CW complex

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Synonym	CW-complex
Related topic	SimplicialComplex
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Related topic	ApproximationTheoremForAnArbitrarySpace
Related topic	SpinNetworksAndSpinFoams
Related topic	CWComplexDefinitionRelatedToSpinNetworksAndSpinFoams
Related topic	GeneralizedHurewiczFundamentalTheorem
Related topic	VariableTopology3
Related topic	QuantumAlgebraicTopologyOfCWComplexRepres
Defines	skeleton
Defines	skeleta
Defines	closure-finite
Defines	cell complex
Defines	CW structure
Defines	CW-structure

A Hausdorff topological space X is said to be a *CW complex* if it the following conditions:

1. There exists a filtration by subspaces
$$X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \cdots$$
with $X = \bigcup_{n \geq -1} X^{(n)}$.

2. $X^{(-1)}$ is empty, and, for $n \geq 0$, $X^{(n)}$ is obtained from $X^{(n-1)}$ by attachment of a collection $\{e_\iota^n : \iota \in I_n\}$ of n-cells.

3. *("closure-finite")* Every closed cell is contained in a finite union of open cells.

4. *("weak topology")* X has the weak topology with respect to the collection of all cells. That is, $A \subset X$ is closed in X if and only if the intersection of A with every closed cell e is closed in e with respect to the subspace topology.

The letters 'C' and 'W' stand for "closure-finite" and "weak topology," respectively. In particular, this means that one shouldn't look too closely at the initials of J.H.C. Whitehead, who invented CW complexes.

The subspace $X^{(n)}$ is called the n-skeleton of X. Note that there normally are many possible choices of a filtration by skeleta for a given CW complex. A particular choice of skeleta and attaching maps for the cells is called a *CW structure* on the space.

Intuitively, X is a CW complex if it can be constructed, starting from a discrete space, by first attaching one-cells, then two-cells, and so on. Note that the definition above does not allow one to attach k-cells before h-cells if $k > h$. While some authors allow this in the definition, it seems to be common usage to restrict CW complexes to the definition given here, and to call a space constructed by cell attachment with unrestricted order of dimensions a *cell complex.* This is not essential for homotopy purposes, since any cell complex is homotopy equivalent to a CW complex.

CW complexes are a generalization of simplicial complexes, and have some of the same advantages. In particular, they allow inductive reasoning on the of skeleta. However, CW complexes are far more flexible than simplicial complexes. For a space X drawn from "everyday" topological spaces, it is a good bet that it is homotopy equivalent, or even homeomorphic, to a CW complex. This includes, for instance, smooth finite-dimensional manifolds, algebraic varieties, certain smooth infinite-dimensional manifolds (such as Hilbert manifolds), and loop spaces of CW complexes. This makes the category of spaces homotopy equivalent to a CW complex a very popular category for doing homotopy theory.

Remark 1. *There is potential for confusion in the way words like "open" and "interior" are used for cell complexes. If* e^k *is a closed* k*-cell in CW complex* X *it does* not *follow that the corresponding open cell* $\mathring{\mathrm{e}}^k$ *is an open set of* X*. It is, however, an open set of the* k*-skeleton. Also, while* $\mathring{\mathrm{e}}^k$ *is often referred to as the "interior" of* e^k*, it is not necessarily the case that it is the interior of* e^k *in the sense of pointset topology. In particular, any closed* 0*-cell is its own corresponding open* 0*-cell, even though it has empty interior in most cases.*