# canonical quantization

| | |
|---|---|
| Canonical name | CanonicalQuantization |
| Date of creation | 2013-03-22 15:53:34 |
| Last modified on | 2013-03-22 15:53:34 |
| Owner | bci1 (20947) |
| Last modified by | bci1 (20947) |
| Numerical id | 18 |
| Author | bci1 (20947) |
| Entry type | Definition |
| Classification | msc 81S10 |
| Classification | msc 53D50 |
| Classification | msc 46L65 |
| Related topic | Quantization |
| Related topic | PoissonBracket |
| Related topic | HamiltonianOperatorOfAQuantumSystem |
| Related topic | SchrodingerOperator |
| Related topic | AsymptoticMorphismsAndWignerWeylMoyalQuantizationProcedures |
| Defines | operator substitution rule |
| Defines | operator ordering problem |

Canonical quantization is a method of relating, or associating, a classical system of the form $(T^*X, \omega, H)$, where $X$ is a manifold, $\omega$ is the canonical symplectic form on $T^*X$, with a (more complex) quantum system represented by $H \in C^\infty(X)$, where $H$ is the `http://planetmath.org/HamiltonianOperatorOfAQuantumSyste` operator. Some of the early formulations of quantum mechanics used such quantization methods under the umbrella of the *correspondence principle or postulate*. The latter states that a correspondence exists between certain classical and quantum operators, (such as the Hamiltonian operators) or algebras (such as Lie or Poisson (brackets)), with the classical ones being in the real ($\mathbb{R}$) domain, and the quantum ones being in the complex ($\mathbb{C}$) domain. Whereas all classical observables and states are specified only by real numbers, the 'wave' amplitudes in quantum theories are represented by complex functions.

Let $(x^i, p_i)$ be a set of Darboux coordinates on $T^*X$. Then we may obtain from each coordinate function an operator on the Hilbert space $\mathcal{H} = L^2(X, \mu)$, consisting of functions on $X$ that are square-integrable with respect to some measure $\mu$, by the *operator substitution* rule:

$$x^i \mapsto \hat{x}^i = x^i \cdot, \tag{1}$$

$$p_i \mapsto \hat{p}_i = -i\hbar \frac{\partial}{\partial x^i}, \tag{2}$$

where $x^i \cdot$ is the "multiplication by $x^i$" operator. Using this rule, we may obtain operators from a larger class of functions. For example,

1. $x^i x^j \mapsto \hat{x}^i \hat{x}^j = x^i x^j \cdot$,
2. $p_i p_j \mapsto \hat{p}_i \hat{p}_j = -\hbar^2 \frac{\partial^2}{\partial x^i x^j}$,
3. if $i \neq j$ then $x^i p_j \mapsto \hat{x}^i \hat{p}_j = -i\hbar x^i \frac{\partial}{\partial x^j}$.

*Remark.* The substitution rule creates an ambiguity for the function $x^i p_j$ when $i = j$, since $x^i p_j = p_j x^i$, whereas $\hat{x}^i \hat{p}_j \neq \hat{p}_j \hat{x}^i$. This is the *operator ordering* problem. One possible solution is to choose

$$x^i p_j \mapsto \frac{1}{2}\left(\hat{x}^i \hat{p}_j + \hat{p}_j \hat{x}^i\right),$$

since this choice produces an operator that is self-adjoint and therefore corresponds to a physical observable. More generally, there is a construction known as *Weyl quantization* that uses Fourier transforms to extend the substitution rules (**??**)-(**??**) to a map

$$\begin{aligned} C^\infty(T^*X) &\to \mathrm{Op}(\mathcal{H}) \\ f &\mapsto \hat{f}. \end{aligned}$$

*Remark.* This procedure is called "canonical" because it preserves the canonical Poisson brackets. In particular, we have that

$$\frac{-i}{\hbar}[\hat{x}^i, \hat{p}_j] := \frac{-i}{\hbar}\left(\hat{x}^i\hat{p}_j - \hat{p}_j\hat{x}^i\right) = \delta^i_j,$$

which agrees with the Poisson bracket $\{x^i, p_j\} = \delta^i_j$.

*Example* 1. Let $X = \mathbb{R}$. The Hamiltonian function for a one-dimensional point particle with mass $m$ is

$$H = \frac{p^2}{2m} + V(x),$$

where $V(x)$ is the potential energy. Then, by operator substitution, we obtain the Hamiltonian operator

$$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x).$$