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quantum operator algebras in quantum field theories

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0.1 Introduction

This is a topic entry that introduces quantum operator algebras and presents concisely the important roles they play in quantum field theories.

Definition 0.1. *Quantum operator algebras* (QOA) in quantum field theories are defined as the algebras of observable operators, and as such, they are also related to the von Neumann algebra; quantum operators are usually defined on Hilbert spaces, or in some QFTs on Hilbert space bundles or other similar families of spaces.

Remark 0.1. Representations of Banach $*$-algebras (that are defined on Hilbert spaces) are closely related to C* -algebra representations which provide a useful approach to defining quantum space-times.

0.2 Quantum operator algebras in quantum field theories: QOA Role in QFTs

Important examples of quantum operators are: the Hamiltonian operator (or Schrödinger operator), the position and momentum operators, Casimir operators, unitary operators and spin operators. The observable operators are also *self-adjoint*. More general operators were recently defined, such as Prigogine's superoperators.

Another development in quantum theories was the introduction of Frechét nuclear spaces or 'rigged' Hilbert spaces (Hilbert *bundles*). The following sections define several types of quantum operator algebras that provide the foundation of modern quantum field theories in mathematical physics.

0.2.1 Quantum groups; quantum operator algebras and related symmetries.

Quantum theories adopted a new lease of life post 1955 when von Neumann beautifully re-formulated quantum mechanics (QM) and quantum theories (QT) in the mathematically rigorous context of Hilbert spaces and operator algebras defined over such spaces. From a current physics perspective, von Neumann' s approach to quantum mechanics has however done much more: it has not only paved the way to expanding the role of symmetry in physics, as for example with the Wigner-Eckhart theorem and its applications, but also revealed the fundamental importance in quantum physics of the state space geometry of quantum operator algebras.

0.3 Basic mathematical definitions in QOA:

- *Von Neumann algebra*
- *Hopf algebra*
- *Groupoids*

- *Haar systems associated to measured groupoids or locally compact groupoids.*
- C*-algebras and quantum groupoids entry (attached).

0.3.1 Von Neumann algebra

Let $\mathcal{H}$ denote a complex (separable) Hilbert space. A *von Neumann algebra* $\mathcal{A}$ acting on $\mathcal{H}$ is a subset of the algebra of all bounded operators $\mathcal{L}(\mathcal{H})$ such that:

- (i) $\mathcal{A}$ is closed under the adjoint operation (with the adjoint of an element T denoted by T^*).
- (ii) $\mathcal{A}$ equals its bicommutant, namely:

$$\mathcal{A} = \{A \in \mathcal{L}(\mathcal{H}) : \forall B \in \mathcal{L}(\mathcal{H}), \forall C \in \mathcal{A},\ (BC = CB) \Rightarrow (AB = BA)\}\ . \tag{0.1}$$

If one calls a *commutant* of a set $\mathcal{A}$ the special set of bounded operators on $\mathcal{L}(\mathcal{H})$ which commute with all elements in $\mathcal{A}$, then this second condition implies that the commutant of the commutant of $\mathcal{A}$ is again the set $\mathcal{A}$.

On the other hand, a von Neumann algebra $\mathcal{A}$ inherits a *unital* subalgebra from $\mathcal{L}(\mathcal{H})$, and according to the first condition in its definition $\mathcal{A}$, it does indeed inherit a $*$-subalgebra structure as further explained in the next section on C* -algebras. Furthermore, one also has available a notable *'bicommutant theorem'* which states that: "*$\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{L}(\mathcal{H})$, closed for the smallest topology defined by continuous maps $(\xi,\eta) \longmapsto (A\xi,\eta)$ for all $< A\xi,\eta) >$ where $< .,. >$ denotes the inner product defined on $\mathcal{H}$* ".

For a well-presented treatment of the geometry of the state spaces of quantum operator algebras, the reader is referred to Aflsen and Schultz (2003; [**?**]).

0.3.2 Hopf algebra

First, a unital associative algebra consists of a linear space A together with two linear maps:

$$\begin{aligned} m &: A \otimes A \longrightarrow A\ ,\ (multiplication) \\ \eta &: \mathbb{C} \longrightarrow A\ ,\ (unity) \end{aligned} \tag{0.2}$$

satisfying the conditions

$$\begin{aligned} m(m \otimes \mathbf{1}) &= m(\mathbf{1} \otimes m) \\ m(\mathbf{1} \otimes \eta) &= m(\eta \otimes \mathbf{1}) = \mathrm{id}\ . \end{aligned} \tag{0.3}$$

This first condition can be seen in terms of a commuting diagram :

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id}\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array} \tag{0.4}$$

Next suppose we consider 'reversing the arrows', and take an algebra A equipped with a linear homorphisms $\Delta : A \longrightarrow A \otimes A$, satisfying, for $a, b \in A$:

$$\begin{aligned} \Delta(ab) &= \Delta(a)\Delta(b) \\ (\Delta \otimes \mathrm{id})\Delta &= (\mathrm{id} \otimes \Delta)\Delta \ . \end{aligned} \tag{0.5}$$

We call Δ a *comultiplication*, which is said to be *coasociative* in so far that the following diagram commutes

$$\begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta\otimes\mathrm{id}"'] \\ A \otimes A \arrow[u, "\mathrm{id}\otimes\Delta"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd} \tag{0.6}$$

There is also a counterpart to η, the *counity* map $\varepsilon : A \longrightarrow \mathbb{C}$ satisfying

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} \ . \tag{0.7}$$

A *bialgebra* $(A, m, \Delta, \eta, \varepsilon)$ is a linear space A with maps $m, \Delta, \eta, \varepsilon$ satisfying the above properties.

Now to recover anything resembling a group structure, we must append such a bialgebra with an antihomomorphism $S : A \longrightarrow A$, satisfying $S(ab) = S(b)S(a)$, for $a, b \in A$. This map is defined implicitly via the property :

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon \quad . \tag{0.8}$$

We call S the *antipode map*.

A *Hopf algebra* is then a bialgebra $(A, m, \eta, \Delta, \varepsilon)$ equipped with an antipode map S .

Commutative and non-commutative Hopf algebras form the backbone of quantum 'groups' and are essential to the generalizations of symmetry. Indeed, in most respects a quantum 'group' is closely related to its dual Hopf algebra; in the case of a finite, commutative quantum group its dual Hopf algebra is obtained via Fourier transformation of the group elements. When Hopf algebras are actually associated with their dual, proper groups of matrices, there is considerable scope for their representations on both finite and infinite dimensional Hilbert spaces.

0.3.3 Groupoids

Recall that a *groupoid* G is, loosely speaking, a small category with inverses over its set of objects $X = Ob(\mathsf{G})$. One often writes G_x^y for the set of morphisms in G from x to y . A *topological groupoid* consists of a space G, a distinguished subspace $\mathsf{G}^{(0)} = \mathrm{Ob}(\mathsf{G}) \subset \mathsf{G}$, called *the space of objects* of G, together with maps

$$r, s \ : \ \mathsf{G} \overset{r}{\underset{s}{\rightrightarrows}} \mathsf{G}^{(0)} \tag{0.9}$$

called the *range* and *source maps* respectively, together with a law of composition

$$\circ \ : \ \mathsf{G}^{(2)} := \mathsf{G} \times_{\mathsf{G}^{(0)}} \mathsf{G} = \{ \ (\gamma_1, \gamma_2) \in \mathsf{G} \times \mathsf{G} \ : \ s(\gamma_1) = r(\gamma_2) \ \} \longrightarrow \mathsf{G} \ , \qquad (0.10)$$

such that the following hold :

(1) $s(\gamma_1 \circ \gamma_2) = r(\gamma_2)$, $r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$, for all $(\gamma_1, \gamma_2) \in \mathsf{G}^{(2)}$.

(2) $s(x) = r(x) = x$, for all $x \in \mathsf{G}^{(0)}$.

(3) $\gamma \circ s(\gamma) = \gamma$, $r(\gamma) \circ \gamma = \gamma$, for all $\gamma \in \mathsf{G}$.

(4) $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$.

(5) Each γ has a two–sided inverse γ^{-1} with $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$. Furthermore, only for topological groupoids the inverse map needs be continuous. It is usual to call $\mathsf{G}^{(0)} = Ob(\mathsf{G})$ *the set of objects* of G . For $u \in Ob(\mathsf{G})$, the set of arrows $u \longrightarrow u$ forms a group G_u, called the *isotropy group of* G *at* u.

Thus, as it is well kown, a topological groupoid is just a groupoid internal to the category of topological spaces and continuous maps. The notion of internal groupoid has proved significant in a number of fields, since groupoids generalise bundles of groups, group actions, and equivalence relations. For a further study of groupoids we refer the reader to Brown (2006).

Several examples of groupoids are:

- (a) locally compact groups, transformation groups , and any group in general (e.g. [59]
- (b) equivalence relations
- (c) tangent bundles
- (d) the tangent groupoid
- (e) holonomy groupoids for foliations
- (f) Poisson groupoids
- (g) graph groupoids.

As a simple, helpful example of a groupoid, consider (b) above. Thus, let R be an *equivalence relation* on a set X. Then R is a groupoid under the following operations: $(x, y)(y, z) = (x, z), (x, y)^{-1} = (y, x)$. Here, $\mathsf{G}^0 = X$, (the diagonal of $X \times X$) and $r((x, y)) = x, s((x, y)) = y$.

Therefore, $R^2 = \{((x, y), (y, z)) : (x, y), (y, z) \in R\}$. When $R = X \times X$, R is called a *trivial* groupoid. A special case of a trivial groupoid is $R = R_n = \{1, 2, ..., n\} \times \{1, 2, ..., n\}$. (So every i is equivalent to every j). Identify $(i, j) \in R_n$ with the matrix unit e_{ij}. Then the groupoid R_n is just matrix multiplication except that we only multiply e_{ij}, e_{kl} when $k = j$, and $(e_{ij})^{-1} = e_{ji}$. We do not really lose anything by restricting the multiplication, since the pairs e_{ij}, e_{kl} excluded from groupoid multiplication just give the 0 product in normal algebra anyway. For a groupoid G_{lc} to be a locally compact groupoid means that G_{lc} is required to be a (second countable) locally compact Hausdorff space, and the product and also inversion maps are required to be continuous. Each G_{lc}^u as well as the unit space G_{lc}^0 is closed in G_{lc}. What replaces the left Haar measure on G_{lc} is a system of measures λ^u ($u \in \mathsf{G}_{lc}^0$), where λ^u is a positive regular Borel measure on G_{lc}^u with dense support. In addition, the λ^u 's are required to vary continuously (when integrated against $f \in C_c(\mathsf{G}_{lc})$) and to form an invariant family in the sense that for each x, the map $y \mapsto xy$ is a measure preserving homeomorphism from $\mathsf{G}_{lc}^s(x)$ onto $\mathsf{G}_{lc}^r(x)$. Such a system $\{\lambda^u\}$ is called a *left Haar system* for the locally compact groupoid G_{lc}.

This is defined more precisely in the next subsection next.

0.3.4 Haar systems for locally compact topological groupoids

Let

$$\mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} = X \tag{0.11}$$

be a locally compact, locally trivial topological groupoid with its transposition into transitive (connected) components. Recall that for $x \in X$, the *costar of* x denoted $\mathrm{CO}^*(\mathrm{x})$ is defined as the closed set $\bigcup\{\mathsf{G}(y, x) : y \in \mathsf{G}\}$, whereby

$$\mathsf{G}(x_0, y_0) \hookrightarrow \mathrm{CO}^*(\mathrm{x}) \longrightarrow \mathrm{X} , \tag{0.12}$$

is a principal $\mathsf{G}(x_0, y_0)$–bundle relative to fixed base points (x_0, y_0) . Assuming all relevant sets are locally compact, then following Seda (1976), a *(left) Haar system on* G denoted (G, τ) (for later purposes), is defined to comprise of i) a measure κ on G, ii) a measure μ on X and iii) a measure μ_x on $\mathrm{CO}^*(\mathrm{x})$ such that for every Baire set E of G, the following hold on setting $E_x = E \cap \mathrm{CO}^*(\mathrm{x})$:

(1) $x \mapsto \mu_x(E_x)$ is measurable.

(2) $\kappa(E) = \int_x \mu_x(E_x)\, d\mu_x$.

(3) $\mu_z(tE_x) = \mu_x(E_x)$, for all $t \in \mathsf{G}(x, z)$ and $x, z \in \mathsf{G}$.

The presence of a left Haar system on G_{lc} has important topological implications: it requires that the range map $r : \mathsf{G}_{lc} \to \mathsf{G}_{lc}^0$ is open. For such a G_{lc} with a left Haar system, the vector space $C_c(\mathsf{G}_{lc})$ is a *convolution *–algebra*, where for $f, g \in C_c(\mathsf{G}_{lc})$:

$f * g(x) = \int f(t)g(t^{-1}x)d\lambda^{r(x)}(t)$, with $f^*(x) = \overline{f(x^{-1})}$.

One has $C^*(\mathsf{G}_{lc})$ to be the *enveloping* C^**–algebra* of $C_c(\mathsf{G}_{lc})$ (and also representations are required to be continuous in the inductive limit topology). Equivalently, it is the completion of $\pi_{univ}(C_c(\mathsf{G}_{lc}))$ where π_{univ} is the *universal representation* of G_{lc}. For example, if $\mathsf{G}_{lc} = R_n$, then $C^*(\mathsf{G}_{lc})$ is just the finite dimensional algebra $C_c(\mathsf{G}_{lc}) = M_n$, the span of the e_{ij}'s.

There exists (e.g.[63, p.91]) a *measurable Hilbert bundle* $(\mathsf{G}^0_{lc}, \mathcal{H}, \mu)$ with $\mathcal{H} = \left\{ \mathcal{H}^u_{u \in \mathsf{G}^0_{lc}} \right\}$ and a G-representation L on $\mathcal{H}$. Then, for every pair ξ, η of square integrable sections of $\mathcal{H}$, it is required that the function $x \mapsto (L(x)\xi(s(x)), \eta(r(x)))$ be ν–measurable. The representation Φ of $C_c(\mathsf{G}_{lc})$ is then given by:
$\langle \Phi(f)\xi|, \eta \rangle = \int f(x)(L(x)\xi(s(x)), \eta(r(x)))d\nu_0(x)$.

The triple $(\mu, \mathcal{H}, L)$ is called a *measurable* G_{lc}*–Hilbert bundle.*

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