degree (map of spheres)

Canonical name	DegreemapOfSpheres
Date of creation	2013-03-22 13:22:12
Last modified on	2013-03-22 13:22:12
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Last modified by	drini (3)
Numerical id	12
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Entry type	Definition
Classification	msc 55M25
Defines	degree

Given a non-negative integer n, let S^n denote the n-dimensional sphere. Suppose $f\colon S^n \to S^n$ is a continuous map. Applying the n^{th} reduced homology functor $\widetilde{H}_n(\_)$, we obtain a homomorphism $f_*\colon \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$. Since $\widetilde{H}_n(S^n) \approx \mathbb{Z}$, it follows that f_* is a homomorphism $\mathbb{Z} \to \mathbb{Z}$. Such a map must be multiplication by an integer d. We define the *degree* of the map f, to be this d.

0.1 Basic Properties

1. If $f, g\colon S^n \to S^n$ are continuous, then $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
2. If $f, g\colon S^n \to S^n$ are homotopic, then $\deg(f) = \deg(g)$.
3. The degree of the identity map is $+1$.
4. The degree of the constant map is 0.
5. The degree of a reflection through an $(n+1)$-dimensional hyperplane through the origin is -1.
6. The antipodal map, sending x to $-x$, has degree $(-1)^{n+1}$. This follows since the map f_i sending $(x_1, \ldots, x_i, \ldots, x_{n+1}) \mapsto (x_1, \ldots, -x_i, \ldots, x_{n+1})$ has degree -1 by (4), and the compositon $f_1 \circ \cdots \circ f_{n+1}$ yields the antipodal map.

0.2 Examples

If we identify $S^1 \subset \mathbb{C}$, then the map $f : S^1 \to S^1$ defined by $f(z) = z^k$ has degree k. It is also possible, for any positive integer n, and any integer k, to construct a map $f\colon S^n \to S^n$ of degree k.

Using degree, one can prove several theorems, including the so-called 'hairy ball theorem', which that there exists a continuous non-zero vector field on S^n if and only if n is odd.