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LU decomposition

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Definition 1 *Let A be an $n \times m$ matrix. An LU decomposition (sometimes also called an LU factorization) of A, if it exists, is an $n \times n$ unit lower triangular matrix L and an $n \times m$ matrix U, in (upper) echelon form, such that*

$$A = LU$$

The LU factorization is closely related to the row reduction algorithm. In a very real sense, the factorization is a record of the steps taken in row reducing a matrix to echelon form. The matrix L "encodes" the sequence of row replacement operations that row reduce the given matrix A to echelon form U.

Proposition 2 *Suppose that $A = LU$ is an LU factorization, and let L_{ij} denote the entries of L. Then, the row reduction $A \xRightarrow{\rho} U$ is accomplished by the following sequence ρ of $\frac{1}{2}n(n-1)$ row replacement operations:*

$$\begin{aligned}
\text{row}_j &\mapsto \text{row}_j - L_{j1}\,\text{row}_1, \quad j = 2, \ldots, n; \\
\text{row}_j &\mapsto \text{row}_j - L_{j2}\,\text{row}_2, \quad j = 3, \ldots, n; \\
&\ \ \vdots \\
\text{row}_j &\mapsto \text{row}_j - L_{jk}\,\text{row}_k, \quad j = k+1, \ldots, n, \quad k = 1, \ldots, n-1 \\
&\ \ \vdots
\end{aligned}$$

Note: the first $n-1$ steps clear out column 1, the next $n-2$ steps clear out column 2, etc.

Proposition 3 *Not every matrix admits an LU factorization. Indeed, an LU factorization exists if and only if A can be reduced to echelon without using row exchange operations. However, if an LU factorization exists, then it is unique.*

In the most general case, one has to employ row exchange operations.

Proposition 4 *Let A be an $n \times m$ matrix. Then, there exists an $n \times n$ permutation matrix P (indeed, many such) such that the matrix PA admits an LU factorization, i.e., there exist matrices P, L, U such that*

$$PA = LU.$$

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The key idea behind LU factorization is that one does not need to employ row scalings to do row reduction until the second half (the back-substitution phase) of the algorithm. This has significant implication for numerical stability of the algorithm.

The LU decomposition of a given matrix A is useful for the solution of the systems of linear equations of the form $Ax = b$. Indeed, it suffices to first solve the linear system $Ly = b$, and second, to solve the system $Ux = y$. This two-step procedure is easy to implement, because owing to the lower and upper-triangular nature of the matrices L and U, the required row operations are determined, more or less, directly from the entries of L and U. Indeed, the first step,

$$[\, L \ b\,] \overset{\rho_1}{\Longrightarrow} [\, I \ y\,]$$

a sequence of row operations ρ_1, and the second step

$$[\, U \ y\,] \overset{\rho_2}{\Longrightarrow} [\, E \ x_0\,]$$

a sequence of row operations ρ_2, are exactly the same row operations one has to perform to row reduce A directly to reduced echelon form E:

$$[\, A \ b\,] \overset{\rho_1}{\Longrightarrow} [\, U \ y\,] \overset{\rho_2}{\Longrightarrow} [\, E \ x_0\,].$$

Note: x_0 is the particular solution of $Ax = b$ that sits in the rightmost colulmn of the augmented matrix at the termination of the row-reduction algorithm.