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(path) connectness as a homotopy invariant

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Theorem. Let X and Y be arbitrary topological spaces with Y (path) connected. If there are maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f : X \to X$ is homotopic to the identity map, then X is (path) connected.

Proof: Let $f : X \to Y$ and $g : Y \to X$ be maps satisfying theorem's assumption. Furthermore let $X = \bigcup X_i$ be a decomposition of X into (path) connected components. Since Y is (path) connected, then $g(Y) \subseteq X_i$ for some i. Thus $(g \circ f)(X) \subseteq X_i$. Now let $H : I \times X \to X$ be the homotopy from $g \circ f$ to the identity map. Let $\alpha_x : I \to X$ be a path defined by the formula: $\alpha_x(t) = H(t, x)$. Since for all $x \in X$ we have $\alpha_x(0) \in X_i$ and I is path connected, then $\alpha_x(I) \subseteq X_i$. Therefore $H(I \times X) \subseteq X_i$, but $H(\{1\} \times X) = X$ which implies that $X_i = X$, so X is (path) connected. $\square$

Straightforward application of this theorem is following:

Corollary. Let X and Y be homotopy equivalent spaces. Then X is (path) connected if and only if Y is (path) connected.