Poincaré 1-form

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Definition 1. *Suppose M is a manifold, and T^*M is its cotangent bundle. Then the , $\alpha \in \Omega^1(T^*M)$, is locally defined as*

$$\alpha = \sum_{i=1}^{n} y_i dx^i$$

where x^i, y_i are canonical local coordinates for T^*M.

Let us show that the Poincaré 1-form is globally defined. That is, α has the same expression in all local coordinates. Suppose $x^i, \tilde{x}^i$ are overlapping coordinates for M. Then we have overlapping local coordinates (x^i, y_i), $(\tilde{x}^i, \tilde{y}_i)$ for T^*M with the transformation rule

$$\tilde{y}_i = \frac{\partial \tilde{x}^j}{\partial x^i} y_j.$$

Hence

$$\begin{aligned}
\sum_{i=1}^{n} \tilde{y}_i d\tilde{x}^i &= \sum_{i=1}^{n} \tilde{y}_i \frac{\partial \tilde{x}^i}{\partial x^k} dx^k \\
&= \sum_{i=1}^{n} \frac{\partial \tilde{x}^j}{\partial x^i} y_j \frac{\partial \tilde{x}^i}{\partial x^k} dx^k \\
&= \sum_{k=1}^{n} y_k dx^k.
\end{aligned}$$

Properties

1. The Poincaré 1-form play a crucial role in symplectic geometry. The form $d\alpha$ is the canonical symplectic form for T^*M.

2. Suppose $\pi \colon T^*M \to M$ is the canonical projection. Then

   $$\alpha(w) = \xi((D\pi)(w)), \quad w \in T_\xi(T^*M),$$

   which is an alternative definition of α without local coordinates.

3. The restriction of this form to the unit cotangent bundle, is a contact form.