law of rare events

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Synonym	Poisson theorem

Let X be distributed as $Bin(n, p)$, a binomial random variable with parameters n and p. Suppose

$$\lim_{n\to\infty} np = \lambda,$$

where λ is a positive real constant, then X is asymptotically distributed as $Poisson(\lambda)$, a Poisson distribution with parameter λ.

Basically, when the size of the population n is very large and the occurrence of certain *event* A is rare, where p, the probability of A is very small, the binomial random variable X can be approximated by a Poisson random variable.

Sketch of Proof. Let $X \sim Bin(n, p)$. So

$$\begin{aligned} P(X = m) &= \frac{n!}{m!(n-m)!} p^m (1-p)^{n-m} \\ &= \frac{n!}{n^m (n-m)!} \frac{(np)^m}{m!} (1 - \frac{np}{n})^{n-m} \\ &= \frac{n!}{n^m (n-m)!} \frac{(np)^m}{m!} (1 - \frac{np}{n})^{n} (1 - \frac{np}{n})^{-m}. \end{aligned}$$

As $n \to \infty$,

$$\frac{n!}{n^m (n-m)!} = \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-m+1}{n} \approx 1,$$

$$(1 - \frac{np}{n})^{-m} \approx (1 - \frac{\lambda}{n})^{-m} \approx 1,$$

$$(1 - \frac{np}{n})^{n} \approx (1 - \frac{\lambda}{n})^{n} \approx e^{-\lambda},$$

and

$$\frac{(np)^m}{m!} \approx \frac{\lambda^m}{m!}.$$

Therefore,

$$P(X = m) \approx \frac{\lambda^m}{m!} e^{-\lambda} = Poisson(\lambda).$$

Example. Suppose in a given year, the number of fatal automobile accidents has a binomial distribution for a particular insuarance company with five hundred automobile insurance policies. On average, there is one policy out of the five hundred that will be involved in a fatal crash. What is the probability that there will be no fatal accidents (out of five hundred policies) in any particular year?

Solution. If X be the number of fatal accidents in a year from a population of 500 auto policies, then $X \sim Bin(n, p)$ with $n = 500$ and $p = 1/500$. $\lambda = 500 \times 1/500 = 1$ and so

$$P(X = 0) \approx e^{-1} \approx 0.368.$$

Using the binomial distribution, we have

$$P(X = 0) = (1 - \frac{1}{500})^{500} \approx 0.367.$$