Lie algebroids

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0.1 Topic on Lie algebroids

This is a topic entry on Lie algebroids that focuses on their quantum applications and extensions of current algebraic theories.

Lie algebroids generalize *Lie algebras*, and in certain quantum systems they represent *extended quantum (algebroid) symmetries*. One can think of a *Lie algebroid* as generalizing the idea of a tangent bundle where the tangent space at a point is effectively the equivalence class of curves meeting at that point (thus suggesting a groupoid approach), as well as serving as a site on which to study infinitesimal geometry (see, for example, ref. [**?**]). The formal definition of a *Lie algebroid* is presented next.

Definition 0.1 Let M be a manifold and let $\mathfrak{X}(M)$ denote the set of vector fields on M. Then, a *Lie algebroid* over M consists of a *vector bundle* $E \longrightarrow M$*, equipped with a Lie bracket* $[\ ,\ ]$ *on the space of sections* $\gamma(E)$*, and a bundle map* $\Upsilon : E \longrightarrow TM$, usually called the *anchor*. Furthermore, there is an induced map $\Upsilon : \gamma(E) \longrightarrow \mathfrak{X}(M)$, which is required to be a map of Lie algebras, such that given sections $\alpha, \beta \in \gamma(E)$ and a differentiable function f, the following Leibniz rule is satisfied :

$$[\alpha, f\beta] = f[\alpha, \beta] + (\Upsilon(\alpha))\beta \ . \tag{0.1}$$

Example 0.1. A typical example of a Lie algebroid is obtained when M is a Poisson manifold and $E = T^*M$, that is E is the cotangent bundle of M.

Now suppose we have a Lie groupoid G:

$$r, s \ : \quad \mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} \ = M \ . \tag{0.2}$$

There is an associated Lie algebroid $\mathcal{A} = \mathcal{A}(\mathsf{G})$, which in the guise of a vector bundle, it is the restriction to M of the bundle of tangent vectors along the fibers of s (ie. the s–vertical vector fields). Also, the space of sections $\gamma(\mathcal{A})$ can be identified with the space of s–vertical, right–invariant vector fields $\mathfrak{X}^s_{inv}(\mathsf{G})$ which can be seen to be closed under $[\ ,\ ]$, and the latter induces a bracket operation on $\gamma(A)$ thus turning $\mathcal{A}$ into a Lie algebroid. Subsequently, a Lie algebroid $\mathcal{A}$ is integrable if there exists a Lie groupoid G inducing $\mathcal{A}$.

Remark 0.1. Unlike Lie algebras that can be integrated to corresponding Lie groups, not all *Lie algebroids* are 'smoothly integrable' to Lie groupoids; the subset of Lie groupoids that have corresponding Lie algebroids are sometimes called *'Weinstein groupoids'*.

Note also the relation of the Lie algebroids to Hamiltonian algebroids, also concerning recent developments in (relativistic) quantum gravity theories.

References

[1] K. C. H. Mackenzie: *General Theory of Lie Groupoids and Lie Algebroids*, London Math. Soc. Lecture Notes Series, **213**, Cambridge University Press: Cambridge,UK (2005).