Poincaré dodecahedral space

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Poincaré originally conjectured [?] that a http://planetmath.org/HomologySpherehomology 3-sphere must be homeomorphic to S^3. He later found a counterexample based on the group of rotations of the http://planetmath.org/RegularPolyhedronregular dodecahedron, and restated his conjecture in of the fundamental group. (See [?]). To be accurate, the restatement took the form of a question. However it has always been referred to as Poincaré's Conjecture.)

This conjecture was one of the http://www.claymath.org/Clay Mathematics Institute's Millennium Problems. It was finally proved by http://planetmath.org/Grigori Perelman as a corollary of his on http://planetmath.org/ThurstonsGeometrizationConjecture geometrization conjecture. Perelman was awarded the http://planetmath.org/FieldsMedalField Medal for this work, but he http://news.bbc.co.uk/2/hi/science/nature/5274040.stmdecline the award. Perelman's manuscripts can be found at the arXiv: [?], [?], [?].

Here we take a quick look at Poincaré's example. Let Γ be the rotations of the http://planetmath.org/RegularPolyhedronregular dodecahedron. It is easy to check that $\Gamma \cong A_5$. (Indeed, Γ http://planetmath.org/GroupActionpermutes transitively the 6 pairs of faces, and the stabilizer of any pair induces a dihedral group of http://planetmath.org/OrderGrouporder 10.) In particular, Γ is perfect. Let P be the quotient space $P = SO_3(\mathbb{R})/\Gamma$. We check that P is a homology sphere.

To do this it is easier to work in the universal cover $SU(2)$ of $SO_3(\mathbb{R})$, since $SU(2) \cong S^3$. The of Γ to $SU(2)$ will be denoted $\hat{\Gamma}$. Hence $P = SU(2)/\hat{\Gamma}$. $\hat{\Gamma}$ is a nontrivial central of A_5 by $\{\pm I\}$, which means that there is no splitting to the surjection $\hat{\Gamma} \to \Gamma$. In fact A_5 has no non-identity 2-dimensional unitary representations. In particular, $\hat{\Gamma}$, like Γ, is http://planetmath.org/PerfectGroupperfect.

Now $\pi_1(P) \cong \hat{\Gamma}$, whence $H^1(P) = 0$ (since it is the abelianization of $\hat{\Gamma}$). By Poincaré duality and the http://planetmath.org/UniversalCoefficentTheoremuniversal coefficient theorem, $H^2(P) \cong 0$ as well. Thus, P is indeed a homology sphere.

The dodecahedron is a fundamental in a tiling of hyperbolic 3-space, and hence P can also be realized by gluing the faces of a dodecahedron. Alternatively, Dehn showed how to construct this same example using surgery around a trefoil. Dale Rolfson's fun book [?] has more on the surgical view of Poincaré's example.

References

[1] G. Perelman, http://arxiv.org/abs/math.DG/0211159/ "The entropy formula for the Ricci flow and its geometric applications",

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[4] H. Poincaré, "Second complément à l'analysis situs", Proceedings of the LMS, 1900.

[5] H. Poincaré, "Cinquième complément à l'analysis situs", Proceedings of the LMS, 1904.

[6] D. Rolfson, Knots and Links. Publish or Perish Press, 1976.