proof that transition functions of cotangent bundle are valid

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In this entry, we shall verify that the transition functions proposed for the cotangent bundle the three criteria required by the classical definition of a manifold.

The first criterion is the easiest to verify. If $\alpha = \beta$, then $\sigma_{\alpha\alpha}$ reduces to the identity and we have

$$\left(\sigma'_{\alpha\alpha}(x_1, \ldots, x_{2n})\right)^i = \left(\sigma_{\alpha\alpha}(x_1, \ldots, x_n)\right)^i = x^i \qquad 1 \le i \le n$$

$$\left(\sigma'_{\alpha\alpha}(x_1, \ldots, x_{2n})\right)^{i+n} = \sum_{j=1}^n \frac{\partial\left(\sigma_{\alpha\alpha}(x_1, \ldots, x_n)\right)^i}{\partial x_j} x^{j+n} = \sum_{j=1}^n \frac{\partial x^i}{\partial x_j} x^{j+n} = x^{i+n} \qquad 1 \le i \le n$$

Thus we see that $\sigma'_{\alpha\alpha}$ is the identity map, as required.

Next, we turn our attention to the third criterion — showing that $\sigma'_{\beta\gamma} \circ \sigma'_{\alpha\beta} = \sigma'_{\alpha\gamma}$. For clarity of notation let us define $y^i = (\sigma'_{\alpha\beta})^i(x^1, \ldots x^{2n})$. Then we have

$$\begin{aligned}
(\sigma'_{\beta\gamma} \circ \sigma'_{\alpha\beta})^i(x^1, \ldots, x^{2n}) &= (\sigma'_{\beta\gamma})^i(y^1, \ldots, y^{2n}) \\
&= (\sigma_{\beta\gamma})^i(y^1, \ldots, y^n) \\
&= (\sigma_{\beta\gamma} \circ \sigma_{\alpha\beta})^i(x^1, \ldots, x^n) \\
&= (\sigma_{\alpha\gamma})^i(x^1, \ldots, x^n) \\
&= (\sigma'_{\alpha\gamma})^i(x^1, \ldots, x^{2n})
\end{aligned}$$

when $1 \le i \le n$.

$$
\begin{aligned}
({\sigma'}_{\beta\gamma} \circ {\sigma'}_{\alpha\beta})^{i+n}(x^1, \ldots, x^{2n}) &= ({\sigma'}_{\beta\gamma})^{i+n}(y^1, \ldots, y^{2n}) \\
&= \sum_{j=1}^{n} \frac{\partial \Big(\sigma_{\beta\gamma}(y_1, \ldots, y_n) \Big)^i}{\partial y_j} y^{j+n} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial \Big(\sigma_{\beta\gamma}(y_1, \ldots, y_n) \Big)^i}{\partial y_j} \frac{\partial \Big(\sigma_{\alpha\beta}(x_1, \ldots, x_n) \Big)^j}{\partial x_k} x^{n+k} \\
&= \sum_{k=1}^{n} \frac{\partial \Big(\sigma_{\beta\gamma} \circ \sigma_{\alpha\beta}(x_1, \ldots, x_n) \Big)^i}{\partial x_k} x^{n+k} \\
&= \sum_{k=1}^{n} \frac{\partial \Big(\sigma_{\alpha\gamma}(x_1, \ldots, x_n) \Big)^i}{\partial x_k} x^{n+k} \\
&= {\sigma'}_{\alpha\gamma}(x^1, \ldots, x^{2n})
\end{aligned}
$$

when $1 \le i \le n$.

Finally, the second criterion does not need to be checked because it is a consequence of the first and third criteria.