first order operators in Riemannian geometry

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On a pseudo-Riemannian manifold M, and in Euclidean space in particular, one can express the gradient operator, the divergence operator, and the curl operator (which makes sense only if M is 3-dimensional) in terms of the exterior derivative. Let $\mathcal{C}^\infty(M)$ denote the ring of smooth functions on M; let $\mathcal{X}(M)$ denote the $\mathcal{C}^\infty(M)$-module of smooth vector fields, and let $\Omega^1(M)$ denote the $\mathcal{C}^\infty(M)$-module of smooth 1-forms. The contraction with the metric tensor g and its inverse g^{-1}, respectively, defines the $\mathcal{C}^\infty(M)$-module isomorphisms

$$\flat : \mathcal{X}(M) \to \Omega^1(M), \quad \sharp \colon \Omega^1(M) \to \mathcal{X}(M).$$

In local coordinates, this isomorphisms is expressed as

$$\left(\frac{\partial}{\partial x^i}\right)^\flat = \sum_j g_{ij} dx^j, \quad \left(dx^j\right)^\sharp = \sum_i g^{ij} \frac{\partial}{\partial x^i}.$$

or as the lowering of an index. To wit, for $V \in \mathcal{X}(M)$, we have

$$V = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i},$$
$$(V^\flat)_j = \sum_{i=1}^n g_{ij} V^i, \quad j = 1, \ldots, n.$$

The gradient operator, which in tensor notation is expressed as

$$(\operatorname{grad} f)^i = g^{ij} \frac{\partial f}{\partial x^j}, \quad f \in \mathcal{C}^\infty(M),$$

can now be defined as

$$\operatorname{grad} f = (df)^\sharp, \quad f \in \mathcal{C}^\infty(M).$$

Another natural structure on an n-dimensional Riemannian manifold is the volume form, $\omega \in \Omega^n(M)$, defined by

$$\omega = \sqrt{\det g_{ij}}\, dx^1 \wedge \ldots \wedge dx^n.$$

Multiplication by the volume form defines a natural isomorphism between functions and n-forms:

$$f \mapsto f\omega, \quad f \in \mathcal{C}^\infty(M).$$

Contraction with the volume form defines a natural isomorphism between vector fields and $(n-1)$-forms:

$$X \mapsto X \rfloor \omega, \quad X \in \mathcal{X}(M),$$

or equivalently

$$\frac{\partial}{\partial x^i} \mapsto (-1)^{i+1} \sqrt{\det g_{ij}}\, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n,$$

where $\widehat{dx^i}$ indicates an omitted factor. The divergence operator, which in tensor notation is expressed as

$$\operatorname{div} X = \nabla_i X^i, \quad X \in \mathcal{X}(M)$$

can be defined in a coordinate-free way by the following relation:

$$(\operatorname{div} X)\, \omega = d(X \rfloor \omega), \quad X \in \mathcal{X}(M).$$

Finally, on a 3-dimensional manifold we may define the curl operator in a coordinate-free fashion by means of the following relation:

$$(\operatorname{curl} X) \rfloor \omega = d(X^\flat), \quad X \in \mathcal{X}(M).$$