random vector

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Defines	random matrix
Defines	distribution of a random vector
Defines	distribution of a random matrix
Defines	mean vector

A *random vector* is a finite-dimensional formal vector of random variables. The random vector can be written either as a column or row of random variables, depending on its context and use. So if $X_1, X_2, \ldots, X_n$ are random variables, then

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = (X_1, X_2, \ldots, X_n)^{\mathrm{T}}$$

is a random (column) vector. Similarly, one defines a *random matrix* to be a formal matrix whose entries are all random variables. The size of a random vector and the size of a random matrix are assumed to be finite fixed constants.

The *distribution of a random vector* $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is defined to be the joint distribution of its coordinates $X_1, \ldots, X_n$:

$$F_{\mathbf{X}}(\mathbf{x}) := F_{X_1,\ldots,X_n}(x_1, \ldots, x_n).$$

Similarly, the *distribution of a random matrix* is the joint distribution of its matrix components.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random vector. If $\mathrm{E}[X_i]$ exists ($< \infty$) for each i, then the expectation of $\mathbf{X}$, called the *mean vector* and denoted by $\mathbf{E}[\mathbf{X}]$, is defined to be:

$$\mathbf{E}[\mathbf{X}] := (\mathrm{E}[X_1], \mathrm{E}[X_2], \ldots, \mathrm{E}[X_n]).$$

Clearly $\mathbf{E}[\mathbf{X}]^T = \mathbf{E}[\mathbf{X}^T]$. The expectation of a random matrix is similarly defined. Note that the definitions of expectations can also be defined via measure theory. Then, using Fubini's Theorem, one can show that the two sets of definitions coincide.

Again, let $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$ be a random vector. If $\boldsymbol{\mu}$=$\mathbf{E}[\mathbf{X}]$ is defined and $\mathrm{E}[X_i X_j]$ are defined for all $1 \leq i, j \leq n$, then the variance of $\mathbf{X}$, denoted by $\mathbf{Var}[\mathbf{X}]$, is defined to be:

$$\mathbf{Var}[\mathbf{X}] := \mathbf{E}\big[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T\big].$$

It is not hard to see that $\mathbf{Var}[\mathbf{X}]$ is an $n \times n$ symmetric matrix and it is equal to the covariance matrix of the X_i's.

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1. If $\mathbf{X}$ is an n-dimensional random vector with $\mathbf{A}$ a $m \times n$ constant matrix and $\boldsymbol{\alpha}$ an m-dimensional constant vector, then

$$\mathbf{E}[\mathbf{A}\mathbf{X} + \boldsymbol{\alpha}] = \mathbf{A}\mathbf{E}[\mathbf{X}] + \boldsymbol{\alpha}.$$

2. Same set up as above. Then

$$\mathbf{Var}[\mathbf{A}\mathbf{X} + \boldsymbol{\alpha}] = \mathbf{A}\mathbf{Var}[\mathbf{X}]\mathbf{A}^T.$$

   If the X_i's are *iid* (independent identically distributed), with variance $\boldsymbol{\sigma}^2$, then

$$\mathbf{Var}[\mathbf{A}\mathbf{X} + \boldsymbol{\alpha}] = \boldsymbol{\sigma}^2\mathbf{A}\mathbf{A}^T.$$

3. Let $\mathbf{X}$ be an n-dimensional random vector with $\boldsymbol{\mu} = \mathbf{E}[\mathbf{X}]$, $\boldsymbol{\Sigma} = \mathbf{Var}[\mathbf{X}]$. $\mathbf{A}$ is an $n \times n$ constant matrix. Then

$$\mathbf{E}[\mathbf{X}^T\mathbf{A}\mathbf{X}] = \mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}^T\mathbf{A}\boldsymbol{\mu}.$$