derivation of a definite integral formula using the method of exhaustion

Canonical name	DerivationOfADefiniteIntegralFormulaUsingTheMethodOfExhaustion
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The area under an arbitrary function $f(x)$ that is piecewise continuous on $[a, b]$ can be "exhausted" with triangles. The first triangle has vertices at $(a, 0)$ and $(b, 0)$, and intersects $f(x)$ at

$$x = a + \frac{b-a}{2},$$

yielding the estimate

$$A_1 = \frac{1}{2}(b-a)f(a + \frac{b-a}{2})$$

The second approximation involves two triangles, each sharing two vertices with the original triangle, and intersecting $f(x)$ at

$$x = a + \frac{b-a}{4}$$

and

$$x = a + \frac{3(b-a)}{4},$$

adding the area:

$$A_2 = \frac{1}{4}(b-a)\{f(a + \frac{b-a}{4}) - f(a + \frac{b-a}{2}) + f(a + \frac{3(b-a)}{4})\}$$

A third such approximation involves four more triangles, adding the area

$$\begin{gathered} A_3 = \tfrac{1}{8}(b-a)\{f(a + \tfrac{b-a}{8}) - f(a + \tfrac{b-a}{4}) \\ +f(a + \tfrac{3(b-a)}{8}) - f(a + \tfrac{b-a}{2}) + f(a + \tfrac{5(b-a)}{8}) \\ -f(a + \tfrac{3(b-a)}{4}) + f(a + \tfrac{7(b-a)}{8})\}. \end{gathered}$$

This procedure eventually leads to the formula

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} A_n = (b-a)\sum_{n=1}^{\infty}\sum_{m=1}^{2^n-1}(-1)^{m+1}2^{-n}f\left(a + m(b-a)/2^n\right)$$

References

1. `http://arxiv.org/abs/math.CA/0011078`http://arxiv.org/abs/math.CA/0011078.

2. Int. J. Math. Math. Sci. 31, 345-351, 2002.