fundamental groupoid

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Definition 1. Given a topological space X the fundamental groupoid $\Pi_1(X)$ of X is defined as follows:

- The objects of $\Pi_1(X)$ are the points of X
$$\mathrm{Obj}(\Pi_1(X)) = X\,,$$
- morphisms are homotopy classes of paths "rel endpoints" that is
$$\mathrm{Hom}_{\Pi_1(X)}(x, y) = \mathrm{Paths}(x, y)/\sim\,,$$
where, $\sim$ denotes homotopy rel endpoints, and,
- composition of morphisms is defined via concatenation of paths.

It is easily checked that the above defined category is indeed a groupoid with the inverse of (a morphism represented by) a path being (the homotopy class of) the "reverse" path. Notice that for $x \in X$, the group of automorphisms of x is the fundamental group of X with basepoint x,

$$\mathrm{Hom}_{\Pi_1(X)}(x, x) = \pi_1(X, x)\,.$$

Definition 2. Let $f\colon\ X \to Y$ be a continuous function between two topological spaces. Then there is an induced functor

$$\Pi_1(f)\colon\ \Pi_1(X) \to \Pi_1(Y)$$

defined as follows

- on objects $\Pi_1(f)$ is just f,
- on morphisms $\Pi_1(f)$ is given by "composing with f", that is if $\alpha\colon\ I \to X$ is a path representing the morphism $[\alpha]\colon\ x \to y$ then a representative of $\Pi_1(f)([\alpha])\colon\ f(x) \to f(y)$ is determined by the following commutative diagram

$$\begin{tikzcd} I \arrow[d, "\alpha"'] \arrow[dr, "\Pi_1(f)(\alpha)"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It is straightforward to check that the above indeed defines a functor. Therefore Π_1 can (and should) be regarded as a functor from the category of topological spaces to the category of groupoids. This functor is not really homotopy invariant but it is "homotopy invariant up to homotopy" in the sense that the following holds.

Theorem 3. *A homotopy between two continuous maps induces a natural transformation between the corresponding functors.*

A reader who understands the meaning of the statement should be able to give an explicit construction and supply the proof without much trouble.