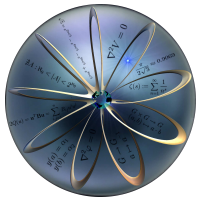

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Whitehead theorem

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Theorem 1 (J.H.C. Whitehead) If $f : X \to Y$ is a weak homotopy equivalence and X and Y are path-connected and of the homotopy type of CW complexes, then f is a strong homotopy equivalence.

Remark 1 It is essential to the theorem that isomorphisms between $\pi_k(X)$ and $\pi_k(Y)$ for all k are induced by a map $f : X \to Y$; if an isomorphism exists which is not induced by a map, it need not be the case that the spaces are homotopy equivalent.

For example, let $X = \mathbb{R}P^m \times S^n$ and $Y = \mathbb{R}P^n \times S^m$. Then the two spaces have isomorphic homotopy groups because they both have a universal covering space homeomorphic to $S^m \times S^n$, and it is a double covering in both cases. However, for $m < n$, X and Y are not homotopy equivalent, as can be seen, for example, by using homology:

$$\begin{aligned} H_m(X; \mathbb{Z}/2\mathbb{Z}) &\cong \mathbb{Z}/2\mathbb{Z}, \quad \text{but} \\ H_m(Y; \mathbb{Z}/2\mathbb{Z}) &\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

(Here, $\mathbb{R}P^n$ is n-dimensional real projective space, and S^n is the n-sphere.)