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exponential family

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Defines	canonical exponential family
Defines	nuisance parameter
Defines	natural parameter

A probability (density) function $f_X(x \mid \theta)$ given a parameter θ is said to belong to the (one parameter) *exponential family* of distributions if it can be written in one of the following two equivalent forms:

1. $a(x)b(\theta)\exp\big[c(x)d(\theta)\big]$
2. $\exp\big[a(x)+b(\theta)+c(x)d(\theta)\big]$

where a, b, c, d are known functions. If $c(x) = x$, then the distribution is said to be in *canonical form*. When the distribution is in canonical form, the function $d(\theta)$ is called a *natural parameter*. Other parameters present in the distribution that are not of any interest, or that are already calculated in advance, are called *nuisance parameters*.

Examples:

- The normal distribution, $N(\mu, \sigma^2)$, treating σ^2 as a nuisance parameter, belongs to the exponential family. To see this, take the natural logarithm of $N(\mu, \sigma^2)$ to get

  $$-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2$$

  Rearrange the above expression and we have

  $$\frac{x\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\Big[\frac{x^2}{\sigma^2} + \ln(2\pi\sigma^2)\Big]$$

  Set $c(x) = x$, $d(\mu) = \mu/\sigma^2$, $b(\mu) = -\mu^2/(2\sigma^2)$, and $a(x) = -1/2\big[x^2/\sigma^2 + \ln(2\pi\sigma^2)\big]$. Then we see that $N(\mu, \sigma^2)$ does indeed belong to the exponential family. Furthermore, it is in canonical form. The natural parameter is $d(\mu) = \mu/\sigma^2$.

- Similarly, the Poisson, binomial, Gamma, and inverse Gaussian distributions all belong to the exponential family and they are all in canonical form.

- Lognormal and Weibull distributions also belong to the exponential family but they are not in canonical form.

Remarks

- If the p.d.f of a random variable X belongs to an exponential family, and it is expressed in the second of the two above forms, then

$$\mathrm{E}[c(X)] = -\frac{b'(\theta)}{d'(\theta)}, \tag{1}$$

and

$$\mathrm{Var}[c(X)] = \frac{d''(\theta)b'(\theta) - d'(\theta)b''(\theta)}{d'(\theta)^3}, \tag{2}$$

provided that functions b and d are appropriately conditioned.

- Given a member from the exponential family of distributions, we have $\mathrm{E}[U] = 0$ and $I = -\mathrm{E}[U']$, where U is the score function and I the Fisher information. To see this, first observe that the log-likelihood function from a member of the exponential family of distributions is given by

$$\ell(\theta \mid x) = a(x) + b(\theta) + c(x)d(\theta),$$

and hence the score function is

$$U(\theta) = b'(\theta) + c(X)d'(\theta).$$

From (1), $\mathrm{E}[U] = 0$. Next, we obtain the Fisher information I. By definition, we have

$$\begin{aligned} I &= \mathrm{E}[U^2] - \mathrm{E}[U]^2 \\ &= \mathrm{E}[U^2] \\ &= d'(\theta)^2 \,\mathrm{Var}[c(X)] \\ &= \frac{d''(\theta)b'(\theta) - d'(\theta)b''(\theta)}{d'(\theta)} \end{aligned}$$

On the other hand,

$$\frac{\partial U}{\partial \theta} = b''(\theta) + c(X)d''(\theta)$$

so

$$\begin{aligned}
\mathrm{E}\left[\frac{\partial U}{\partial \theta}\right] &= b''(\theta) + \mathrm{E}[c(X)]d''(\theta) \\
&= b''(\theta) - \frac{b'(\theta)}{d'(\theta)} d''(\theta) \\
&= \frac{b''(\theta)d'(\theta) - b'(\theta)d''(\theta)}{d'(\theta)} \\
&= -I
\end{aligned}$$

- For example, for a Poisson distribution

$$f_X(x \mid \theta) = \frac{\theta^x e^{-\theta}}{x!},$$

the natural parameter $d(\theta)$ is $\ln \theta$ and $b(\theta) = -\theta$. $c(x) = x$ since Poisson is in canonical form. Then

$$U(\theta) = -1 + \frac{X}{\theta} \text{ and } I = -\,\mathrm{E}\left[\frac{-X}{\theta^2}\right] = \frac{1}{\theta}$$

as expected.