section of a fiber bundle

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Let $p : E \to B$ be a fiber bundle, denoted by ξ.

A *section* of ξ is a continuous map $s : B \to E$ such that the composition $p \circ s$ equals the identity. That is, for every $b \in B$, $s(b)$ is an element of the fiber over b. More generally, given a topological subspace A of B, a section of ξ over A is a section of the restricted bundle $p|_A : p^{-1}(A) \to A$.

The set of sections of ξ over A is often denoted by $\Gamma(A; \xi)$, or by $\Gamma(\xi)$ for sections defined on all of B. Elements of $\Gamma(\xi)$ are sometimes called *global sections,* in contrast with the *local sections* $\Gamma(U; \xi)$ defined on an open set U.

Remark 1 If E and B have, for example, smooth structures, one can talk about smooth sections of the bundle. According to the context, the notation $\Gamma(\xi)$ often denotes smooth sections, or some other set of suitably restricted sections.

Example 1 If ξ is a trivial fiber bundle with fiber F, so that $E = F \times B$ and p is projection to B, then sections of ξ are in a natural bijective correspondence with continuous functions $B \to F$.

Example 2 If B is a smooth manifold and $E = TB$ its tangent bundle, a (smooth) section of this bundle is precisely a (smooth) tangent vector field.

In fact, any tensor field on a smooth manifold M is a section of an appropriate vector bundle. For instance, a contravariant k-tensor field is a section of the bundle $TM^{\otimes k}$ obtained by repeated tensor product from the tangent bundle, and similarly for covariant and mixed tensor fields.

Example 3 If B is a smooth manifold which is smoothly embedded in a Riemannian manifold M, we can let the fiber over $b \in B$ be the orthogonal complement in T_bM of the tangent space T_bB of B at b. These choices of fiber turn out to make up a vector bundle $\nu(B)$ over B, called the of B. A section of $\nu(B)$ is a normal vector field on B.

Example 4 If ξ is a vector bundle, the *zero section* is defined simply by $s(b) = 0$, the zero vector on the fiber.

It is interesting to ask if a vector bundle admits a section which is nowhere zero. The answer is yes, for example, in the case of a trivial vector bundle, but in general it depends on the topology of the spaces involved. A well-known case of this question is the *hairy ball theorem,* which says that there are no nonvanishing tangent vector fields on the sphere.

Example 5 If ξ is a `http://planetmath.org/PrincipalBundle`principal G-`http://planetmath.org/PrincipalBundle`bundle, the existence of *any* section is equivalent to the bundle being trivial.

Remark 2 The correspondence taking an open set U in B to $\Gamma(U;\xi)$ is an example of a sheaf on B.