Lie superalgebra

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Synonym	Lie super algebra
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Related topic	CartanCalculus
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Related topic	SuperfieldsSuperspace
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Related topic	LieAlgebroids
Related topic	JordanBanachAndJordanLieAlgebras
Related topic	LieAlgebra
Related topic	LieAlgebraCohomology
Related topic	SuperAlgebra
Related topic	CartanCalculus
Related topic	QuantumGravityTheories
Related topic	Fu
Defines	vector superspace
Defines	Lie superbracket
Defines	supercommutator bracket

Definition 1. A *Lie superalgebra* is a vector superspace equipped with a bilinear map

$$\begin{aligned} [\cdot,\cdot] : V \otimes V &\to V, \\ v \otimes w &\mapsto [v, w], \end{aligned} \tag{1}$$

satisfying the following properties:

1. If v and w are homogeneous vectors, then $[v, w]$ is a homogeneous vector of degree $|v| + |w| \pmod 2$,
2. For any homogeneous vectors v, w, $[v, w] = (-1)^{|v||w|+1}[w, v]$,
3. For any homogeneous vectors u, v, w, $(-1)^{|u||w|}[u, [v, w]]+(-1)^{|v||u|}[v, [w, u]]+(-1)^{|w||v|}[w, [u, v]] = 0$.

The map $[\cdot,\cdot]$ is called a *Lie superbracket*.

Example 1. A Lie algebra V can be considered as a Lie superalgebra by setting $V = V_0$ and, therefore, $V_1 = \{0\}$.

Example 2. Any associative superalgebra A has a Lie superalgebra structure where, for any homogeneous elements $a, b \in A$, the Lie superbracket is defined by the equation

$$[a, b] = ab - (-1)^{|a||b|}ba. \tag{2}$$

The Lie superbracket (**??**) is called the *supercommutator bracket* on A.

Example 3. The space of graded derivations of a supercommutative superalgebra, equipped with the supercommutator bracket, is a Lie superalgebra.

Definition 2. A *vector superspace* is a vector space V equipped with a decomposition $V = V_0 \oplus V_1$.

Let $V = V_0 \oplus V_1$ be a vector superspace. Then any element of V_0 is said to be *even*, and any element of V_1 is said to be *odd*. By the definition of the direct sum, any element v of V can be uniquely written as $v = v_0 + v_1$, where $v_0 \in V_0$ and $v_1 \in V_1$.

Definition 3. A vector $v \in V$ is *homogeneous* of degree i if $v \in V_i$ for $i = 0$ or 1.

If $v \in V$ is homogeneous, then the degree of v is denoted by $|v|$. In other words, if $v \in V_i$, then $|v| = i$ by definition.

Remark. The vector 0 is homogeneous of both degree 0 and 1, and thus $|0|$ is not well-defined.