# differential form

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# 1 Notation and Preliminaries.

Let $M$ be an $n$-dimensional differential manifold. Let $TM$ denote the manifold's tangent bundle, $C^\infty(M)$ the algebra of smooth functions, and $V(M)$ the Lie algebra of smooth vector fields. The directional derivative makes $C^\infty(M)$ into a $V(M)$ module. Using local coordinates, the directional derivative operation can be expressed as

$$v(f) = v^i \partial_i f, \quad v \in V(M),\ f \in C^\infty(M).$$

# 2 Definitions.

**Differential forms.** Let $A$ be a $C^\infty(M)$ module. An $\mathbb{R}$-linear mapping $\alpha : V(M) \to A$ is said to be *tensorial* if it is a $C^\infty(M)$-homomorphism, in other words, if it satisfies

$$\alpha(fv) = f\alpha(v)$$

for all for all vector fields $v \in V(M)$ and functions $f \in C^\infty(M)$. More generally, a multilinear map $\alpha : V(M) \times \cdots \times V(M) \to A$ is called tensorial if it satisfies

$$\alpha(fu, \ldots, v) = \cdots = \alpha(u, \ldots, fv) = f\alpha(u, \ldots, v)$$

for all vector fields $u, \ldots, v$ and all functions $f \in C^\infty(M)$.

We now define a differential 1-form to be a tensorial linear mapping from $V(M)$ to $C^\infty(M)$. More generally, for $k = 0, 1, 2, \ldots$, we define a differential $k$-form to be a tensorial multilinear, antisymmetric, mapping from $V(M) \times \cdots \times V(M)$ ($k$ times) to $C^\infty(M)$. Using slightly fancier language, the above amounts to saying that a 1-form is a section of the cotangent bundle $T^*M = \mathrm{Hom}(TM, \mathbb{R})$, while a differential $k$-form as a section of $\mathrm{Hom}(\Lambda^k TM, \mathbb{R})$.

Henceforth, we let $\Omega^k(M)$ denote the $C^\infty(M)$-module of differential $k$-forms. In particular, a differential 0-form is the same thing as a function. Since the tangent spaces of $M$ are $n$-dimensional vector spaces, we also have $\Omega^k(M) = 0$ for $k > n$. We let

$$\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$$

denote the vector space of all differential forms. There is a natural operator, called the exterior product, that endows $\Omega(M)$ with the structure of a graded algebra. We describe this operation below.

**Exterior and Interior Product.** Let $v \in V(M)$ be a vector field and $\alpha \in \Omega^k(M)$ a differential form. We define $\iota_v(\omega)$, the interior product of $v$ and $\alpha$, to be the differential $k-1$ form given by

$$\iota_v(\alpha)(u_1, \dots, u_{k-1}) = \alpha(v, v_1, \dots, v_{k-1}), \quad v_1, \dots, v_{k-1} \in V(M).$$

The interior product of a vector field with a 0-form is defined to be zero.

Let $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$ be differential forms. We define the exterior, or wedge product $\alpha \wedge \beta \in \Omega^{k+\ell}(M)$ to be the unique differential form such that

$$\iota_v(\alpha \wedge \beta) = \iota_v(\alpha) \wedge \beta + (-1)^k \alpha \wedge \iota_v(\beta)$$

for all vector fields $v \in V(M)$. Equivalently, we could have defined

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \sum_\pi \operatorname{sgn}(\pi) \alpha(v_{\pi_1}, \dots, v_{\pi_k}) \beta(v_{\pi_{k+1}}, \dots, v_{\pi_{k+\ell}}),$$

where the sum is taken over all permutations $\pi$ of $\{1, 2, \dots, k+\ell\}$ such that $\pi_1 < \pi_2 < \cdots \pi_k$ and $\pi_{k+1} < \cdots < \pi_{k+\ell}$, and where $\operatorname{sgn} \pi = \pm 1$ according to whether $\pi$ is an even or odd permutation.

**Exterior derivative.** The exterior derivative is a first-order differential operator $d : \Omega^*(M) \to \Omega^*(M)$, that can be defined as the unique linear mapping satisfying

$$\begin{aligned} d(d\alpha) &= 0, \qquad \alpha \in \Omega^k(M); \\ \iota_V(df) &= v(f), \qquad v \in V(M),\ f \in C^\infty(M); \\ d(\alpha \wedge \beta) &= d(\alpha) \wedge \beta + (-1)^k \alpha \wedge d(\beta), \qquad \alpha \in \Omega^k(M),\ \beta \in \Omega^\ell(M). \end{aligned}$$

# 3 Local coordinates.

Let $(x^1, \dots, x^n)$ be a system of local coordinates on $M$, and let $\partial_1, \dots, \partial_n$ denote the corresponding frame of coordinate vector fields. In other words,

$$\partial_i(x^j) = \delta_i{}^j,$$

where the right hand side is the usual Kronecker delta symbol. By the definition of the exterior derivative,

$$\iota_{\partial_i}(dx^j) = \delta_i{}^j;$$

In other words, the 1-forms $dx^1, \dots, dx^n$ form the dual coframe.

Locally, the $\partial_i$ freely generate $V(M)$, meaning that every vector field $v \in V(M)$ has the form

$$v = v^i \partial_i,$$

where the coordinate components $v^i$ are uniquely determined as

$$v^i = v(x^i).$$

Similarly, locally the $dx^i$ freely generate $\Omega^1(M)$. This means that every one-form $\alpha \in \Omega^1(M)$ takes the form

$$\alpha = \alpha_i dx^i,$$

where

$$\alpha_i = \iota_{\partial_i}(\alpha).$$

More generally, locally $\Omega^k(M)$ is a freely generated by the differential $k$-forms

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \qquad 1 \le i_1 < i_2 < \cdots < i_k \le n.$$

Thus, a differential form $\alpha \in \Omega^k(M)$ is given by

$$\begin{aligned} \alpha &= \sum_{i_1 < \ldots < i_k} \alpha_{i_1 \ldots i_k} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \\ &= \frac{1}{k!} \, \alpha_{i_1 \ldots i_k} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \end{aligned} \tag{1}$$

where

$$\alpha_{i_1 \ldots i_k} = \alpha(\partial_{i_1}, \ldots, \partial_{i_k}).$$

Consequently, for vector fields $u, v, \ldots, w \in V(M)$, we have

$$\alpha(u, v, \ldots, w) = \alpha_{i_1 i_2 \ldots i_k} u^{i_1} v^{i_2} \cdots w^{i_k}.$$

In terms of local coordinates and the skew-symmetrization index notation, the interior and exterior product, and the exterior derivative take the following expressions:

$$(\iota_v(\alpha))_{i_1 \ldots i_k} = v^j \alpha_{j i_1 \ldots i_k}, \quad v \in V(M), \ \alpha \in \Omega^{k+1}(M); \tag{2}$$

$$(\alpha \wedge \beta)_{i_1 \ldots i_{k+\ell}} = \binom{k+\ell}{k} \alpha_{[i_1 \ldots i_k} \beta_{i_{k+1} \ldots i_{k+\ell}]}, \quad \alpha \in \Omega^k(M), \ \beta \in \Omega^\ell(M); \tag{3}$$

$$(d\alpha)_{i_0 i_1 \ldots i_k} = (k+1) \, \partial_{[i_0} \alpha_{i_1 \ldots i_k]}, \quad \alpha \in \Omega^k(M). \tag{4}$$

Note that some authors prefer a different definition of the components of a differential. According to this alternate convention, a factor of $k!$ placed before the summation sign in (**??**), and the leading factors are removed from (**??**) and (**??**).