proof that normal distribution is a distribution

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$$
\begin{aligned}
\int\limits_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\,dx &= \sqrt{\left(\int\limits_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\,dx\right)^2} \\
&= \sqrt{\int\limits_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\,dx \int\limits_{-\infty}^{\infty} \frac{e^{-\frac{(y-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\,dy} \\
&= \sqrt{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2+(y-\mu)^2}{2\sigma^2}}}{\sigma^2 2\pi}\,dx\,dy}
\end{aligned}
$$

Substitute $x' = x - \mu$ and $y' = y - \mu$. Since the bounds are infinite, they do not change, and $dx' = dx$ and $dy' = dy$. Thus, we have

$$
\sqrt{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2+(y-\mu)^2}{2\sigma^2}}}{\sigma^2 2\pi}\,dx\,dy} = \sqrt{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{e^{-\frac{(x')^2+(y')^2}{2\sigma^2}}}{\sigma^2 2\pi}\,dx'\,dy'}.
$$

Converting to polar coordinates, we obtain

$$
\begin{aligned}
\sqrt{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{e^{-\frac{(x')^2+(y')^2}{2\sigma^2}}}{\sigma^2 2\pi}\,dx'\,dy'} &= \sqrt{\int\limits_{0}^{\infty}\int\limits_{0}^{2\pi} \frac{re^{-\frac{r^2}{2\sigma^2}}}{\sigma^2 2\pi}\,dr\,d\theta} \\
&= \sqrt{\int\limits_{0}^{2\pi} \frac{d\theta}{2\pi}}\sqrt{\int\limits_{0}^{\infty} \frac{re^{-\frac{r^2}{2\sigma^2}}}{\sigma^2}\,dr} \\
&= \sqrt{\frac{\theta}{2\pi}\bigg|_0^{2\pi}}\sqrt{\frac{1}{\sigma^2}\int\limits_{0}^{\infty} re^{-\frac{r^2}{2\sigma^2}}\,dr} \\
&= \sqrt{\frac{2\pi}{2\pi}}\sqrt{\frac{\sigma^2}{\sigma^2}\left(-e^{-\frac{r^2}{2\sigma^2}}\right)\bigg|_0^{\infty}} \\
&= \sqrt{1}\sqrt{1} \\
&= 1.
\end{aligned}
$$