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proof of invariance of dimension

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An application of the invariance of dimension theorem shows that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $m = n$. Already this is a difficult question. (We will assume $n \leq m$ throughout this article.)

Simple arguments suffice for small dimensions.

- If $n = 0$ cardinality is sufficient: there can be no bijection between $\mathbb{R}^0 = \{0\}$ and $\mathbb{R}^n$, $m > 0$, as the latter is uncountable.

- If $n = 1$, then suppose $f : \mathbb{R} \to \mathbb{R}^m$ is a homeomorphism with $m > 1$. Then certainly the following restriction of f

$$f : \mathbb{R} - \{0\} \to \mathbb{R}^m - \{f(0)\}$$

  is also a homeomorphism. Yet, as $m > 1$, $\mathbb{R}^m - \{f(0)\}$ is (path) connected but $\mathbb{R} - \{0\}$ is not connected. Thus this restriction of f cannot be a homeomorphism so indeed the original f could not be a homeomorphism.

Unfortunately neither of these two arguments extends well to the cases where $n, m > 1$. Indeed even the case for $n = 1$ requires a reasonable amount of work to fill in the details. However, the latter approach does provide the necessary hint for a full solution.

To solve the problem outright depends on algebraic invariants from homology, a surprisingly big hammer for such a basic topological question. But the conceptual steps are still basic, and we will attempt to highlight them in our exposition of the proof.

Let U and V be non-empty open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Assume that $f : U \to V$ is a homeomorphism.

Choose a point $x \in U$ (akin to the point we removed when $n = 1$.) Then consider the relative homology groups $H_i(U, U - \{x\})$, $i \in \mathbb{N}$. As U is open we may apply the Excision Theorem (axiom) to claim $H_i(U, U - \{x\}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ – basically, to look at a punctured open disk it to look at a punctured $\mathbb{R}^n$. Now we look at the induced long exact sequence from the relative pair $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ and find $H_i(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ is isomorphic to the reduced homology $\tilde{H}_i(\mathbb{R}^n - \{x\})$. But $\mathbb{R}^n - \{x\}$ contracts to the sphere S^{n-1} – and homologoy preserves homotopy type – so we now have $H_i(U, U - \{x\}) \cong H_i(S^{n-1})$. (Puncture a disk, and it deflates to a sphere of lower dimension.)

Now it is an exercise in homology to prove that $H_i(S^{n-1}) = 0$ if $i \neq 0, n-1$ and $\mathbb{Z}$ otherwise. In particular we are using the fact that the invariance of dimension of spheres is (more) easily established by the homology groups.

We now repeat the process with V. If U and V are indeed homeomorphic, then this process will result in isomorphic homology groups for every $i \in \mathbb{N}$. In particular,

$$\mathbb{Z} \cong H_{m-1}(S^{m-1}) \cong H_{m-1}(V, V - \{f(x)\}) \cong H_{m-1}(U, U - \{x\}) \cong H_{m-1}(S^{n-1}).$$

Thus either $m = 1$ which implies $n = 0, 1$ as $n \leq m$, or $m = n$. If $n = 0$ we have already seen $m = n$. So the result stands for all m, n.

For a detailed accounting of this theorem together with the necessary lemmas refer to:

Allen Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002. Available on-line at: `http://www.math.cornell.edu/ hatcher/AT/ATpage.html`http://ww