Lehmer mean

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Let p be a real number. *Lehmer mean* of the positive numbers $a_1,\,\ldots,\,a_n$ is defined as

$$L_p(a_1,\,\ldots,\,a_n) \;:=\; \frac{a_1^p+\ldots+a_n^p}{a_1^{p-1}+\ldots+a_n^{p-1}}. \tag{1}$$

This definition fulfils both requirements set for `http://planetmath.org/Mean3`means. In the case of Lehmer mean of two positive numbers a and b we see for $a \leqq b$ that

$$a \;=\; \frac{a^p\!+\!ab^{p-1}}{a^{p-1}\!+\!b^{p-1}} \;\leqq\; \frac{a^p\!+\!b^p}{a^{p-1}\!+\!b^{p-1}} \;\leqq\; \frac{a^{p-1}b\!+\!b^p}{a^{p-1}\!+\!b^{p-1}} \;=\; b.$$

The Lehmer mean of certain numbers is the greater the greater is the parametre p, i.e.

$$L_p(a_1,\,\ldots,\,a_n) \;\geqq\; L_q(a_1,\,\ldots,\,a_n) \quad \forall\, p \;>\; q.$$

This turns out from the nonnegativeness of the partial derivative of L_p with respect to p; in the case $n=2$ it writes

$$\frac{\partial L_p}{\partial p} \;=\; \frac{a^{p-1}b^{p-1}(a\!-\!b)(\ln a-\ln b)}{(a^{p-1}\!+\!b^{p-1})^2} \;\geqq\; 0.$$

Thus in the below list containing special cases of Lehmer mean, the is the least and the contraharmonic the greatest (cf. the comparison of Pythagorean means).

E.g. for two arguments a and b, one has

- $L_0(a,\,b) \;=\; \dfrac{2ab}{a\!+\!b}$, harmonic mean,
- $L_{1/2}(a,\,b) \;=\; \sqrt{ab}$, geometric mean,
- $L_1(a,\,b) \;=\; \dfrac{a\!+\!b}{2}$, arithmetic mean,
- $L_2(a,\,b) \;=\; \dfrac{a^2\!+\!b^2}{a\!+\!b}$, contraharmonic mean.

Note. The `http://planetmath.org/LeastNumber`least and the `http://planetmath.org/Gre` of the numbers $a_1,\,\ldots,\,a_n$ may be regarded as borderline cases of the Lehmer mean, since

$$\lim_{p\to-\infty} L_p(a_1,\,\ldots,\,a_n) \;=\; \min\{a_1,\,\ldots,\,a_n\}, \quad \lim_{p\to+\infty} L_p(a_1,\,\ldots,\,a_n) \;=\; \max\{a_1,\,\ldots,\,a_n\}.$$

For proving these equations, suppose that there are exactly k greatest (resp. least) ones among the numbers and that those are $a_1 = \ldots = a_k$. Then we can write

$$L_p(a_1, \ldots, a_n) \;=\; \frac{a_1^p \left[k + \left(\frac{a_{k+1}}{a_1}\right)^p + \ldots + \left(\frac{a_n}{a_1}\right)^p\right]}{a_1^{p-1} \left[k + \left(\frac{a_{k+1}}{a_1}\right)^{p-1} + \ldots + \left(\frac{a_n}{a_1}\right)^{p-1}\right]}.$$

Letting $p \to +\infty$ (resp. $p \to -\infty$), this equation yields

$$L_p(a_1, \ldots, a_n) \;\longrightarrow\; a_1.$$