finite difference

Canonical name	FiniteDifference
Date of creation	2013-03-22 15:35:00
Last modified on	2013-03-22 15:35:00
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	11
Author	CWoo (3771)
Entry type	Definition
Classification	msc 65Q05
Related topic	Equation
Related topic	RecurrenceRelation
Related topic	IndefiniteSum
Related topic	DifferentialPropositionalCalculus
Defines	forward difference
Defines	backward difference
Defines	difference equation

Definition of Δ.

The derivative of a function $f\colon \mathbb{R} \to \mathbb{R}$ is defined to be the expression

$$\frac{df}{dx} := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

which makes sense whenever f is differentiable (at least at x). However, the expression

$$\frac{f(x+h)-f(x)}{h}$$

makes sense even without f being continuous, as long as $h \neq 0$. The expression is called a *finite difference*. The simplest case when $h = 1$, written

$$\Delta f(x) := f(x+1) - f(x),$$

is called the *forward difference* of f. For other non-zero h, we write

$$\Delta_h f(x) := \frac{f(x+h)-f(x)}{h}.$$

When $h = -1$, it is called a *backward difference* of f, sometimes written $\nabla f(x) := \Delta_{-1} f(x)$. Given a function $f(x)$ and a real number $h \neq 0$, if we define $y = \frac{x}{h}$ and $g(y) = \frac{f(hy)}{h}$, then we have

$$\Delta g(y) = \Delta_h f(x).$$

Conversely, given $g(y)$ and $h \neq 0$, we can find $f(x)$ such that $\Delta g(y) = \Delta_h f(x)$.

Some Properties of Δ.

It is easy to see that the forward difference operator Δ is linear:

1. $\Delta(f+g) = \Delta(f) + \Delta(g)$
2. $\Delta(cf) = c\Delta(f)$, where $c \in \mathbb{R}$ is a constant.

Δ also has the properties

1. $\Delta(c) = 0$ for any real-valued constant function c, and
2. $\Delta(I) = 1$ for the identity function $I(x) = x$. constant.

The behavior of Δ in this respect is similar to that of the derivative operator. However, because the continuity of f is not assumed, $\Delta f = 0$ does not imply that f is a constant. f is merely a periodic function $f(x+1) = f(x)$. Other interesting properties include

1. $\Delta a^x = (a-1)a^x$ for any real number a
2. $\Delta x^{(n)} = nx^{(n-1)}$ where $x^{(n)}$ denotes the falling factorial polynomial
3. $\Delta b_n(x) = nx^{n-1}$, where $b_n(x)$ is the Bernoulli polynomial of order n.

From Δ, we can also form other operators. For example, we can iteratively define

$$\Delta^1 f := \Delta f \tag{1}$$
$$\Delta^k f := \Delta(\Delta^{k-1} f), \quad \text{where } k > 1. \tag{2}$$

Of course, all of the above can be readily generalized to Δ_h. It is possible to show that $\Delta_h f$ can be written as a linear combination of

$$\Delta f, \Delta^2 f, \ldots, \Delta^h f.$$

Difference Equation.

Suppose $F\colon \mathbb{R}^n \to \mathbb{R}$ is a real-valued function whose domain is the n-dimensional Euclidean space. A *difference equation* (in one variable x) is the equation of the form

$$F(x, \Delta_{h_1}^{k_1} f, \Delta_{h_2}^{k_2} f, \ldots, \Delta_{h_n}^{k_n} f) = 0,$$

where $f := f(x)$ is a one-dimensional real-valued function of x. When h_i are all integers, the expression on the left hand side of the difference equation can be re-written and simplified as

$$G(x, f, \Delta f, \Delta^2 f, \ldots, \Delta^m f) = 0.$$

Difference equations are used in many problems in the real world, one example being in the study of traffic flow.