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cotangent bundle

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Overview

Let M be a differentiable manifold. Analogously to the construction of the tangent bundle, we can make the set of covectors on a given manifold into a vector bundle over M, denoted T^*M and called the *cotangent* bundle of M.

Rigorous Definition

To make this definition precise it is convenient to use the `http://planetmath.org/NotesOnTheC` definition of a manifold. Let M be an n-dimensional differentiable manifold, let $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ (each V_α is an open subset of $\mathbb{R}^n$) be an atlas of M with transition functions $\sigma_{\alpha\beta}$.

As an atlas for $T^*(M)$, we may take $\{V_\alpha \times \mathbb{R}^n \mid \alpha \in \mathcal{A}\}$. We may construct transition functions ${\sigma'}_{\alpha\beta}$ as follows:

$$\left({\sigma'}_{\alpha\beta}(x^1, \dots, x^{2n})\right)^i = \left(\sigma_{\alpha\beta}(x^1, \dots, x^n)\right)^i \qquad 1 \le i \le n$$

$$\left({\sigma'}_{\alpha\beta}(x^1, \dots, x^{2n})\right)^{i+n} = \sum_{j=1}^n \frac{\partial\left(\sigma_{\alpha\beta}(x^1, \dots, x^n)\right)^i}{\partial x^j} x^{j+n} \qquad 1 \le i \le n$$

For these to be valid transition functions, they must satisfy the three criteria. For a verification that these criteria are satisfied, please see the attachment.

Bundle Structure

The cotangent bundle is a $GL(n)$ vector bundle over the manifold M. To substantiate this claim, we must specify a projection map onto the manifold M and local trivializations and transition functions and verify that they satisfies the defining properties of a bundle. In terms of the local coordinates used above, it is easy to describe the projection map π:

$$\pi(x^1, \dots, x^{2n})^i = x^i$$

The local trivializations are also somewhat trivial:

$$\phi_\alpha(x^1, \dots, x^{2n}) = x^{i+n}$$

Finally, the transition functions are given as follows:

$$g_{\alpha\beta}(x^1, \dots, x^{2n})^i_j = \frac{\partial\left(\sigma_{\alpha\beta}(x^1, \dots x^n)\right)^i}{\partial x^j}$$

For a verification that $(T^*M, \pi, \phi_\alpha, g_{\alpha\beta})$ satisfies the three criteria for a bundle, please see the attachment.

Properties

The cotangent bundle T^*M is the vector bundle dual to the tangent bundle TM. On any differentiable manifold, $T^*M \cong TM$ (for example, by the existence of a Riemannian metric), but this identification is by no means canonical, and thus it is useful to distinguish between these two objects.

The cotangent bundle to any manifold has a natural symplectic structure given in terms of the Poincaré 1-form, which is in some sense unique. This is not true of the tangent bundle. The existence of a symplectic structure implies that the cotangent bundle is always orientable, even if the original manifold is not.