particle moving on a cardioid at constant frequency

Canonical name	ParticleMovingOnACardioidAtConstantFrequency
Date of creation	2013-03-22 17:14:20
Last modified on	2013-03-22 17:14:20
Owner	perucho (2192)
Last modified by	perucho (2192)
Numerical id	6
Author	perucho (2192)
Entry type	Topic
Classification	msc 70B05

This is another elementary example[1] about particle kinematics. In this case we will use polar coordinates. Let us consider the cardioid [2]

$$r = 4R\cos^2\frac{\omega t}{2},$$

[3] with $R, \omega > 0$ given constants and $t \in [0, \infty)$ means time parameter. The position vector of a particle, respect to an orthonormal reference basis $\{\hat{\mathbf{r}}, \hat{\theta}\}$, moving on the cardioid is

$$\mathbf{r} = 4R\cos^2\frac{\omega t}{2}\,\hat{\mathbf{r}},$$

and its velocity [4]

$$\mathbf{v} = \dot{\mathbf{r}} = -4R\omega\sin\frac{\omega t}{2}\cos\frac{\omega t}{2}\,\hat{\mathbf{r}} + 4R\omega\cos^2\frac{\omega t}{2}\,\hat{\theta}.$$

Therefore the speed is

$$v = 4R\omega\cos\frac{\omega t}{2},$$

and the tangent vector

$$\mathbf{T} = -\sin\frac{\omega t}{2}\,\hat{\mathbf{r}} + \cos\frac{\omega t}{2}\,\hat{\theta}.$$

Next we use the formula

$$\frac{v}{\rho} := \left\|\dot{\mathbf{T}}\right\| = \left\| -\frac{\omega}{2}\cos\frac{\omega t}{2}\,\hat{\mathbf{r}} - \sin\frac{\omega t}{2}\,\dot{\hat{\mathbf{r}}} - \frac{\omega}{2}\sin\frac{\omega t}{2}\,\hat{\theta} + \cos\frac{\omega t}{2}\,\dot{\hat{\theta}} \right\|,$$

[1] C.F. particle moving on the astroid at constant frequency

[2] the locus of the points of the plane described by a circle (or disc) boundary point which it is rolling over another one with the same radius R.

[3] indeed the native polar equation of the cardioid is

$$r = 2R(1+\cos\theta), \quad \theta = \omega t.$$

[4] in polar coordinates we have

$$\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\theta},$$

because the base vectors $\hat{\mathbf{r}}, \hat{\theta}$ are changing on direction and sense according the formulas

$$\frac{d\hat{\mathbf{r}}}{d\theta} = \hat{\theta}, \qquad \frac{d\hat{\theta}}{d\theta} = -\hat{\mathbf{r}}.$$

We are using the chain rule with $\dot{\theta} = \omega$. Overdot denotes time differentiation everywhere.

and by using the time derivative of base vectors

$$\frac{v}{\rho} = \left\| -\frac{3\omega}{2}\cos\frac{\omega t}{2}\,\hat{\mathbf{r}} - \frac{3\omega}{2}\sin\frac{\omega t}{2}\,\hat{\theta} \right\|,$$

getting the equation

$$v = \frac{3}{2}\omega\rho.$$