superfields, superspace and supergravity

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0.1 Superspace, superfields, supergravity and Lie superalgebras.

In general, a *superfield*–or *quantized gravity field*- has a highly reducible representation of the supersymmetry algebra, and the problem of specifying a supergravity theory can be defined as a search for those representations that allow the construction of consistent local actions, perhaps considered as either quantum group, or quantum groupoid, actions. Extending quantum symmetries to include quantized gravity fields–specified as 'superfields'– is called *supersymmetry* in current theories of Quantum Gravity. Graded 'Lie' algebras (or Lie superalgebras) represent the quantum operator supersymmetries by defining these simultaneously for both *fermion* (spin 1/2) and *boson* (integer or 0 spin particles).

The quantized physical space with supersymmetric properties is then called a *'superspace'*, (another name for *'quantized space with supersymmetry'*) in Quantum Gravity. The following subsection defines these physical concepts in precise mathematical terms.

0.1.1 Mathematical definitions and propagation equations for superfields in superspace: Graded Lie algebras

Supergravity, in essence, is an extended supersymmetric theory of both matter and gravitation (*viz.* Weinberg, 1995 [?]). A first approach to supersymmetry relied on a curved 'superspace' (Wess and Bagger,1983 [?]) and is analogous to supersymmetric gauge theories (see, for example, Sections 27.1 to 27.3 of Weinberg, 1995). Unfortunately, a complete non–linear supergravity theory might be forbiddingly complicated and furthermore, the constraints that need be made on the graviton superfield appear somewhat subjective, (according to Weinberg, 1995). In a different approach to supergravity, one considers the physical components of the gravitational superfield which can be then identified based on 'flat-space' superfield methods (Chs. 26 and 27 of Weinberg, 1995). By implementing the *gravitational weak-field approximation* one obtains several of the most important consequences of supergravity theory, including masses for the hypothetical 'gravitino' and 'gaugino particles' whose existence might be expected from supergravity theories. Furthermore, by adding on the higher order terms in the gravitational constant to the supersymmetric transformation, the general coordinate transformations form a *closed algebra* and the Lagrangian that describes the interactions of the physical fields is then *invariant* under such transformations.The first quantization of such a flat-space superfield would obviously involve its 'deformation', and as a result its corresponding *supersymmetry algebra* becomes *non–commutative.*

0.1.2 Metric superfield

Because in supergravity both spinor and tensor fields are being considered, the gravitational fields are represented in terms of *tetrads*, $e^a_\mu(x)$, rather than in terms of Einstein's general relativistic metric $g_{\mu\nu}(x)$. The connections between these two distinct representations are as follows:

$$g_{\mu\nu}(x) = \eta_{ab}\ e^a_\mu(x) e^b_\gamma(x)\ , \tag{0.1}$$

with the general coordinates being indexed by μ, ν, etc., whereas local coordinates that are being defined in a locally inertial coordinate system are labeled with superscripts a, b, etc.; η_{ab} is the diagonal matrix with elements +1, +1, +1 and -1. The tetrads are invariant to two distinct types of symmetry transformations–the local Lorentz transformations:

$$e^a_\mu(x) \longmapsto \Lambda^a_b(x) e^b_\mu(x)\ , \tag{0.2}$$

(where Λ^a_b is an arbitrary real matrix), and the general coordinate transformations:

$$x^\mu \longmapsto (x')^\mu(x)\ . \tag{0.3}$$

In a weak gravitational field the tetrad may be represented as:

$$e^a_\mu(x) = \delta^a_\mu(x) + 2\kappa \Phi^a_\mu(x)\ , \tag{0.4}$$

where $\Phi^a_\mu(x)$ is small compared with $\delta^a_\mu(x)$ for all x values, and $\kappa = \sqrt{8\pi G}$, where G is Newton's gravitational constant. As it will be discussed next, the supersymmetry algebra (SA) implies that the graviton has a fermionic superpartner, the hypothetical *'gravitino'*, with helicities $\pm$ 3/2. Such a self-charge-conjugate massless particle as the 'gravitiono' with helicities $\pm$ 3/2 can only have *low-energy* interactions if it is represented by a Majorana field $\psi_\mu(x)$ which is invariant under the gauge transformations:

$$\psi_\mu(x) \longmapsto \psi_\mu(x) + \delta_\mu \psi(x)\ , \tag{0.5}$$

with $\psi(x)$ being an arbitrary Majorana field as defined by Grisaru and Pendleton (1977). The tetrad field $\Phi_{\mu\nu}(x)$ and the graviton field $\psi_\mu(x)$ are then incorporated into a term $H_\mu(x,\theta)$ defined as the *metric superfield*. The relationships between $\Phi_{\mu_\nu}(x)$ and $\psi_\mu(x)$, on the one hand, and the components of the metric superfield $H_\mu(x,\theta)$, on the other hand, can be derived from the transformations of the whole metric superfield:

$$H_\mu(x,\theta) \longmapsto H_\mu(x,\theta) + \Delta_\mu(x,\theta)\ , \tag{0.6}$$

by making the simplifying– and physically realistic– assumption of a weak gravitational field (further details can be found, for example, in Ch.31 of vol.3. of Weinberg, 1995). The interactions of the entire superfield $H_\mu(x)$ with matter would be then described by considering how a weak gravitational field, h_{μ_ν} interacts with an energy-momentum tensor $T^{\mu\nu}$ represented as a linear combination of components of a real vector superfield Θ^μ. Such interaction terms would, therefore, have the form:

$$I_{\mathcal{M}} = 2\kappa \int dx^4 [H_\mu \Theta^\mu]_D\ , \tag{0.7}$$

($\mathcal{M}$ denotes 'matter') integrated over a four-dimensional (Minkowski) spacetime with the metric defined by the superfield $H_\mu(x, \theta)$. The term Θ^μ, as defined above, is physically a *supercurrent* and satisfies the conservation conditions:

$$\gamma^\mu \mathbf{D}\Theta_\mu = \mathbf{D} \ , \tag{0.8}$$

where $\mathbf{D}$ is the four-component super-derivative and X denotes a real chiral scalar superfield. This leads immediately to the calculation of the interactions of matter with a weak gravitational field as:

$$I_{\mathcal{M}} = \kappa \int d^4x T^{\mu\nu}(x) h_{\mu\nu}(x) \ , \tag{0.9}$$

It is interesting to note that the gravitational actions for the superfield that are invariant under the generalized gauge transformations $H_\mu \longmapsto H_\mu + \Delta_\mu$ lead to solutions of the Einstein field equations for a homogeneous, non-zero vacuum energy density ρ_V that correspond to either a de Sitter space for $\rho_V > 0$, or an anti-de Sitter space for $\rho_V < 0$. Such spaces can be represented in terms of the hypersurface equation

$$x_5^2 \pm \eta_{\mu,\nu} x^\mu x^\nu = R^2 \ , \tag{0.10}$$

in a *quasi-Euclidean five-dimensional space* with the metric specified as:

$$ds^2 = \eta_{\mu,\nu} x^\mu x^\nu \pm dx_5^2 \ , \tag{0.11}$$

with '+' for de Sitter space and '−' for anti-de Sitter space, respectively.

The spacetime symmetry groups, or extended symmetry groupoids, as the case may be– are different from the 'classical' Poincaré symmetry group of translations and Lorentz transformations. Such spacetime symmetry groups, in the simplest case, are therefore the $\mathrm{O}(4,1)$ group for the *de Sitter space* and the $\mathrm{O}(3,2)$ group for the *anti–de Sitter space*. A detailed calculation indicates that the transition from ordinary flat space to a bubble of anti-de Sitter space is *not* favored energetically and, therefore, the ordinary (de Sitter) flat space is stable (viz. Coleman and De Luccia, 1980), even though quantum fluctuations might occur to an anti–de Sitter bubble within the limits permitted by the Heisenberg uncertainty principle.

0.2 Supersymmetry algebras and Lie (graded) superalgebras.

It is well known that *continuous symmetry transformations* can be represented in terms of a *Lie algebra* of linearly independent *symmetry generators* t_j that satisfy the commutation relations:

$$[t_j, t_k] = \iota \Sigma_l C_{jk} t_l \ , \tag{0.12}$$

Supersymmetry is similarly expressed in terms of the symmetry generators t_j of a *graded ('Lie') algebra* which is in fact defined as a *superalgebra*) by satisfying relations of the general form:

$$t_j t_k - (-1)^{\eta_j \eta_k} t_k t_j = \iota \Sigma_l C^l_{jk} t_l \ . \tag{0.13}$$

The generators for which $\eta_j = 1$ are fermionic whereas those for which $\eta_j = 0$ are bosonic. The coefficients C^l_{jk} are structure constants satisfying the following conditions:

$$C^l_{jk} = -(-1)^{\eta_j \eta_k} C^l_{jk} \ . \tag{0.14}$$

If the generators $_j$ are quantum Hermitian operators, then the structure constants satisfy the reality conditions $C^*_{jk} = -C_{jk}$. Clearly, such a graded algebraic structure is a superalgebra and not a proper Lie algebra; thus graded Lie algebras are often called *'Lie superalgebras'*.

The standard computational approach in QM utilizes the S-matrix approach, and therefore, one needs to consider the general, *graded* 'Lie algebra' of *supersymmetry generators* that commute with the S-matrix. If one denotes the fermionic generators by Q, then $U^{-1}(\Lambda) Q U(\Lambda)$ will also be of the same type when $U(\Lambda)$ is the quantum operator corresponding to arbitrary, homogeneous Lorentz transformations Λ^{μ_ν} . Such a group of generators provide therefore a representation of the homogeneous Lorentz group of transformations $\mathbb{L}$. The irreducible representation of the homogeneous Lorentz group of transformations provides therefore a classification of such individual generators.

0.2.1 Graded 'Lie Algebras'/Superalgebras.

A set of quantum operators Q^{AB}_{jk} form an $\mathbf{A}, \mathbf{B}$ representation of the group $\mathbf{L}$ defined above which satisfy the commutation relations:

$$[\mathbf{A}, Q^{AB}_{jk}] = -[\Sigma'_j J^A_{jj'}, Q^{AB}_{j'k}] \ , \tag{0.15}$$

and

$$[\mathbf{B}, Q^{AB}_{jk}] = -[\Sigma_{j'} J^A_{kk'}, Q^{AB}_{jk'}] \ , \tag{0.16}$$

with the generators $\mathbf{A}$ and $\mathbf{B}$ defined by $\mathbf{A} \equiv (1/2)(\mathbf{J} \pm i\mathbf{K})$ and $\mathbf{B} \equiv (1/2)(\mathbf{J} - i\mathbf{K})$, with $\mathbf{J}$ and $\mathbf{K}$ being the Hermitian generators of rotations and 'boosts', respectively.

In the case of the two-component Weyl-spinors Q_{jr} the Haag–Lopuszanski–Sohnius (HLS) theorem applies, and thus the fermions form a *supersymmetry algebra* defined by the anti-commutation relations:

$$\begin{aligned} [Q_{jr}, Q^*_{ks}] &= 2\delta_{rs} \sigma^\mu_{jk} P_\mu \ , \\ [Q_{jr}, Q_{ks}] &= e_{jk} Z_{rs} \ , \end{aligned} \tag{0.17}$$

where P_μ is the 4–momentum operator, $Z_{rs} = -Z_{sr}$ are the bosonic symmetry generators, and σ_μ and $\mathbf{e}$ are the usual 2×2 Pauli matrices. Furthermore, the fermionic generators commute with both energy and momentum operators:

$$[P_\mu, Q_{jr}] = [P_\mu, Q^*_{jr}] = 0 \ . \tag{0.18}$$

The bosonic symmetry generators Z_{ks} and Z^*_{ks} represent the set of *central charges* of the supersymmetric algebra:

$$[Z_{rs}, Z^*_{tn}] = [Z^*_{rs}, Q_{jt}] = [Z^*_{rs}, Q^*_{jt}] = [Z^*_{rs}, Z^*_{tn}] = 0 \ . \tag{0.19}$$

From another direction, the Poincaré symmetry mechanism of special relativity can be extended to new algebraic systems (Tanasă, 2006). In Moultaka et al. (2005) in view of such extensions, consider invariant-free Lagrangians and bosonic multiplets constituting a symmetry that interplays with (Abelian) U(1)–gauge symmetry that may possibly be described in categorical terms, in particular, within the notion of a *cubical site* (Grandis and Mauri, 2003).

We shall proceed to introduce in the next section generalizations of the concepts of Lie algebras and graded Lie algebras to the corresponding Lie *algebroids* that may also be regarded as C*–convolution representations of *quantum gravity groupoids* and superfield (or supergravity) supersymmetries. This is therefore a novel approach to the proper representation of the *non-commutative geometry of quantum spacetimes*–that are *curved* (or 'deformed') by the presence of *intense* gravitational fields–in the framework of *non-Abelian, graded Lie algebroids*. Their correspondingly *deformed quantum gravity groupoids* (QGG) should, therefore, adequately represent supersymmetries modified by the presence of such intense gravitational fields on the Planck scale. Quantum fluctuations that give rise to quantum 'foams' at the Planck scale may be then represented by *quantum homomorphisms* of such QGGs. If the corresponding graded Lie algebroids are also *integrable*, then one can reasonably expect to recover in the limit of $\hbar \to 0$ the Riemannian geometry of General Relativity and the *globally hyperbolic spacetime* of Einstein's classical gravitation theory (GR), as a result of such an integration to the *quantum gravity fundamental groupoid* (QGFG). The following subsection will define the precise mathematical concepts underlying our novel quantum supergravity and extended supersymmetry notions.

References

[1] S. Weinberg.: *The Quantum Theory of Fields.* Cambridge, New York and Madrid: Cambridge University Press, Vols. 1 to 3, (1995–2000).

[2] A. Weinstein : Groupoids: unifying internal and external symmetry, *Notices of the Amer. Math. Soc.* **43** (7): 744-752 (1996).

[3] J. Wess and J. Bagger: *Supersymmetry and Supergravity*, Princeton University Press, (1983).