every map into sphere which is not onto is nullhomotopic

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Proposition. Let X be a topological space and $f : X \to \mathbb{S}^n$ a continous map from X to n-dimensional sphere which is not onto. Then f is nullhomotopic.

Proof. Assume that there is $y_0 \in \mathbb{S}^n$ such that $y_0 \notin \operatorname{im}(f)$. It is well known that there is a homeomorphism $\phi : \mathbb{S}^n \setminus \{y_0\} \to \mathbb{R}^n$. Then we have an induced map

$$\phi \circ f : X \to \mathbb{R}^n.$$

Since $\mathbb{R}^n$ is contractible, then there is $c \in \mathbb{R}^n$ such that $\phi \circ f$ is homotopic to the constant map in c (denoted with the same symbol c). Let $\psi : \mathbb{R}^n \to \mathbb{S}^n$ be a map such that $\psi(x) = \phi^{-1}(x)$ (note that ψ is not the inverse of ϕ because ψ is not onto) and take any homotopy $H : \mathrm{I} \times X \to \mathbb{R}^n$ from $\phi \circ f$ to c. Then we have a homotopy $F : \mathrm{I} \times X \to \mathbb{S}^n$ defined by the formula $F = \psi \circ H$. It is clear that

$$F(0, x) = \psi(H(0, x)) = \psi(\phi(f(x))) = f(x);$$

$$F(1, x) = \psi(H(1, x)) = \psi(c) \in \mathbb{S}^n.$$

Thus F is a homotopy from f to a constant map. $\square$

Corollary. If $A \subseteq \mathbb{S}^n$ is a deformation retract of $\mathbb{S}^n$, then $A = \mathbb{S}^n$.

Proof. If $A \subseteq X$ then by deformation retraction (associated to A) we understand a map $R : \mathrm{I} \times X \to X$ such that $R(0, x) = x$ for all $x \in X$, $R(1, a) = a$ for all $a \in A$ and $R(1, x) \in A$ for all $x \in X$. Thus a deformation retract is a subset $A \subseteq X$ such that there is a deformation retraction $R : \mathrm{I} \times X \to X$ associated to A.

Assume that A is a deformation retract of $\mathbb{S}^n$ and $A \neq \mathbb{S}^n$. Let $R : \mathrm{I} \times \mathbb{S}^n \to \mathbb{S}^n$ be a deformation retraction. Then $r : \mathbb{S}^n \to \mathbb{S}^n$ such that $r(x) = R(1, x)$ is homotopic to the identity map (by definition of a deformation retract), but on the other hand it is homotopic to a constant map (it follows from the proposition, since r is not onto, because A is a proper subset of $\mathbb{S}^n$). Thus the identity map is homotopic to a constant map, so $\mathbb{S}^n$ is contractible. Contradiction. $\square$