Hamiltonian algebroids

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0.1 Introduction

Hamiltonian algebroids are generalizations of the Lie algebras of canonical transformations, but cannot be considered just a special case of Lie algebroids. They are instead a special case of a `http://planetphysics.org/encyclopedia/QuantumAlgebroid.html`quantum algebroid.

Definition 0.1. Let X and Y be two vector fields on a smooth manifold M, represented here as operators acting on functions. Their commutator, or Lie bracket, L, is :

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Moreover, consider the classical configuration space $Q = \mathbb{R}^3$ of a classical, mechanical system, or particle whose phase space is the cotangent bundle $T^*\mathbb{R}^3 \cong \mathbb{R}^6$, for which the space of (classical) observables is taken to be the real vector space of smooth functions on M, and with T being an element of a `http://planetmath.org/JordanBanachAndJordanLieAlgebras`Jordan-Lie (Poisson) algebra whose definition is also recalled next. Thus, one defines as in classical dynamics the *Poisson algebra* as a Jordan algebra in which $\circ$ is associative. We recall that one needs to consider first a specific algebra (defined as a vector space E over a ground field (typically $\mathbb{R}$ or $\mathbb{C}$)) equipped with a bilinear and distributive multiplication $\circ$. Then one defines a *Jordan algebra* (over $\mathbb{R}$), as a a specific algebra over $\mathbb{R}$ for which:

$$S \circ T = T \circ S \ ,$$
$$S \circ (T \circ S^2) = (S \circ T) \circ S^2,'$$

for all elements S, T of this algebra.

Then, the usual algebraic types of morphisms automorphism, isomorphism, etc.) apply to a `http://planetmath.org/JordanBanachAndJordanLieAlgebras`Jordan-Lie (Poisson) algebra defined as a real vector space $U_{\mathbb{R}}$ together with a *Jordan product* $\circ$ and *Poisson bracket*

$\{\ ,\ \}$, satisfying :

1. for all $S, T \in U_{\mathbb{R}}$,
   $S \circ T = T \circ S$
   $\{S, T\} = -\{T, S\}$
2. the *Leibniz rule* holds
   $$\{S, T \circ W\} = \{S, T\} \circ W + T \circ \{S, W\}$$
   for all $S, T, W \in U_{\mathbb{R}}$, along with
3. the *Jacobi identity* :
   $$\{S, \{T, W\}\} = \{\{S, T\}, W\} + \{T, \{S, W\}\}$$
4. for some $\hbar^2 \in \mathbb{R}$, there is the *associator identity* :
   $$(S \circ T) \circ W - S \circ (T \circ W) = \frac{1}{4}\hbar^2\{\{S, W\}, T\} \ .$$

Thus, the canonical transformations of the Poisson sigma model phase space specified by the http://planetmath.org/JordanBanachAndJordanLieAlgebrasJordan-Lie (Poisson) algebra (also Poisson algebra), which is determined by both the Poisson bracket and the *Jordan product* $\circ$, define a *Hamiltonian algebroid* with the Lie brackets L related to such a Poisson structure on the target space.