orthogonal Latin squares

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Synonym	mutually orthogonal Latin squares
Synonym	MOLS
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Defines	complete set of Latin squares

Given two Latin squares $L_1 = (A, B, C_1, f_1)$ and $L_2 = (A, B, C_2, f_2)$ of the same order n, we can combine them coordinate-wise to form a single square, whose cells are ordered pairs of elements from C_1 and C_2 respectively. Formally, we can form a function $f : A \times B \to C_1 \times C_2$ given by

$$f(i,j) = (f_1(i,j), f_2(i,j)).$$

This function f says that we have created a new square $A \times B$, whose cell (i,j) contains the ordered pair of values, the first coordinate of which corresponds to the value in cell (i,j) of L_1, and the second to the value in cell (i,j) of L_2. We may write the combined square $L_1 * L_2$.

For example,

$$\begin{pmatrix} a & b & c & d \\ c & d & a & b \\ d & c & b & a \\ b & a & d & c \end{pmatrix} * \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} (a,1) & (b,2) & (c,3) & (d,4) \\ (c,4) & (d,3) & (a,2) & (b,1) \\ (d,2) & (c,1) & (b,4) & (a,3) \\ (b,3) & (a,4) & (d,1) & (c,2) \end{pmatrix}$$

In general, the combined square is not a Latin square unless the original two squares are equivalent: $f_1(i,j) = f_1(k,\ell)$ iff $f_2(i,j) = f_2(k,\ell)$. Nevertheless, the more interesting aspect of pairing up two Latin squares (of the same order) lies in the function f:

> **Definition**. We say that two Latin squares are *orthogonal* if f is a bijection.

Since there are n^2 cells in the combined square, and $|C_1 \times C_2| = n^2$, the function f is a bijection if it is either one-to-one or onto. It is therefore easy to see that the two Latin squares in the example above are orthogonal.

Remarks.

- The combined square is usually known as a *Graeco-Latin square*, originated from statisticians Fischer and Yates.
- It can be shown that if $L_1, \ldots, L_m$ are Latin squares of order $n \geq 3$ such that each pair of them are orthogonal, then $m \leq n-1$. If the equality occurs, then the set of Latin squares are said to be *complete*.
- (Bose) If $n \geq 3$, then $L_1, \ldots, L_m$ form a complete set of pairwise orthogonal Latin squares of order n iff there exists a finite projective plane of order n.

References

[1] H. J. Ryser, *Combinatorial Mathematics*, The Carus Mathematical Monographs, 1963