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group cohomology (topological definition)

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Let G be a topological group. Suppose some contractible space X admits a fixed point free action of G, so that the quotient map $p : X \to X/G$ is a fibre map. Then X/G, denoted BG is called the classifying space of G. Classifying spaces always exist and are unique up to homotopy. Further, if G has the structure of a CW- complex, we can choose BG to have one too.

The group (co)homology of G is defined to be the (co)homology of BG. From the long-exact sequence associated to the fibre map, p, we know that $\pi_n(G) = \pi_{n+1}(BG)$ for $n \geq 0$. In particular the fundamental group of BG is $\pi_0(G)$, which inherits a group structure as a quotient of G. Let H denote $\pi_0(G)$. Then H acts freely on the cells of BG^*, the universal over of BG. Hence the cellular resolution for BG^*, denoted, $C_*(BG^*)$, is a sequence of free ZH- modules and ZH- linear maps. Taking coefficients in some ZH-module A, we have

$$H^n(G;A) = H^n(C_*(BG^*);A) \text{ and } \mathrm{H_n(G;A) = H_n(C_*(BG^*);A)}$$

In particular, when G is discrete, p must be the covering map associated to a universal cover. Hence $X = BG^*$ and $C_*(BG^*)$ is exact, as X is contractible and hence has trivial homology. Note in this case $H = G$. So for a discrete group G, we have,

$$H^n(G;A) = Ext^n_{ZG}(Z,A) \text{ and } \mathrm{H_n(G;A) = Tor^n_{ZG}(Z,A)}$$

Also, as passing to the universal cover preserves π_n for $n > 1$, we know that $\pi_n(BG) = 0$ for $n > 1$. BG is always connected and for a discrete group $\pi_0(G) = G$ so we have $BG = k(G,1)$, the Eilenberg - Maclane space.

As an example take $G = SU_1$. Note topologically, $SU_1 = S^1 = k(Z,1)$. As $\pi_n(G) = \pi_{n+1}(BG)$ for $n \geq 0$, we know that $BSU_1 = k(Z,2) = CP^\infty$.

More explicitly, we may identify SU_1 with the unit complex numbers. This acts freely on the infinite complex sphere (which is contractible) leaving a quotient of CP^∞.

Hence $H^n(SU_1, Z) = Z$ if 2 divides n and 0 otherwise.

Similiarly $BC_2 = RP^\infty$ and $BSU_2 = HP^\infty$, as C_2 and SU_2 are isomorphic to U(R) and U(H) respectively. So $H^n(C_2, Z_2) = Z_2$ for all n and $H^n(SU_2, Z) = Z$ if 4 divides n and 0 otherwise.