QR decomposition

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1 QR Decomposition

Orthogonal matrix triangularization (QR decomposition) reduces a real $m \times n$ matrix A with $m \geq n$ and full rank to a much simpler form. It guarantees numerical stability by minimizing errors caused by machine roundoffs. A suitably chosen orthogonal matrix Q will triangularize the given matrix:

$$A = Q \begin{bmatrix} R \\ 0 \end{bmatrix}$$

with the $n \times n$ right triangular matrix R. One only has then to solve the triangular system $Rx = Pb$, where P consists of the first n rows of Q.

The least squares problem $Ax \approx b$ is easy to solve with $A = QR$ and Q an orthogonal matrix (here and henceforth R is the entire $m \times n$ augmented matrix from above). The solution

$$x = (A^T A)^{-1} A^T b$$

becomes

$$x = (R^T Q^T Q R)^{-1} R^T Q^T b = (R^T R)^{-1} R^T Q^T b = R^{-1} Q^T b$$

This is a matrix-vector multiplication $Q^T b$, followed by the solution of the triangular system $Rx = Q^T b$ by back-substitution. The QR factorization saves us the formation of $A^T A$ and the solution of the normal equations.

Many different methods exist for the QR decomposition, e.g. the Householder transformation, the Givens rotation, or the Gram-Schmidt decomposition.

References

[1] The Data Analysis Briefbook. `http://rkb.home.cern.ch/rkb/titleA.html`http://rkb.home.c