proof of Euler-Maclaurin summation formula

Canonical name	ProofOfEulerMaclaurinSummationFormula
Date of creation	2013-03-22 13:28:41
Last modified on	2013-03-22 13:28:41
Owner	pbruin (1001)
Last modified by	pbruin (1001)
Numerical id	5
Author	pbruin (1001)
Entry type	Proof
Classification	msc 65B15

Let a and b be integers such that $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous. We will prove by induction that for all integers $k \geq 0$, if f is a C^{k+1} function,

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)\mathrm{d}t + \sum_{r=0}^{k} \frac{(-1)^{r+1}B_{r+1}}{(r+1)!}(f^{(r)}(b) - f^{(r)}(a)) + \frac{(-1)^k}{(k+1)!}\int_a^b B_{k+1}(t)f^{(k+1)}(t)\mathrm{d}t \tag{1}$$

where B_r is the rth Bernoulli number and $B_r(t)$ is the rth Bernoulli periodic function.

To prove the formula for $k = 0$, we first rewrite $\int_{n-1}^{n} f(t)\mathrm{d}t$, where n is an integer, using integration by parts:

$$\begin{aligned}
\int_{n-1}^{n} f(t)\mathrm{d}t &= \int_{n-1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}(t - n + \frac{1}{2})f(t)\mathrm{d}t \\
&= (t - n + \frac{1}{2})f(t)\Big|_{n-1}^{n} - \int_{n-1}^{n} (t - n + \frac{1}{2})f'(t)\mathrm{d}t \\
&= \frac{1}{2}(f(n) + f(n-1)) - \int_{n-1}^{n} (t - n + \frac{1}{2})f'(t)\mathrm{d}t.
\end{aligned}$$

Because $t - n + \frac{1}{2} = B_1(t)$ on the interval $(n-1, n)$, this is equal to

$$\int_{n-1}^{n} f(t)\mathrm{d}t = \frac{1}{2}(f(n) + f(n-1)) - \int_{n-1}^{n} B_1(t)f'(t)\mathrm{d}t.$$

From this, we get

$$f(n) = \int_{n-1}^{n} f(t)\mathrm{d}t + \frac{1}{2}(f(n) - f(n-1)) + \int_{n-1}^{n} B_1(t)f'(t)\mathrm{d}t.$$

Now we take the sum of this expression for $n = a+1, a+2, \ldots, b$, so that the middle term on the right telescopes away for the most part:

$$\sum_{n=a+1}^{b} f(n) = \int_a^b f(t)\mathrm{d}t + \frac{1}{2}(f(b) - f(a)) + \int_a^b B_1(t)f'(t)\mathrm{d}t$$

which is the Euler-Maclaurin formula for $k = 0$, since $B_1 = -\frac{1}{2}$.

Suppose that $k > 0$ and the formula is correct for $k - 1$, that is

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)\mathrm{d}t + \sum_{r=0}^{k-1} \frac{(-1)^{r+1}B_{r+1}}{(r+1)!}(f^{(r)}(b) - f^{(r)}(a)) + \frac{(-1)^{k-1}}{k!}\int_a^b B_k(t)f^{(k)}(t)\mathrm{d}t. \tag{2}$$

We rewrite the last integral using integration by parts and the facts that B_k is continuous for $k \geq 2$ and $B'_{k+1}(t) = (k+1)B_k(t)$ for $k \geq 0$:

$$\begin{aligned}\int_a^b B_k(t) f^{(k)}(t)\mathrm{d}t &= \int_a^b \frac{B'_{k+1}(t)}{k+1} f^{(k)}(t)\mathrm{d}t \\ &= \frac{1}{k+1} B_{k+1}(t) f^{(k)}(t)\Big|_a^b - \frac{1}{k+1}\int_a^b B_{k+1}(t) f^{(k+1)}(t)\mathrm{d}t.\end{aligned}$$

Using the fact that $B_k(n) = B_k$ for every integer n if $k \geq 2$, we see that the last term in Eq. **??** is equal to

$$\frac{(-1)^{k+1}B_{k+1}}{(k+1)!}(f^{(k)}(b) - f^{(k)}(a)) + \frac{(-1)^k}{(k+1)!}\int_a^b B_{k+1}(t) f^{(k+1)}(t)\mathrm{d}t.$$

Substituting this and absorbing the left term into the summation yields Eq. **??**, as required.