Gram-Schmidt orthogonalization

Canonical name	GramSchmidtOrthogonalization
Date of creation	2013-03-22 12:06:14
Last modified on	2013-03-22 12:06:14
Owner	akrowne (2)
Last modified by	akrowne (2)
Numerical id	9
Author	akrowne (2)
Entry type	Algorithm
Classification	msc 65F25
Synonym	Gram-Schmidt decomposition
Synonym	Gram-Schmidt orthonormalization
Synonym	Gram-Schmidt process
Related topic	HouseholderTransformation
Related topic	GivensRotation
Related topic	QRDecomposition
Related topic	AnExampleForSchurDecomposition

Any set of linearly independent vectors $v_1, \ldots, v_n$ can be converted into a set of orthogonal vectors $q_1, \ldots, q_n$ by the Gram-Schmidt process. In three dimensions, v_1 determines a line; the vectors v_1 and v_2 determine a plane. The vector q_1 is the unit vector in the direction v_1. The (unit) vector q_2 lies in the plane of v_1, v_2, and is normal to v_1 (on the same side as v_2. The (unit) vector q_3 is normal to the plane of v_1, v_2, on the same side as v_3, etc.

In general, first set $u_1 = v_1$, and then each u_i is made orthogonal to the preceding $u_1, \ldots u_{i-1}$ by subtraction of the projections of v_i in the directions of $u_1, \ldots, u_{i-1}$:

$$u_i = v_i - \sum_{j=1}^{i-1} \frac{u_j^T v_i}{u_j^T u_j} u_j$$

The i vectors u_i span the same subspace as the v_i. The vectors $q_i = u_i/||u_i||$ are orthonormal. This leads to the following theorem:

Theorem.

Any $m \times n$ matrix A with linearly independent columns can be factorized into a product, $A = QR$. The columns of Q are orthonormal and R is upper triangular and invertible.

This "classical" Gram-Schmidt method is often numerically unstable, see [Golub89] for a "modified" Gram-Schmidt method.

References

- Originally from The Data Analysis Briefbook (`http://rkb.home.cern.ch/rkb/titleA.html`

Golub89 Gene H. Golub and Charles F. van Loan: Matrix Computations, 2nd edn., The John Hopkins University Press, 1989.