Morse lemma

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Defines	index of a bilinear map

Let M be a smooth n-dimensional manifold, and $f : M \to \mathbb{R}$ a smooth map. We denote by $\mathrm{Crit}(f)$ the set of critical points of f, i.e.

$$\mathrm{Crit}(f) = \{p \in M \mid (f_*)_p = 0\}$$

For each $p \in \mathrm{Crit}(f)$ we denote by $f_{**} : T_pM \times T_pM \to \mathbb{R}$ (or $(f_{**})_p$ if p need to be specified) the bilinear map

$$f_{**}(v, w) = v(\tilde{w}(f)) = w(\tilde{v}(f)), \quad \forall v, w \in T_pM,$$

where $\tilde{v}, \tilde{w} \in \mathcal{T}(M)$ are smooth vector fields such that $\tilde{v}_p = v$ and $\tilde{w}_p = w$. This is a good definition. In fact $p \in \mathrm{Crit}(f)$ implies

$$v(\tilde{w}(f)) - w(\tilde{v}(f)) = (\tilde{v}(f), \tilde{w}(f))_p = f_*(\tilde{v}, \tilde{w})_p = 0.$$

In smooth local coordinates $x^1, ..., x^n$ on a neighborhood U of p we have

$$f_{**}\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right) = \frac{\partial^2 f}{\partial x^i\, \partial x^j}(p).$$

A critical point $p \in \mathrm{Crit}(f)$ is called *non degenerate* when the matrix

$$\left(\frac{\partial^2 f}{\partial x^i\, \partial x^j}(p)\right)_{i,j \in \{1,...,n\}}$$

is non singular. We can equivalently express this condition without the use of local coordinates saying that $p \in \mathrm{Crit}(f)$ is non degenerate when for each $v \in T_pM \setminus \{0\}$ the linear functional $f_{**}(v, \cdot) \in \mathrm{Hom}(T_pM, \mathbb{R})$ is not zero, i.e. there exists w such that $f_{**}(v, w) \neq 0$.

We recall that the *index* of a bilinear functional $H : V \times V \to \mathbb{R}$ is the dimension $\mathrm{Index}(H)$ of a maximal linear subspace $W \subseteq V$ such that H is negative definite on $W \times W$.

Theorem 1 (Morse lemma) *Let $f : M \to \mathbb{R}$ be a smooth map. For each non degenerate $p \in \mathrm{Crit}(f)$ there exists a neighborhood U of p and smooth coordinates $x = (x^1, ..., x^n)$ on U such that $x(p) = 0$ and*

$$f|_U = f(p) - (x^1)^2 - ... - (x^\lambda)^2 + (x^{\lambda+1})^2 + ... + (x^n)^2,$$

where $\lambda = \mathrm{Index}((f_{**})_p)$.