Fréchet space

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We consider two classes of topological vector spaces, one more general than the other. Following Rudin [?] we will define a Fréchet space to be an element of the smaller class, and refer to an instance of the more general class as an *F-space*. After giving the definitions, we will explain why one definition is stronger than the other.

Definition 1. An F-space is a complete topological vector space whose topology is induced by a translation invariant metric. To be more precise, we say that U is an F-space if there exists a metric function

$$d : U \times U \to \mathbb{R}$$

such that

$$d(x, y) = d(x + z, y + z), \quad x, y, z \in U;$$

and such that the collection of balls

$$B_\epsilon(x) = \{y \in U : d(x, y) < \epsilon\}, \quad x \in U, \ \epsilon > 0$$

is a base for the topology of U.

Note 1. Recall that a topological vector space is a uniform space. The hypothesis that U is complete is formulated in reference to this uniform structure. To be more precise, we say that a sequence $a_n \in U, \ n = 1, 2, \ldots$ is Cauchy if for every neighborhood O of the origin there exists an $N \in \mathbb{N}$ such that $a_n - a_m \in O$ for all $n, m > N$. The completeness condition then takes the usual form of the hypothesis that all Cauchy sequences possess a limit point.

Note 2. It is customary to include the hypothesis that U is Hausdorff in the definition of a topological vector space. Consequently, a Cauchy sequence in a complete topological space will have a unique limit.

Note 3. Since U is assumed to be complete, the pair (U, d) is a complete metric space. Thus, an equivalent definition of an F-space is that of a vector space equipped with a complete, translation-invariant (but not necessarily `http://planetmath.org/NormedVectorSpace`homogeneous) metric, such that the operations of scalar multiplication and vector addition are continuous with respect to this metric.

Definition 2. A Fréchet space is a complete topological vector space (either real or complex) whose topology is induced by a countable family of semi-norms. To be more precise, there exist semi-norm functions

$$\| - \|_n : U \to \mathbb{R}, \quad n \in \mathbb{N},$$

such that the collection of all balls

$$B_\epsilon^{(n)}(x) = \{y \in U : \|x - y\|_n < \epsilon\}, \quad x \in U, \ \epsilon > 0, \ n \in \mathbb{N},$$

is a base for the topology of U.

Proposition 1 *Let U be a complete topological vector space. Then, U is a Fréchet space if and only if it is a locally convex F-space.*

Proof. First, let us show that a Fréchet space is a locally convex F-space, and then prove the converse. Suppose then that U is Fréchet. The semi-norm balls are convex; this follows directly from the semi-norm axioms. Therefore U is locally convex. To obtain the desired distance function we set

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}, \quad x, y \in U. \tag{1}$$

We now show that d satisfies the metric axioms. Let $x, y \in U$ such that $x \neq y$ be given. Since U is Hausdorff, there is at least one seminorm such

$$\|x - y\|_n > 0.$$

Hence $d(x, y) > 0$.

Let $a, b, c > 0$ be three real numbers such that

$$a \leq b + c.$$

A straightforward calculation shows that

$$\frac{a}{1 + a} \leq \frac{b}{1 + b} + \frac{c}{1 + c}, \tag{2}$$

as well. The above trick underlies the definition (**??**) of our metric function. By the seminorm axioms we have that

$$\|x - z\|_n \leq \|x - y\|_n + \|y - z\|_n, \quad x, y, z \in U$$

for all n. Combining this with (**??**) and (**??**) yields the triangle inequality for d.

Next let us suppose that U is a locally convex F-space, and prove that it is Fréchet. For every $n = 1, 2, \dots$ let U_n be an open convex neighborhood of the origin, contained inside a ball of radius $1/n$ about the origin. Let $\| - \|_n$ be the seminorm with U_n as the unit ball. By definition, the unit balls of these seminorms give a neighborhood base for the topology of U. QED.

References

[1] W.Rudin, *Functional Analysis.*