homotopy invariance

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Let $\mathcal{F}$ be a functor from the category of topological spaces to some category $\mathcal{C}$. Then $\mathcal{F}$ is called *homotopy invariant* if for any two homotopic maps $f, g\colon X \to Y$ between topological spaces X and Y the morphisms $\mathcal{F}f$ and $\mathcal{F}g$ in $\mathcal{C}$ induced by $\mathcal{F}$ are identical.

Suppose $\mathcal{F}$ is a homotopy invariant functor, and X and Y are homotopy equivalent topological spaces. Then there are continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$ (i.e. $g \circ f$ and $f \circ g$ are homotopic to the identity maps on X and Y, respectively). Assume that $\mathcal{F}$ is a covariant functor. Then the homotopy invariance of $\mathcal{F}$ implies

$$\mathcal{F}g \circ \mathcal{F}f = \mathcal{F}(g \circ f) = \mathrm{id}_{\mathcal{F}X}$$

and

$$\mathcal{F}f \circ \mathcal{F}g = \mathcal{F}(f \circ g) = \mathrm{id}_{\mathcal{F}Y}.$$

From this we see that $\mathcal{F}X$ and $\mathcal{F}Y$ are isomorphic in $\mathcal{C}$. (The same argument clearly holds if $\mathcal{F}$ is contravariant instead of covariant.)

An important example of a homotopy invariant functor is the fundamental group π_1; here $\mathcal{C}$ is the category of groups.