Cartan calculus

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Defines	Cartan's magic formula
Defines	supercommutation relation
Defines	graded derivation

Suppose M is a smooth manifold, and denote by $\Omega(M)$ the algebra of differential forms on M. Then, the *Cartan calculus* consists of the following three types of linear operators on $\Omega(M)$:

1. the exterior derivative d,
2. the space of Lie derivative operators $\mathcal{L}_X$, where X is a vector field on M, and
3. the space of contraction operators ι_X, where X is a vector field on M.

The above operators satisfy the following identities for any vector fields X and Y on M:

$$d^2 = 0, \tag{1}$$
$$d\mathcal{L}_X - \mathcal{L}_X d = 0, \tag{2}$$
$$d\iota_X + \iota_X d = \mathcal{L}_X, \tag{3}$$
$$\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]}, \tag{4}$$
$$\mathcal{L}_X\iota_Y - \iota_Y\mathcal{L}_X = \iota_{[X,Y]}, \tag{5}$$
$$\iota_X\iota_Y + \iota_Y\iota_X = 0, \tag{6}$$

where the brackets on the right hand side denote the Lie bracket of vector fields.

The identity (**??**) is known as *Cartan's magic formula* or *Cartan's identity*

Interpretation as a Lie Superalgebra

Since $\Omega(M)$ is a graded algebra, there is a natural grading on the space of linear operators on $\Omega(M)$. Under this grading, the exterior derivative d is degree 1, the Lie derivative operators $\mathcal{L}_X$ are degree 0, and the contraction operators ι_X are degree -1.

The identities (**??**)-(**??**) may each be written in the form

$$AB \pm BA = C, \tag{7}$$

where a plus sign is used if A and B are both of odd degree, and a minus sign is used otherwise. Equations of this form are called *supercommutation relations* and are usually written in the form

$$[A, B] = C, \tag{8}$$

where the bracket in (**??**) is a *Lie superbracket*. A Lie superbracket is a generalization of a Lie bracket.

Since the Cartan Calculus operators are closed under the Lie superbracket, the vector space spanned by the Cartan Calculus operators has the structure of a *Lie superalgebra*.

Graded derivations of $\Omega(M)$

Definition 1. A degree k linear operator A on $\Omega(M)$ is a *graded derivation* if it satisfies the following property for any p-form ω and any differential form η:

$$A(\omega \wedge \eta) = A(\omega) \wedge \eta + (-1)^{kp} \omega \wedge A(\eta). \tag{9}$$

All of the Calculus operators are graded derivations of $\Omega(M)$.