Thom space

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Defines	Thom space
Defines	disk bundle
Defines	sphere bundle

Let $\xi \to X$ be a vector bundle over a topological space X. Assume that ξ has a Riemannian metric. We can form its associated disk bundle $D(\xi)$ and its associated sphere bundle $S(\xi)$, by letting

$$D(\xi) = \{v \in \xi : \|v\| \leq 1\}, \quad S(\xi) = \{v \in \xi : \|v\| = 1\}.$$

The *Thom space* of ξ is defined to be the quotient space $D(\xi)/S(\xi)$, obtained by taking the disk bundle and collapsing the sphere bundle to a point. Notice that this makes the Thom space naturally into a based topological space.

Two common forms of notation for the Thom space are $\mathrm{Th}(\xi)$ and X^ξ.

Remark 1 If $\xi = X \times \mathbb{R}^d$ is a trivial vector bundle, then its Thom space is homeomorphic to $\Sigma^d X_+$, where X_+ stands for X with an added disjoint basepoint, and Σ^d stands for the based suspension iterated d times. Thus, we may think of X^ξ as a "twisted suspension" of X_+.

Remark 2 If X is compact, then X^ξ is homeomorphic as a based space to the one-point compactification of ξ. Even if X is not compact, X^ξ can be obtained by doing a one-point compactification on each fiber and then collapsing the resulting section of points at infinity to a point.

Remark 3 The choice of Riemannian metric on ξ does not change the homeomorphism type of X^ξ, and, by the previous remark, the Thom space can be described without reference to associated disk and sphere bundles.