deformation retract is transitive

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Proposition. *Let $Z \subset Y \subset X$ be nested topological spaces. If there exist a* `http://planetmath.org/DeformationRetraction`*deformation retraction of X onto Y and a deformation retraction of Y onto Z, then there also exists a deformation retraction of X onto Z. In other words, "being a deformation retract of" is a transitive relation.*

Proof. Since Y is a deformation retract of X, there is a homotopy $F : I \times X \to X$ between id_X and a retract $r : X \to Y$ of X onto Y. Similarly, there is a homotopy $G : I \times Y \to Y$ between id_Y and a retract $s : Y \to Z$ of Y onto Z.

First notice that since both r and s fix Z, the map $sr : X \to Z$ is a retraction.

Now define a map $\widetilde{G} : I \times X \to X$ by $\widetilde{G} = iG(\mathrm{id}_I \times r)$, where $i : Y \hookrightarrow X$ is inclusion. Observe that

- $\widetilde{G}(0, x) = r(x)$ for any $x \in X$;
- $\widetilde{G}(1, x) = sr(x)$ for any $x \in X$; and
- $\widetilde{G}(t, a) = a$ for any $a \in Z$.

Hence $\widetilde{G}$ is a homotopy between the retractions r and sr.

Finally we must `http://planetmath.org/GluingTogentherContinuousFunctions`glue together the homotopies F and $\widetilde{G}$ to get a homotopy between id_X and sr. To do this, define a function $H : I \times X \to X$ by

$$H(t, x) = \begin{cases} F(2t, x), & 0 \leq t \leq \frac{1}{2} \\ \widetilde{G}(2t - 1, x), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $F(1, x) = \widetilde{G}(0, x) = r(x)$, the gluing yieds a continuous map. By construction,

- $H(0, x) = x$ for all $x \in X$;
- $H(1, x) = sr(x)$ for all $x \in X$; and
- $H(t, a) = a$ for any $a \in Z$.

Hence H is a homotopy between the identity map on X and a retraction of X onto Z. We conclude that H is a deformation retraction of X onto Z. $\square$