particle moving on the astroid at constant frequency

Canonical name	ParticleMovingOnTheAstroidAtConstantFrequency
Date of creation	2013-03-22 17:14:09
Last modified on	2013-03-22 17:14:09
Owner	perucho (2192)
Last modified by	perucho (2192)
Numerical id	9
Author	perucho (2192)
Entry type	Topic
Classification	msc 70B05

In parametric Cartesian equations, the astroid can be represented by

$$x = a\cos^3\omega t, \quad y = a\sin^3\omega t,$$

where $a > 0$ is a known constant, $\omega > 0$ is the constant angular frequency, and $t \in [0, \infty)$ is the time parameter. Thus the position vector of a particle, moving over the astroid, is

$$\mathbf{r} = a\cos^3\omega t\,\mathbf{i} + a\sin^3\omega t\,\mathbf{j},$$

and its velocity

$$\mathbf{v} = -3a\omega\sin\omega t\cos^2\omega t\,\mathbf{i} + 3a\omega\sin^2\omega t\cos\omega t\,\mathbf{j},$$

where $\{\mathbf{i}, \mathbf{j}\}$ is a reference basis. Hence for the particle speed we have

$$v = 3a\omega\sin\omega t\cos\omega t.$$

From the last two equations we get the tangent vector

$$\mathbf{T} = -\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j},$$

and by using the well known formula [1]

$$\left\|\frac{d\mathbf{T}}{dt}\right\| = \frac{v}{\rho},$$

$\rho > 0$ being the radius of curvature at any instant t, we arrive to the useful equation

$$v = \omega\rho.$$

[1]By applying the chain rule,

$$\left\|\frac{d\mathbf{T}}{dt}\right\| = \left\|\frac{d\mathbf{T}}{ds}\right\|\left|\frac{ds}{dt}\right| = \left\|\frac{\mathbf{N}}{\rho}\right\|v = \frac{v}{\rho},$$

by Frenet-Serret. $\mathbf{N}$ is the normal vector.