de Rham cohomology

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Defines	de Rham cohomology group

Let X be a paracompact $\mathcal{C}^\infty$ differential manifold. Let

$$\Omega X = \bigoplus_{i=0}^{\infty} \Omega^i X$$

denote the graded-commutative $\mathbb{R}$-algebra of differential forms on X. Together with the exterior derivative

$$d^i \colon \Omega^i X \to \Omega^{i+1} X \quad (i = 0, 1, \ldots),$$

ΩX forms a chain complex $(\Omega X, d)$ of $\mathbb{R}$-vector spaces. The $\mathrm{H}^i_{\mathrm{dR}} X$ of X are defined as the homology groups of this complex, that is to say

$$\mathrm{H}^i_{\mathrm{dR}} X := (\ker d^i)/(\operatorname{im} d^{i-1}) \quad (i = 0, 1, \ldots),$$

where $\Omega^{-1} X$ is taken to be 0, so $d^{-1} \colon 0 \to \Omega^0 X$ is the zero map. The wedge product in ΩX induces the structure of a graded-commutative $\mathbb{R}$-algebra on

$$\mathrm{H}_{\mathrm{dR}} X := \bigoplus_{i=0}^{\infty} \mathrm{H}^i_{\mathrm{dR}} X.$$

If X and Y are both paracompact $\mathcal{C}^\infty$ manifolds and $f \colon X \to Y$ is a differentiable map, there is an induced map

$$f^* \colon \mathrm{H}_{\mathrm{dR}} Y \to \mathrm{H}_{\mathrm{dR}} X,$$

defined by

$$f^*[\omega] := [f^*\omega] \quad \text{for } \omega \in \ker d.$$

Here $[\omega]$ denotes the class of ω modulo $\operatorname{im} d$, and the second f^* is the map $\Omega Y \to \Omega X$ induced by the functor Ω. This action on differentiable maps makes the de Rham cohomology into a contravariant functor from the category of paracompact $\mathcal{C}^\infty$ manifolds to the category of graded-commutative $\mathbb{R}$-algebras. It turns out to be homotopy invariant; this implies that homotopy equivalent manifolds have isomorphic de Rham cohomology.