consistent estimator

Canonical name	ConsistentEstimator
Date of creation	2013-03-22 15:26:34
Last modified on	2013-03-22 15:26:34
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	5
Author	CWoo (3771)
Entry type	Definition
Classification	msc 62F12
Defines	consistent sequence of estimators

Given a set of samples $X_1, \ldots, X_n$ from a given probability distribution f with an unknown parameter $\theta \in \Theta$, where Θ is the parameter space that is a subset of $\mathbb{R}^m$. Let $U(= U(X_1, \ldots, X_n))$ be an estimator of θ. Allowing the sample size n to vary, we get a sequence of estimators for θ:

$$\begin{array}{rcl} U_1 & = & U(X_1), \\ & \vdots & \\ U_n & = & U(X_1, \ldots, X_n), \\ & \vdots & \end{array}$$

We say that the sequence of estimators $\{U_n\}$ *consistent* (or that U is a *consistent estimator* of θ), if U_i converges in probability to θ for *every* $\theta \in \Theta$. That is, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|h_n - \theta| \geq \varepsilon) = 0$$

for all $\theta \in \Theta$.

Remark. Suppose U is an estimator of θ such that the sequence $\{U_n\}$ is consistent. If $\alpha_n \to \alpha \in \mathbb{R}$ and $\beta_n \to \beta \in \mathbb{R}^m$ are two convergent sequences of constants with $0 < |\alpha| < \infty$ and $|\beta| < \infty$, then the sequence $\{V_n\}$, defined by $V_n := \alpha_n U_n + \beta_n$, is consistent, V is an estimator of $\alpha\theta + \beta$.

Proof. First, observe that

$$\begin{array}{rcl} |V_n - (\alpha\theta + \beta)| & = & |\alpha_n U_n + \beta_n - \alpha\theta - \beta| \\ & \leq & |\alpha_n U_n - \alpha\theta| + |\beta_n - \beta| \\ & = & |\alpha_n U_n - \alpha_n\theta + \alpha_n\theta - \alpha\theta| + |\beta_n - \beta| \\ & \leq & |\alpha_n U_n - \alpha_n\theta| + |\alpha_n\theta - \alpha\theta| + |\beta_n - \beta| \\ & = & |\alpha_n||U_n - \theta| + |\alpha_n - \alpha||\theta| + |\beta_n - \beta|. \end{array}$$

This implies

$$\begin{array}{rl} & P(|V_n - (\alpha\theta + \beta)| \geq \varepsilon) \\ \leq & P(|\alpha_n||U_n - \theta| + |\alpha_n - \alpha||\theta| + |\beta_n - \beta| \geq \varepsilon) \\ = & P(|U_n - \theta| \geq \dfrac{\varepsilon - |\beta_n - \beta| - |\alpha_n - \alpha||\theta|}{|\alpha_n|}). \end{array}$$

As $n \to \infty$, $|\beta_n - \beta| \to 0$, $|\alpha_n - \alpha||\theta| \to 0$, and $|\alpha_n| \to |\alpha| \neq 0$. So the last expression goes to 0 as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} P(|V_n - (\alpha\theta + \beta)| \geq \varepsilon) = 0,$$

and thus $\{V_n\}$ is a consistent sequence of estimators of $\alpha\theta + \beta$. $\square$