formulas for differential forms of small valence

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Coboundary formulas. Given a function f (same thing as a differential 0-form), a differential 1-form α and a differential 2-form β, and for vector fields u, v, w, we have

$$\begin{aligned} df(u) =& u(f), \\ d\alpha(u,v) =& u(\alpha(v)) - v(\alpha(u)) - \alpha([u,v]); \\ d\beta(u,v,w) =& u(\beta(v,w)) + v(\beta(w,u)) + w(\beta(u,v)) \\ & - \beta([u,v],w) - \beta([v,w],u) - \beta([w,u],v). \end{aligned}$$

Local coordinate formulas. Let f be a function, $v = v^i\,\partial_i$ a vector field, and $\alpha = \alpha_i\,dx^i$ and $\beta = \beta_i\,dx^i$ be 1-forms, and $\gamma = \frac{1}{2}\gamma_{ij}\,dx^i \wedge dx^j$ a 2-form, expressed relative to a system of local coordinates. The corresponding interior product expressions are:

$$\begin{aligned} \iota_v(\alpha) &= v^i\alpha_i, \\ \iota_v(\gamma) &= v^i\gamma_{ij}\,dx^j. \end{aligned}$$

The exterior product formulas are:

$$\begin{aligned} \alpha \wedge \beta &= \alpha_i\beta_j\,dx^i \wedge dx^j \\ &= \tfrac{1}{2}(\alpha_i\beta_j - \alpha_j\beta_i)\,dx^i \wedge dx^j \\ &= \sum_{i<j}(\alpha_i\beta_j - \alpha_j\beta_i)\,dx^i \wedge dx^j; \\ \alpha \wedge \gamma &= \tfrac{1}{2}\,\alpha_i\gamma_{jk}\,dx^i \wedge dx^j \wedge dx^k \\ &= \tfrac{1}{6}(\alpha_i\gamma_{jk} + \alpha_j\gamma_{ki} + \alpha_k\gamma_{ij})\,dx^i \wedge dx^j \wedge dx^k \\ &= \sum_{i<j<k}(\alpha_i\gamma_{jk} + \alpha_j\gamma_{ki} + \alpha_k\gamma_{ij})\,dx^i \wedge dx^j \wedge dx^k. \end{aligned}$$

The exterior derivative formulas are:

$$
\begin{aligned}
df &= \partial_i f \, dx^i, \\
d\alpha &= \partial_i \alpha_j \, dx^i \wedge dx^j \\
&= \tfrac{1}{2} \left(\partial_i \alpha_j - \partial_j \alpha_i \right) dx^i \wedge dx^j \\
&= \sum_{i<j} (\partial_i \alpha_j - \partial_j \alpha_i) \, dx^i \wedge dx^j; \\
d\gamma &= \tfrac{1}{2} \, \partial_i \gamma_{jk} \, dx^i \wedge dx^j \wedge dx^k \\
&= \tfrac{1}{6} \left(\partial_i \gamma_{jk} + \partial_j \gamma_{ki} + \partial_k \gamma_{ij} \right) dx^i \wedge dx^j \wedge dx^k \\
&= \sum_{i<j<k} \left(\partial_i \gamma_{jk} + \partial_j \gamma_{ki} + \partial_k \gamma_{ij} \right) dx^i \wedge dx^j \wedge dx^k.
\end{aligned}
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