Jensen's inequality

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If f is a convex function on the interval $[a,b]$, for each $\{x_k\}_{k=1}^n \in [a,b]$ and each $\{\mu_k\}_{k=1}^n$ with $\mu_k \geq 0$ one has:

$$f\left(\frac{\sum_{k=1}^n \mu_k x_k}{\sum_k^n \mu_k}\right) \leq \frac{\sum_{k=1}^n \mu_k f\left(x_k\right)}{\sum_k^n \mu_k}.$$

A common situation occurs when $\mu_1 + \mu_2 + \cdots + \mu_n = 1$; in this case, the inequality simplifies to:

$$f\left(\sum_{k=1}^n \mu_k x_k\right) \leq \sum_{k=1}^n \mu_k f(x_k)$$

where $0 \leq \mu_k \leq 1$.

If f is a concave function, the inequality is reversed.

Example:
$f(x) = x^2$ is a convex function on $[0, 10]$. Then

$$(0.2 \cdot 4 + 0.5 \cdot 3 + 0.3 \cdot 7)^2 \leq 0.2(4^2) + 0.5(3^2) + 0.3(7^2).$$

A very special case of this inequality is when $\mu_k = \frac{1}{n}$ because then

$$f\left(\frac{1}{n}\sum_{k=1}^n x_k\right) \leq \frac{1}{n}\sum_{k=1}^n f(x_k)$$

that is, the value of the function at the mean of the x_k is less or equal than the mean of the values of the function at each x_k.

There is another formulation of Jensen's inequality used in probability: Let X be some random variable, and let $f(x)$ be a convex function (defined at least on a segment containing the range of X). Then the expected value of $f(X)$ is at least the value of f at the mean of X:

$$\mathrm{E}[f(X)] \geq f(\mathrm{E}[X]).$$

With this approach, the weights of the first form can be seen as probabilities.