orientation

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Defines	orientable manifold
Defines	oriented manifold
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Defines	local orientation

There are many definitions of an orientation of a manifold. The most general, in the sense that it doesn't require any extra on the manifold, is based on (co-)homology theory. For this article manifold means a connected, *topological* manifold possibly with boundary.

Theorem 1. *Let M be a closed, n–dimensional* manifold. *Then $H_n(M\,;\mathbb{Z})$ the top dimensional homology group of M, is either trivial ($\{0\}$) or isomorphic to $\mathbb{Z}$.*

Definition 2. A closed n–manifold is called *orientable* if its top homology group is isomorphic to the integers. An *orientation* of M is a choice of a particular isomorphism

$$\mathfrak{o}\colon\ \mathbb{Z} \to H_n(M\,;\mathbb{Z}).$$

An oriented manifold is a (necessarily orientable) manifold M endowed with an orientation. If $(M, \mathfrak{o})$ is an oriented manifold then $\mathfrak{o}(1)$ is called the *fundamental class* of M , or the *orientation class* of M, and is denoted by $[M]$.

Remark 3. Notice that since $\mathbb{Z}$ has exactly two automorphisms an orientable manifold admits two possible orientations.

Remark 4. The above definition could be given using cohomology instead of homology.

The top dimensional homology of a non-closed manifold is always trivial, so it is trickier to define orientation for those beasts. One approach (which we will not follow) is to use special kind of homology (for example relative to the boundary for compact manifolds with boundary). The approach we follow defines (global) orientation as compatible fitting together of local orientations. We start with manifolds without boundary.

Theorem 5. *Let M be an n-manifold without boundary and $x \in M$. Then the relative homology group*

$$H_n(M, M \setminus x\,;\mathbb{Z}) \cong \mathbb{Z}$$

Definition 6. Let M be an n-manifold and $x \in M$. An *orientation of M at x* is a choice of an isomorphism

$$\mathfrak{o}_x\colon\ \mathbb{Z} \to H_n(M, M \setminus x\,;\mathbb{Z}).$$

to make precise the notion of nicely fitting together of orientations at points, is to require that for nearby points the orientations are defined in a way.

Theorem 7. *Let U be an open subset of M that is homeomorphic to $\mathbb{R}^n$ (e.g. the domain of a chart). Then,*

$$H_n(M, M \setminus U\,; \mathbb{Z}) \cong \mathbb{Z}.$$

Definition 8. Let U be an open subset of M that is homeomorphic to $\mathbb{R}^n$. A *local orientation* of M on U is a choice of an isomorphism

$$\mathfrak{o}_U\colon\ H_n(M, M \setminus U\,; \mathbb{Z}) \to \mathbb{Z}.$$

Now notice that with U as above and $x \in U$ the inclusion

$$\imath_x^U\colon\ M \setminus U \hookrightarrow M \setminus x$$

a map (actually isomorphism)

$$\imath_{x\,*}^U\colon\ H_n(M, M \setminus U\,; \mathbb{Z}) \to H_n(M, M \setminus x\,; \mathbb{Z})$$

and therefore a local orientation at U (by composing with the above isomorphism) an orientation at each point $x \in U$. It is to declare that all these orientations fit nicely together.

Definition 9. Let M be a manifold with non-empty boundary, $\partial M \neq \emptyset$. M is called *orientable* if its double

$$\hat{M} := M \bigcup_{\partial M} M$$

is orientable, where $\bigcup_{\partial M}$ denotes gluing along the boundary.
An orientation of M is determined by an orientation of $\hat{M}$.