Morse homology

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Morse homology is a tool developed by Thom, Smale, and Milnor for homology theory.

Take M to be a smooth compact manifold. Throughout we assume that f is a suitable Morse function, that is, all critical points of f are nondegenerate. We must first make some definitions before defining the Morse homology. Choose a Riemannian metric on M so that the notion of a gradient vector field makes sense. The map $\phi\colon \mathbb{R} \times M \to M$ such that

$$\frac{d}{dt}\phi(t,x) = -\nabla f(\phi(t,x)),$$

with $\phi(0,x) = \mathrm{Id}$, is called the negative gradient flow of f. Let p be a critical point of f, and define

$$W_p^s := \{x \in M | \lim_{t\to\infty} \phi(t,x) = p\} \text{ and } W_p^u := \{x \in M | \lim_{t\to-\infty} \phi(t,x) = p\}$$

to be the stable and unstable manifolds respectively. Thom realized that one could decompose M into its unstable manifolds and arrive at something that is homologically equivalent to its handle decomposition, but this decomposition was not a CW complex, hence it was hard to say anything about the homotopy type of M. But Smale realized that if we impose more conditions on the metric itself, then we can make this into a CW complex.

The pair (f, g), where f is a Morse function and g is the Riemannian metric, is called Morse-Smale pair, if for every pair p, q of critical points of f, W_p^u is transverse to W_q^s. This is known as the Morse-Smale condition. This condition actually holds for a generic Riemannian metric on M. With this restriction, this makes Thom's decomposition into a CW complex.

We can define a complex called the Morse complex as follows:

Let $\mathrm{Crit}_k(f)$ be the set of critical points of f of index k. We define the chain group , $C_k(f)$ to be the formal linear combination with integer coefficients of elements of $\mathrm{Crit}_k(f)$. We must also keep track of the signs of the flow lines. (However, it is true if you count mod 2, the Morse complex computes homology with coefficients in $\frac{Z}{2}$.) To make this a chain complex we must define the differential map. The map $\delta_k : C_k \to C_{k-1}$ applied to a critical point p is a formal sum of critical points with q given by this number. It is possible to prove that $\delta^2 = 0$, making this into a chain complex.

The homology of this complex is called the Morse homology. It can be shown to be isomorphic to the singular homology of M.

Note: There is another way of realizing the Morse homology using Hodge theory, an idea pioneered by Edward Witten. His idea is essentially to conjugate the d operator by e^{sf} and it can be shown that this conjugation again leads to another isomorphism between the set of harmonic forms and the De Rham cohomology. This parameter s is like a curve of chain complexes and Witten claimed that if s is large enough, then we can obtain a space whose dimension is the number of critical points of a given index and the boundary operator induced on d is the number of critical paths between critical points, as before. Witten did not prove this idea rigorously, but it was done later by Helffer and Sjostrand.