Thom class

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Owner	antonio (1116)
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Author	antonio (1116)
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Let h^* be a generalized cohomology theory (for example, let $h^* = H^*$, singular cohomology with integer coefficients). Let $\xi \to X$ be a vector bundle of dimension d over a topological space X. Assume for convenience that ξ has a Riemannian metric, so that we may speak of its associated sphere and disk bundles, $S(\xi)$ and $D(\xi)$ respectively.

Let $x \in X$, and consider the fibers $S(\xi_x)$ and $D(\xi_x)$. Since $D(\xi_x)/S(\xi_x)$ is homeomorphic to the d-sphere, the Eilenberg-Steenrod axioms for h^* imply that $h^{*+d}(D(\xi_x), S(\xi_x))$ is isomorphic to the coefficient group $h^*(\mathrm{pt})$ of h^*. In fact, $h^*(D(\xi_x), S(\xi_x))$ is a free module of rank one over the ring $h^*(\mathrm{pt})$.

Definition 1 An element $\tau \in h^*(D(\xi), S(\xi))$ is said to be a *Thom class* for ξ if, for every $x \in X$, the restriction of τ to $h^*(D(\xi_x), S(\xi_x))$ is an $h^*(\mathrm{pt})$-module generator.

Note that τ lies necessarily in $h^d(D(\xi), S(\xi))$.

Definition 2 If a Thom class for ξ exists, ξ is said to be *orientable* with respect to the cohomology theory h^*.

Remark 1 Notice that we may consider τ as an element of the reduced h^*-cohomology group $\tilde{h}^*(X^\xi)$, where X^ξ is the Thom space $D(\xi)/S(\xi)$ of ξ. As is the case in the definition of the Thom space, the Thom class may be defined without reference to associated disk and sphere bundles, and hence to a Riemannian metric on ξ. For example, the pair $(\xi, \xi - X)$ (where X is included in ξ as the zero section) is homotopy equivalent to $(D(\xi), S(\xi))$.

Remark 2 If h^* is singular cohomology with integer coefficients, then ξ has a Thom class if and only if it is an orientable vector bundle in the ordinary sense, and the choices of Thom class are in one-to-one correspondence with the orientations.