proof of general Stokes theorem

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We divide the proof in several steps.

Step One.

Suppose $M = (0,1] \times (0,1)^{n-1}$ and

$$\omega(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$

(i.e. the term dx_j is missing). Hence we have

$$\begin{aligned} d\omega(x_1, \ldots, x_n) &= \left(\frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n \right) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= (-1)^{j-1} \frac{\partial f}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n \end{aligned}$$

and from the definition of integral on a manifold we get

$$\int_M d\omega = \int_0^1 \cdots \int_0^1 (-1)^{j-1} \frac{\partial f}{\partial x_j} dx_1 \cdots dx_n.$$

From the fundamental theorem of Calculus we get

$$\int_M d\omega = (-1)^{j-1} \int_0^1 \cdots \widehat{\int_0^1} \cdots \int_0^1 f(x_1, \ldots, 1, \ldots, x_n) - f(x_1, \ldots, 0, \ldots, x_n) dx_1 \cdots \widehat{dx_j} \cdots dx_n.$$

Since ω and hence f have compact support in M we obtain

$$\int_M d\omega = \begin{cases} \int_0^1 \cdots \int_0^1 f(1, x_2, \ldots, x_n) dx_2 \cdots dx_n & \text{if} \quad j = 1 \\ \\ 0 & \text{if} \quad j > 1. \end{cases}$$

On the other hand we notice that $\int_{\partial M} \omega$ is to be understood as $\int_{\partial M} i^* \omega$ where $i : \partial M \to M$ is the inclusion map. Hence it is trivial to verify that when $j \neq 1$ then $i^* \omega = 0$ while if $j = 1$ it holds

$$i^* \omega(x) = f(1, x_2, \ldots, x_n) dx_2 \wedge \ldots \wedge dx_n$$

and hence, as wanted

$$\int_{\partial M} i^* \omega = \int_0^1 \cdots \int_0^1 f(1, x_2, \ldots, x_n) dx_2 \cdots dx_n.$$

Step Two.

Suppose now that $M = (0,1] \times (0,1)^{n-1}$ and let ω be any differential form. We can always write

$$\omega(x) = \sum_j f_j(x) dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$

and by the additivity of the integral we can reduce ourself to the previous case.

Step Three.

When $M = (0,1)^n$ we could follow the proof as in the first case and end up with $\int_M d\omega = 0$ while, in fact, $\partial M = \emptyset$.

Step Four.

Consider now the general case.

First of all we consider an oriented atlas (U_i, ϕ_i) such that either U_i is the cube $(0,1] \times (0,1)^{n-1}$ or $U_i = (0,1)^n$. This is always possible. In fact given any open set U in $[0, +\infty) \times \mathbb{R}^{n-1}$ and a point $x \in U$ up to translations and rescaling it is possible to find a "cubic" neighbourhood of x contained in U.

Then consider a partition of unity α_i for this atlas.

From the properties of the integral on manifolds we have

$$\begin{aligned} \int_M d\omega &= \sum_i \int_{U_i} \alpha_i \phi^* d\omega = \sum_i \int_{U_i} \alpha_i d(\phi^* \omega) \\ &= \sum_i \int_{U_i} d(\alpha_i \cdot \phi^* \omega) - \sum_i \int_{U_i} (d\alpha_i) \wedge (\phi^* \omega). \end{aligned}$$

The second integral in the last equality is zero since $\sum_i d\alpha_i = d \sum_i \alpha_i = 0$, while applying the previous steps to the first integral we have

$$\int_M d\omega = \sum_i \int_{\partial U_i} \alpha_i \cdot \phi^* \omega.$$

On the other hand, being $(\partial U_i, \phi_{|\partial U_i})$ an oriented atlas for ∂M and being $\alpha_{i|\partial U_i}$ a partition of unity, we have

$$\int_{\partial M} \omega = \sum_i \int_{\partial U_i} \alpha_i \phi^* \omega$$

and the theorem is proved.