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generalized eigenvector

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Defines	cycle of generalized eigenvectors

Let V be a vector space over a field k and T a linear transformation on V (a linear operator). A non-zero vector $v \in V$ is said to be a *generalized eigenvector* of T (corresponding to λ) if there is a $\lambda \in k$ and a positive integer m such that

$$(T - \lambda I)^m(v) = 0,$$

where I is the identity operator.

In the equation above, it is easy to see that λ is an eigenvalue of T. Suppose that m is the least such integer satisfying the above equation. If $m = 1$, then λ is an eigenvalue of T. If $m > 1$, let $w = (T - \lambda I)^{m-1}(v)$. Then $w \neq 0$ (since $v \neq 0$) and $(T - \lambda I)(w) = 0$, so λ is again an eigenvalue of T.

Let v be a generalized eigenvector of T corresponding to the eigenvalue λ. We can form a sequence

$$v, (T - \lambda I)(v), (T - \lambda I)^2(v), \ldots, (T - \lambda I)^i(v), \ldots, (T - \lambda I)^m(v) = 0, 0, \ldots$$

The set $C_\lambda(v)$ of all non-zero terms in the sequence is called a *cycle of generalized eigenvectors* of T corresponding to λ. The cardinality m of $C_\lambda(v)$ is its . For any $C_\lambda(v)$, write $v_\lambda = (T - \lambda I)^{m-1}(v)$.

Below are some properties of $C_\lambda(v)$:

- v_λ is the only eigenvector of λ in $C_\lambda(v)$, for otherwise $v_\lambda = 0$.

- $C_\lambda(v)$ is linearly independent.

  Proof. Let $v_i = (T - \lambda I)^{i-1}(v)$, where $i = 1, \ldots, m$. Let $0 = \sum_{i=1}^{m} r_i v_i$ with $r_i \in k$. Induct on i. If $i = 1$, then $v_1 = v \neq 0$, so $r_1 = 0$ and $\{v_1\}$ is linearly independent. Suppose the property is true when $i = m-1$. Apply $T - \lambda I$ to the equation, and we have $0 = \sum_{i=1}^{m} r_i(T - \lambda I)(v_i) = \sum_{i=1}^{m-1} r_i v_{i+1}$. Then $r_1 = \cdots = r_{m-1} = 0$ by induction. So $0 = r_m v_m = r_m v_\lambda$ and thus $r_m = 0$ since v_λ is an eigenvector and is non-zero. $\square$

- More generally, it can be shown that $C_\lambda(v_1) \cup \cdots \cup C_\lambda(v_k)$ is linearly independent whenever $\{v_{1\lambda}, \ldots, v_{k\lambda}\}$ is.

- Let $E = \operatorname{span}(C_\lambda(v))$. Then E is a $(m+1)$-dimensional subspace of the generalized eigenspace of T corresponding to λ. Furthermore, let $T|_E$

be the restriction of T to E, then $[T|_E]_{C_\lambda(v)}$ is a Jordan block, when $C_\lambda(v)$ is ordered (as an ordered basis) by setting

$$(T-\lambda I)^i(v) < (T-\lambda I)^j(v) \qquad \text{whenever} \qquad i > j.$$

Indeed, for if we let $w_i = (T-\lambda I)^{m+1-i}(v)$ for $i = 1, \ldots m+1$, then

$$T(w_i) = (T-\lambda I+\lambda I)(T-\lambda I)^{m+1-i}(v) \quad = \quad \begin{cases} \lambda w_i & \text{if } i = 1, \\ w_{i-1} + \lambda w_i & \text{otherwise.} \end{cases}$$

so that $[T|_E]_{C_\lambda(v)}$ is the $(m+1) \times (m+1)$ matrix given by

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

- A cycle of generalized eigenvectors is called *maximal* if $v \notin (T-\lambda I)(V)$. If V is finite dimensional, any cycle of generalized eigenvectors $C_\lambda(v)$ can always be extended to a maximal cycle of generalized eigenvectors $C_\lambda(w)$, meaning that $C_\lambda(v) \subseteq C_\lambda(w)$.

- In particular, any eigenvector v of T can be extended to a maximal cycle of generalized eigenvectors. Any two maximal cycles of generalized eigenvectors extending v span the same subspace of V.

References

[1] Friedberg, Insell, Spence. *Linear Algebra*. Prentice-Hall Inc., 1997.