the topologist's sine curve has the fixed point property

Canonical name	TheTopologistsSineCurveHasTheFixedPointProperty
Date of creation	2013-03-22 16:59:37
Last modified on	2013-03-22 16:59:37
Owner	Mathprof (13753)
Last modified by	Mathprof (13753)
Numerical id	6
Author	Mathprof (13753)
Entry type	Proof
Classification	msc 55M20
Classification	msc 54H25
Classification	msc 47H10

The typical example of a connected space that is not path connected (the topologist's sine curve) has the fixed point property.

Let $X_1 = \{0\} \times [-1, 1]$ and $X_2 = \{(x, \sin(1/x)) : 0 < x \leq 1\}$, and $X = X_1 \cup X_2$.

If $f : X \to X$ is a continuous map, then since X_1 and X_2 are both path connected, the image of each one of them must be entirely contained in another of them.

If $f(X_1) \subset X_1$, then f has a fixed point because the interval has the fixed point property. If $f(X_2) \subset X_1$, then $f(X) = f(cl(X_2)) \subset cl(f(X_2)) \subset X_1$, and in particular $f(X_1) \subset X_1$and again f has a fixed point.

So the only case that remains is that $f(X) \subset X_2$. And since X is compact, its projection to the first coordinate is also compact so that it must be an interval $[a, b]$ with $a > 0$. Thus $f(X)$ is contained in $S = \{(x, \sin(1/x)) : x \in [a, b]\}$. But S is homeomorphic to a closed interval, so that it has the fixed point property, and the restriction of f to S is a continuous map $S \to S$, so that it has a fixed point.

This proof is due to Koro.