monodromy

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Defines	monodromy
Defines	monodromy action
Defines	monodromy homomorphism

Let $(X, *)$ be a connected and locally connected based space and $p\colon E \to X$ a covering map. We will denote $p^{-1}(*)$, the fiber over the basepoint, by F, and the fundamental group $\pi_1(X, *)$ by π. Given a loop $\gamma\colon I \to X$ with $\gamma(0) = \gamma(1) = *$ and a point $e \in F$ there exists a unique $\tilde{\gamma}\colon I \to E$, with $\tilde{\gamma}(0) = e$ such that $p \circ \tilde{\gamma} = \gamma$, that is, a lifting of γ starting at e. Clearly, the endpoint $\tilde{\gamma}(1)$ is also a point of the fiber, which we will denote by $e \cdot \gamma$.

Theorem 1. *With notation as above we have:*

1. *If γ_1 and γ_2 are homotopic relative ∂I then*
$$\forall e \in F \quad e \cdot \gamma_1 = e \cdot \gamma_2.$$

2. *The map*
$$F \times \pi \to F, \quad (e, \gamma) \mapsto e \cdot \gamma$$
defines a right action of π on F.

3. *The stabilizer of a point e is the image of the fundamental group $\pi_1(E, e)$ under the map induced by p:*
$$\mathrm{Stab}(x) = p_* \left(\pi_1(E, e)\right) .$$

Proof. 1. Let $e \in F$, $\gamma_1, \gamma_2\colon I \to X$ two loops homotopic relative ∂I and $\tilde{\gamma}_1, \tilde{\gamma}_2\colon I \to E$ their liftings starting at e. Then there is a homotopy $H\colon I \times I \to X$ with the following properties:

- $H(\bullet, 0) = \gamma_1$,
- $H(\bullet, 1) = \gamma_2$,
- $H(0, t) = H(1, t) = *, \quad \forall t \in I.$

According to the lifting theorem H lifts to a homotopy $\tilde{H}\colon I \times I \to E$ with $H(0, 0) = e$. Notice that $\tilde{H}(\bullet, 0) = \tilde{\gamma}_1$ (respectively $\tilde{H}(\bullet, 1) = \tilde{\gamma}_2$) since they both are liftings of γ_1 (respectively γ_2) starting at e. Also notice that that $\tilde{H}(1, \bullet)$ is a path that lies entirely in the fiber (since it lifts the constant path $*$). Since the fiber is discrete this means that $\tilde{H}(1, \bullet)$ is a constant path. In particular $\tilde{H}(1, 0) = \tilde{H}(1, 1)$ or equivalently $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$.

2. By (1) the map is well defined. To prove that it is an action notice that firstly the constant path $*$ lifts to constant paths and therefore

$$\forall e \in F, \quad e \cdot 1 = e\,.$$

Secondly the concatenation of two paths lifts to the concatenation of their liftings (as is easily verified by projecting). In other words, the lifting of $\gamma_1\gamma_2$ that starts at e is the concatenation of $\tilde{\gamma}_1$, the lifting of γ_1 that starts at e, and $\tilde{\gamma}_2$ the lifting of γ_2 that starts in $\gamma_1(1)$. Therefore

$$e \cdot (\gamma_1\gamma_2) = (e \cdot \gamma_1) \cdot \gamma_2\,.$$

3. This is a tautology: γ fixes e if and only if its lifting starting at e is a loop.

$\square$

Definition 2. The action described in the above theorem is called the *monodromy action* and the corresponding homomorphism

$$\rho\colon\ \pi \to \mathrm{Sym}(F)$$

is called the *monodromy* of p.