quantization

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0.1 Introduction

Quantization is understood as the process of defining a *formal* correspondence between a quantum system operator (such as the quantum Hamiltonian operator) or quantum algebra and a classical system operator (such as the Hamiltonian) or a classical algebra, such as the Poisson algebra. Theoretical quantum physicists often proceed in two 'stages', so that both first and second quantization procedures were reported in QFT, for example. Generalized quantization procedures involve *asymptotic morphisms and Wigner–Weyl–Moyal quantization procedures* or noncommutative 'deformations' of `http://planetmath.org/CAlgebra3`C*-algebras associated with quantum operators on Hilbert spaces (as in noncommutative geometry). The *non-commutative* algebra of quantum observable operators is a `http://planetmath.org/CliffordAlgebra`Clifford algebra, and the associated `http://planetmath.org/CCliffordAlgebra`C^*-Clifford algebra is a fundamental concept of modern mathematical treatments of quantum theories. Note, however, that classical systems, including Einstein's general relativity are commutative (or Abelian) theories, whereas quantum theories are intrinsically non-commutative (or non-Abelian), most likely as a consequnece of the non-comutativity of quantum logics and the Heisenberg uncertainty principle of quantum mechanics.

This definition is quite broad, and as a result there are many approaches to quantization, employing a variety of techniques. It should be emphasized the result of quantization is not unique; in fact, methods of quantization usually possess inherent ambiguities, in the sense that, while performing quantization, one usually must make choices at certain points of the process.

Classical systems

Definition 1. A *classical system* is a triplet (M, ω, H), where (M, ω) (the *phase space*) is a symplectic manifold and H (the *Hamiltonian*) is a smooth function on M.

In most physical examples the phase space M is the cotangent bundle T^*X of a manifold X. In this case, X is called the *configuration space.*

Definition 2.

1. A *classical state* is a point x in M.
2. A *classical observable* is a function f on M.

In classical mechanics, one studies the time-evolution of a classical system. The time-evolution of an observable is described the equation

$$\frac{df}{dt} = -\{H, f\}, \tag{1}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. Equation (**??**) is equivalent to the Hamilton equations.

Remark. A classical system is sometimes defined more generally as a triplet (M, π, H), where π is a Poisson structure on M.

Quantum systems

Definition 3. A *quantum system* is a pair $(\mathcal{H}, \hat{H})$, where $\mathcal{H}$ is a Hilbert space and $\hat{H}$ is a self-adjoint linear operator on $\mathcal{H}$.

If $(\mathcal{H}, \hat{H})$ is a quantum system, $\mathcal{H}$ is referred to as the (quantum) phase space and $\hat{H}$ is referred to as the *Hamiltonian operator*.

Definition 4. 1. A *quantum state* is a vector Ψ in $\mathcal{H}$.

2. A *quantum observable* is a self-adjoint linear operator A on $\mathcal{H}$.

The space of quantum observables is denoted $\mathcal{O}(\mathcal{H})$. If A and B are in $\mathcal{O}(\mathcal{H})$, then

$$(i\hbar)^{-1}[A, B] := (i\hbar)^{-1}(AB - BA) \tag{2}$$

is in $\mathcal{O}(\mathcal{H})$ (Planck's constant $\hbar$ appears as a scaling factor arising from physical considerations). The operation $(i\hbar)^{-1}[\cdot, \cdot]$ thus gives $\mathcal{O}(\mathcal{H})$ the structure of a Lie algebra.

The time evolution of a quantum observable is described by the equation

$$\frac{dA}{dt} = \frac{i}{\hbar}[\hat{H}, A]. \tag{3}$$

Equation (**??**) is equivalent to the *time-dependent* Schrödinger's equation

$$i\hbar\frac{d\Psi}{dt} = \hat{H}\Psi. \tag{4}$$

The problem of quantization

The problem of quantization is to find a correspondence between a quantum system and a classical system; this is clearly not always possible. Thus, specific methods of quantization describe several ways of constructing a pair $(\mathcal{H}, \hat{H})$ from a triplet (M, ω, H). Furthermore, in order to give physical meaning to the observables in the quantum system, there should be a map

$$q\colon C^\infty(M) \to \mathcal{O}(\mathcal{H}), \tag{5}$$

satisfying the following conditions:

1. q is a Lie algebra homomorphism,
2. $q(H) = \hat{H}$.

Remark. Note that q is not an algebra homomorphism. Much of the complexity of quantization lies in the fact that, while $C^\infty(M)$ is a commutative algebra, its image in $\mathcal{O}(\mathcal{H})$ necessarily does not commute.

The following is a list of some well-known methods of quantization:

- Canonical quantization
- Geometric quantization
- Deformation quantization
- Path-integral quantization

A detailed example of geometric quantization on quantum Riemannian spaces can be found in ref. [**?**].

References

[1] Abhay Ashtekar and Jerzy Lewandowski. 2005. Quantum Geometry and Its Applications. `http://cgpg.gravity.psu.edu/people/Ashtekar/articles/qgfinal.pdf`Available PDF download.