current

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Defines	mass
Defines	support

Let $\Lambda^m_c(\mathbb{R}^n)$ denote the space of C^∞ differentiable m-forms with compact support in $\mathbb{R}^n$. A continuous linear operator $T\colon \Lambda^m_c(\mathbb{R}^n) \to \mathbb{R}$ is called an *m-current*. Let $\mathcal{D}_m$ denote the space of m-currents in $\mathbb{R}^n$. We define a boundary operator $\partial\colon \mathcal{D}_{m+1} \to \mathcal{D}_m$ by

$$\partial T(\omega) := T(d\omega).$$

We will see that currents represent a generalization of m-surfaces. In fact if M is a compact m-dimensional oriented manifold with boundary, we can associate to M the current $[[M]]$ defined by

$$[[M]](\omega) = \int_M \omega.$$

So the definition of boundary ∂T of a current, is justified by Stokes Theorem:

$$\int_{\partial M} \omega = \int_M d\omega.$$

The space $\mathcal{D}_m$ of m-dimensional currents is a real vector space with operations defined by

$$(T+S)(\omega) := T(\omega) + S(\omega), \qquad (\lambda T)(\omega) := \lambda T(\omega).$$

The sum of two currents represents the *union* of the surfaces they represents. Multiplication by a scalar represents a change in the *multiplicity* of the surface. In particular multiplication by -1 represents the change of orientation of the surface.

We define the *support* of a current T, denoted by $\mathrm{spt}(T)$, the smallest closed set C such that

$$T(\omega) = 0 \text{ whenever } \omega = 0 \text{ on } C.$$

We denote with $\mathcal{E}_m$ the vector subspace of $\mathcal{D}_m$ of currents with compact support.

Topology

The space of currents is naturally endowed with the *weak-star* topology, which will be further simply called *weak convergence*. We say that a sequence T_k of currents, weakly converges to a current T if

$$T_k(\omega) \to T(\omega), \qquad \forall \omega.$$

A stronger norm on the space of currents is the *mass norm*. First of all we define the mass norm of a m-form ω as

$$||\omega|| := \sup\{|\langle\omega,\xi\rangle|: \xi \text{ is a unit, simple, } m\text{-vector}\}.$$

So if ω is a simple m-form, then its mass norm is the usual norm of its coefficient. We hence define the *mass* of a current T as

$$\mathbf{M}(T) := \sup\{T(\omega): \sup_x ||\omega(x)|| \le 1\}.$$

The mass of a currents represents the *area* of the generalized surface.

An intermediate norm, is the *flat norm* defined by

$$\mathbf{F}(T) := \inf\{\mathbf{M}(A) + \mathbf{M}(B): T = A + \partial B,\ A \in \mathcal{E}_m,\ B \in \mathcal{E}_{m+1}\}.$$

Notice that two currents are close in the mass norm if they coincide apart from a small part. On the other hand the are close in the flat norm if they coincide up to a small deformation.

Examples

Recall that $\Lambda^0_c(\mathbb{R}^n) \equiv C^\infty_c(\mathbb{R}^n)$ so that the following defines a 0-current:

$$T(f) = f(0).$$

In particuar every signed measure μ with finite mass is a 0-current:

$$T(f) = \int f(x)\, d\mu(x).$$

Let (x, y, z) be the coordinates in $\mathbb{R}^3$. Then the following defines a 2-current:

$$T(a\, dx \wedge dy + b\, dy \wedge dz + c\, dx \wedge dz) = \int_0^1 \int_0^1 b(x, y, 0)\, dx\, dy.$$