associated bundle construction

Canonical name	AssociatedBundleConstruction
Date of creation	2013-03-22 13:26:46
Last modified on	2013-03-22 13:26:46
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Last modified by	rspuzio (6075)
Numerical id	9
Author	rspuzio (6075)
Entry type	Definition
Classification	msc 55R10
Defines	associated bundle

Let G be a topological group, $\pi\colon\ P \to X$ a (right) principal G-bundle, F a topological space and $\rho\colon\ G \to \mathrm{Aut}(F)$ a representation of G as homeomorphisms of F. Then the fiber bundle *associated* to P by ρ, is a fiber bundle $\pi_\rho\colon\ P \times_\rho F \to X$ with fiber F and group G that is defined as follows:

- The total space is defined as

$$P \times_\rho F := P \times F/G$$

  where the (left) action of G on $P \times F$ is defined by

$$g \cdot (p, f) := \big(pg^{-1}, \rho(g)(f)\big)\,, \quad \forall g \in G, p \in P, F \in F\,.$$

- The projection π_ρ is defined by

$$\pi_\rho[p, f] := \pi(p)\,,$$

  where $[p, f]$ denotes the G–orbit of $(p, f) \in P \times F$.

Theorem 1. *The above is well defined and defines a G–bundle over X with fiber F. Furthermore $P \times_\rho F$ has the same transition functions as P.*

Sketch of proof. To see that π_ρ is well defined just notice that for $p \in P$ and $g \in G$, $\pi(pg) = \pi(p)$. To see that the fiber is F notice that since the principal action is simply transitive, given $p \in P$ any orbit of the G–action on $P \times F$ contains a *unique* representative of the form (p, f) for some $f \in F$. It is clear that an open cover that trivializes P trivializes $P \times_\rho F$ as well. To see that $P \times_\rho F$ has the same transition functions as P notice that transition functions of P act on the left and thus commute with the principal G–action on P. $\square$

Notice that if G is a Lie group, P a smooth principal bundle and F is a smooth manifold and ρ maps inside the diffeomorphism group of F, the above construction produces a smooth bundle. Also quite often F has extra structure and ρ maps into the homeomorphisms of F that preserve that structure. In that case the above construction produces a "bundle of such structures." For example when F is a vector space and $\rho(G) \subset \mathrm{GL}(F)$, i.e. ρ is a linear representation of G we get a vector bundle; if $\rho(G) \subset \mathrm{SL}(F)$ we get an oriented vector bundle, etc.