homotopy category

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0.1 Homotopy category, fundamental groups and fundamental groupoids

Let us consider first the category Top whose objects are topological spaces X with a chosen basepoint $x \in X$ and whose morphisms are continuous maps $X \to Y$ that associate the basepoint of Y to the basepoint of X. The fundamental group of X specifies a functor $Top \to \mathbf{G}$, with $\mathbf{G}$ being the category of groups and group homomorphisms, which is called *the fundamental group functor*.

0.2 Homotopy category

Next, when one has a suitably defined relation of homotopy between morphisms, or maps, in a category U, one can define the *homotopy category hU* as the category whose objects are the same as the objects of U, but with morphisms being defined by the homotopy classes of maps; this is in fact the homotopy category of *unbased spaces*.

0.3 Fundamental groups

We can further require that homotopies on Top map each basepoint to a corresponding basepoint, thus leading to the definition of the *homotopy category hTop of based spaces*. Therefore, the fundamental group is a *homotopy invariant* functor on Top, with the meaning that the latter functor factors through a functor $hTop \to \mathbf{G}$. A homotopy equivalence in U is an isomorphism in $hTop$. Thus, based homotopy equivalence induces an isomorphism of fundamental groups.

0.4 Fundamental groupoid

In the general case when one does not choose a basepoint, a *fundamental groupoid* $\Pi_1(X)$ of a topological space X needs to be defined as the category whose objects are the base points of X and whose morphisms $x \to y$ are the equivalence classes of paths from x to y.

- Explicitly, the objects of $\Pi_1(X)$ are the points of X

$$\mathrm{Obj}(\Pi_1(X)) = X \, ,$$

- morphisms are homotopy classes of paths "rel endpoints" that is

$$\mathrm{Hom}_{\Pi_1(x)}(x,y) = \mathrm{Paths}(x,y)/\sim,$$

  where, $\sim$ denotes homotopy rel endpoints, and,

- composition of morphisms is defined *via* piecing together, or concatenation, of paths.

0.5 Fundamental groupoid functor

Therefore, the set of endomorphisms of an object x is precisely the fundamental group $\pi(X,x)$. One can thus construct the *groupoid of homotopy equivalence classes*; this construction can be then carried out by utilizing functors from the category Top, or its subcategory hU, to the *category of groupoids and groupoid homomorphisms*, *Grpd*. One such functor which associates to each topological space its fundamental (homotopy) groupoid is appropriately called the *fundamental groupoid functor*.

0.6 An example: the category of simplicial, or CW-complexes

As an important example, one may wish to consider the category of simplicial, or CW-complexes and homotopy defined for CW-complexes. Perhaps, the simplest example is that of a one-dimensional CW-complex, which is a graph. As described above, one can define a functor from the category of graphs, **Grph**, to *Grpd* and then define the fundamental homotopy groupoids of graphs, hypergraphs, or pseudographs. The case of freely generated graphs (one-dimensional CW-complexes) is particularly simple and can be computed with a digital computer by a finite algorithm using the finite groupoids associated with such finitely generated CW-complexes.

0.6.1 Remark

Related to this concept of homotopy category for unbased topological spaces, one can then prove the *approximation theorem for an arbitrary space* by considering a functor

$$\Gamma : \mathbf{hU} \longrightarrow \mathbf{hU},$$

and also the construction of an approximation of an arbitrary space X as the colimit ΓX of a sequence of cellular inclusions of CW-complexes $X_1, ..., X_n$, so that one obtains $X \equiv colim[X_i]$.

Furthermore, the homotopy groups of the CW-complex ΓX are the colimits of the homotopy groups of X_n, and $\gamma_{n+1} : \pi_q(X_{n+1}) \longmapsto \pi_q(X)$ is a group epimorphism.

References

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[2] R. Brown and G. Janelidze.(2004). Galois theory and a new homotopy double groupoid of a map of spaces.(2004). *Applied Categorical Structures*,**12**: 63-80. Pdf file in arxiv: math.AT/0208211