homotopy with a contractible domain

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Theorem. Assume that Y is an arbitrary topological space and X is a contractible topological space. Then all maps $f : X \to Y$ are homotopic if and only if Y is path connected.

Proof: Assume that all maps are homotopic. In particular constant maps are homotopic, so if $y_1, y_2 \in Y$, then there exists a continous map $H : I \times Y \to Y$ such that $H(0, y) = y_1$ and $H(1, y) = y_2$ for all $y \in Y$. Thus the map $\alpha : I \to Y$ defined by the formula $\alpha(t) = H(t, y_0)$ for a fixed $y_0 \in Y$ is the wanted path.

On the other hand assume that Y is path connected. Since X is contractible, then for any $c \in X$ there exists a continous homotopy $H : I \times X \to X$ connecting the identity map and a constant map c. Let $f : X \to Y$ be an arbitrary map. Define a map $F : I \times X \to Y$ by the formula: $F(t, x) = f(H(t, x))$. This map is a homotopy from f to a constant map $f(c)$. Thus every map is homotopic to some constant map.

The space Y is path connected, so for all $y_1, y_2 \in Y$ there exists a path $\alpha : I \to Y$ from y_1 to y_2. Therefore constant maps are homotopic via the homotopy $H(t, x) = \alpha(t)$.

Finaly for any continous maps $f, g : X \to Y$ and any point $c \in X$ we get:

$$f \simeq f(c) \simeq g(c) \simeq g,$$

which completes the proof. $\square$

Corollary. If X is a contractible space, then for any topological space Y there exists a bijection between the set $[X, Y]$ of homotopy classes of maps from X to Y and the set $\pi_0(Y)$ of path components of Y.

Proof: Assume that $Y = \bigcup Y_i$, where Y_i are path components of Y. It is well known that contractible spaces are path connected, thus the image of any continous map $f : X \to Y$ is contained in Y_i for some i. It follows from the theorem that two maps from X to Y are homotopic if and only if their images are contained in the same Y_i. Thus we have a well defined, injective map

$$\psi : [X, Y] \to \pi_0(Y)$$
$$\psi([f]) = Y_i,$$

where i is such that $f(X) \subseteq Y_i$. This map is also surjective, since for any i there exists $y \in Y_i$, so the class of the constant map $f(x) = y$ is mapped into Y_i. $\square$