reduction of structure group

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Given a fiber bundle $p : E \to B$ with typical fiber F and structure group G (henceforth called an (F, G)-bundle over B), we say that the bundle admits a *reduction of its structure group to H,* where $H < G$ is a subgroup, if it is isomorphic to an (F, H)-bundle over B.

Equivalently, E admits a reduction of structure group to H if there is a choice of local trivializations covering E such that the transition functions all belong to H.

Remark 1 Here, the action of H on F is the restriction of the G-action; in particular, this means that an (F, H)-bundle is automatically an (F, G)-bundle. The bundle isomorphism in the definition then becomes meaningful in the category of (F, G)-bundles over B.

Example 1 Let H be the trivial subgroup. Then, the existence of a reduction of structure group to H is equivalent to the bundle being trivial.

For the following examples, let E be an n-dimensional vector bundle, so that $F \cong \mathbb{R}^n$ with $G = GL(n, \mathbb{R})$, the general linear group acting as usual.

Example 2 Set $H = GL^+(n, \mathbb{R})$, the subgroup of $GL(n, \mathbb{R})$ consisting of matrices with positive determinant. A reduction to H is equivalent to an orientation of the vector bundle. In the case where B is a smooth manifold and $E = TB$ is its tangent bundle, this coincides with other definitions of an orientation of B.

Example 3 Set $H = O(n)$, the orthogonal group. A reduction to H is called a *Riemannian* or *Euclidean structure* on the vector bundle. It coincides with a continuous fiberwise choice of a positive definite inner product, and for the case of the tangent bundle, with the usual notion of a Riemannian metric on a manifold.

When B is paracompact, an argument with partitions of unity shows that a Riemannian structure always exists on any given vector bundle. For this reason, it is often convenient to start out assuming the structure group to be $O(n)$.

Example 4 Let $n = 2m$ be even, and let $H = GL(m, \mathbb{C})$, the group of invertible complex matrices, embedded in $GL(n, \mathbb{R})$ by means of the usual identification of $\mathbb{C}$ with $\mathbb{R}^2$. A reduction to H is called a *complex structure* on the vector bundle, and it is equivalent to a continuous fiberwise choice of an endomorphism J satisfying $J^2 = -I$.

A complex structure on a tangent bundle is called an *almost-complex structure* on the manifold. This is to distinguish it from the more restrictive notion of a complex structure on a manifold, which requires the existence of an atlas with charts in $\mathbb{C}^m$ such that the transition functions are holomorphic.

Example 5 Let $H = GL(1, \mathbb{R}) \times GL(n-1, \mathbb{R})$, embedded in $GL(n, \mathbb{R})$ by $(A, B) \mapsto A \oplus B$. A reduction to H is equivalent to the existence of a splitting $E \cong E_1 \oplus E_2$, where E_1 is a line bundle. More generally, a reduction to $GL(k, \mathbb{R}) \times GL(n-k, \mathbb{R})$ is equivalent to a splitting $E \cong E_1 \oplus E_2$, where E_1 is a k-plane bundle.

Remark 2 These examples all have two features in common, namely:

- the subgroup H can be interpreted as being precisely the subgroup of G which preserves a particular structure, and,
- a reduction to H is equivalent to a continuous fiber-by-fiber choice of a structure of the same kind.

For example, $O(n)$ is the subgroup of $GL(n, \mathbb{R})$ which preserves the standard inner product of $\mathbb{R}^n$, and reduction of structure to $O(n)$ is equivalent to a fiberwise choice of inner products.

This is not a coincidence. The intuition behind this is as follows. There is no obstacle to choosing a fiberwise inner product in a neighborhood of any given point $x \in B$: we simply choose a neighborhood U on which the bundle is trivial, and with respect to a trivialization $p^{-1}(U) \cong \mathbb{R}^n \times U$, we can let the inner product on each $p^{-1}(y)$ be the standard inner product. However, if we make these choices locally around every point in B, there is no guarantee that they "glue together" properly to yield a global continuous choice, *unless* the transition functions preserve the standard inner product. But this is precisely what reduction of structure to $O(n)$ means.

The same explanation holds for subgroups preserving other kinds of structure.