# example of differentiable function which is not continuously differentiable

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Let $f$ be defined in the following way:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then if $x \neq 0$, $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ using the usual rules for calculating derivatives. If $x = 0$, we must compute the derivative by evaluating the limit

$$\lim_{\epsilon \to 0} \frac{f(\epsilon) - f(0)}{\epsilon}$$

which we can simplify to

$$\lim_{\epsilon \to 0} \epsilon \sin\left(\frac{1}{\epsilon}\right).$$

We know $|\sin(x)| \leq 1$ for every $x$, so this limit is just 0. Combining this with our previous calculation, we see that

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

This is just a slightly modified version of the topologist's sine curve; in particular,

$$\lim_{x \to 0} f'(x)$$

diverges, so that $f'(x)$ is not continuous, even though it is defined for every real number. Put another way, $f$ is differentiable but not $C^1$.