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hairy ball theorem

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Theorem. *If X is a vector field on S^{2n}, then X has a zero. Alternatively, there are no continuous unit vector field on the sphere. Moreover, the tangent bundle of the sphere is nontrivial as a bundle, that is, it is not simply a product.*

There are two proofs for this. The first proof is based on the fact that the antipodal map on S^{2n} is not homotopic to the identity map. The second proof gives the as a corollary of the Poincaré-Hopf index theorem.

Near a zero of a vector field, we can consider a small sphere around the zero, and restrict the vector field to that. By normalizing, we get a map from the sphere to itself. We define the index of the vector field at a zero to be the degree of that map.

Theorem (Poincaré-Hopf index theorem)**.** *If X is a vector field on a compact manifold M with isolated zeroes, then $\chi(M) = \sum_{v \in Z(X)} \iota(v)$ where $Z(X)$ is the set of zeroes of X, and $\iota(v)$ is the index of x at v, and $\chi(M)$ is the Euler characteristic of M.*

It is not difficult to show that S^{2n+1} has non-vanishing vector fields for all n. A much harder result of Adams shows that the tangent bundle of S^m is trivial if and only if $n = 0, 1, 3, 7$, corresponding to the unit spheres in the 4 real division algebras.

Proof. First, the low tech proof. Assume that S^{2n} has a unit vector field X. Then the `http://planetmath.org/AntipodalMapOnSnIsHomotopicToTheIdentityIfAndOnlyIfN` map is homotopic to the identity. But this cannot be, since the degree of the antipodal map is -1 and the degree of the identity map is $+1$. We therefore reject the assumption that X is a unit vector field.

This also implies that the tangent bundle of S^{2n} is non-trivial, since any trivial bundle has a non-zero section. □

Proof. Now for the sledgehammer proof. Suppose X is a nonvanishing vector field on S^{2n}. Then by the Poincaré-Hopf index theorem, the Euler characteristic of S^{2n} is $\chi(X) = \sum_{v \in X^{-1}(0)} \iota(v) = 0$. But the Euler characteristic of S^{2k} is 2. Hence X must have a zero. □