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surface bundle over the circle

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A bundle over S^1 is a closed 3-manifold which is constructed as a fiber bundle over the circle with fiber a closed surface.

$$F \subset E \to S^1.$$

This construction it is also a particular case of a more general concept called **mapping torus**.

The precise construction is as follows: Take any surface F and multiply by the unit interval I to get $F \times I$. Choose any ϕ autohomeomorphism of F. Then the quotient space

$$E_\phi = \frac{F \times I}{(x,0) \sim (\phi(x),1)}$$

defines a 3-manifold, characterized by the isotopy class of ϕ, that is, any other representative of the same class is going to produce a bundle homeomorphic to the original one. The isotopy class is called the monodromy for the bundle. It is also used for E_ϕ the symbol:

$$F \times_\phi S^1$$

This construction is an important source of examples in low dimensional topology as well in geometric group theory, because the geometry associated to the monodromy's action and because the bundle's fundamental group can be viewed as a particular kind of HNN extension: the fundamenal group of F extended by the integers. More precisely, if $\pi^* = g_1 g_2 g_1^{-1} g_2^{-1} \cdots g_{k-1} g_k g_{k-1}^{-1} g_k^{-1}$ or $\pi^* = g_1^2 \cdots g_k^2$ then

$$\pi_1(E_\phi) = \langle g_1, ..., g_k, x \mid \pi^* = 1, x g_k x^{-1} = \phi_*(g_k) \rangle,$$

depending on F is orientable or non-orientable.

When one considers periodic monodromies it is an amusing situation since, in this case, the bundles can be seen as **Seifert fiber space** i.e. bundles of the form

$$S^1 \subset E \to G$$

where G may be, perhaps, an orbifold.

For example, it is known that the extended mapping class group of the torus is $GL_2(\mathbb{Z})$, so there are only seven periodic elements, corresponding to seven Seifert fiber space already studied by J.Hempel.

Seven torus bundles $T \subset M \to S^1$.

It is known that the following matrices generate $\mathcal{M}^*(T)$

$$t_a = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad t_b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which obey

$$\langle t_a, t_b, y \ : \ (t_a t_b t_a)^4 = 1, t_a t_b t_a = t_b t_a t_b, y^2 = 1, y t_a y^{-1} = t_a^{-1}, y t_b y^{-1} = t_b^{-1} \rangle$$

The first two are left twists from a a simple meridian curve and b a simple longitude curve. The matrix for y represents a autohomeomorphism which is not a twist and inverts orientation. It is obtained by inverting curve b's direction and extending in a regular neighborhood $N(b)$, then extending to $N(a \cup b)$ and finally in a disk to the whole torus.

Now we can represent the periodic monodromies of example 12.4 in [Hempel, pp.122-123] in terms of those generators as

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ (t_b t_a)^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ t_a y = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$(t_b t_a)^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \ (t_a t_b t_a)^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ t_a t_b = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

of periods 1, 2, 2, 2, 3, 4, 6 respectively [Hempel, pp.123].

And in turn give the Seifert fiber spaces

$(Oo, 1|\ 0) = T \times S^1$
$(On, 2|\ 0) = (Oo, 0|\ (1, -2), (2, 1), (2, 1), (2, 1), (2, 1))$
$(No, 1|\ 0) = (NnI, 2|\ 0) = K \times S^1$
$(No, 1|\ 1) = (NnI, 2|\ 1)) = K \times_\tau S^1$
$(Oo, 0|\ (3, 2), (3, -1), (3, -1))$
$(Oo, 0|\ (2, 1), (4, -1), (4, -1))$
$(Oo, 0|\ (2, 1), (3, -1), (6, -1))$
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