proof of Simpson's rule

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We want to derive Simpson's rule for

$$\int_a^b f(x)\,dx.$$

We will use Newton and Cotes formulas for $n = 2$. In this case, $x_0 = a$, $x_2 = b$ and $x_1 = (a+b)/2$. We use Lagrange's interpolation formula to get a polynomial $p(x)$ such that $p(x_j) = f(x_j)$ for $j = 0, 1, 2$.

The corresponding interpolating polynomial is

$$p(x) = f(x_1)\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + f(x_2)\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}.$$

and thus

$$\int_a^b f(x)\,dx \approx \int_a^b f(x_1)\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + f(x_2)\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}\,dx.$$

Since integration is linear, we are concerned only with integrating each term in the sum. Now, taking $x_j = a + hj$ where $j = 0, 1, 2$ and $h = |b-a|/2$, we can rewrite the quotients on the last integral as

$$\int_a^b p(x)\,dx = hf(x_0)\int_0^2 \frac{(t-1)(t-2)}{(0-1)(0-2)}\,dt + hf(x_1)\int_0^2 \frac{(t-0)(t-2)}{(1-0)(1-2)}\,dt + hf(x_2)\int_0^2 \frac{(t-0)(t-1)}{(2-0)(2-1)}\,dt$$

and if we calculate the integrals on the last expression we get

$$\int_a^b p(x)\,dx = hf(x_0)\frac{1}{3} + hf(x_1)\frac{4}{3} + hf(x_2)\frac{1}{3},$$

which is Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)).$$