# Van Kampen's theorem

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Van Kampen's theorem for fundamental groups may be stated as follows:

**Theorem 1.** *Let $X$ be a topological space which is the union of the interiors of two path connected subspaces $X_1, X_2$. Suppose $X_0 := X_1 \cap X_2$ is path connected. Let further $* \in X_0$ and $i_k \colon \pi_1(X_0, *) \to \pi_1(X_k, *)$, $j_k \colon \pi_1(X_k, *) \to \pi_1(X, *)$ be induced by the inclusions for $k = 1, 2$. Then $X$ is path connected and the natural morphism*

$$\pi_1(X_1, *) \bigstar_{\pi_1(X_0,*)} \pi_1(X_2, *) \to \pi_1(X, *)\,,$$

*is an isomorphism, that is, the fundamental group of $X$ is the free product of the fundamental groups of $X_1$ and $X_2$ with amalgamation of $\pi_1(X_0, *)$.*

Usually the morphisms induced by inclusion in this theorem are not themselves injective, and the more precise version of the statement is in terms of pushouts of groups.

The notion of pushout in the category of groupoids allows for a version of the theorem for the non path connected case, using the fundamental groupoid $\pi_1(X, A)$ on a set $A$ of base points, [**?**]. This groupoid consists of homotopy classes rel end points of paths in $X$ joining points of $A \cap X$. In particular, if $X$ is a contractible space, and $A$ consists of two distinct points of $X$, then $\pi_1(X, A)$ is easily seen to be isomorphic to the groupoid often written $\mathcal{I}$ with two vertices and exactly one morphism between any two vertices. This groupoid plays a role in the theory of groupoids analogous to that of the group of integers in the theory of groups.

**Theorem 2.** *Let the topological space $X$ be covered by the interiors of two subspaces $X_1, X_2$ and let $A$ be a set which meets each path component of $X_1, X_2$ and $X_0 := X_1 \cap X_2$. Then $A$ meets each path component of $X$ and the following diagram of morphisms induced by inclusion*

$$\begin{array}{ccc}
\pi_1(X_0, A) & {\scriptstyle \pi_1(i_1)} & \pi_1(X_1, A) \\
{\scriptstyle \pi_1(i_2)} & & {\scriptstyle \pi_1(j_1)} \\
\pi_1(X_2, A) & {\scriptstyle \pi_1(j_2)} & \pi_1(X, A)
\end{array}$$

*is a pushout diagram in the category of groupoids.*

The interpretation of this theorem as a calculational tool for fundamental groups needs some development of 'combinatorial groupoid theory', [**?**, **?**].

This theorem implies the calculation of the fundamental group of the circle as the group of integers, since the group of integers is obtained from the groupoid $\mathcal{I}$ by identifying, in the category of groupoids, its two vertices.

There is a version of the last theorem when $X$ is covered by the union of the interiors of a family $\{U_\lambda : \lambda \in \Lambda\}$ of subsets, [**?**]. The conclusion is that if $A$ meets each path component of all 1,2,3-fold intersections of the sets $U_\lambda$, then A meets all path components of $X$ and the diagram

$$\bigsqcup_{(\lambda,\mu)\in\Lambda^2} \pi_1(U_\lambda \cap U_\mu, A) \rightrightarrows \bigsqcup_{\lambda\in\Lambda} \pi_1(U_\lambda, A) \to \pi_1(X, A)$$

of morphisms induced by inclusions is a coequaliser in the category of groupoids.

# References


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