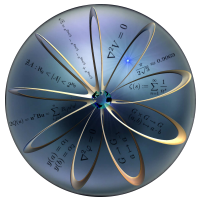

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fixed point property

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Let X be a topological space. If every continuous function $f\colon X \to X$ has a `http://planetmath.org/FixedPoint`fixed point, then X is said to have the *fixed point property*.

The fixed point property is obviously preserved under homeomorphisms. If $h\colon X \to Y$ is a homeomorphism between topological spaces X and Y, and X has the fixed point property, and $f\colon Y \to Y$ is continuous, then $h^{-1} \circ f \circ h$ has a fixed point $x \in X$, and $h(x)$ is a fixed point of f.

Examples

1. A space with only one point has the fixed point property.
2. A closed interval $[a, b]$ of $\mathbb{R}$ has the fixed point property. `http://planetmath.org/BrouwerFi` can be seen using the mean value theorem.
3. The extended real numbers have the fixed point property, as they are homeomorphic to $[0, 1]$.
4. The topologist's sine curve has the fixed point property.
5. The real numbers $\mathbb{R}$ do not have the fixed point property. For example, the map $x \mapsto x + 1$ on $\mathbb{R}$ has no fixed point.
6. An open interval (a, b) of $\mathbb{R}$ does not have the fixed point property. This follows since any such interval is homeomorphic to $\mathbb{R}$. Similarly, an open ball in $\mathbb{R}^n$ does not have the fixed point property.
7. Brouwer's Fixed Point Theorem states that in $\mathbb{R}^n$, the closed unit ball with the subspace topology has the fixed point property. (Equivalently, $[0, 1]^n$ has the fixed point property.) The Schauder Fixed Point Theorem generalizes this result further.
8. For each $n \in \mathbb{N}$, the real projective space $\mathbb{RP}^{2n}$ has the fixed point property.
9. Every simply-connected plane continuum has the fixed-point property.
10. The Alexandroff–Urysohn square (also known as the Alexandroff square) has the fixed point property.

Properties

1. http://planetmath.org/AnyTopologicalSpaceWithTheFixedPointPropertyIsConnected topological space with the fixed point property is connected and http://planetmath.org/TOS

2. Suppose X is a topological space with the fixed point property, and Y is a retract of X. Then Y has the fixed point property.

3. Suppose X and Y are topological spaces, and $X \times Y$ has the fixed point property. Then X and Y have the fixed point property. (Proof: If $f\colon X \to X$ is continuous, then $(x, y) \mapsto (f(x), y)$ is continuous, so f has a fixed point.)

References

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[3] L. E. Ward, *Topology, An Outline for a First Course*, Marcel Dekker, Inc., 1972.

[4] Charles Hagopian, The Fixed-Point Property for simply-connected plane continua, Trans. Amer. Math. Soc. 348 (1996) 4525–4548.