copula

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Set-up

An n-dimensional rectangle S is a subset of $\mathbb{R}^n$ of the form $I_1 \times \cdots \times I_n$, where each I_k is an interval, with end points $a_k \leq b_k \in \mathbb{R}^*$, where $\mathbb{R}^*$ is the set of extended real numbers (so that $\mathbb{R}$ itself may be considered as an interval).

Groundedness. A function $C : S \to \mathbb{R}$ is said to be *grounded* if for each $1 \leq k \leq n$, and each $r_j \in I_j$ where $j \neq k$, the function $C_k : I_k \to \mathbb{R}$ defined by

$$C_k(x) := C(r_1, \ldots, r_{j-1}, x, r_{j+1}, \ldots, r_n)$$

is right-continuous at a_k, the lower end point of I_k.

Margin. Note that C_k defined above may or may not exist as each $r_j \to b_j$, the upper end point of I_j ($j \neq k$). If the limit exists, then we call this limiting function, also written C_k, a (one-dimensional) *margin* of C:

$$C_k(x) := \lim_{r_j \to b_j} C(r_1, \ldots, r_{j-1}, x, r_{j+1}, \ldots, r_n), \text{ where } j \in \{1, \ldots, n\},\ j \neq i.$$

Given an n-dimensional rectangle $S = I_1 \times \cdots \times I_n$, let's call each I_k a *side* of S. A *vertex* of S is a point $v \in \mathbb{R}^n$ such that each of its coordinates is an end point. Clearly S is a convex set and the sides and vertices lie on the boundary of S.

C-volume. Suppose we have a function $C : S \to \mathbb{R}$, with S defined as above. Let T be a closed n-dimensional rectangle in S ($T \subseteq S$), with sides $J_k = [c_k, d_k]$, $1 \leq k \leq n$. The *C-volume* of T is the sum

$$\mathrm{Vol}_C(T) = \sum (-1)^{n(v)} C(v)$$

where v is a vertex of T, $n(v)$ is the number of lower end points that occur in the coordinate representation of v, and the sum is taken over all vertices of T.

The name is derived from the fact that if $C(x_1, \ldots, x_n) = x_1 \cdots x_n$, then for each closed rectangle T, $\mathrm{Vol}_C(T)$ is the volume of T in the traditional sense.

Note, however, depending on the function C, $\mathrm{Vol}_C(T)$ may be 0 or even negative. For example, if C is a linear function, then the C-volume is identically 0 for every closed rectangle T, whenever n is even. An example where $\mathrm{Vol}_C(T)$ is negative is given by the function $C(x, y) = -xy$, and T is the unit square.

n-increasing. A function $C : S \to \mathbb{R}$ where S is an open n-dimensional rectange is said to be *n-increasing* if Vol_C is non-negative evaluated at each closed rectangle $T \subseteq S$.

Any multivariate distribution function is both grounded and n-increasing.

Definition

A copula, introduced by Sklar, is both a variant and a generalization of a multivariate distribution function.

Formally, a *copula* is a function C from the n-dimensional unit cube I^n ($I = [0, 1]$) to $\mathbb{R}$ satisfying the following conditions:

1. C is n-increasing,

2. C is grounded,

3. every margin C_k of C is the identity function.

If we replace the domain by any n-dimensional rectangle S, then the resulting function is called a *subcopula*.

For example, the functions $C(x, y, z) = xyz$, $C(x, y, z) = \min(x, y, z)$, and $C(x, y, z) = \max(0, (x + y + z - 2))$ defined on the unit cube are all copulas.

(This entry is in the process of being expanded, more to come shortly).

References

[1] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, Dover Publications, (2005).