one-term formal sum, and thus as a point in $\mathbb{R}^V$.

The *geometric realization* of K, denoted $|K|$, is the subset of $\mathbb{R}^V$ consisting of the union, over all $\sigma \in K$, of the convex hull of $\sigma \subset \mathbb{R}^V$. If we fix a bijection $\phi \colon V \to \{1, \dots, k\}$, then the vector space $\mathbb{R}^V$ is isomorphic to the Euclidean vector space $\mathbb{R}^k$ via ϕ, and the set $|K|$ inherits a metric from $\mathbb{R}^k$ making it into a metric space and topological space. The isometry class of K is independent of the choice of the bijection ϕ.

Examples:

1. $\Delta_2 = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$ has $V = 3$, so its realization $|\Delta_2|$ is a subset of $\mathbb{R}^3$, consisting of all points on the hyperplane $x + y + z = 1$ that are inside or on the boundary of the first octant. These points form a triangle in $\mathbb{R}^3$ with one face, three edges, and three vertices (for example, the convex hull of $\{0,1\} \in \Delta_2$ is the edge of this triangle that lies in the xy–plane).

2. Similarly, the realization of the standard n–simplex Δ_n is an n–dimensional tetrahedron contained inside $\mathbb{R}^{n+1}$.

3. A triangle without interior (a "wire frame" triangle) can be geometrically realized by starting from the simplicial complex $\{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}$.

Notice that, under this procedure, an element of K of cardinality 1 is geometrically a vertex; an element of cardinality 2 is an edge; cardinality 3, a face; and, in general, an element of cardinality n is realized as an n-face inside $\mathbb{R}^V$.

In general, a *triangulation* of a topological space X is a simplicial complex K together with a homeomorphism from $|K|$ to X.

2 Homology of a simplicial complex

In this section we define the homology and cohomology groups associated to a simplicial complex K. We do so not because the homology of a simplicial complex is so intrinsically interesting in and of itself, but because the resulting homology theory is identical to the singular homology of the associated topological space $|K|$, and therefore provides an accessible way to calculate the latter homology groups (and, by extension, the homology of any space X admitting a triangulation by K).

As before, let K be a simplicial complex, and let V be the set of vertices in K. Let the chain group $C_n(K)$ be the subgroup of the exterior algebra $\Lambda(\mathbb{R}^V)$ generated by all elements of the form $V_0 \wedge V_1 \wedge \cdots \wedge V_n$ such that $V_i \in V$ and $\{V_0, V_1, \ldots, V_n\} \in K$. Note that we are ignoring here the $\mathbb{R}$–vector space structure of $\mathbb{R}^V$; the group $C_n(K)$ under this definition is merely a free abelian group, generated by the alternating products of the above form and with the relations that are implied by the properties of the wedge product.

Define the boundary map $\partial_n : C_n(K) \longrightarrow C_{n-1}(K)$ by the formula

$$\partial_n(V_0 \wedge V_1 \wedge \cdots \wedge V_n) := \sum_{j=0}^{n} (-1)^j (V_0 \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_n),$$

where the hat notation means the term under the hat is left out of the product, and extending linearly to all of $C_n(K)$. Then one checks easily that $\partial_{n-1} \circ \partial_n = 0$, so the collection of chain groups $C_n(K)$ and boundary maps

∂_n forms a chain complex $\mathcal{C}(K)$. The simplicial homology and cohomology groups of K are defined to be that of $\mathcal{C}(K)$.

Theorem: The simplicial homology and cohomology groups of K, as defined above, are canonically isomorphic to the singular homology and cohomology groups of the geometric realization $|K|$ of K.

The proof of this theorem is considerably more difficult than what we have done to this point, requiring the techniques of barycentric subdivision and simplicial approximation, and we refer the interested reader to [**?**].

References

[1] Munkres, James. *Elements of Algebraic Topology*, Addison–Wesley, New York, 1984.