common formulas in calculus of differential forms

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1 Euclidean forms

To begin with we have the total differential for scalars $f\colon D \to \mathbb{R}$ where D is a domain in $\mathbb{R}^n$:

$$df = \sum_s \frac{\partial f}{\partial x^s} dx^s$$

or by the Einstein summation convention

$$df = \frac{\partial f}{\partial x^s} dx^s$$

which are a special case of the so-called Euclidean 1-forms. Here we reconize the covariant form of the gradient of f in contravaiant "state":

$$\nabla f = \frac{\partial f}{\partial x^s}$$

being the components of df.

Here the symbols dx^s are linear functionals $\mathbb{R}^n \to \mathbb{R}$ dual to the derivations $\frac{\partial}{\partial x^s}$, that is

$$dx^s\Big(\frac{\partial}{\partial x^t}\Big) = \delta^s_t$$

this coincides with the calculation $dx^s(\frac{\partial}{\partial x^t}) = \frac{\partial x^s}{\partial x^t} = \delta^s_t$.

If X is a vector field and f a scalar field then one has for the directional derivative

$$Xf = X^s \frac{\partial}{\partial x^s} f = X^s df(\frac{\partial}{\partial x^s}) = df(X)$$

For a pair of functions $g, f\colon D \to \mathbb{R}$ we can check Leibniz's rule

$$d(fg) = gdf + fdg$$

Let $\Omega^0(D) = C^\infty(D)$ be the set of 0-forms in D and let $\Omega^1(D) = \{w = w_s dx^s\colon w_s \in \Omega^0\}$ (where $w_s dx^s = \sum_s w_s dx^s$) be the set of 1-forms in D.

Then the operator d can be seen as a linear operator $d\colon \Omega^0(D) \to \Omega^1(D)$.

This can be generalized by defining $\Omega^k(D)$ to be the set of k-forms; that is, expressions of the type:

$$A_{s_1 \ldots s_k} dx^{s_1} \wedge \cdots \wedge dx^{s_k}$$

where $A_{s_1 \ldots s_k}$ are in $\Omega^0(D)$ i.e. they are scalars and they are multi-indexed sums. Further, the symbols $dx^{s_1} \wedge \cdots \wedge dx^{s_k}$ are the wedge products of the dx^s.

So $d\colon \Omega^k(D) \to \Omega^{k+1}(D)$ is calculated by

$$d(A_{s_1 \dots s_k} dx^{s_1} \wedge \cdots \wedge dx^{s_k}) = d(A_{s_1 \dots s_k}) \wedge dx^{s_1} \wedge \cdots \wedge dx^{s_k}$$

For example, if $A = A_s dx^s$ then $dA = dA_s \wedge dx^s$, hence

$$dA = \frac{\partial A_s}{\partial x^t} dx^t \wedge dx^s$$

which is rearranged as

$$dA = \Big(\frac{\partial A_s}{\partial x^t} - \frac{\partial A_t}{\partial x^s} \Big) dx^t \wedge dx^s,$$

and for two forms, if $B = B_{st} dx^s \wedge dx^t$ then

$$dB = \frac{\partial B_{st}}{\partial x^u} dx^u \wedge dx^s \wedge dx^t.$$

Now if we have a map between two domains $F\colon D \to E$ and $F = (F^1, ..., F^n)$, we can pullback forms as $F^*\colon \Omega^k(E) \to \Omega^k(D)$, beginnig with the observation that at basics dx^k, we pullback it as

$$F^*(dx^k) = d(x^k \circ F) = dF^k = \frac{\partial F^k}{\partial x^s} dx^s$$

then, if we want $\omega \mapsto F^*(\omega)$, where $\omega = \omega_{s_1 \dots s_k} dx^{s_1} \wedge \cdots \wedge dx^{s_k}$, we are going to receive

$$F^*(\omega) = \omega_{s_1 \dots s_k} \circ f \ \frac{\partial F^{s_1}}{\partial x^{t_1}} \cdots \frac{\partial F^{s_k}}{\partial x^{t_k}} dx^{t_1} \wedge \cdots \wedge dx^{t_k}$$

Here the t_i-sums must be taken between all indexes obeying $1 \leq t_1 < t_2 < \cdots < t_k \leq n$.

So if $\omega \in \Omega^n(D)$, $F^*(\omega) = \omega_{1 \dots n} \circ F \ \det(F') dx^1 \wedge \cdots \wedge dx^n$

We also have

$$F^*(v \wedge w) = F^*(v) \wedge F^*(w)$$

Obviously there are no $n+1, n+2, \dots$ forms in D and usually one set $\Omega^k(D) = 0$ if $k \geq n$.

2 The de Rham complex.

The collection of mappings

$$0 \longrightarrow \Omega^0(D) \xrightarrow{d} \Omega^1(D) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(D) \longrightarrow 0$$

give us a chain complex due that $dd = 0$, so one can measure how much this differs from exactness via its homology

$$H^k(D) = \frac{\ker(d)}{\operatorname{im}(d)}$$

called the cohomological k-group for D.

Some with the fear of being confused with the giving of the same name to the operator $\Omega^k(D) \xrightarrow{d} \Omega^{k+1}(D)$, would like to write

$$\Omega^k(D) \xrightarrow{d^k} \Omega^{k+1}(D)$$

and then one should modify the above conventions with

$$d^{k+1}d^k = 0$$

and

$$H^k(D) = \frac{\ker(d^k)}{\operatorname{im}(d^{k-1})}$$

3 Manifold's Forms.

One had seen that for mappings $F\colon D \to E$ between $\mathbb{R}^n$'s domains behave as $F^*\colon \Omega^k(E) \to \Omega^k(D)$. Then we can assign k-forms in each chart (U, Φ) of a n-manifold M by means of the coordinated functions $u^i = x^i \circ \Phi$ on the neighborhood U. Then

$$du^i = d(x^i \circ \Phi) = \Phi^* dx^i$$

which will be the duals of the derivations $\frac{\partial}{\partial u^j}$.

Observe that if $\Phi^*\colon \Omega^0(\phi(U)) \to \Omega^0(U)$ then $\Phi(g) = g \circ \Phi$ is a scalar in U.

If $\Phi^*\colon \Omega^1(\phi(U)) \to \Omega^1(U)$ then

$$\Phi^*(w_s dx^s) = w_s \circ \Phi \Phi^*(dx^s) = w_s \circ \Phi du^s$$

For k-forms

$$w_{s_1 s_2 \dots s_k} du^{s_1} \wedge \cdots \wedge du^{s_k} = w_{s_1 s_2 \dots s_k} \circ \Phi^{-1} \circ \Phi d(x^{s_1} \circ \Phi) \wedge \cdots \wedge d(x^{s_k} \circ \Phi)$$

$$= \Phi^*(w_{s_1 s_2 \dots s_k} \circ \Phi^{-1}) \Phi^*(dx^{s_1} \wedge \cdots \wedge dx^{s_k})$$
$$= \Phi^*(w_{s_1 s_2 \dots s_k} \circ \Phi^{-1} dx^{s_1} \wedge \cdots \wedge dx^{s_k})$$

where $w_{s_1 s_2 \dots s_k} \circ \Phi^{-1} dx^{s_1} \wedge \cdots \wedge dx^{s_k}$ is a k-form in $\Phi(U)$.

4 Forms and connections

A connection is a bi-linear operator $\nabla : \Gamma(TM)^2 \to \Gamma(TM)$ where $\Gamma(TM)$ is the space of differentiable sections in the tangent bundle.

The Chistoffel symbols Γ^s_{ij} are the components of $\nabla_{\partial_i} \partial_j$ through the equation

$$\nabla_{\partial_i} \partial_j = \Gamma^s_{ij} \partial_s$$

where the ∂_s are the coordinated tangent vectors.

The curvature tensor is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

which is a tri-linear map $\Gamma(TM)^3 \to \Gamma(TM)$, so the Riemann-Chistoffel symbols are defined by the components $R^s{}_{ijk}$ of

$$R(\partial_i, \partial_j) \partial_k = R^s{}_{ijk} \partial_s$$

With these one define the connection forms and the curvature forms as

$$\nabla_X \partial_j = \omega^s{}_j(X) \partial_s$$

and

$$R(X,Y) \partial_j = \Omega^s{}_j(X,Y) \partial_s$$

these $\omega^s{}_j$ and $\Omega^s{}_j$ define a 1-form and a 2-form viewed as a sections $M \to \Omega^1(TM)$ and $M \to \Omega^2(TM)$ respectively.

Observe that $\nabla_{\partial_k} \partial_j = \omega^s{}_j(\partial_k) \partial_s$ which compared with $\nabla_{\partial_k} \partial_j = \Gamma^s_{kj} \partial_s$, it implies $\omega^s{}_j(\partial_k) = \Gamma^s_{kj}$ and for an arbitrary vector field $X = X^k \partial_k$ (in the tangent coordinated basis)

$$\omega^s{}_j(X) = X^k \Gamma^s_{kj}$$

Let $X_1, X_2, ..., X_n$ be another frame field (the ∂_i are the coordinated frame field) , i.e. a system of n-tangent vectors which are linearly independent in the tangent space, i.e, they span each T_pM.

Define thru

$$\nabla_{X_i} X_j = \hat{\Gamma}^s_{ij} X_s$$

a an-holonomic connection coefficients

and

$$R(X_i, X_j)X_k = \hat{R}^s{}_{ijk} X_s$$

as the an-holonomic.

Remember that in the coordinated frame field $[\partial_i, \partial_j] = 0$, but since $\nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j]$ this define the structural "constants"

$$c^s{}_{ij} X_s = [X_i, X_j]$$

and the give relation

$$c^s{}_{ij} = \hat{\Gamma}^s_{ij} - \hat{\Gamma}^s_{ji}$$

5 Cartan Structural Equations

The connection and the curvature forms satisfy the premiere $d\theta^i = -\hat{\omega^i}_s \wedge \theta^s$, where the θ^i are the 1-forms dual to the X_j and the deuxieme $\hat{\Omega^i}_j = d\hat{\omega^i}_j + \hat{\omega^i}_s \wedge \hat{\omega^s}_j$ where the corresponding connection forms are calculated by $\nabla_Y X_j = \hat{\omega^s}_j(Y) X_s$ i.e.

$$\hat{\omega^l}_j = \hat{\Gamma}^l_{js} \theta_s.$$

All that fits perfectly to give

$$\hat{\Omega^i}_j = \frac{1}{2} \hat{R^i}_{jkl} \theta^k \wedge \theta^l$$

with $k < l$.

This shows that the calculations of $\hat{R^i}_{jkl}$ are very easy objects to put into an algorithm (Debever).