Thom isomorphism theorem

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Let $\xi \to X$ be a d-dimensional vector bundle over a topological space X, and let h^* be a multiplicative generalized cohomology theory, such as ordinary cohomology. Let $\tau \in h^d(D(\xi), S(\xi))$ be a Thom class for ξ, where $D(\xi)$ and $S(\xi)$ are the associated disk and sphere bundles of ξ.

Since h^* is a multiplicative theory, there is a generalized cup product map

$$h^*(D(\xi)) \otimes_{h^*} h^*(D(\xi), S(\xi)) \to h^*(D(\xi), S(\xi)),$$

where the tensor product is over the coefficient ring $h^*(\mathrm{pt})$ of the theory. Using the isomorphism $p^* : h^*(X) \cong h^*(D(\xi))$ induced by the homotopy equivalence $p : D(\xi) \to X$, we obtain a homomorphism

$$T : h^n(X) \to h^{n+d}(D(\xi), S(\xi)) \cong \tilde{h}^{n+d}(X^\xi)$$

taking α to $p^*(\alpha) \cdot \tau$. Here X^ξ stands for the Thom space $D(\xi)/S(\xi)$ of ξ.

Thom isomorphism theorem T is an isomorphism $h^*(X) \cong \tilde{h}^{*+d}(X^\xi)$ of graded modules over $h^*(\mathrm{pt})$.

Remark 1 When ξ is a trivial bundle of dimension 1, this generalizes the suspension isomorphism. In fact, a typical proof of this theorem for compact X proceeds by induction over the number of open sets in a trivialization of ξ, using the suspension isomorphism as the base case and the Mayer-Vietoris sequence to carry out the inductive step.

Remark 2 There is also a homology Thom isomorphism $\tilde{h}_{*+d}(X^\xi) \cong h_*(X)$, in which the map is given by cap product with the Thom class rather than cup product.