Jacobi determinant

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Let

$$f = f(x) = f(x_1, \ldots, x_n)$$

be a function of n variables, and let

$$u = u(x) = (u_1(x), \ldots, u_n(x))$$

be a function of x, where inversely x can be expressed as a function of u,

$$x = x(u) = (x_1(u), \ldots, x_n(u))$$

The formula for a change of variable in an n-dimensional integral is then

$$\int_\Omega f(x) d^n x = \int_{u(\Omega)} f(x(u)) |\det(dx/du)| d^n u$$

Ω is an integration region, and one integrates over all $x \in \Omega$, or equivalently, all $u \in u(\Omega)$. $dx/du = (du/dx)^{-1}$ is the Jacobi matrix and

$$|\det(dx/du)| = |\det(du/dx)|^{-1}$$

is the absolute value of the *Jacobi determinant* or *Jacobian*.
As an example, take $n = 2$ and

$$\Omega = \{(x_1, x_2) | 0 < x_1 \leq 1, 0 < x_2 \leq 1\}$$

Define

$$\begin{array}{cc} \rho = \sqrt{-2\log(x_1)} & \varphi = 2\pi x_2 \\ u_1 = \rho\cos\varphi & u_2 = \rho\sin\varphi \end{array}$$

Then by the chain rule and definition of the Jacobi matrix,

$$\begin{aligned} du/dx &= \partial(u_1, u_2)/\partial(x_1, x_2) \\ &= (\partial(u_1, u_2)/\partial(\rho, \varphi))(\partial(\rho, \varphi)/\partial(x_1, x_2)) \\ &= \begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix} \begin{pmatrix} -1/\rho x_1 & 0 \\ 0 & 2\phi \end{pmatrix} \end{aligned}$$

The Jacobi determinant is

$$\begin{aligned}\det(du/dx) &= \det\{\partial(u_1,u_2)/\partial(\rho,\varphi)\}\det\{\partial(\rho,\varphi)/\partial(x_1,x_2)\} \\ &= \rho(-2\pi/\rho x_1) = -2\pi/x_i\end{aligned}$$

and

$$\begin{aligned}d^2x &= |\det(dx/du)|d^2u = |\det(du/dx)|^{-1}d^2u \\ &= (x_1/2\pi) = (1/2\pi)\exp(-(u_1^2+u_2^2/2))d^2u\end{aligned}$$

This shows that if x_1 and x_2 are independent random variables with uniform distributions between 0 and 1, then u_1 and u_2 as defined above are independent random variables with standard normal distributions.

References

- Originally from The Data Analysis Briefbook (`http://rkb.home.cern.ch/rkb/titleA.html`