proof of existence and uniqueness of singular value decomposition

Canonical name	ProofOfExistenceAndUniquenessOfSingularValueDecomposition
Date of creation	2013-03-22 17:07:46
Last modified on	2013-03-22 17:07:46
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Numerical id	12
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Entry type	Proof
Classification	msc 65-00
Classification	msc 15-00

Proof. To prove existence of the SVD, we isolate the direction of the largest action of $A \in \mathbb{C}^{m\times n}$, and then proceed by induction on the dimension of A. We will denote hermitian conjugation by ^{T}. Norms for vectors in $\mathbb{C}^n$ will be the usual euclidean 2-norm $\|\cdot\| = \|\cdot\|_2$ and for matrix the induced by norm of vectors.

Let $\sigma_1 = \|A\|$. By a compactness argument, there must be vectors $v_1 \in \mathbb{C}^n, u_1^* \in \mathbb{C}^m$ with $\|v_1\| = 1, \|u_1^*\| = \sigma_1$ and $u_1^* = Av_1$. Normalize u_1^* by setting $u_1 = u_1^* / \|u_1^*\|$ and consider any extensions of v_1 to an orthonormal basis $\{v_i\}$ of $\mathbb{C}^n$ and of u_1 to an orthonormal basis $\{u_j\}$ of $\mathbb{C}^m$; let U_1 and V_1 denote the unitary matrices with columns $\{v_i\}$ and $\{u_j\}$ respectively. Then we have

$$U_1^T A V_1 = S = \begin{pmatrix} \sigma_1 & w^T \\ 0 & B \end{pmatrix}$$

where 0 is a column vector of dimension $m-1$, w^T is a row vector of dimension $n-1$, and B is a matrix of dimension $(m-1)\times(n-1)$. Now,

$$\left\| \begin{pmatrix} \sigma_1 & w^T \\ 0 & B \end{pmatrix} \begin{pmatrix} \sigma_1 \\ w \end{pmatrix} \right\| \geq \sigma_1^2 + w^2 = (\sigma_1^2 + w^2)^{1/2} \left\| \begin{pmatrix} \sigma_1 \\ w \end{pmatrix} \right\|$$

so that $\|S\| \geq (\sigma_1^2 + w^2)^{1/2}$. But U_1 and V_1 are unitary matrix, hence $\|S\| = \sigma_1$; it therefore implies $w = 0$.
If $n = 1$ or $m = 1$ we are done. Otherwise the submatrix B describes the action of A on the subspace orthogonal to v_1. By the induction hypothesis B has an SVD $B = U_2 \Sigma_2 V_2^T$. Now it is easily verified that

$$A = U_1 \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix}^T V_1^T$$

is an SVD of A. completing the proof of existence.

For the uniqueness let $A = U\Sigma V^T$ a SVD for A and let e_i denote the i-th, $i = 1 \cdots min(m,n)$ vector of the canonical base of $\mathbb{C}^n$. As U and V are unitary, $\|Ae_i\| = \sigma_i^2$, so each σ_i is uniquely determined.

$\square$