winding number

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Winding numbers are a basic notion in algebraic topology, and play an important role in connection with analytic functions of a complex variable. Intuitively, given a closed curve $t \mapsto S(t)$ in an oriented Euclidean plane (such as the complex plane $\mathbb{C}$), and a point p not in the image of S, the winding number (or index) of S with respect to p is the net number of times S surrounds p. It is not altogether easy to make this notion rigorous.

Let us take $\mathbb{C}$ for the plane. We have a continuous mapping $S : [a, b] \to \mathbb{C}$ where a and b are some reals with $a < b$ and $S(a) = S(b)$. Denote by $\theta(t)$ the angle from the positive real axis to the ray from z_0 to $S(t)$. As t moves from a to b, we expect θ to increase or decrease by a multiple of 2π, namely $2\omega\pi$ where ω is the winding number. One therefore thinks of using integration. And indeed, in the theory of functions of a complex variable, it is proved that the value

$$\frac{1}{2\pi i} \int_S \frac{dz}{z - z_0}$$

is an integer and has the expected properties of a winding number around z_0. To define the winding number in this way, we need to assume that the closed path S is rectifiable (so that the path integral is defined). An equivalent condition is that the real and imaginary parts of the function S are of bounded variation.

But if S is any continuous mapping $[a, b] \to \mathbb{C}$ having $S(a) = S(b)$, the winding number is still definable, without any integration. We can break up the domain of S into a finite number of intervals such that the image of S, on any of those intervals, is contained in a disc which does not contain z_0. Then $2\omega\pi$ emerges as a finite sum: the sum of the angles subtended at z_0 by the sides of a polygon.

Let A, B, and C be any three distinct rays from z_0. The three sets

$$S^{-1}(A) \qquad S^{-1}(B) \qquad S^{-1}(C)$$

are closed in $[a, b]$, and they *determine* the winding number of S around z_0. This result can provide an alternative definition of winding numbers in $\mathbb{C}$, and a definition in some other spaces also, but the details are rather subtle.

For one more variation on the theme, let S be any topological space homeomorphic to a circle, and let $f : S \to S$ be any continuous mapping. Intuitively we expect that if a point x travels once around S, the point $f(x)$ will travel around S some integral number of times, say n times. The notion can be made precise. Moreover, the number n is determined by the three closed sets

$$f^{-1}(a) \qquad f^{-1}(b) \qquad f^{-1}(c)$$

where a, b, and c are any three distinct points in S.