Kunneth theorem

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Let X, Y be topological spaces. One can ask a question: how homology of $X \times Y$ are related to homologies of X and Y. The answer to this question depends on the homology theory we're talking about and also the coefficients ring. On the other hand, it is well known, that all homology theories are isomorphic on CW-complexes. Thus we may restrict to CW-complexes. Nevertheless the following theorem is more general:

Theorem. (Kunneth) *Assume, that X, Y are topological spaces and R is a principal ideal domain. Denote by $H_*(X, R)$ the singular homology with coefficients in R. Then, for any $k > 0$ there exists following short exact sequence in the category of R-modules:*

$$0 \to \bigoplus_{i+j=k} H_i(X, R) \otimes_R H_j(Y, R) \to H_k(X \times Y, R) \to \bigoplus_{i+j=k-1} \mathrm{Tor}_1^R(H_i(X, R), H_j(Y, R)) \to 0,$$

where Tor *denotes the Tor functor. Furthermore this sequence splits, i.e. the middle term is a direct sum (up to an isomorphism) of left and right term.*

It should be mentioned, that if $R = \mathbb{F}$ is a field, then the Tor functor is always trivial (i.e. $\mathrm{Tor}_1^{\mathbb{F}}(M, N) = 0$ for all vector spaces M, N over $\mathbb{F}$) and in this case Kunneth formula can be stated as

$$H_k(X \times Y, \mathbb{F}) \simeq \bigoplus_{i+j=k} H_i(X, \mathbb{F}) \otimes_{\mathbb{F}} H_j(Y, \mathbb{F})$$

for any $k > 0$.