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Jones Polynomial

Canonical name	JonesPolynomial
Date of creation	2013-03-22 18:51:58
Last modified on	2013-03-22 18:51:58
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Last modified by	Stephaninos (23208)
Numerical id	17
Author	Stephaninos (23208)
Entry type	Topic
Classification	msc 57M25
Related topic	KnotTheory

Introduction...

Definition 0.1. An N -component link is the image of a C^∞ embedding $f : \underbrace{S^1 \times \cdots \times S^1}_{N\text{times}} \rightarrow \mathbb{R}^3$. If $N = 1$, we call the link a **knot**.

Using links and knots as embeddings is not very convenient, as visualising the bends and curves in \mathbb{R}^3 is very hard. Therefore, we use the notion of a knot diagram.

Definition 0.2. Given a knot K , a knot projection $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear surjective map that satisfies:

1. $\pi^2 = \pi$
2. $\text{card}\{\pi^{-1}(x)\} \leq 2, \forall x \in \pi(K)$
3. There is a finite number of points in K for which $\text{card}\{\pi^{-1}(x)\} = 2$.

A **knot diagram** is the image of a projection of a knot.

Note that there is no universal definition of a knot, but this one is the one we use here. Specifically, this definition is used to rule out singular knots, whose projection has an infinite number of crossings.

In this way, we have the set D of all possible knot diagrams. Here, again one would like to study properties of truly distinct knot diagrams, so naturally we study D / \sim' , where \sim' now represents an equivalence relation on D , consisting of 2-dimensional ambient isotopies and the *Reidemeister moves*.

Necessary is:

Definition 0.3. The *writhe (or Tait number)* $w(L)$ is the sum of **all** crossing numbers of a given projection, $w(L) = \sum_i \text{sign}(c_i)$.

Knot invariants

Definition 0.4. Given a link projection D , let y be a crossing, and \hat{y}, \hat{y}' be that crossing opened vertically and horizontally. Then there exists a polynomial $P_L(\xi, \eta, \psi)$ that satisfies:

1. $P_{\text{unknot}} = 1$

$$2. P_y = \xi P_{\hat{y}} + \eta P_{\hat{y}'}$$

$$3. P_{L \cup \text{unknot}} = \psi P_L$$

where $\xi, \eta, \psi \in \mathbb{R}$. This P_L is called the *bracket*, also denoted by $[..]$.

An important remark here is that this definition of the polynomial is through a recursion on the number of crossings of a link diagram. This means that we can construct the polynomial by performing as many recursions as there are crossings and so this polynomial is well defined, because there are only a finite number of crossings in our knots.

Corollary 0.1. *From this also follows that $P_{L_1 \cup L_2} = \psi P_{L_1} P_{L_2}$*

Example 0.1. For the Hopf link, we see that the bracket polynomial $[\text{Hopf}] = \eta\xi + \xi^2\psi + \eta\xi + \eta^2\psi$.

Conjecture 0.1. $\hat{P}_L = \alpha P_L, \alpha \in \mathbb{R}$, is a knot invariant for a suitable chosen α .

This is not true (reason will be added, just check the Reidemeister moves), but this is:

Proposition 0.2. $\hat{P}_L(\xi) := (-\xi^{-3})^{w(L)} P_L(\xi, \xi^{-1}, -\xi^2 - \xi^{-2})$ is a knot invariant.

Proof. Since $w(L)$ and P_L are invariant under Ω_2, Ω_3 , $\hat{P}_L(\xi)$ naturally is invariant under those moves too. Under Ω_1 , a positive crossing attains an extra term $-\xi^3$ which is compensated by the extra $-\xi^{-3}$ of the prefactor, since Ω_1 changes the writhe by $+1$. For a negative crossing, the result is analogous. Therefore, $\hat{P}_L(\xi)$ is invariant under Ω_1 also. Finally, because all projections of a link can be obtained through a finite number of Reidemeister moves, $\hat{P}_L(\xi)$ is a knot invariant. \square

Now the Jones polynomial, first conceived by *Vaughan Jones* in 1984, is exactly $J_L(t) := \hat{P}_L(t^{-\frac{1}{4}})$.

Example 0.2. For the Hopf link, $w(L) = 2$ and $P_{\text{Hopf}} = \xi^2(-\xi^2 - \xi^{-2}) + 1 + 1 + \xi^{-2}(-\xi^2 - \xi^{-2}) = -\xi^4 - \xi^{-4}$. Therefore, $J_{\text{Hopf}}(t) = (-t^{-\frac{1}{4}})^{-3})^2((-t^{-\frac{1}{4}})^4) - (t^{-\frac{1}{4}})^{-4} = -(t^{\frac{1}{2}} + t^{\frac{5}{2}})$

Having defined the Jones polynomial, this allows us once again to classify knots. Notice that the Jones polynomial allows the powers of t to be negative. Each knot has a Jones polynomial associated to it and it has been shown that the Jones polynomial distinguishes between more knots than simple knot invariants, such as tricolorability. In this sense, the Jones polynomial is a better invariant to determine nonequivalency between two knots. Although it is known that there are pairs of knots that have the same Jones polynomial, the question whether or not there exists a nontrivial knot which has Jones polynomial 1 (same as the unknot), but is not equivalent to the unknot, has still not been answered. Above all, it should be clear that the Jones polynomial still does not suffice to distinguish all possible knots.