groupoid and group representations related to quantum symmetries

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1 Groupoid representations

Whereas http://planetmath.org/GroupRepresentationgroup representations of quantum unitary operators are extensively employed in standard quantum mechanics, the applications of http://planetmath.org/RepresentationsOfLocallyCompactGroupoidsgroupoid representations are still under development. For example, a description of stochastic quantum mechanics in curved spacetime (Drechsler and Tuckey, 1996) involving a Hilbert bundle is possible in terms of *groupoid representations* which can indeed be defined on such a Hilbert bundle $(X * \mathcal{H}, \pi)$, but cannot be expressed as the simpler group representations on a Hilbert space $\mathcal{H}$. On the other hand, as in the case of group representations, unitary groupoid representations induce associated C*-algebra representations. In the next subsection we recall some of the basic results concerning groupoid representations and their associated groupoid *-algebra representations. For further details and recent results in the mathematical theory of groupoid representations one has also available the succint monograph by Buneci (2003) and references cited therein (*www.utgjiu.ro/math/mbuneci/preprint.html*).

Let us consider first the relationships between these mainly algebraic concepts and their extended quantum symmetries, also including relevant computation examples; then let us consider several further extensions of symmetry and algebraic topology in the context of local quantum physics/algebraic quantum field theory, symmetry breaking, quantum chromodynamics and the development of novel supersymmetry theories of quantum gravity. In this respect one can also take spacetime 'inhomogeneity' as a criterion for the comparisons between physical, partial or local, symmetries: on the one hand, the example of paracrystals reveals thermodynamic disorder (entropy) within its own spacetime framework, whereas in spacetime itself, whatever the selected model, the inhomogeneity arises through (super) gravitational effects. More specifically, in the former case one has the technique of the generalized Fourier–Stieltjes transform (along with convolution and Haar measure), and in view of the latter, we may compare the resulting 'broken'/paracrystal–type symmetry with that of the supersymmetry predictions for weak gravitational fields (e.g., 'ghost' particles) along with the broken supersymmetry in the presence of intense gravitational fields. Another significant extension of quantum symmetries may result from the superoperator algebra/algebroids of Prigogine's quantum *superoperators* which are defined only for irreversible, infinite-dimensional systems (Prigogine, 1980).

1.1 Definition of extended quantum groupoid and algebroid symmetries

Quantum groups $\rightarrow$ Representations $\rightarrow$ Weak Hopf algebras $\rightarrow$ Quantum groupoids and algebroids

Our intention here is to view the latter scheme in terms of *weak Hopf C*–algebroid*– and/or other– extended symmetries, which we propose to do, for example, by incorporating the concepts of *rigged Hilbert spaces* and *sectional functions for a small category.* We note, however, that an alternative approach to quantum 'groupoids' has already been reported (Maltsiniotis, 1992), (perhaps also related to noncommutative geometry); this was later expressed in terms of deformation-quantization: the Hopf algebroid deformation of the universal enveloping algebras of Lie algebroids (Xu, 1997) as the classical limit of a quantum 'groupoid'; this also parallels the introduction of quantum 'groups' as the deformation-quantization of Lie bialgebras. Furthermore, such a Hopf algebroid approach (Lu, 1996) leads to categories of Hopf algebroid modules (Xu, 1997) which are monoidal, whereas the links between Hopf algebroids and monoidal bicategories were investigated by Day and Street (1997).

As defined under the following heading on groupoids, let (G_{lc}, τ) be a *locally compact groupoid* endowed with a (left) Haar system, and let $A = C^*(\mathsf{G}_{lc}, \tau)$ be the convolution C^*–algebra (we append A with $\mathbf{1}$ if necessary, so that A is unital). Then consider such a *groupoid representation*
$\Lambda : (\mathsf{G}_{lc}, \tau) \longrightarrow \{\mathcal{H}_x, \sigma_x\}_{x \in X}$ that respects a compatible measure σ_x on $\mathcal{H}_x$ (cf Buneci, 2003). On taking a state ρ on A, we assume a parametrization

$$(\mathcal{H}_x, \sigma_x) := (\mathcal{H}_\rho, \sigma)_{x \in X} \ . \tag{1.1}$$

Furthermore, each $\mathcal{H}_x$ is considered as a *rigged Hilbert space* Bohm and Gadella (1989), that is, one also has the following nested inclusions:

$$\Phi_x \subset (\mathcal{H}_x, \sigma_x) \subset \Phi_x^\times \ , \tag{1.2}$$

in the usual manner, where Φ_x is a dense subspace of $\mathcal{H}_x$ with the appropriate locally convex topology, and $\Phi_x^\times$ is the space of continuous antilinear functionals of Φ . For each $x \in X$, we require Φ_x to be invariant under Λ and Im $\Lambda | \Phi_x$ is a continuous representation of G_{lc} on Φ_x . With these conditions, representations of (proper) quantum groupoids that are derived for weak C*–Hopf algebras (or algebroids) modeled on rigged Hilbert spaces could be suitable generalizations in the framework of a Hamiltonian generated semigroup of time evolution of a quantum system via integration of Schrödinger's equation $\iota\hbar \frac{\partial \psi}{\partial t} = H\psi$ as studied in the case of Lie groups (Wickramasekara and Bohm, 2006). The adoption of the rigged Hilbert spaces is also based on how the latter are recognized as reconciling the Dirac and von Neumann approaches to quantum theories (Bohm and Gadella, 1989).

Next, let G be a *locally compact Hausdorff groupoid* and X a locally compact Hausdorff space. (G will be called a *locally compact groupoid, or lc- groupoid* for short). In order to achieve a small C*–category we follow a suggestion of A. Seda (private communication) by using a general principle in the context of Banach bundles (Seda, 1976, 982)). Let $q = (q_1, q_2) : \mathsf{G} \longrightarrow X \times X$ be a continuous, open and surjective map. For each $z = (x, y) \in X \times X$, consider the fibre $\mathsf{G}_z = \mathsf{G}(x, y) = q^{-1}(z)$, and set $\mathcal{A}_z = C_0(\mathsf{G}_z) = C_0(\mathsf{G}(x, y))$ equipped with a uniform norm $\| \ \|_z$. Then we set $\mathcal{A} = \bigcup_z \mathcal{A}_z$. We form a Banach bundle $p : \mathcal{A} \longrightarrow X \times X$ as follows. Firstly, the projection is defined via the typical fibre $p^{-1}(z) = \mathcal{A}_z = \mathcal{A}_{(x,y)}$. Let $C_c(\mathsf{G})$ denote the continuous complex valued functions on G with compact support. We obtain a sectional function $\widetilde{\psi} : X \times X \longrightarrow \mathcal{A}$ defined via restriction as $\widetilde{\psi}(z) = \psi|\mathsf{G}_z = \psi|\mathsf{G}(x,y)$. Commencing from the vector space $\gamma = \{\widetilde{\psi} : \psi \in C_c(\mathsf{G})\}$, the set $\{\widetilde{\psi}(z) : \widetilde{\psi} \in \gamma\}$ is dense in $\mathcal{A}_z$. For each $\widetilde{\psi} \in \gamma$, the function $\|\widetilde{\psi}(z)\|_z$ is continuous on X, and each $\widetilde{\psi}$ is a continuous section of $p : \mathcal{A} \longrightarrow X \times X$. These facts follow from Seda (1982, Theorem 1). Furthermore, under the convolution product $f * g$, *the space* $C_c(\mathsf{G})$ *forms an associative algebra over* $\mathbb{C}$ (cf. Seda, 1982, Theorem 3).

1.2 Groupoids

Recall that a groupoid G is, loosely speaking, a small category with inverses over its set of objects $X = Ob(\mathsf{G})$. One often writes G_x^y for the set of morphisms in G from x to y . *A topological groupoid* consists of a space G, a distinguished subspace $\mathsf{G}^{(0)} = \mathrm{Ob}(\mathsf{G}) \subset \mathsf{G}$, called *the space of objects* of G, together with maps

$$r, s \ : \ \mathsf{G} \overset{r}{\underset{s}{\rightrightarrows}} \mathsf{G}^{(0)} \tag{1.3}$$

called the *range* and *source maps* respectively, together with a law of composition

$$\circ \ : \ \mathsf{G}^{(2)} := \mathsf{G} \times_{\mathsf{G}^{(0)}} \mathsf{G} = \{ \ (\gamma_1, \gamma_2) \in \mathsf{G} \times \mathsf{G} \ : \ s(\gamma_1) = r(\gamma_2) \ \} \longrightarrow \mathsf{G} \ , \tag{1.4}$$

such that the following hold :

(1) $s(\gamma_1 \circ \gamma_2) = r(\gamma_2)$, $r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$, for all $(\gamma_1, \gamma_2) \in \mathsf{G}^{(2)}$.

(2) $s(x) = r(x) = x$, for all $x \in \mathsf{G}^{(0)}$.

(3) $\gamma \circ s(\gamma) = \gamma$, $r(\gamma) \circ \gamma = \gamma$, for all $\gamma \in \mathsf{G}$.

(4) $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$.

(5) Each γ has a two–sided inverse γ^{-1} with $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$.

Furthermore, only for topological groupoids the inverse map needs be continuous. It is usual to call $\mathsf{G}^{(0)} = Ob(\mathsf{G})$ *the set of objects* of G . For $u \in Ob(\mathsf{G})$, the set of arrows $u \longrightarrow u$ forms a group G_u, called the *isotropy group of* G *at* u.

Thus, as is well kown, a topological groupoid is just a groupoid internal to the category of topological spaces and continuous maps. The notion of internal groupoid has proved significant in a number of fields, since groupoids generalise bundles of groups, group actions, and equivalence relations. For a further study of groupoids we refer the reader to Brown (2006).

Several examples of groupoids are:

- (a) locally compact groups, transformation groups, and any group in general (e.g. [59]

- (b) equivalence relations

- (c) tangent bundles
- (d) the tangent groupoid (e.g. [4])
- (e) holonomy groupoids for foliations (e.g. [4])
- (f) Poisson groupoids (e.g. [81])
- (g) graph groupoids (e.g. [47, 64]).

As a simple example of a groupoid, consider (b) above. Thus, let R be an *equivalence relation* on a set X. Then R is a groupoid under the following operations: $(x,y)(y,z) = (x,z), (x,y)^{-1} = (y,x)$. Here, $\mathsf{G}^0 = X$, (the diagonal of $X \times X$) and $r((x,y)) = x, s((x,y)) = y$.

So $R^2 = \{((x,y),(y,z)) : (x,y),(y,z) \in R\}$. When $R = X \times X$, R is called a *trivial* groupoid. A special case of a trivial groupoid is $R = R_n = \{1,2,...,n\} \times \{1,2,...,n\}$. (So every i is equivalent to every j). Identify $(i,j) \in R_n$ with the matrix unit e_{ij}. Then the groupoid R_n is just matrix multiplication except that we only multiply e_{ij}, e_{kl} when $k = j$, and $(e_{ij})^{-1} = e_{ji}$. We do not really lose anything by restricting the multiplication, since the pairs e_{ij}, e_{kl} excluded from groupoid multiplication just give the 0 product in normal algebra anyway.

Definition 1.1. For a groupoid G_{lc} to be a *locally compact groupoid* means that G_{lc} is required to be *a (second countable) locally compact Hausdorff space*, and the product and also inversion maps are required to be continuous. Each G_{lc}^u as well as the unit space G_{lc}^0 is closed in G_{lc}.

Remark 1.1. What replaces the left Haar measure on G_{lc} is a system of measures λ^u ($u \in \mathsf{G}_{lc}^0$), where λ^u is a positive regular Borel measure on G_{lc}^u with dense support. In addition, the λ^u 's are required to vary continuously (when integrated against $f \in C_c(\mathsf{G}_{lc})$) and to form an invariant family in the sense that for each x, the map $y \mapsto xy$ is a measure preserving homeomorphism from $\mathsf{G}_{lc}^s(x)$ onto $\mathsf{G}_{lc}^r(x)$. Such a system $\{\lambda^u\}$ is called a *left Haar system* for the locally compact groupoid G_{lc}.

This is defined more precisely next.

1.3 Haar systems for locally compact topological groupoids

Let

$$\mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} = X \tag{1.5}$$

be a locally compact, locally trivial topological groupoid with its transposition into transitive (connected) components. Recall that for $x \in X$, the *costar of* x denoted $\mathrm{CO}^*(\mathrm{x})$ is defined as the closed set $\bigcup\{\mathsf{G}(y,x) : y \in \mathsf{G}\}$, whereby

$$\mathsf{G}(x_0, y_0) \hookrightarrow \mathrm{CO}^*(\mathrm{x}) \longrightarrow \mathrm{X}\ , \tag{1.6}$$

is a principal $\mathsf{G}(x_0, y_0)$–bundle relative to fixed base points (x_0, y_0) . Assuming all relevant sets are locally compact, then following Seda (1976), a *(left) Haar system on* G denoted (G, τ) (for later purposes), is defined to comprise of i) a measure κ on G, ii) a measure μ on X and iii) a measure μ_x on CO*(x) such that for every Baire set E of G, the following hold on setting $E_x = E \cap \mathrm{CO}^*(\mathrm{x})$:

(1) $x \mapsto \mu_x(E_x)$ is measurable.

(2) $\kappa(E) = \int_x \mu_x(E_x)\ d\mu_x$.

(3) $\mu_z(tE_x) = \mu_x(E_x)$, for all $t \in \mathsf{G}(x, z)$ and $x, z \in \mathsf{G}$.

The presence of a left Haar system on G_{lc} has important topological implications: it requires that the range map $r : \mathsf{G}_{lc} \to \mathsf{G}^0_{lc}$ is open. For such a G_{lc} with a left Haar system, the vector space $C_c(\mathsf{G}_{lc})$ is a *convolution *–algebra*, where for $f, g \in C_c(\mathsf{G}_{lc})$:

$f * g(x) = \int f(t) g(t^{-1}x) d\lambda^{r(x)}(t)$, with $f^*(x) = \overline{f(x^{-1})}$.

One has $C^*(\mathsf{G}_{lc})$ to be the *enveloping C^*–algebra* of $C_c(\mathsf{G}_{lc})$ (and also representations are required to be continuous in the inductive limit topology). Equivalently, it is the completion of $\pi_{univ}(C_c(\mathsf{G}_{lc}))$ where π_{univ} is the *universal representation* of G_{lc}. For example, if $\mathsf{G}_{lc} = R_n$, then $C^*(\mathsf{G}_{lc})$ is just the finite dimensional algebra $C_c(\mathsf{G}_{lc}) = M_n$, the span of the e'_{ij}s.

There exists (cf. [?]) a *measurable Hilbert bundle* $(\mathsf{G}^0_{lc}, \mathcal{H}, \mu)$ with $\mathcal{H} = \left\{ \mathcal{H}^u_{u \in \mathsf{G}^0_{lc}} \right\}$ and a G-representation L on $\mathcal{H}$. Then, for every pair ξ, η of square integrable sections of $\mathcal{H}$, it is required that the function $x \mapsto (L(x)\xi(s(x)), \eta(r(x)))$ be ν–measurable. The representation Φ of $C_c(\mathsf{G}_{lc})$ is then given by:
$\langle \Phi(f)\xi|, \eta \rangle = \int f(x)(L(x)\xi(s(x)), \eta(r(x))) d\nu_0(x)$.

The triple $(\mu, \mathcal{H}, L)$ is called a *measurable G_{lc}–Hilbert bundle.*

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