design

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Synonym	block design
Synonym	tau-design
Synonym	τ-design
Synonym	BIBD
Defines	block
Defines	simple design
Defines	square design
Defines	symmetric design
Defines	tactical configuration
Defines	balanced incomplete block design

A τ-(ν, κ, λ) **design**, aka τ**-design** or **block design**, is an incidence structure $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ with

- $|\mathcal{P}| = \nu$ points in all,
- $|\mathcal{P}_B| = \kappa$ points in each block B, and such that
- any set $T \subseteq \mathcal{P}$ of $|T| = \tau$ points occurs as subset $T \subseteq \mathcal{P}_B$ in exactly λ blocks.

The numbers $\tau, \nu, \kappa, lambda$ are called the parameters of a design. They are often called t, v, k, λ (in mixed Latin and Greek alphabets) by some authors.

Given parameters $\tau, \nu, \kappa, lambda$, there may be several non-isomorphic designs, or no designs at all.

Designs need not be simple (they can have **repeated blocks**), but they usually are (and don't) in which case B can again be used as synonym for $\mathcal{P}_B$.

- 0-designs ($\tau = 0$) are allowed.
- 1-designs ($\tau = 1$) are known as **tactical configurations**.
- 2-designs are called **balanced incomplete block designs** or **BIBD**.
- 3, 4, 5... -designs have all been studied.

Being a τ-(ν, κ, λ) design implies also being an ι-$(\nu, \kappa, \lambda_\iota)$ design for every $0 \leq \iota \leq \tau$ (on the same ν points and with the same block size κ), with λ_ι given by $\lambda_\tau = \lambda$ and recursively

$$\lambda_\iota = \frac{\nu - \iota}{\kappa - \iota} \lambda_{\iota+1}$$

from which we get the number of blocks as

$$\lambda_0 = \frac{\nu! \,/\, (\nu - \tau)!}{\kappa! \,/\, (\kappa - \tau)!} = \binom{\nu}{\tau} \Big/ \binom{\kappa}{\tau}$$

Being a 0-design says nothing more than all blocks having the same size. As soon as we have $\tau \geq 1$ however we also have a 1-design, so the number $\lambda_1 = |\mathcal{B}_P|$ of blocks per point P is constant throughout the structure. Note now

$$\lambda_0 \, \kappa = \lambda_1 \, \nu$$

which is also evident from their interpretation.

As an example: designs (simple designs) with $\kappa = 2$ are multigraphs (simple **graphs**), now

- $\tau = 0$ implies no more than that,
- $\tau = 1$ gives **regular graphs**, and
- $\tau = 2$ gives **complete graphs**.

A more elaborate "lambda calculus" (pun intended) can be introduced as follows. Let $I \subseteq P$ and $O \subseteq P$ with $|I| = \iota$ and $|O| = o$. The number of blocks B such that all the points of I are inside B and all the points of O are outside B is independent of the choice of I and O, only depending on ι and o, provided $\iota + o \leq \tau$. Call this number λ_ι^o. It satisfies a kind of reverse Pascal triangle like recursion

$$\lambda_\iota^o \;=\; \lambda_{\iota+1}^o + \lambda_\iota^{o+1}$$

that starts off for $o = 0$ with $\lambda_\iota^0 = \lambda_\iota$. An important quantity (for designs with $\tau \geq 2$) is the **order** $\lambda_1^1 = \lambda_1^0 - \lambda_2^0 = \lambda_1 - \lambda_2$.

Finally, the dual of a design can be a design but need not be.

- A **square design** aka **symmetric design** is one where $\tau = 2$ and $|\mathcal{P}| = |\mathcal{B}|$, now also $|\mathcal{P}_B| = |\mathcal{B}_P|$. Here the dual is also a square design.

Note that for $\tau \geq 3$ no designs exist with $|\mathcal{P}| = |\mathcal{B}|$ other than trivial ones (where any $\kappa = \nu - 1$ points form a block).