Eilenberg-MacLane space

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Let π be a discrete group. A based topological space X is called an *Eilenberg-MacLane space* of type $K(\pi, n)$, where $n \geq 1$, if all the homotopy groups $\pi_k(X)$ are trivial except for $\pi_n(X)$, which is isomorphic to π. Clearly, for such a space to exist when $n \geq 2$, π must be abelian.

Given any group π, with π abelian if $n \geq 2$, there exists an Eilenberg-MacLane space of type $K(\pi, n)$. Moreover, this space can be constructed as a CW complex. It turns out that any two Eilenberg-MacLane spaces of type $K(\pi, n)$ are weakly homotopy equivalent. The Whitehead theorem then implies that there is a unique $K(\pi, n)$ space up to homotopy equivalence in the category of topological spaces of the homotopy type of a CW complex. We will henceforth restrict ourselves to this category. With a slight abuse of notation, we refer to any such space as $K(\pi, n)$.

An important property of $K(\pi, n)$ is that, for π abelian, there is a natural isomorphism

$$H^n(X; \pi) \cong [X, K(\pi, n)]$$

of contravariant set-valued functors, where $[X, K(\pi, n)]$ is the set of homotopy classes of based maps from X to $K(\pi, n)$. Thus one says that the $K(\pi, n)$ are *representing spaces* for cohomology with coefficients in π.

Remark 1. *Even when the group π is nonabelian, it can be seen that the set $[X, K(\pi, 1)]$ is naturally isomorphic to $\mathrm{Hom}(\pi_1(X), \pi)/\pi$; that is, to conjugacy classes of homomorphisms from $\pi_1(X)$ to π. In fact, this is a way to define $H^1(X; \pi)$ when π is nonabelian.*

Remark 2. *Though the above description does not include the case $n = 0$, it is natural to define a $K(\pi, 0)$ to be any space homotopy equivalent to π. The above statement about cohomology then becomes true for the reduced zeroth cohomology functor.*