discrete cosine transform

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Synonym	DCT
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Related topic	DiscreteSineTransform
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Related topic	DiscreteFourierTransform
Defines	DCT-I
Defines	DCT-II
Defines	DCT-III
Defines	DCT-IV
Defines	DCT-V
Defines	DCT-VI
Defines	DCT-VII
Defines	DCT-VIII

The *discrete cosine transforms (DCT)* are a family of transforms closely related to the discrete sine transform and the discrete Fourier transform. The DCT-II is the most commonly used form and plays an important role in coding signals and images [?], e.g. in the widely used standard JPEG compression. The discrete cosine transform was first introduced by Ahmed, Natarajan, and Rao [?]. Later Wang and Hunt [?] introduced the set of variants.

The DCT is included in many mathematical packages, such as Matlab, Mathematica and GNU Octave.

1 Definition

The orthonormal variants of the DCT, where x_n is the original vector of N real numbers, C_k is the transformed vector of N real numbers and δ is the Kronecker delta, are defined by the following equations:

1.1 DCT-I

$$\begin{aligned} C_k^I &= p_k \sum_{n=0}^{N-1} x_n q_n \cos \frac{\pi n k}{N-1} \qquad k = 0, 1, 2, \ldots, N-1 \\ p_k &= \sqrt{\frac{2 - \delta_{k,0} - \delta_{k,N-1}}{N-1}} \\ q_n &= \sqrt{\frac{1}{1 + \delta_{n,0} + \delta_{n,N-1}}} \end{aligned}$$

The DCT-I is its own inverse.

1.2 DCT-II

$$\begin{aligned} C_k^{II} &= p_k \sum_{n=0}^{N-1} x_n \cos \frac{\pi \left(n + \frac{1}{2}\right) k}{N} \qquad k = 0, 1, 2, \ldots, N-1 \\ p_k &= \sqrt{\frac{2 - \delta_{k,0}}{N}} \end{aligned}$$

The inverse of DCT-II is DCT-III.

1.3 DCT-III

$$
\begin{aligned}
C_k^{III} &= p \sum_{n=0}^{N-1} x_n q_n \cos \frac{\pi n \left(k + \frac{1}{2}\right)}{N} \qquad k = 0, 1, 2, \ldots, N-1 \\
p &= \sqrt{\frac{2}{N}} \\
q_n &= \sqrt{\frac{1}{1 + \delta_{n,0}}}
\end{aligned}
$$

The inverse of DCT-III is DCT-II.

1.4 DCT-IV

$$
\begin{aligned}
C_k^{IV} &= p \sum_{n=0}^{N-1} x_n \cos \frac{\pi \left(n + \frac{1}{2}\right) \left(k + \frac{1}{2}\right)}{N} \qquad k = 0, 1, 2, \ldots, N-1 \\
p &= \sqrt{\frac{2}{N}}
\end{aligned}
$$

The DCT-IV is its own inverse.

1.5 DCT-V

$$
\begin{aligned}
C_k^{V} &= p_k \sum_{n=0}^{N-1} x_n q_n \cos \frac{\pi n k}{N - \frac{1}{2}} \qquad k = 0, 1, 2, \ldots, N-1 \\
p_k &= \sqrt{\frac{2 - \delta_{k,0}}{N - \frac{1}{2}}} \\
q_n &= \sqrt{\frac{1}{1 + \delta_{n,0}}}
\end{aligned}
$$

The DCT-V is its own inverse.

1.6 DCT-VI

$$
\begin{aligned}
C_k^{VI} &= p_k \sum_{n=0}^{N-1} x_n q_n \cos \frac{\pi \left(n + \frac{1}{2}\right) k}{N - \frac{1}{2}} \qquad k = 0, 1, 2, \ldots, N-1 \\
p_k &= \sqrt{\frac{2 - \delta_{k,0}}{N - \frac{1}{2}}} \\
q_n &= \sqrt{\frac{1}{1 + \delta_{n,N-1}}}
\end{aligned}
$$

The inverse of DCT-VI is DCT-VII.

1.7 DCT-VII

$$
\begin{aligned}
C_k^{VII} &= p_k \sum_{n=0}^{N-1} x_n q_n \cos \frac{\pi n \left(k + \frac{1}{2}\right)}{N - \frac{1}{2}} \qquad k = 0, 1, 2, \ldots, N-1 \\
p_k &= \sqrt{\frac{2 - \delta_{k,N-1}}{N - \frac{1}{2}}} \\
q_n &= \sqrt{\frac{1}{1 + \delta_{n,0}}}
\end{aligned}
$$

The inverse of DCT-VII is DCT-VI.

1.8 DCT-VIII

$$
\begin{aligned}
C_k^{VII} &= p \sum_{n=0}^{N-1} x_n \cos \frac{\pi \left(n + \frac{1}{2}\right) \left(k + \frac{1}{2}\right)}{N + \frac{1}{2}} \qquad k = 0, 1, 2, \ldots, N-1 \\
p &= \sqrt{\frac{2}{N + \frac{1}{2}}}
\end{aligned}
$$

The DCT-VIII is its own inverse.

2 Two-dimensional DCT

The DCT in two dimensions is simply the one-dimensional transform computed in each row and each column. For example, the DCT-II of a $N_1 \times N_2$ matrix is given by

$$C^{II}_{k_1,k_2} = p_{k_1} p_{k_2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x_{n_1,n_2} \cos \frac{\pi \left(n_1 + \frac{1}{2}\right) k_1}{N_1} \cos \frac{\pi \left(n_2 + \frac{1}{2}\right) k_2}{N_2}$$

References

[1] This entry is based on content from The Data Analysis Briefbook (`http://rkb.home.cern.ch/rkb/titleA.html`http://rkb.home.cern.ch/rkb/titleA.html)

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[6] Z. Wang and B. Hunt, The Discrete W Transform, Applied Mathematics and Computation, 16. 1985.