transversality

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Transversality is a fundamental concept in differential topology. We say that two smooth submanifolds A, B of a smooth manifold M intersect *transversely*, if at any point $x \in A \cap B$, we have

$$T_x A + T_x B = T_x X,$$

where T_x denotes the tangent space at x, and we naturally identify $T_x A$ and $T_x B$ with subspaces of $T_x X$.

In this case, A and B intersect properly in the sense that $A \cap B$ is a submanifold of M, and

$$\operatorname{codim}(A \cap B) = \operatorname{codim}(A) + \operatorname{codim}(B).$$

A useful generalization is obtained if we replace the inclusion $A \hookrightarrow M$ with a smooth map $f : A \to M$. In this case we say that f is transverse to $B \subset M$, if for each point $a \in f^{-1}(B)$, we have

$$df_a(T_a A) + T_{f(a)} B = T_{f(a)} M.$$

In this case it turns out, that $f^{-1}(B)$ is a submanifold of A, and

$$\operatorname{codim}(f^{-1}(B)) = \operatorname{codim}(B).$$

Note that if B is a single point b, then the condition of f being transverse to B is precisely that b is a regular value for f. The result is that $f^{-1}(b)$ is a submanifold of A. A further generalization can be obtained by replacing the inclusion of B by a smooth function as well. We leave the details to the reader.

The importance of transversality is that it's a stable and generic condition. This means, in broad terms that if $f : A \to M$ is transverse to $B \subset M$, then small perturbations of f are also transverse to B. Also, given any smooth map $A \to M$, it can be perturbed slightly to obtain a smooth map which is transverse to a given submanifold $B \subset M$.