Donaldson Freedman exotic R4

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Let K denote the simply connected closed 4- manifold given by

$$K = \{x : y : z : w \in \mathbb{C}P^3 | x^4 + y^4 + z^4 + w^4 = 0\}$$

Let E_8 denote the unique rank 8 unimodular symmetric bilinear form over $\mathbb{Z}$, which is positive definite and with respect to which, the norm of any vector is even. Let B denote the rank 2 bilinear form over $\mathbb{Z}$ which may be represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then we may regard $H_2(K;\mathbb{Z})$ as a direct sum $M \oplus N$, where the cup product induces the form $E_8 \oplus E_8$ on M and $B \oplus B \oplus B$ on N and we have M orthogonal to N. (This does not contradict Donaldson's theorem as B has 1 and -1 as eigenvalues.)

We may choose a (topological) open ball, U, in $\#_3 S^2 \times S^2$ which contains a (topological) closed ball, V, such that we have a smooth embedding, $f : \#_3 S^2 \times S^2 \; - \; V \to K$ satisfying the following property:

The map f induces an isomorphism from $H_2(\#_3 S^2 \times S^2 \; - \; U;\mathbb{Z})$ into the summand N .

If we could smoothly embed S^3 into $U - V$, enclosing V, then by replacing the outside of the embedded S^3 with a copy of B^4, and regarding $U - V$ as lying in K, we obtain a smooth simply connected closed 4- manifold, with bilinear form $E_8 \oplus E_8$ induced by the cup product. This contradicts Donaldson's theorem.

Therefore, U has the property of containing a compact set which is not enclosed by any smoothly embedded S^3. Hence U is an exotic $\mathbb{R}^4$.

By considering the three copies of B one at a time, we could have obtained our exotic $\mathbb{R}^4$ as an open subset of $S^2 \times S^2$.