round function

Canonical name	RoundFunction
Date of creation	2013-03-22 15:44:12
Last modified on	2013-03-22 15:44:12
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Last modified by	juanman (12619)
Numerical id	11
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Entry type	Definition
Classification	msc 57R70
Synonym	functions with critical loops
Related topic	DifferntiableFunction

Let M be a manifold. By a *round function* we a function $M \to \mathbb{R}$ whose critical points form connected components, each of which is homeomorphic to the circle S^1.

For example, let M be the torus. Let $K =]0, 2\pi[\times]0, 2\pi[$. Then we know that a map $X \colon K \to \mathbb{R}^3$ given by

$$X(\theta, \phi) = ((2 + \cos\theta)\cos\phi, (2 + \cos\theta)\sin\phi, \sin\theta)$$

is a parametrization for almost all of M. Now, via the projection $\pi_3 \colon \mathbb{R}^3 \to \mathbb{R}$ we get the restriction $G = \pi_3|_M \colon M \to \mathbb{R}$ whose critical sets are determined by

$$\nabla G(\theta, \phi) = \left(\tfrac{\partial G}{\partial \theta}, \tfrac{\partial G}{\partial \phi} \right)(\theta, \phi) = (0, 0)$$

if and only if $\theta = \frac{\pi}{2},\ \frac{3\pi}{2}$.

These two values for θ give the critical set

$$X\left(\tfrac{\pi}{2}, \phi \right) = (2\cos\phi, 2\sin\phi, 1)$$

$$X\left(\tfrac{3\pi}{2}, \phi \right) = (2\cos\phi, 2\sin\phi, -1)$$

which represent two extremal circles over the torus M.

Observe that the Hessian for this function is $d^2(G) = \begin{pmatrix} -\sin\theta & 0 \\ 0 & 0 \end{pmatrix}$ which clearly it reveals itself as of $\text{rank}(d^2(G)) = 1$ at the tagged circles, making the critical point degenerate, that is, showing that the critical points are not isolated.