discrete Fourier transform

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Defines	IDFT

The *discrete Fourier transform (DFT)* is an invertible transform widely employed in signal processing and analysis. It can be computed using stable efficient algorithms known as Fast Fourier Transform (FFT) algorithms.

1 Definition

Given a sequence $\{f_n\}_{n=0}^{N-1}$ of N complex numbers, the DFT is defined as a sequence $\{F_k\}_{k=0}^{N-1}$ of N complex numbers according to the equation

$$F_k = \frac{1}{N^{(1-p)/2}} \sum_{n=0}^{N-1} f_n \, W_N^{-k,n} \qquad k = 0, 1, 2, \ldots, N-1$$

where $W_N^{k,n} = e^{2\pi i q k n / N}$ are the basis functions, p is an arbitrary real number and q is either 1 or -1.

Similarly, we can reconstruct $\{f_n\}_{n=0}^{N-1}$ from $\{F_k\}_{k=0}^{N-1}$ via the *inverse discrete Fourier transform (IDFT)*:

$$f_n = \frac{1}{N^{(1+p)/2}} \sum_{k=0}^{N-1} F_k \, W_N^{k,n} \qquad n = 0, 1, 2, \ldots, N-1$$

2 Explanation

Generically, if we have some vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ that we wish to project on to a set of unit basis vectors $\hat{\mathbf{B}}_1, \hat{\mathbf{B}}_2, \ldots, \hat{\mathbf{B}}_m$, we find that the kth component of the projection $\mathbf{v}'$ is given by

$$v_k' = \langle \mathbf{v}, \hat{\mathbf{B}}_k \rangle$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ is an inner product. Using the standard dot product of complex vectors, we have

$$v_k' = \sum_{j=0}^{m} v_j \bar{B}_{k,j}$$

(where $\bar{z}$ represents the complex conjugate of z).

We may use a procedure to project a function onto basis functions. Here, the components of our vectors are the functions sampled at certain time intervals.

The idea of the Fourier transform is to project our discrete function $\{f_n\}$ onto a basis made up of sinusoidal functions of varying frequency.

We know from Euler's relation that

$$e^{ix} = \cos x + i \sin x$$

so we can construct a sinusoidal function of any frequency k:

$$W_N^{k,n} = e^{2\pi i q k n / N} = \cos(2\pi i q k n / N) + i \sin(2\pi i q k n / N)$$

By Nyquist's theorem, any frequency components of $\{f_n\}$ above the Nyquist frequency $\frac{N}{2}$ will be aliased into lower frequencies, so we will only concern ourselves with the frequency components $\frac{-N}{2} \leq k \leq \frac{N}{2}$.

Now we substitute these results into the standard formula for projection to reveal that

$$F_k = \sum_{n=0}^{N-1} f_n \, W_N^{-k,n} \qquad k = 0, 1, 2, \ldots, N-1$$

3 Group theoretic interpretation and generalization

This procedure may be interpreted in terms of group theory as follows: The basis functions $\chi_k(j) e^{-2\pi \nu_k i j / f}$ are the characters of the irreducible representations of the group $\mathbb{Z}_M$. The orthogonality relation $\langle \chi_m, \chi_n \rangle = \delta_{mn}$ which was used to obtain the transform as a projection is the orthogonality relation for characters.

Once we view the discrete Fourier transform in this fashion, it is not hard to generalize it by replacing $\mathbb{Z}_M$ with any discrete group G. As it turns out, there are two versions of the generalization which turn out to coincide exactly in the case of commutative groups (such as $\mathbb{Z}_M$).

First generalization: Let $f \colon G \to \mathbb{C}$ be conjugation invariant, which means that, for every $y \in G$, it is the case that $f(yxy^{-1}) = f(x)$. Then one has

$$f(x) = \sum_k \frac{1}{d_k} \langle f, \chi_k \rangle \chi_k(x)$$

where the index k runs over all irreducible representations and d_k is the dimension of the representation k. The inner product on vectors is the usual one:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{g \in G} \mathbf{u}(g) \bar{\mathbf{v}}(g)$$

The justification for this comes from the fact that the set of irreducible characters spans the space of conjugation invariant functions and the orthogonality of these characters.

Second generalization: If $f \colon G \to \mathbb{C}$ is any function on a group (not necessarily conjugation invariant), then we have the transform

$$f(x) = \sum_{k} \sum_{i,j=1}^{d_k} \langle f, \rho_{mn}^k \rangle \rho_{mn}^k(x)$$

where k and d_k are as before and ρ_{mn}^k are the matrix elements of the representation k. The justification for this comes from the fact that the set of matrix elements of representations spans the space of functions on the group and the orthogonality relation for matrix elements.

Remarks: DFT is also generalized by two-dimensional Fourier transforms (2D-FT), Fourier-Stieltjes transforms, and via groupoid representation to anharmonic analysis that can also be numerically computed by utilizing FFT, but tend to be very much slower that the one-dimensional FFT because many series of sequential one-dimensional FFTs and also additional matrix transformations are involved in 2D-FT, or 2D-FFT.

References: Paul Bourke, `http://astronomy.swin.edu.au/ pbourke/analysis/dft/`"Discr Fourier Transform"